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两参数非线性反应扩散积分微分方程的内层激波渐近解*

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提要 研究了一类两参数非线性反应扩散积分微分奇摄动问题. 利用奇摄动方法, 构造了问题的外部解、内部激波层、边界层及初始层校正项, 由此得到了问题解的形式渐近展开式. 最后利用积分微分方程的比较定理证明了该问题解的渐近展开式的一致有效性.

关键词 反应扩散, 奇异摄动, 初边值问题
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1 引 言

非线性奇异摄动问题在国内外学术界十分重视 [1−2]. 近年来奇异摄动渐近方法有很大改进, 包括边界层校正法、匹配法、平均法和多重尺度法等等, 许多学者作了很多的工作 [1−8], 利用比较定理和其它方法, 作者等也研究了一些奇异摄动问题 [9−23]. 本文讨论一类具有两参数的非线性积分微分反应扩散系统的奇异摄动初边值问题.

考虑如下形式两参数非线性反应扩散系统初边值问题:

$$\mu\frac{\partial u_i}{\partial t}-\varepsilon^2 Lu_i+Tu_i=f_i(t,x,u_i),\quad 0<t\leqslant T_0,\ x\in\Omega,\ i=1,2,\cdots,m,\tag{1.1}$$

$$u_i=g_i(t,x),\quad x\in\partial\Omega,\ i=1,2,\cdots,m,\tag{1.2}$$

$$u_i(0,x)=h_i(x),\quad i=1,2,\cdots,m,\tag{1.3}$$

其中

$$L=\sum_{j,k=1}^{n}\alpha_{jk}(x)\frac{\partial^2}{\partial x_j\partial x_k},\quad \sum_{j,k=1}^{n}\alpha_{jk}(x)\xi_j\xi_k\geqslant\lambda\sum_{j=1}^{n}\xi_j^2,\ \lambda>0,$$

$$Tu_i=\int_\Omega K(t,x,y)u_i\mathrm{d}y,$$

且 ε 和 μ 为正的小参数, $x=(x_1,x_2\cdots,x_n)\in\Omega$, Ω 为 $\mathbb{R}^n$ 中的有界区域, T_0 为正常数, $\frac{\partial}{\partial n}$ 表示在边界 $\partial\Omega$ 上的外法向导数. 假设:

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$[\mathrm{H}_1]$ 线性算子 $\mathbf{L}$ 的系数及 $\mathbf{f}_i$ (除 $\mathbf{f}_i(\mathbf{t},\mathbf{x}_0,\mathbf{u}_i(\mathbf{t},\mathbf{x}_0)$ 外), $\mathbf{g}_i,\mathbf{h}_i$ 和 $\mathbf{K}$ 在各自的定义域内为充分光滑的函数, 且 $\mathbf{g}_i(0,\mathbf{x})=\mathbf{A}_i(\mathbf{x}),\ \mathbf{x}\in\partial\Omega$, 并存在常数 $\mathbf{N}_i>0$ 和 $\mathbf{M}_i>0$, 使得 $-\mathbf{N}_i\leqslant\mathbf{f}_i\mathbf{u}_i\leqslant-\mathbf{M}_i$ $(\mathbf{i}=1,2,\cdots,\mathbf{m})$;

$[\mathrm{H}_2]$ $\mathbf{K}(\mathbf{t},\mathbf{x},\mathbf{y})\geqslant 0,\ \int_\Omega\mathbf{K}(\mathbf{t},\mathbf{x},\mathbf{y})\mathbf{u}_i\mathrm{d}\mathbf{y}\leqslant\mathbf{d}$, 其中 $\mathbf{d}>0$ 为常数.

2 积分微分方程的比较定理

定义 2.1 若 $\overline{\mathbf{u}}=(\overline{\mathbf{u}}_1,\overline{\mathbf{u}}_2,\cdots,\overline{\mathbf{u}}_m)$, $\underline{\mathbf{u}}=(\underline{\mathbf{u}}_1,\underline{\mathbf{u}}_2,\cdots,\underline{\mathbf{u}}_m)$ 是 $(\mathbf{t},\mathbf{x})\in[0,\mathbf{T}_0]\times(\Omega+\partial\Omega)$ 上的光滑函数, $\underline{\mathbf{u}}_i\leqslant\overline{\mathbf{u}}_i$ $(\mathbf{i}=1,2,\cdots,\mathbf{m})$, 满足

$$\mu(\underline{\mathbf{u}}_i)_t-\varepsilon^2\mathbf{L}\underline{\mathbf{u}}_i+\mathbf{T}\underline{\mathbf{u}}_i-\mathbf{f}_i(\mathbf{t},\mathbf{x},\underline{\mathbf{u}}_i)\leqslant 0\leqslant\mu(\overline{\mathbf{u}}_i)_t-\varepsilon^2\mathbf{L}\overline{\mathbf{u}}_i+\mathbf{T}\overline{\mathbf{u}}_i-\mathbf{f}_i(\mathbf{t},\mathbf{x},\overline{\mathbf{u}}_i),$$
$$\underline{\mathbf{u}}_i\leqslant\mathbf{g}_i(\mathbf{t},\mathbf{x})\leqslant\overline{\mathbf{u}}_i,\quad\mathbf{x}\in\partial\Omega,\quad\underline{\mathbf{u}}_i\leqslant\mathbf{A}_i(\mathbf{x})\leqslant\overline{\mathbf{u}}_i,\quad\mathbf{t}=0,$$

则称 $\overline{\mathbf{u}}$ 和 $\underline{\mathbf{u}}$ 分别为积分微分方程初边值问题 (1.1)–(1.3) 的上解和下解.

