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退化抛物-双曲方程动力学解的唯一性*

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提要 主要研究系数显含有时间和空间变量的退化抛物-双曲型方程柯西问题动力学解的唯一性. 首先推广了这种类型方程的动力学公式, 在给定系数适当的光滑性条件下, 得到了动力学解的唯一性.

关键词 退化抛物-双曲方程, 动力学解, 熵解, 唯一性
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1 引 言

本文研究下列拟线性退化抛物-双曲型方程的柯西问题

$$\partial_t u+\nabla_x\cdot f(u,t,x)-\nabla_x\cdot(A(u,t,x)\nabla_x u)=g(u,t,x), \tag{1.1}$$

$$u(0,x)=u_0(x), \tag{1.2}$$

其中 $(t,x)\in\mathbf{R}^+\times\mathbf{R}^n$, $f(u,t,x)=(f_1(u,t,x),f_2(u,t,x),\cdots,f_n(u,t,x))$ 是向量值函数, $g(u,t,x)$ 是源项, $A(u,t,x)=(a_{ij}(u,t,x))_{n\times n}$ 半正定对称阵, 即对任意的

$$\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\mathbf{R}^n,$$

都有

$$\sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j\geqslant 0,$$

其元素可以写为

$$a_{ij}(u,t,x)=\sigma^\top(u,t,x)\sigma(u,t,x)=\sum_{k=1}^K\sigma_{ik}(u,t,x)\sigma_{jk}(u,t,x),$$

这里 $\sigma(u,t,x)=(\sigma_{ij}(u,t,x))$ 是 $A(u,t,x)$ 的平方根矩阵, 额外指标 K 是 A 的最大秩. 特别地, 当 $A=0$ 时方程就强退化为双曲型方程.

方程 (1.1) 在很多领域都有重要的应用. 例如, 多孔介质的两相流方程[1−3]

$$\partial_t u+\nabla_x\cdot(k(x,t)f(u))-\nabla_x\cdot(A(u,x)\nabla_x u)=0, \tag{1.3}$$

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再如沉降 - 固化过程中的热传导过程[4−5]

$$\partial_t u+\partial_x(l(t)u+f(u))-\partial_x^2(A(u))=0, \tag{1.4}$$

在金融数学中也有这类方程[6].

由于 (1.1) 具有广泛而重要的应用, 多年来这类方程一直受到许多关注, 并用不同形式的数值方法来研究它[2,7−16]. 而作为研究的基础, 其解的适定性是首先要考虑的. 由于方程 (1.1) 可以退化为双曲型方程, 它的解会有间断, 从而必须在适当的函数空间寻找唯一的弱解. 在 1969 年, Volpert 和 Hugjaev[17]最先在 BV 空间得到了熵解的存在性, 但其唯一性的证明却经历了较长时间. 对于一维的情形, Zhao[18], Bénilan 和 Touré [19]分别证明了 BV 和 L^1 熵解的唯一性. 对于多维的情形, Carrillio[20]在 1999 年证明了各项同性方程的 L^∞ 熵解的唯一性, 对于各项异性的情形, Pertham 和 Chen[21]利用动力学技巧首次得到了动力学解的适定性.

以上结果都是在方程 (1.1) 中 f 和 A 不显含 (t,x), 而在实际的应用中, 它们往往与 (t,x) 有关, 正如 (1.3) 和 (1.4) 一样, 因此, (1.1)–(1.2) 的研究更有实际意义和一般意义. 在这方面, Chen 和 Karlsen[22]证明了熵解的唯一性, Li 和 Wang[23]则得到了无界重整化解唯一性. 这些结果都是借助于双变量方法[24−25]得到的, 作为 (1.1)–(1.2) 更一般解的动力学解, 其唯一性仍然没有结果.

作为研究波尔兹曼方程的动力学解, 现已在双曲型方程中得到应用[26−27]. 在文 [21] 中, Chen 和 Perthame 将这种方法应用到系数不显含 (t,x) 的退化抛物-双曲型方程中. 他们通过引入抛物耗散测度与链式法则, 克服了这类方程研究中的困难. 本文将利用文 [21] 中的方法, 证明一般形式的退化抛物 - 双曲方程的柯西问题 (1.1)–(1.2) 动力学解的唯一性. 为此, 首先要给出 (1.1)–(1.2) 的动力学公式与链式法则. 除此之外还要仔细处理系数产生的余项.

在接下来的第 2 部分, 首现给出动力学解的定义与主要定理. 在第 3 部分证明主要定理.

2 动力学解的定义与主要结果

首先, 简单介绍 (1.1) 的熵解的定义. 令

$$f_i'=\frac{\partial f_i(u,\cdot,\cdot)}{\partial u},\quad f_{ix_i}=\frac{\partial f_i(\cdot,\cdot,x)}{\partial x_i},\quad \partial_{x_i}f_i(u,t,x)=\frac{\partial f_i}{\partial u}\cdot\frac{\partial u}{\partial x_i}+\frac{\partial f_i(\cdot,\cdot,x)}{\partial x_i},$$

$$\eta_i(u,t,x)=\int_0^u S'(\xi)f_i'(\xi,t,x)\,\mathrm{d}\xi,\qquad r_{ij}(u,t,x)=\int_0^u S'(\xi)a_{ij}(\xi,t,x)\,\mathrm{d}\xi,$$

$$\eta_{ix_i}=\int_0^u S'(\xi)f_{ix_i}'(\xi,t,x)\,\mathrm{d}\xi,\qquad r_{ijx_j}=\int_0^u S'(\xi)a_{ijx_j}(\xi,t,x)\,\mathrm{d}\xi,$$

$$i,j=1,\cdots,n,$$

其中 S 是熵, 它是一个 C^2 的凸函数. (S,η_i,r_{ij}) 是熵 - 熵流对. 记

$$\kappa_{ik}(u,t,x)=\int_0^u \sigma_{ik}(\xi,t,x)\,\mathrm{d}\xi, \tag{2.1}$$

$$\kappa_{ik}^{\psi}(u,t,x)=\int_0^u \psi(\xi)\sigma_{ik}(\xi,t,x)\,\mathrm{d}\xi, \quad \psi\in C(\mathbf{R}). \tag{2.2}$$

(1.1)–(1.2) 的熵解定义如下.