定理 2.1 (比较定理) 假设 $[\mathrm{H}_1],[\mathrm{H}_2]$ 成立, 有两个充分小的正常数 ε_0,μ_0, $\forall\varepsilon\in(0,\varepsilon_0)$, $\mu\in(0,\mu_0)$, 若初边值问题 (1.1)–(1.3) 有一对上、下解 $(\overline{\mathbf{u}},\underline{\mathbf{u}})$, 则积分微分方程初边值问题 (1.1)–(1.3) 存在解 $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_1,\cdots,\mathbf{u}_m)$, 且 $\underline{\mathbf{u}}_i\leqslant\mathbf{u}_i\leqslant\overline{\mathbf{u}}_i,(\mathbf{t},\mathbf{x})\in[0,\mathbf{T}_0]\times(\Omega+\partial\Omega)$.

证 设 $\overline{\mathbf{u}}_i^0=\overline{\mathbf{u}}_i\geqslant\underline{\mathbf{u}}_i^0=\underline{\mathbf{u}}_i$ $(\mathbf{i}=1,2,\cdots,\mathbf{m})$ 为两个初始函数, 则能按下列线性系统分别构造出两个迭代序列 $\{\overline{\mathbf{u}}_i^k\},\{\underline{\mathbf{u}}_i^k\}$ $(\mathbf{i}=1,2,\cdots,\mathbf{m})$:

$$\mu(\overline{\mathbf{u}}_i^k)_t-\varepsilon^2\mathbf{L}\overline{\mathbf{u}}_i^k+\mathbf{T}\overline{\mathbf{u}}_i^k+\sum_{l=1}^m\mathbf{N}_l\overline{\mathbf{u}}_i^k=\sum_{l=1}^m\mathbf{N}_l\overline{\mathbf{u}}_i^{k-1}+\mathbf{f}_i(\mathbf{t},\mathbf{x},\overline{\mathbf{u}}_i^{k-1}),\quad\mathbf{x}\in\Omega,$$
$$\overline{\mathbf{u}}_i^k=\mathbf{g}_i(\mathbf{t},\mathbf{x}),\quad\mathbf{x}\in\partial\Omega,\quad\overline{\mathbf{u}}_i^k(0,\mathbf{x})=\mathbf{h}_i(\mathbf{x}),\quad\mathbf{x}\in\Omega,$$
$$\mu(\underline{\mathbf{u}}_i^k)_t-\varepsilon^2\mathbf{L}\underline{\mathbf{u}}_i^k+\mathbf{T}\underline{\mathbf{u}}_i^k+\sum_{l=1}^m\mathbf{N}_l\underline{\mathbf{u}}_i^k=\sum_{l=1}^m\mathbf{N}_l\underline{\mathbf{u}}_i^{k-1}+\mathbf{f}_i(\mathbf{t},\mathbf{x},\underline{\mathbf{u}}_i^{k-1}),\quad\mathbf{x}\in\Omega,$$
$$\underline{\mathbf{u}}_i^k=\mathbf{g}_i(\mathbf{t},\mathbf{x}),\quad\mathbf{x}\in\partial\Omega,\quad\underline{\mathbf{u}}_i^k(0,\mathbf{x})=\mathbf{h}_i(\mathbf{x}),\quad\mathbf{x}\in\Omega.$$

事实上, 由假设, 线性反应扩散微分积分方程组初边值问题的存在唯一性理论知, 由上述两组决定的线性反应扩散微分积分方程组初边值问题均依次地存在相应的解 [3].