定义 2.1 可测函数 $u\in L^\infty(\mathbf{R}^+\times\mathbf{R}^n)$ 称为 (1.1)–(1.2) 的熵解, 如果

(1) $\displaystyle\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}-\kappa_{ikx_i})(u,t,x)\in L^2(\mathbf{R}^+\times\mathbf{R}^n), \quad k=1,2,\cdots,K;$

(2) 对任意的 $k\in\{1,2,\cdots,K\}$ 和非负函数 $\psi\in C(\mathbf{R})$, 下列等式成立:

$$\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}^{\psi}-\kappa_{ikx_i}^{\psi})(u,t,x)=\psi(u)\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}-\kappa_{ikx_i})(u,t,x)\in L^2(\mathbf{R}^+\times\mathbf{R}^n),$$

(3) 对任意的光滑凸函数 $S(u)$, 存在一个抛物耗散测度

$$n^{S''}(t,x)=\sum_{k=1}^K S''(u)\Big(\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}-\kappa_{ikx_i})(u,t,x)\Big)^2,$$

以及对某一个给定的测度 $m(\xi,t,x)\geqslant 0$, 存在一个熵耗散测度

$$m^{S''}(t,x)=\int_{\mathbf{R}} S''(\xi)m(\xi,t,x)\,\mathrm{d}\xi, \tag{2.3}$$

使得对任意的 $u\in\mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^n)$, 有

$$\begin{aligned}&\partial_t S(u)+\sum_{i=1}^n(\partial_{x_i}\eta_i-\eta_{ix_i})+\sum_{i=1}^n S'(u)f_{ix_i}(u,t,x)-\sum_{i,j=1}^n\partial_{x_i}(\partial_{x_j}r_{ij}-r_{ijx_j})\\&=S'(u)g(u,t,x)-(m^{S''}+n^{S''})(t,x),\end{aligned} \tag{2.4}$$

$$S(u)|_{t=0}=S(u_0). \tag{2.5}$$

注 2.1 如果将抛物耗散测度 $n^{S''}(t,x)$ 写成 $(C_0(\mathbf{R});\mathcal{M}(\mathbf{R}))$ 对偶的形式, 则

$$n^{S''}(t,x)=\int_{\mathbf{R}} S''(\xi)n(\xi,t,x)\,\mathrm{d}\xi, \quad S''\in C_0(\mathbf{R}),$$

其中 n 称为抛物亏损测度, 具有形式

$$n(\xi,t,x)=\delta(\xi-u)\sum_{k=1}^K\Big(\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}-\kappa_{ikx_i})(\xi,t,x)\Big)^2. \tag{2.6}$$

注 2.2 如果选取特殊的熵函数

$$S(u)=\begin{cases}(u-\xi)_+, & \xi>0,\\ (u-\xi)_-, & \xi<0,\end{cases}$$

经过计算可知存在某个 $\mu(\xi)\in L_0^\infty(\mathbf{R})$, 使得

$$\int_0^\infty\int_{\mathbf{R}^n}(m(\xi,t,x)+n(\xi,t,x))\,\mathrm{d}x\,\mathrm{d}t\leqslant\mu(\xi). \tag{2.7}$$

如文 [21, 26–27] 一样, 引入 $\mathbf{R}^2$ 上的函数 $\chi(\xi,u)$

$$\chi(\xi,u)=\begin{cases}1, & 0<\xi<u,\\ -1, & u<\xi<0,\\ 0, & \text{其它},\end{cases}\tag{2.8}$$

它具有如下性质

(i) 如果 $u\in L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^n))$, 那么

$$\chi(\xi,u)\in L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^{n+1}));$$

(ii) 对任意的 $S(u)\in C^1(\mathbf{R})$, 有

$$S(u)=\int_{\mathbf{R}}S'(\xi)\chi(\xi,u)\,\mathrm{d}\xi+S(0);$$

(iii) $\partial_\xi\chi(\xi,u)=\delta(\xi)-\delta(\xi-u),\quad \partial_u\chi(\xi,u)=\delta(\xi-u)$.

利用 $\chi(\xi,u)$ 的性质 (ii) 及 (2.4), 可以得到 (1.1)–(1.2) 的动力学公式, 即

$$\begin{aligned}&\partial_t\chi(\xi,u)+\sum_{i=1}^n f_i'(\xi,t,x)\partial_{x_i}\chi(\xi,u)-\sum_{i,j=1}^n a_{ijx_i}\partial_{x_j}\chi(\xi,u)-\sum_{i,j=1}^n a_{ij}\partial^2_{x_ix_j}\chi(\xi,u)\\ =&(g(\xi,t,x)-\sum_{i=1}^n f_{ix_i}(\xi,t,x))\partial_u\chi(\xi,u)+\partial_\xi(m+n)(\xi,t,x)\end{aligned}\tag{2.9}$$

在 $\mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^{n+1})$ 上成立, 并满足初值

$$\chi(\xi,u)|_{t=0}=\chi(\xi,u_0).\tag{2.10}$$

这样, (1.1)–(1.2) 的动力学解可定义如下.