令 $\mathbf{w}_i=\overline{\mathbf{u}}_i^0-\overline{\mathbf{u}}_i^1$, 由假设, 有

$$\begin{aligned}&\mu(\mathbf{w}_i)_t-\varepsilon^2\mathbf{L}\mathbf{w}_i+\mathbf{T}\mathbf{w}_i+\sum_{l=1}^m\mathbf{N}_l\mathbf{w}_i\\&=\mu(\overline{\mathbf{u}}_i)_t-\varepsilon^2\mathbf{L}\overline{\mathbf{u}}_i+\mathbf{T}\overline{\mathbf{u}}_i+\mathbf{f}_i(\mathbf{t},\mathbf{x},\overline{\mathbf{u}}_i)\geqslant 0,\quad\mathbf{x}\in\Omega,\\&\mathbf{w}_i=0,\quad\mathbf{x}\in\partial\Omega,\quad\mathbf{w}_i(0,\mathbf{x})=0,\quad\mathbf{x}\in\Omega.\end{aligned}$$

于是 $\mathbf{w}_i\geqslant 0$[1,2], 即

$$\overline{\mathbf{u}}_i^1\leqslant\overline{\mathbf{u}}_i^0,\quad\mathbf{x}\in\Omega+\partial\Omega,\ \mathbf{i}=1,2,\cdots,\mathbf{m}.$$

同理可得

$$\underline{\mathbf{u}}_i^1\geqslant\underline{\mathbf{u}}_i^0,\quad\mathbf{x}\in\Omega+\partial\Omega,\ \mathbf{i}=1,2,\cdots,\mathbf{m}.$$

现证 $\bar{u}_i^1 \geqslant \underline{u}_i^1$. 设 $\bar{w} = \bar{u}_i^1 - \underline{u}_i^1$,

$$\begin{aligned}&\mu(\bar{w}_i)_t - \varepsilon^2 L\bar{w}_i + T\bar{w}_i + \sum_{l=1}^{m} N_l\bar{w}_i\\&= N_i(\bar{u}_i^0 - \underline{u}_i^0) + [f_i(t,x,\bar{u}_i^0) - f_i(t,x,\underline{u}_i^0) \geqslant 0, \quad x \in \Omega,\\&\bar{w}_i = 0, \quad x \in \partial\Omega, \quad \bar{w}_i(0,x) = 0, \quad x \in \Omega,\end{aligned}$$

则 $\bar{w}_i \geqslant 0$, 即

$$\underline{u}_i^1 \leqslant \bar{u}_i^1, \quad x \in \Omega + \partial\Omega,\ i = 1,2,\cdots,m.$$

类似地可得

$$\begin{aligned}&\underline{u}_i = \underline{u}_i^0 \leqslant \underline{u}_i^1 \leqslant \cdots \leqslant \underline{u}_i^k \leqslant \cdots \leqslant \bar{u}_i^k \leqslant \cdots \leqslant \bar{u}_i^1 \leqslant \bar{u}_i^0 = \underline{u}_i,\\&0 \leqslant t \leqslant T_0, \quad x \in \Omega + \partial\Omega,\ i = 1,2,\cdots,m.\end{aligned}$$

由文 [1–2], 在本文的假设下, 成立

$$\lim_{k\to\infty} \underline{u}_i^k = u_i, \quad 0 \leqslant t \leqslant T_0,\ x \in \Omega + \partial\Omega,\ i = 1,2,\cdots,m,$$

且 $u = (u_1,u_2,\cdots,u_m)$ 是积分微分方程初边值问题 (1.1)–(1.3) 的唯一解. 定理 2.1 证毕.

3 初边值问题的外部解

积分微分方程初边值问题 (1.1)–(1.3) 的退化系统为

$$Tu_i = f_i(t,x,u_i), \quad 0 < t \leqslant T_0,\ x \in \Omega,\ i = 1,2,\cdots,m. \tag{3.1}$$

由假设知, Fredholm 型积分方程组 (3.1) 有唯一的一组光滑解 $(U_{100},U_{200},\cdots,U_{m00})$. 设系统的外部解为 $U = (U_1,U_2,\cdots,U_m)$, 且

$$U_i(t,x) = \sum_{j,k=0}^{\infty} U_{ijk}\varepsilon^j\mu^k, \quad i = 1,2,\cdots,m. \tag{3.2}$$

将 (3.2) 代入系统 (1.1), 取 $\varepsilon = \mu = 0$ 便为系统 (3.1), 故 (3.2) 中的 $(U_{100},U_{200},\cdots,U_{m00})$ 就是积分系统 (3.1) 的一组解. 由 (3.1), 对于 $j,k = 0,1,\cdots,\ j+k \neq 0$, 由 $\varepsilon^j\mu^k$ 的同次幂的系数, 有

$$\begin{aligned}&TU_{ijk}(t,x)\\&= f_{iu_i}(t,x,U_{i00})U_{ijk} + F_{ijk}, \quad j,k = 0,1,\cdots,\ j+k \neq 0,\ i = 1,2,\cdots,m,\end{aligned} \tag{3.3}$$

其中 F_{ijk} 为逐次已知的函数, 其结构从略. 同样地, 由假设, Fredholm 型积分方程组 (3.3) 有一组解 $(U_{1jk},U_{2jk},\cdots,U_{mjk})$, $j,k = 0,1,\cdots,j+k \neq 0$, $i = 1,2,\cdots,m$. 由 (3.2), 得到原非线性反应扩散系统初边值问题的外部解 $U = (U_1,U_2,\cdots,U_m)$. 但它未必在 $x_0 \in \Omega$ 处连续, 且未必满足原非线性反应扩散系统的边界条件和初始条件 (1.2)–(1.3), 所以尚需构造在 $x_0 \in \Omega$ 附近的内部激波层校正项 $Z = (Z_1,Z_1,\cdots,Z_m)$ 和边界层校正项 $V = (V_1,V_1,\cdots,V_m)$ 以及初始层校正项 $W = (W_1,W_1,\cdots,W_m)$.