定义 2.2 函数 $u\in L^\infty(\mathbf{R}^+;L^1(\mathbf{R}^{n+1}))$ 称为 (1.1)–(1.2) 的动力学解, 如果它满足:

(1′) $\displaystyle\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}^{\psi}-\kappa_{ikx_i}^{\psi})(u,t,x)\in L^2(\mathbf{R}^+\times\mathbf{R}^n)$;

(2′) 对任意两个非负函数 $\psi_1,\psi_2\in\mathcal{D}(\mathbf{R})$,

$$\sqrt{\psi_1}(u)\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}^{\sqrt{\psi_2}}-\kappa_{ikx_i}^{\sqrt{\psi_2}})(u,t,x)=\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}^{\sqrt{\psi_1\psi_2}}-\kappa_{ikx_i}^{\sqrt{\psi_1\psi_2}})(u,t,x);\tag{2.11}$$

(3′) 对非负熵亏损测度 $m(\xi,t,x)$ 和抛物亏损测度 $n(\xi,t,x)$, 使动力学公式 (2.9) 在 $\mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^{n+1})$ 成立. $n(\xi,t,x)$ 通过下式给出

$$\int_{\mathbf{R}}\psi(\xi)n(\xi,t,x)\,\mathrm{d}\xi=\sum_{k=1}^K\Big(\sum_{i=1}^n(\partial_{x_i}\kappa_{ik}^{\sqrt{\psi}}-\kappa_{ikx_i}^{\sqrt{\psi}})(u,t,x)\Big)^2,\tag{2.12}$$

其中 $\psi\in\mathcal{D}(\mathbf{R})$ 且 $\psi\geqslant 0$;

(4′) $\displaystyle\int_0^\infty\int_{\mathbf{R}^n}(m(\xi,t,x)+n(\xi,t,x))\,\mathrm{d}x\,\mathrm{d}t\leqslant\mu(\xi)\in L_0^\infty(\mathbf{R})$.

注 2.3 函数 $\chi(\xi,u)\in\mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^{n+1})$ 是 (2.9)–(2.10) 的一个解, 是指对任意的试验函数 $\phi(\xi,t,x)\in C_c^\infty(\mathbf{R}^+\times\mathbf{R}^{n+1})$, 都有

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\chi(\xi,u)\partial_t\phi(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbf{R}^{n+1}}\chi(\xi,u_0)\phi(\xi,0,x)\,\mathrm{d}\xi\,\mathrm{d}x$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\chi(\xi,u)[f_i'\partial_{x_i}\phi(\xi,t,x)+f_{ix_i}'\phi(\xi,t,x)]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{i,j=1}^{n}\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\chi(\xi,u)[a_{ij}\partial^2_{x_ix_j}\phi(\xi,t,x)+a_{ijx_j}\partial_{x_i}\phi(\xi,t,x)]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
=&\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\Big(\sum_{i=1}^{n}f_{ix_i}-g\Big)\partial_u\chi(\xi,u)\phi(\xi,t,x)+(m+n)\partial_\xi\phi(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.13}
$$

现在给出 f, σ_{ik} 和 g 满足的条件. 它们都属于 $L^1(\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^n)$. 除此之外, 对任意可测函数 $u(t,x)$, 函数 f 满足:

$$
\begin{aligned}
&f_i(\cdot,t,x)\in W^{1,1}(\mathbf{R})\cap W^{1,\infty}(\mathbf{R}) \quad \text{对 } (t,x)\in\mathbf{R}^+\times\mathbf{R}^n \text{ 一致成立},\\
&f_i(u,t,\cdot)\in W^{1,\infty}(\mathbf{R}^n)\cap W^{1,1}(\mathbf{R}^n) \quad \text{对 } (u,t)\in\mathbf{R}\times\mathbf{R}^+ \text{ 一致成立},\\
&f_i'(u,\cdot,\cdot)\in W^{1,1}(\mathbf{R}^+\times\mathbf{R}^n)\cap W^{1,\infty}(\mathbf{R}^+\times\mathbf{R}^n) \quad \text{对 } u\in\mathbf{R} \text{ 一致成立}.
\end{aligned}
$$

矩阵 A 的元素 σ_{ik} 要满足:

$$
\begin{aligned}
&\sigma_{ik}(\cdot,t,x)\in W^{1,1}(\mathbf{R})\cap W^{1,\infty}(\mathbf{R}) \quad \text{对 } (t,x)\in\mathbf{R}^+\times\mathbf{R}^n \text{ 一致成立},\\
&\sigma_{ik}(u,\cdot,\cdot)\in W^{1,\infty}(\mathbf{R}^+\times\mathbf{R}^n) \quad \text{对 } u\in\mathbf{R} \text{ 一致成立},\\
&\sigma_{ik}(u,t,\cdot)\in W^{1,\infty}(\mathbf{R}^n) \quad \text{对 } (u,t)\in\mathbf{R}\times\mathbf{R}^+ \text{ 一致成立}.
\end{aligned}
$$

源项 g 满足:

$$
\begin{aligned}
&g(\cdot,t,x)\in W^{1,1}(\mathbf{R})\cap W^{1,\infty}(\mathbf{R}) \quad \text{对 } (t,x)\in\mathbf{R}^+\times\mathbf{R}^n \text{ 一致成立},\\
&g'(u,\cdot,\cdot)\in W^{1,\infty}(\mathbf{R}^+\times\mathbf{R}^n)\cap W^{1,1}(\mathbf{R}^+\times\mathbf{R}^n) \quad \text{对 } u\in\mathbf{R} \text{ 一致成立},\\
&g(u,\cdot,\cdot)\in L^1(\mathbf{R}^+\times\mathbf{R}^n)\cap L^\infty(\mathbf{R}^+\times\mathbf{R}^n) \quad \text{对 } u\in\mathbf{R} \text{ 一致成立}.
\end{aligned}
$$

如果 $u_0(x)\in L^1(\mathbf{R}^n)$ 且 f, σ_{ik} 和 g 满足上述的条件, 那么柯西问题 (1.1)–(1.2) 存在一个动力学解 $u\in C(\mathbf{R}^+,L^1(\mathbf{R}^n))$. 其证明可以像文 [17, 21] 等一样经过一个标准的过程得到, 在此忽略. 本文的主要结果是下述定理.