4 在 x_0 邻域的内部激波解

在 $x_0 \in \Omega$ 的邻域引入伸长变量 $\sigma = \frac{|x-x_0|}{\varepsilon}$. 设内部激波层校正项 $Z = (Z_1, Z_1, \cdots, Z_m)$ 为

$$Z_i = \sum_{j,k=0}^{\infty} z_{ijk}\varepsilon^j\mu^k, \quad i = 1, 2, \cdots, m, \tag{4.1}$$

并令

$$u_i = U_i + Z_i, \quad i = 1, 2, \cdots, m. \tag{4.2}$$

在 x_0 的邻域上, 将 (4.1)–(4.2) 代入积分微分方程初边值问题的系统 (1.1), 可得

$$\mu\frac{\partial z_i}{\partial t} - \varepsilon^2 L z_i + T z_i = f_i(t, x_0 \pm \varepsilon\sigma, U_i + z_i) - f_i(t, x_0 \pm \varepsilon\sigma, z_i), \quad i = 1, 2, \cdots, m,$$

并对非线性项按 ε, μ 的幂展开, 对应的 $\varepsilon^j\mu^k$ $(j, k = 0, 1, \cdots)$ 的各次幂的系数, 有

$$T z_{i00} = f_i(t, x, z_{i00}), \quad i = 1, 2, \cdots, m, \tag{4.3}$$

$$\lim_{x\to x_0} z_{i00} = U_{i00}(t, x_0), \quad i = 1, 2, \cdots, m, \tag{4.4}$$

$$T z_{ijk} = \overline{F}_{ijk}, \quad j, k = 0, 1, \cdots,\ j + k \neq 0,\ i = 1, 2, \cdots, m, \tag{4.5}$$

$$\lim_{x\to x_0} z_{ijk} = U_{ijk}(t, x_0), \quad j, k = 0, 1, \cdots,\ j + k \neq 0,\ i = 1, 2, \cdots, m, \tag{4.6}$$

其中 $\overline{F}_{ijk}$ 为逐次已知的函数, 它们的结构在此从略. 由 (4.3)–(4.6) 可依次得到 z_{ijk} $(i = 1, 2, \cdots, m)$, 并由假设知, 它们具有内部激波层性态

$$z_{ijk} = O\Big(\exp\Big(-\overline{k}_{ijk}\frac{|x - x_0|}{\varepsilon}\Big)\Big), \quad j, k = 0, 1, \cdots,\ i = 1, 2, \cdots, m, \tag{4.7}$$

其中 $\overline{k}_{ijk}$ 为适当小的正常数.

引入一个充分光滑的分隔函数 $\psi(x)$, 使得

$$\psi(x) = \begin{cases} 1, & 0 \leqslant |x - x_0| \leqslant \frac{1}{3}\delta, \\ 0, & \frac{2}{3}\delta \leqslant |x - x_0|, \end{cases}$$

其中 δ 为足够小的常数. 取 $\overline{z}_{ijk} = \psi z_{ijk}$ $(j, k = 0, 1, \cdots,\ i = 1, 2, \cdots, m)$. 为了方便起见, 下面仍然用 z_{ijk} 来代替 $\overline{z}_{ijk}$. 将得到的 z_{ijk} 代入 (4.1), 便得到了在 $x_0 \in \Omega$ 的邻域的激波层校正项 $Z = (Z_1, Z_2, \cdots, Z_m)$.

5 边界层校正项

在 Ω 的边界 $\partial\Omega$ 的邻域建立一个局部坐标系 (ρ, φ), 在 Ω 的边界 $\partial\Omega$ 的邻域内的每一点 P 的坐标 ρ $(\rho \leqslant \rho_0)$ 为点 P 到 $\partial\Omega$ 的距离, 这里 ρ_0 为足够小的正常数, 并使在边界 $\partial\Omega$ 上的每一点的内法线在边界 $\partial\Omega$ 的邻域 $0 \leqslant \rho \leqslant \rho_0$ 内互不相交. 而 $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{n-1})$ 为在 $\partial\Omega$ 上的一个 $n-1$ 维非奇局部坐标系, 且设点 P 的坐标 φ 为通过点 P 的内法线和

边界 $\partial\Omega$ 的交点 Q 的坐标 φ 相同. 由此, 在 $\partial\Omega$ 的邻域 $0 \leqslant \rho \leqslant \rho_0$ 中, 有

$$L = a_{nn}\frac{\partial^2}{\partial\rho^2} + \sum_{j,i=1}^{n-1} a_{jn}(x)\frac{\partial^2}{\partial\rho\partial\varphi_i} + \sum_{j,i=1}^{n-1} a_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j} + b_{in}\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial\varphi_i},$$

其中 $a_{nn}, a_{jn}, a_{ij}, b_{in}, b_i$ 为已知函数, 它们的结构从略.