定理 2.1　设 u, v 是分别满足初值 u_0 和 v_0 的两个动力学解, 且 $u_0,v_0\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)\cap BV(\mathbf{R}^n)$, 函数 f, σ_{ik} 和 g 分别满足上述的条件, 则对任意的 $T>0$, 存在常数 C(依赖于 T 和 g), 使得

$$||u(t,\cdot)-v(t,\cdot)||_{L^1(\mathbf{R}^n)}\leqslant C||u_0(\cdot)-v_0(\cdot)||_{L^1(\mathbf{R}^n)}.\tag{2.14}$$

由此可得 (1.1)–(1.2) 动力学解的唯一性.

3　定理 2.1 的证明

设 $u(\xi,t,x)$, $v(\xi,t,x)$ 是两个动力学解, $m(\xi,t,x)$, $p(\xi,t,x)$ 分别是它们的熵亏损测度, 而 $n(\xi,t,x)$, $q(\xi,t,x)$ 是它们对应的抛物亏损测度. 为了后面应用的方便, 首先引入一些

术语. 令 $\varphi_j \geqslant 0$ $(j=1,2)$ 分别是关于时间和空间的光滑子, 即 $\varphi_j \geqslant 0$ $(j=1,2)$ 满足 $\int \varphi_j = 1$, $\operatorname{supp}\varphi_1 \subseteq (-1,0)$ 且 $\operatorname{supp}\varphi_2 \subseteq (-1,1)$. 对任意给定的正数 ε_1 和 ε_2, 定义

$$\varphi_\varepsilon(t,x) = \varphi_{1,\varepsilon_1}(t)\varphi_{2,\varepsilon_2}(x) = \frac{1}{\varepsilon_1}\varphi_1\Big(\frac{t}{\varepsilon_1}\Big)\frac{1}{\varepsilon_2^n}\varphi_2\Big(\frac{x}{\varepsilon_2}\Big).$$

设 $\psi(\xi)$ 是关于 ξ 的光滑子, 令

$$\psi_\delta(\xi) = \frac{1}{\delta}\psi\Big(\frac{\xi}{\delta}\Big), \quad \varphi_{\varepsilon,\delta} = \varphi_\varepsilon(t,x)\psi_\delta(\xi).$$

令 $\rho_j(\cdot)$ $(j=1,2,3)$ 分别是关于时间, 空间和 ξ 的截断函数, 它们都是有紧支撑的非负光滑函数, 满足

(i) $0 \leqslant \rho_j(s) \leqslant 1,\ s \in (-\infty,+\infty)$;

(ii) $\rho_j(0) = 1$;

(iii) $\rho_j(s) = 0, \quad |s| \geqslant 1$;

(iv) $\operatorname{sgn}(s)\rho_j'(s) \leqslant 0$.

令

$$\begin{aligned}
\rho(\xi,t,x) &= \rho_1\Big(\frac{t}{\mathrm{R}_1}\Big)\rho_2\Big(\frac{x}{\mathrm{R}_2}\Big)\rho_3\Big(\frac{\xi}{\mathrm{R}_3}\Big),\\
\rho(\xi,x) &= \rho_2\Big(\frac{x}{\mathrm{R}_2}\Big)\rho_3\Big(\frac{\xi}{\mathrm{R}_3}\Big),\\
\rho(t,x) &= \rho_1\Big(\frac{t}{\mathrm{R}_1}\Big)\rho_2\Big(\frac{x}{\mathrm{R}_2}\Big),\\
\rho(\xi,t) &= \rho_1\Big(\frac{t}{\mathrm{R}_1}\Big)\rho_3\Big(\frac{\xi}{\mathrm{R}_3}\Big).
\end{aligned}$$

易证当 $\mathrm{R}_i \to +\infty \quad (i=1,2,3)$ 时,

$$\rho_i\Big(\frac{\cdot}{\mathrm{R}_i}\Big) \to 1\ (i=1,2,3), \quad \rho(\xi,t,x) \to 1.$$

同时也将用到以下记号

$$\chi^u(\xi) := \chi(\xi,u), \quad \chi_\varepsilon^u(\xi) := \chi(\xi,u)*\varphi_\varepsilon, \quad \chi_{\varepsilon,\delta}^u(\xi) := \chi_\varepsilon^u * \psi_\delta,$$

同理, $\chi^v(\xi)$, $\chi_\varepsilon^v(\xi)$, $\chi_{\varepsilon,\delta}^v(\xi)$ 可以类似地定义. 下文中, 约定 $\Theta_{\varepsilon,\delta}$ 都表示函数 Θ 按上述方式定义. 易见 χ_ε^u 具有下列性质:

$$\begin{aligned}
&(1)\ \operatorname{sgn}\xi\cdot\chi_\varepsilon^u = |\chi^u|*\varphi_\varepsilon = |\chi_\varepsilon^u| = |\chi^u|^2*\varphi_\varepsilon \leqslant 1,\\
&(2)\ |\chi_\varepsilon^u|^2 \leqslant |\chi_\varepsilon^u|.
\end{aligned}\tag{3.1}$$

命题 3.1 对 $\chi_\varepsilon^u \in \mathcal{D}'(R^+\times R^{n+1})$, 下式成立

$$\begin{aligned}
&\partial_t\chi_\varepsilon^u + \sum_{i=1}^n \partial_{x_i}(\chi^u f_i')*\varphi_\varepsilon - \sum_{i=1}^n (\chi^u f_{ix_i}')*\varphi_\varepsilon - \sum_{i,j=1}^n \partial_{x_ix_j}^2(a_{ij}\chi^u)*\varphi_\varepsilon\\
&+\sum_{i,j=1}^n \partial_{x_i}(a_{ijx_j}\chi^u)*\varphi_\varepsilon = \Big[\Big(g-\sum_{i=1}^n f_{ix_i}\Big)\partial_u\chi^u\Big]*\varphi_\varepsilon + \partial_\xi(m_\varepsilon+n_\varepsilon).
\end{aligned}\tag{3.2}$$

证　在 (2.13) 中取 $\phi(\xi,t,x)=\varphi_\varepsilon(t-s,x-y)\varphi(\xi,t,x)$, 积分后计算可得

$$
\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\partial_t\chi^u_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\sum_{i=1}^n\partial_{x_i}(\chi^u f_i')*\varphi_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\Big(\sum_{i=1}^n(\chi^u f'_{ix_i})*\varphi_\varepsilon+\sum_{i,j=1}^n\partial^2_{x_ix_j}(a_{ij}\chi^u)*\varphi_\varepsilon\Big)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\sum_{i,j=1}^n\partial_{x_i}(a_{ijx_j}\chi^u)*\varphi_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
=&\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\Big[\Big(g-\sum_{i=1}^n f_{ix_i}\Big)\partial_u\chi^u\Big]*\varphi_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\mathbf{R}^+}\int_{\mathbf{R}^{n+1}}\varphi(\xi,t,x)\partial_\xi(m_\varepsilon+n_\varepsilon)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

此式即 (3.2), 证毕.

考虑非负函数

$$
Q_\varepsilon(\xi,t,x)=|\chi^u_\varepsilon|+|\chi^v_\varepsilon|-2\chi^u_\varepsilon\chi^v_\varepsilon. \tag{3.3}
$$

选择 $\rho(\xi,t,x)$ 作为实验函数. 对任意的 $T>0$, 由 (3.2) 可得

$$
\begin{aligned}
&\int_{\mathbf{R}^{n+1}}(|\chi^u_\varepsilon(\xi,T,x)|\rho(\xi,T,x)-|\chi^u_\varepsilon(\xi,0,x)|\rho(\xi,0,x))\,\mathrm{d}\xi\,\mathrm{d}x\\
&-\int_0^T\int_{\mathbf{R}^{n+1}}\Big\{\frac{1}{\mathrm{R}_1}|\chi^u_\varepsilon|\rho_1'\Big(\frac{t}{\mathrm{R}_1}\Big)\rho(\xi,x)+\rho(\xi,t)\Big[\frac{1}{\mathrm{R}_2}\sum_{i=1}^n(|\chi^u|f_i')*\varphi_\varepsilon\rho_2'\Big(\frac{x}{\mathrm{R}_2}\Big)\\
&+\frac{1}{\mathrm{R}_2^2}\sum_{i,j=1}^n(|\chi^u|a_{ij})*\varphi_\varepsilon\rho_2''\Big(\frac{x}{\mathrm{R}_2}\Big)+\frac{1}{\mathrm{R}_2}\sum_{i,j=1}^n(|\chi^u|a_{ijx_j})*\varphi_\varepsilon\rho_2'\Big(\frac{x}{\mathrm{R}_2}\Big)\Big]\Big\}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_{\mathbf{R}^{n+1}}\sum_{i=1}^n\Big(|\chi^u|f'_{ix_i})*\varphi_\varepsilon\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
=&\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\Big(g-\sum_{i=1}^n f_{ix_i}\Big)\partial_u|\chi^u|\Big]*\varphi_\varepsilon\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&-2\int_0^T\int_{\mathbf{R}^n}(m_\varepsilon+n_\varepsilon)(0,t,x)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{1}{\mathrm{R}_3}\int_0^T\int_{\mathbf{R}^{n+1}}\mathrm{sgn}(\xi)(m_\varepsilon+n_\varepsilon)\rho_3'\Big(\frac{\xi}{\mathrm{R}_3}\Big)\rho(t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.4}
$$

由于

$$
\int_0^T\int_{\mathbf{R}^{n+1}}|\chi^u_\varepsilon|\rho_1'\Big(\frac{t}{\mathrm{R}_1}\Big)\rho(\xi,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\leqslant c\int_0^T\int_{\mathbf{R}^{n+1}}|\chi^u_\varepsilon|\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t<\infty,
$$

当 $\mathrm{R}_1\to+\infty$ 时, 可得

$$
\frac{1}{\mathrm{R}_1}\int_0^T\int_{\mathbf{R}^{n+1}}|\chi^u_\varepsilon|\rho_1'\Big(\frac{t}{\mathrm{R}_1}\Big)\rho(\xi,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\to 0.
$$