再在 $0 \leqslant \rho \leqslant \rho_0$ 上引入一组多重尺度变量 [1−2]：

$$\zeta = \frac{h(\rho,\varphi)}{\varepsilon}, \quad \bar{\rho} = \rho,\ \varphi = \varphi, \tag{5.1}$$

其中 $h(\rho,\varphi)$ 为适当确定的函数. 不失一般性, 下面仍用 ρ 来代替 $\bar{\rho}$. 于是有

$$L = \frac{1}{\varepsilon^2}K_0 + \frac{1}{\varepsilon}K_1 + K_2, \tag{5.2}$$

其中 $K_0 = a_{nn}h_\rho^2\frac{\partial^2}{\partial\zeta^2}$, 而 K_1, K_2 的结构在此从略. 设积分微分方程初边值问题 (1.1)–(1.3) 的解 $(u_1, u_2, \cdots, u_n)$ 为

$$u_i = U_i + Z_i + V_i, \quad i = 1, 2, \cdots, m, \tag{5.3}$$

其中

$$V_i = \sum_{j,k=0}^{\infty} v_{ijk}(t,\rho,\varphi)\varepsilon^j\mu^k, \quad i = 1, 2, \cdots, m. \tag{5.4}$$

将 (3.2), (4.1), (5.2)–(5.4) 代入 (1.1)–(1.2), 按 ε, μ 展开非线性项, 再由 $\varepsilon^j\mu^k$ $(j, k = 0, 1, \cdots)$ 同次幂的系数, 得到

$$\begin{aligned} &K_0 v_{i00} - T v_{i00} \\ &= -[f_i(t,\rho,\varphi,U_{i00} + z_{i00} + v_{i00}) - f_i(t,\rho,\varphi,U_{i00} + z_{i00})], \quad i = 0, 1, \cdots, m, \end{aligned} \tag{5.5}$$

$$v_{i00} = g_i(t, 0, \varphi) - -(U_{i00} + z_{i00}), \quad i = 0, 1, \cdots, m \tag{5.6}$$

和

$$\begin{aligned} K_0 v_{ijk} - T v_{ijk} &= -f_{iu_i}(t,\rho,\varphi,U_{i00} + z_{i00} + v_{ijk})v_{ijk} + G_{ijk}, \\ &\qquad j, k = 0, 1, \cdots,\ j + k \neq 0,\ i = 1, 2, \cdots, m, \end{aligned} \tag{5.7}$$

$$v_{ijk} = -(U_{ijk} + z_{ijk}), \quad j, k = 0, 1, \cdots,\ j + k \neq 0,\ i = 1, 2, \cdots, m, \tag{5.8}$$

其中 G_{ijk} 为逐次已知的函数, 它们的结构从略.

由 (5.5)–(5.8) 和假设知, 能够依次得到具有衰减性态的解 $v_{ijk}(t,\rho,\varphi)$:

$$v_{ijk}(t,\rho,\varphi) = O\Big(\exp\Big(-\widehat{k}_{ijk}\frac{\rho}{\varepsilon}\Big)\Big), \quad j, k = 0, 1, \cdots,\ i = 1, 2, \cdots, m, \tag{5.9}$$

其中 $\widehat{k}_{ijk}$ 为适当小的正常数.

再引入一个充分光滑的分隔函数 $\overline{\psi}(\rho)$, 使得

$$\overline{\psi}(\rho) = \begin{cases} 1, & 0 \leqslant \rho \leqslant \frac{1}{3}\delta, \\ 0, & \frac{2}{3}\delta \leqslant \rho. \end{cases}$$

取 $\overline{v}_{ijk} = \overline{\psi} v_{ijk}$ $(j, k = 0, 1, \cdots,\ i = 1, 2, \cdots, m)$. 为了方便起见, 下面仍然用 v_{ijk} 来代替 $\overline{v}_{ijk}$. 由 (5.4), 便得到具有边界层校正性质的函数 $V = (V_1, V_2, \cdots, V_m)$.

6 初始层校正项

设

$$\mathbf{u}_i = \mathbf{U}_i + \mathbf{Z}_i + \mathbf{V}_i + \mathbf{W}_i, \quad \mathbf{i} = 1, 2, \cdots, \mathbf{m} \tag{6.1}$$

和

$$\mathbf{W}_i(\boldsymbol{\tau}, \mathbf{x}) = \sum_{j,k=0}^{\infty} \mathbf{w}_{ijk}(\boldsymbol{\tau}, \mathbf{x}) \boldsymbol{\varepsilon}^j \boldsymbol{\mu}^k, \quad \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \tag{6.2}$$

其中 $\boldsymbol{\tau} = \frac{t}{\mu}$ 为伸长变量 [1−2].