同理, 令 $\mathrm{R}_2,\mathrm{R}_3\to+\infty$, 则 (3.4) 变为

$$\begin{aligned}&\int_{\mathbf{R}^{n+1}}|\chi_\varepsilon^u(\xi,T,x)|\,\mathrm{d}\xi\,\mathrm{d}x-\int_{\mathbf{R}^{n+1}}|\chi_\varepsilon^u(\xi,0,x)|\,\mathrm{d}\xi\,\mathrm{d}x\\&=-2\int_0^T\int_{\mathbf{R}^n}(m_\varepsilon+n_\varepsilon)(0,t,x)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbf{R}^{n+1}}g'|\chi_\varepsilon^u|\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+T_\varepsilon^u,\end{aligned}\tag{3.5}$$

类似地

$$\begin{aligned}&\int_{\mathbf{R}^{n+1}}|\chi_\varepsilon^v(\xi,T,x)|\,\mathrm{d}\xi\,\mathrm{d}x-\int_{\mathbf{R}^{n+1}}|\chi_\varepsilon^v(\xi,0,x)|\,\mathrm{d}\xi\,\mathrm{d}x\\&=-2\int_0^T\int_{\mathbf{R}^n}(p_\varepsilon+q_\varepsilon)(0,t,x)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbf{R}^{n+1}}g'|\chi_\varepsilon^v|\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+T_\varepsilon^v,\end{aligned}\tag{3.6}$$

其中

$$\begin{aligned}T_\varepsilon^u&=\int_0^T\int_{\mathbf{R}^{n+1}}((g'|\chi^u|)*\varphi_\varepsilon-g'|\chi_\varepsilon^u|)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\\T_\varepsilon^v&=\int_0^T\int_{\mathbf{R}^{n+1}}((g'|\chi^v|)*\varphi_\varepsilon-g'|\chi_\varepsilon^v|)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

引理 3.1 当 $\varepsilon_i\to0$(i=1, 2) 时, T^u 与 T^v 的极限都为 0.

证 这两个极限的证明类似, 只须证明 $T_\varepsilon^u\to0$. 事实上

$$\begin{aligned}T_\varepsilon^u&=\int_0^T\int_{\mathbf{R}^{n+1}}((g'|\chi^u|)*\varphi_\varepsilon-g'|\chi_\varepsilon^u|)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\int_0^T\int_{\mathbf{R}^{n+1}}\int_{\mathbf{R}^n\times\mathbf{R}^+}|g'(\xi,s,y)-g'(\xi,t,x)|\,|\chi(\xi,u(s,y))|\,\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_0^T\int_{\mathbf{R}^{n+1}}|g'(\xi,t-\varepsilon_1s,x-\varepsilon_2y)-g'(\xi,t,x)|\varphi(s,y)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

由于

$$\int_0^T\int_{\mathbf{R}^{n+1}}|(g'(\xi,t-\varepsilon_1s,x-\varepsilon_2y)-g'(\xi,t,x))|\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\leqslant2||g||_{L^1(\mathbf{R}^{n+1}\times\mathbf{R}^+)},$$

当 $\varepsilon_i\to0(i=1,2)$ 时, 由勒贝格控制收敛定理, 可得

$$\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_0^T\int_{\mathbf{R}^{n+1}}|g'(\xi,t-\varepsilon_1s,x-\varepsilon_2y)-g'(\xi,t,x)|\varphi(s,y)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\to0,$$

证毕.

下面计算 (3.3) 的第 3 项. 由命题 3.1 可知

$$\begin{aligned}&\partial_t\chi_{\varepsilon,\delta}^u+\sum_{i=1}^n\partial_{x_i}(\chi^uf_i')*\varphi_{\varepsilon,\delta}-\sum_{i=1}^n(\chi^uf_{ix_i}')*\varphi_{\varepsilon,\delta}-\sum_{i,j=1}^n\partial_{x_ix_j}^2(a_{ij}\chi^u)*\varphi_{\varepsilon,\delta}\\&+\sum_{i,j=1}^n\partial_{x_i}(a_{ijx_j}\chi^u)*\varphi_{\varepsilon,\delta}=\Big[\Big(g-\sum_{i=1}^nf_{ix_i}\Big)\partial_u\chi^u\Big]*\varphi_{\varepsilon,\delta}+\partial_\xi(m_{\varepsilon,\delta}+n_{\varepsilon,\delta}).\end{aligned}\tag{3.7}$$

同样对于 $\chi^v_{\varepsilon,\delta}$ 有下式成立

$$\begin{aligned}&\partial_t\chi^v_{\varepsilon,\delta}+\sum_{i=1}^n\partial_{x_i}(\chi^v f'_i)*\varphi_{\varepsilon,\delta}-\sum_{i=1}^n(\chi^v f'_{ix_i})*\varphi_{\varepsilon,\delta}-\sum_{i,j=1}^n\partial^2_{x_ix_j}(a_{ij}\chi^v)*\varphi_{\varepsilon,\delta}\\&+\sum_{i,j=1}^n\partial_{x_i}(a_{ijx_j}\chi^v)*\varphi_{\varepsilon,\delta}=\Big[\Big(g-\sum_{i=1}^n f_{ix_i}\Big)\partial_v\chi^v\Big]*\varphi_{\varepsilon,\delta}+\partial_\xi(p_{\varepsilon,\delta}+q_{\varepsilon,\delta}).\end{aligned}\tag{3.8}$$

在接下来的计算中, 依次令 $\delta\to0$, $\mathrm{R}_i\to\infty$, $\varepsilon_1\to0$, $\varepsilon_2\to0$. 易知下列等式成立:

$$\begin{aligned}&\partial_\xi\chi^u_{\varepsilon,\delta}=\psi_\delta-\delta(\xi-u)*(\varphi_\varepsilon\psi_\delta),\\&\partial_\xi\chi^v_{\varepsilon,\delta}=\psi_\delta-\delta(\xi-v)*(\varphi_\varepsilon\psi_\delta),\\&(\partial_u\chi^u)*(\varphi_\varepsilon\psi_\delta)=\psi_\delta-\partial_\xi\chi^u_{\varepsilon,\delta},\\&(\partial_v\chi^v)*(\varphi_\varepsilon\psi_\delta)=\psi_\delta-\partial_\xi\chi^v_{\varepsilon,\delta}.\end{aligned}$$

将 (3.7) 乘以 $\chi^u_{\varepsilon,\delta}$, (3.8) 乘以 $\chi^v_{\varepsilon,\delta}$, 相加后结果写成如下形式

$$\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}=\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}.$$

下面将给出这个等式中的每一项, 并依次计算它们.

$$\begin{aligned}\mathrm{I}&=\int_0^T\int_{\mathbf{R}^{n+1}}\partial_t(\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta})\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{\mathbf{R}^{n+1}}(\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta})(\xi,T,x)\rho(\xi,T,x)\,\mathrm{d}\xi\,\mathrm{d}x-\int_{\mathbf{R}^{n+1}}(\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta})(\xi,0,x)\rho(\xi,0,x)\,\mathrm{d}\xi\,\mathrm{d}x-\mathrm{R}^\chi,\end{aligned}$$

其中

$$R^\chi=\frac{1}{\mathrm{R}_1}\int_0^T\int_{\mathbf{R}^{n+1}}(\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta})\rho'\Big(\frac{t}{\mathrm{R}_1}\Big)\rho(\xi,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

当 $\delta\to0$ 和 $\mathrm{R}_i\to+\infty$ 时, 易知 $R^\chi\to0$, 这样

$$\mathrm{I}\to\int_{\mathbf{R}^{n+1}}(\chi^u_\varepsilon\chi^v_\varepsilon)(\xi,T,x)\,\mathrm{d}\xi\,\mathrm{d}x-\int_{\mathbf{R}^{n+1}}(\chi^u_\varepsilon\chi^v_\varepsilon)(\xi,0,x)\,\mathrm{d}\xi\,\mathrm{d}x.\tag{3.9}$$

对于 II, 有

$$\begin{aligned}\mathrm{II}&=\int_0^T\int_{\mathbf{R}^{n+1}}\sum_{i=1}^n[\partial_{x_i}(\chi^u f'_i)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+\partial_{x_i}(\chi^v f'_i)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=-\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[(\chi^u f'_i)*\varphi_{\varepsilon,\delta}\partial_{x_i}\chi^v_{\varepsilon,\delta}+(\chi^v f'_i)*\varphi_{\varepsilon,\delta}\partial_{x_i}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\frac{1}{\mathrm{R}_2}\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[(\chi^u f'_i)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+(\chi^v f'_i)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho'_2\Big(\frac{x}{\mathrm{R}_2}\Big)\rho(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\mathrm{R}^f_1+\mathrm{R}^f_2+\mathrm{R}^f_3+\mathrm{R}^f_4,\end{aligned}$$

其中

$$\mathrm{R}^f_1=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_i(\xi,t,x)\chi^u_{\varepsilon,\delta}\partial_{x_i}\chi^v_{\varepsilon,\delta}-(f'_i\chi^u)*\varphi_{\varepsilon,\delta}\partial_{x_i}\chi^v_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_2^f=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_i'(\xi,t,x)\chi_{\varepsilon,\delta}^v\partial_{x_i}\chi_{\varepsilon,\delta}^u-(f_i'\chi^v)*\varphi_{\varepsilon,\delta}\partial_{x_i}\chi_{\varepsilon,\delta}^u]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_3^f=-\frac{1}{\mathrm{R}_2}\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[(\chi^uf_i')*\varphi_{\varepsilon,\delta}\chi_{\varepsilon,\delta}^v+(\chi^vf_i')*\varphi_{\varepsilon,\delta}\chi_{\varepsilon,\delta}^u]\rho_2'(\frac{x}{\mathrm{R}_2})\rho(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_4^f=\frac{1}{\mathrm{R}_2}\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_i'(\xi,t,x)\chi_{\varepsilon,\delta}^u\chi_{\varepsilon,\delta}^v]\rho_2'\Big(\frac{x}{\mathrm{R}_2}\Big)\rho(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

引理 3.2 当 $\delta\to0$ 且 $\mathrm{R}_i\to+\infty$, $\mathrm{R}_3^f,\mathrm{R}_4^f\to0$, R_1^f, R_2^f 收敛到函数 L_1^f, L_2^f. 进一步, 当 $\varepsilon_1,\varepsilon_2\to0$ 时, $\mathrm{L}_i^f\to0\quad(i=1,2)$.

证 $\mathrm{R}_3^f,\mathrm{R}_4^f\to0$ 可以很容易得到证明, 只须证明 R_1^f 和 R_2^f 的结论.

$$\begin{aligned}\mathrm{R}_1^f&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(f_i'(\xi,t,x)-f_i'(\eta,s,y))\chi^u(\eta,s,y)\\&\qquad\cdot\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big]\partial_{x_i}\chi_{\varepsilon,\delta}^v\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\mathrm{R}_{11}^f+\mathrm{R}_{12}^f,\end{aligned}$$