将 (6.1)–(6.2) 代入 (1.1), (1.3), 按 $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ 展开非线性项, 再由 $\boldsymbol{\varepsilon}^j \boldsymbol{\mu}^k$ $(\mathbf{j}, \mathbf{k} = 0, 1, \cdots)$ 同次幂的系数, 得到

$$\begin{aligned} \frac{\partial \mathbf{w}_{i00}}{\partial \boldsymbol{\tau}} + \mathbf{T} \mathbf{w}_{i00} &= \mathbf{f}_i(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00} + \mathbf{w}_{i00}) \\ &\quad - \mathbf{f}_i(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00}), \quad \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \end{aligned} \tag{6.3}$$

$$\mathbf{w}_{i00} = \mathbf{h}_i(\mathbf{x}) - \mathbf{U}_{i00} - \mathbf{z}_{i00} - \mathbf{v}_{i00}, \quad \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \tag{6.4}$$

$$\begin{aligned} \frac{\partial \mathbf{w}_{ijk}}{\partial \boldsymbol{\tau}} &= \mathbf{f}_{iu_i}(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{Z}_{i00} + \mathbf{V}_{i00} + \mathbf{W}_{i00}) \mathbf{w}_{ijk} + \widetilde{\mathbf{F}}_{ijk}, \\ &\qquad \mathbf{j}, \mathbf{k} = 0, 1, \cdots, \ \mathbf{j} + \mathbf{k} \neq 0, \ \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \end{aligned} \tag{6.5}$$

$$\mathbf{w}_{ijk} = -(\mathbf{U}_{ijk} + \mathbf{z}_{ijk} + \mathbf{v}_{ijk}), \quad \mathbf{j}, \mathbf{k} = 0, 1, \cdots, \ \mathbf{j} + \mathbf{k} \neq 0, \ \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \tag{6.6}$$

其中 $\widetilde{\mathbf{F}}_{ijk}$ 为逐次已知的函数, 其结构也从略. 由 (6.3)–(6.6), 我们能依次得到 $\mathbf{w}_{ijk}$ $(\mathbf{j}, \mathbf{k} = 0, 1, \cdots, \ \mathbf{i} = 0, 1, \cdots, \mathbf{m})$, 且它们具有初始层性态

$$\mathbf{w}_{ijk} = \mathbf{O}\Big(\exp\Big(-\widetilde{\mathbf{k}}_{ijk} \frac{\mathbf{t}}{\boldsymbol{\mu}}\Big)\Big), \quad \mathbf{j}, \mathbf{k} = 0, 1, \cdots, \ \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \tag{6.7}$$

其中 $\widetilde{\mathbf{k}}_{ijk}$ 为适当小的正常数.

再引入一个充分光滑的分隔函数 $\widetilde{\boldsymbol{\psi}}(\boldsymbol{\tau})$, 使得

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{\tau}) = \begin{cases} 1, & 0 \leqslant \boldsymbol{\tau} \leqslant \frac{1}{3}\boldsymbol{\delta}, \\ 0, & \frac{2}{3}\boldsymbol{\delta} \leqslant \boldsymbol{\tau}. \end{cases}$$

取 $\overline{\mathbf{w}}_{ijk} = \widetilde{\boldsymbol{\psi}} \mathbf{w}_{ijk}$ $(\mathbf{j}, \mathbf{k} = 0, 1, \cdots, \mathbf{i} = 1, 2, \cdots, \mathbf{m})$. 为了方便起见, 下面仍然用 $\mathbf{w}_{ijk}$ 来代替 $\overline{\mathbf{w}}_{ijk}$. 再将 $\mathbf{w}_{ijk}(\boldsymbol{\tau}, \mathbf{x})$ 代入 (6.2), 便得到具有初始层校正性质的函数 $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_m)$.

由 (6.1), 两参数非线性反应扩散系统初边值问题 (1.1)–(1.3) 的解 $(\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m)$ 有如下形式的渐近展开式

$$\begin{aligned} \mathbf{u}_i \sim \sum_{j=0}^{M} \sum_{k=0}^{N} (\mathbf{U}_{ijk} + \mathbf{z}_{ijk} + \mathbf{v}_{ijk} + \mathbf{w}_{ijk}) \boldsymbol{\varepsilon}^j \boldsymbol{\mu}^k + \mathbf{O}(\boldsymbol{\lambda}), \\ \mathbf{i} = 1, 2, \cdots, \mathbf{m}, \ 0 < \boldsymbol{\varepsilon}, \boldsymbol{\mu} \ll 1, \end{aligned} \tag{6.8}$$

其中 $\boldsymbol{\lambda} = \max(\boldsymbol{\varepsilon}^{M+1} \boldsymbol{\mu}^N, \boldsymbol{\varepsilon}^M \boldsymbol{\mu}^{N+1}), \ 0 < \boldsymbol{\varepsilon}, \boldsymbol{\mu} \ll 1$.

7 解的一致有效性

定理 7.1 在假设 $[\mathrm{H}_1]$, $[\mathrm{H}_2]$ 下, 非线性反应扩散系统初边值问题 (1.1)–(1.3) 存在解 $u=(u_1,u_2,\cdots,u_m)$, 并在 $[0,T_0]\times(\Omega+\partial\Omega)$ 上关于小参数 ε,μ 成立一致有效的渐近展开式 (6.8).

证 首先构造辅助函数 α_i,β_i :

$$\alpha_i=Y_i-r_i\lambda,\quad \beta_i=Y_i+r_i\lambda,\quad i=1,2,\cdots,m,\ 0<\varepsilon,\mu\ll 1,\tag{7.1}$$

其中 r_i $(i=1,2,\cdots,m)$ 为足够大的正常数, 它们将在下面决定, 而

$$Y_i=\sum_{j=0}^{M}\sum_{k=0}^{N}(U_{ijk}+z_{ijk}+v_{ijk}+w_{ijk})\varepsilon^j\mu^k.$$

显然

$$\alpha_i\leqslant\beta_i.\tag{7.2}$$

由假设不难看出, 存在正常数 D_{i1} $(i=1,2\cdots,m)$, 在 $x\in\partial\Omega$ 上成立

$$\alpha_i|_{\partial\Omega}\leqslant g_i(t,0,\varphi)+\Big[\sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N}(U_{ijk}+z_{ijk}+v_{ijk}+w_{ijk})\varepsilon^j\mu^k\Big]_{\partial\Omega}-r_i\lambda$$
$$\leqslant g_i(t,x)+(D_{i1}-r_i),\quad x\in\partial\Omega.$$

选取 $r_i\geqslant D_{i1}$, 便有

$$\alpha_i\leqslant g_i(t,x),\quad x\in\partial\Omega,\ i=1,2,\cdots,m.\tag{7.3}$$

由 (4.7), (5.9), (6.7), 存在正常数 D_{i2} $(i=1,2,\cdots,m)$, 有

$$\alpha_i|_{t=0}=\sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N}[U_{ijk}+z_{ijk}+v_{ijk}+w_{ijk}]_{t=0}\varepsilon^j\mu^k-r_i\lambda\leqslant h_i(x)+(D_{i2}-r_i)\lambda.$$

于是, 当 $r_i\geqslant D_{i2}$ 时, 有

$$\alpha_i\leqslant h_i(x),\quad t=0,\ i=1,2,\cdots,m.\tag{7.4}$$

同理可得

$$\beta_i|_{\partial\Omega}\geqslant g_i(t,x),\quad \beta_i|_{t=0}\geqslant h_i(x),\ i=1,2,\cdots,m.\tag{7.5}$$

现证

$$\mu\frac{\partial\alpha_i}{\partial t}-\varepsilon^2L\alpha_i+T\alpha_i-f_i(t,x,\alpha_i)\leqslant 0,\quad 0<t\leqslant T_0,\ x\in\Omega,\ i=1,2,\cdots,m,\tag{7.6}$$
$$\mu\frac{\partial\beta_i}{\partial t}-\varepsilon^2L\beta_i+T\beta_i-f_i(t,x,\beta_i)\geqslant 0,\quad 0<t\leqslant T_0,\ x\in\Omega,\ i=1,2,\cdots,m.\tag{7.7}$$

我们区分如下 3 种情形

(i) 当 $\Omega\backslash(\rho\geqslant(\frac{2}{3})\delta)$ 时; (ii) 当 $(\frac{1}{3})\delta\leqslant\rho\leqslant(\frac{2}{3})\delta$ 时; (iii) 当 $0\leqslant\rho\leqslant(\frac{1}{3})\delta$ 时. 现只证明情形 (iii), 其余的情形类似.

当 $0 \leqslant \boldsymbol{\rho} \leqslant \left(\frac{1}{3}\right)\boldsymbol{\delta}$ 时. 由中值定理和关系式 (4.7), (5.9), (6.7) 对于 $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ 足够地小, 存在正常数 $\mathbf{D}_{i3}$ ($\mathbf{i} = 1, 2, \cdots, \mathbf{m}$), 使得

$$
\begin{aligned}
&\boldsymbol{\mu}\frac{\partial \boldsymbol{\alpha}_i}{\partial \mathbf{t}} - \boldsymbol{\varepsilon}^2 \mathbf{L}\boldsymbol{\alpha}_i + \mathbf{T}\boldsymbol{\alpha}_i - \mathbf{f}_i(\mathbf{t}, \mathbf{x}, \boldsymbol{\alpha}_i) \\
&= \boldsymbol{\mu}\frac{\partial \mathbf{Y}_i}{\partial \mathbf{t}} - \boldsymbol{\varepsilon}^2 \mathbf{L}\mathbf{Y}_i + \mathbf{T}\mathbf{Y}_i - \mathbf{f}_i(\mathbf{t}, \mathbf{x}, \mathbf{Y}_i) + [\mathbf{f}_i(\mathbf{t}, \mathbf{x}, \mathbf{Y}_i) - \mathbf{f}_i(\mathbf{t}, \mathbf{x}, \mathbf{Y}_i - \mathbf{r}_i\boldsymbol{\lambda})] \\
&\leqslant [\mathbf{T}\mathbf{U}_{i00} - \mathbf{f}_i(\mathbf{t}, \mathbf{x}, \mathbf{U}_{i00})] \\
&\quad + \sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N} [\mathbf{T}\mathbf{U}_{ijk}(\mathbf{t}, \mathbf{x}) - \mathbf{f}_{iu_i}(\mathbf{t}, \mathbf{x}, \mathbf{U}_{i00})\mathbf{U}_{ijk} - \mathbf{F}_{ijk}]\boldsymbol{\varepsilon}^j\boldsymbol{\mu}^k \\
&\quad + [\mathbf{T}\mathbf{z}_{i00} - \mathbf{f}_i(\mathbf{t}, \mathbf{x}, \mathbf{z}_{i00})] + \sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N} [\mathbf{T}\mathbf{z}_{ijk} - \overline{\mathbf{F}}_{ijk}]\boldsymbol{\varepsilon}_j\boldsymbol{\mu}^k \\
&\quad + [\mathbf{K}_0\mathbf{v}_{i00} - \mathbf{T}\mathbf{v}_{i00} + \mathbf{f}_i(\mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\varphi}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00}) - \mathbf{f}_i(\mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\varphi}, \mathbf{U}_{i00} + \mathbf{z}_{i00})] \\
&\quad + \sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N} [\mathbf{K}_0\mathbf{v}_{ijk} - \mathbf{T}\mathbf{v}_{ijk} + [\mathbf{f}_i(\mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\varphi}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00}) \\
&\quad - \mathbf{f}_i(\mathbf{t}, \boldsymbol{\rho}, \boldsymbol{\varphi}, \mathbf{U}_{i00} + \mathbf{z}_{i00})]]\boldsymbol{\varepsilon}^j\boldsymbol{\mu}^k \\
&\quad + \Big[\frac{\partial \mathbf{w}_{i00}}{\partial \boldsymbol{\tau}} + \mathbf{T}\mathbf{w}_{i00} - \mathbf{f}_i(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00} + \mathbf{w}_{i00}) - \mathbf{f}_i(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{z}_{i00} + \mathbf{v}_{i00})\Big] \\
&\quad + \sum_{j=0}^{M}\sum_{\substack{k=0\\ j+k\neq 0}}^{N} \Big[\frac{\partial \mathbf{w}_{ijk}}{\partial \boldsymbol{\tau}} - \mathbf{f}_{iu_i}(0, \mathbf{x}, \mathbf{U}_{i00} + \mathbf{Z}_{i00} + \mathbf{V}_{i00} + \mathbf{W}_{i00})\mathbf{w}_{ijk} - \widetilde{\mathbf{F}}_{ijk}\Big]\boldsymbol{\varepsilon}^j\boldsymbol{\mu}^k \\
&\quad + [\mathbf{D}_{i3} - (\mathbf{M} + \mathbf{N})\mathbf{r}_i]\boldsymbol{\lambda} \leqslant [\mathbf{D}_{i3} - (\mathbf{M} + \mathbf{N})\mathbf{r}_i]\boldsymbol{\lambda}.
\end{aligned}
$$

最后, 选取 $\mathbf{r}_i$ ($\mathbf{i} = 1, 2, \cdots, \mathbf{m}$) 足够大, 使得 $\mathbf{r}_i \geqslant \frac{D_{i3}}{M+N}$, 这时便证明了不等式 (7.6). 同理, 不等式 (7.7) 也成立. 由 (7.2)–(7.7) 及比较定理 (定理 2.1), 得到 $\boldsymbol{\alpha}_i \leqslant \mathbf{u}_i \leqslant \boldsymbol{\beta}_i$ ($\mathbf{i} = 1, 2, \cdots, \mathbf{m}$), $(\mathbf{t}, \mathbf{x}) \in [0, \mathbf{T}_0 \times [\Omega + \boldsymbol{\partial}\Omega]$. 于是, 由 (7.1), 有 (6.8). 定理 7.1 证毕.

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Interior Shock Asymptotic Solutions to Nonlinear Reaction Diffusion Integral Differential Equations with Two Parameters

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Abstract The singular perturbation problem for the nonlinear reaction diffusion integral differential problem with two parameters is considered. By using the singular perturbation method, the outer solution, interior shock layer, boundary layer and initial layer corrective terms are constructed, then the formal asymptotic expansion of solution is obtained. Finally, the uniform validity of asymptotic expansion for solution to this problem is proved by using the comparison theorem for integral differential equation.

Keywords Reaction diffusion, Singular perturbation, Initial boundary value problem

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