这里

$$\begin{aligned}\mathrm{R}_{11}^f&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(f_i'(\xi,t,x)-f_i'(\xi,t,y))\chi^u(\eta,s,y)\\&\qquad\cdot\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big]\partial_{x_i}\chi_{\varepsilon,\delta}^v\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(f_i'(\xi,t,y)-f_i'(\xi,s,y))\chi^u(\eta,s,y)\\&\qquad\cdot\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big]\partial_{x_i}\chi_{\varepsilon,\delta}^v\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\\\mathrm{R}_{12}^f&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(f_i'(\xi,s,y)-f_i'(\eta,s,y))\chi^u(\eta,s,y)\\&\qquad\cdot\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big]\partial_{x_i}\chi_{\varepsilon,\delta}^v\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

由勒贝格控制收敛定理, $\mathrm{R}_{12}^f\to0,(\delta\to0)$, 而 $\delta\to0$, $\mathrm{R}_i\to+\infty$ 时, R_{11}^f 收敛到函数 L_1^f. 这样 $\varepsilon_1\to0$ 时,

$$\begin{aligned}&\lim_{\varepsilon_1\to0}\mathrm{L}_1^f\\&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\int_{\mathbf{R}^n}(f_i'(\xi,t,x)-f_i'(\xi,t,y))\chi^u(\xi,t,y)\varphi_{\varepsilon_2}(x-y)\,\mathrm{d}y\Big]\partial_{x_i}\chi_{\varepsilon_2}^v\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

注意到如果 $v_0\in BV(\mathbf{R}^n)$, 则

$$||v||_{L^\infty(\mathbf{R}^+;BV(\mathbf{R}^n))}\leqslant||v_0||_{BV(\mathbf{R}^n)}$$

(见 [17]). 经过计算, 可得

$$\lim_{\varepsilon_1\to 0}\mathrm{L}_1^f=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(f_i'(v,t,x)-f_i'(v,t,y))\chi(v(t,\widetilde{y}),u(t,y))v_{\widetilde{y}_i}$$

$$\cdot\varphi_{\varepsilon_2}(x-y)\varphi_{\varepsilon_2}(x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant\varepsilon_2\sum_{i=1}^n||f'_{x_i}||_{L^\infty}\int_0^T\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}v_{\widetilde{y}_i}\varphi_{\varepsilon_2}(x-y)\varphi_{\varepsilon_2}(x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}\widetilde{y}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant c\varepsilon_2\sum_{i=1}^n||f'_{x_i}||_{L^\infty}||v_0||_{BV(\mathbf{R}^n)}.$$

所以当 $\varepsilon\to 0$ 时, $\mathrm{L}_1^f\to 0$. 用相同的方法可以证明 R_2^f.

接下来计算 III.

$$\mathrm{III}=-\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[(f'_{ix_i}\chi^u)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+(f'_{ix_i}\chi^v)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=-2\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta},]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\mathrm{R}_5^f,$$

这里

$$\mathrm{R}_5^f=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}-(f'_{ix_i}\chi^u)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^v_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}-(f'_{ix_i}\chi^v)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

引理 3.3　当 $\delta\to 0$, $\mathrm{R}_i\to+\infty,(i=1,2,3)$ 时, 上式给出的 R_5^f 收敛到 L_5^f, 这里

$$\mathrm{L}_5^f=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^u_\varepsilon\chi^v_\varepsilon-(f'_{ix_i}\chi^u)*\varphi_\varepsilon\chi^v_\varepsilon]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}(\xi,t,x)\chi^v_\varepsilon\chi^u_\varepsilon-(f'_{ix_i}\chi^v)*\varphi_\varepsilon\chi^u_\varepsilon]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

进一步, 当 $\varepsilon\to 0$ 时, $\mathrm{L}_5^f\to 0$.

这一引理可以类似的证明, 省略其过程. 下面计算VI. 它可以写成

$$\mathrm{VI}=-\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[(f_{ix_i}\partial_u\chi^u)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+(f_{ix_i}\partial_v\chi^v)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=-\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\mathrm{R}_6^f+\mathrm{R}_7^f+\mathrm{R}_8^f,$$

这里

$$\mathrm{R}_6^f=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_{ix_i}\partial_u\chi^u_{\varepsilon,\delta}-(f_{ix_i}\partial_u\chi^u)*\varphi_{\varepsilon,\delta}]\chi^v_{\varepsilon,\delta}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_{ix_i}\partial_v\chi^v_{\varepsilon,\delta}-(f_{ix_i}\partial_v\chi^v)*\varphi_{\varepsilon,\delta}]\chi^u_{\varepsilon,\delta}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_7^f=-\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_{ix_i}\varphi_\delta\chi_{\varepsilon,\delta}^v+f_{ix_i}\varphi_\delta\chi_{\varepsilon,\delta}^u]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_8^f=-\frac{1}{\mathrm{R}_3}\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f'_{ix_i}\chi_{\varepsilon,\delta}^u\chi_{\varepsilon,\delta}^v]\rho'\Big(\frac{\xi}{\mathrm{R}_3}\Big)\rho(t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

引理 3.4 当 $\delta\to 0$ 及 $\mathrm{R}_i\to+\infty,(i=1,2,3)$ 时, $\mathrm{R}_7^f,\mathrm{R}_8^f\to 0$. R_6^f 收敛到函数 L_6^f. 进一步, 在 $\varepsilon\to 0$ 时它趋于 0.

证 易见当 $\delta\to 0$ 和 $\mathrm{R}_i\to+\infty,(i=1,2,3)$ 时, $\mathrm{R}_7^f,\mathrm{R}_8^f\to 0$. 只需要证明 R_6^f 的结论. 为此将其写成 $\mathrm{R}_6^f=\mathrm{R}_{61}^f+\mathrm{R}_{62}^f$,

$$\begin{aligned}\mathrm{R}_{61}^f&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}[f_{ix_i}\partial_u\chi_{\varepsilon,\delta}^u-(f_{ix_i}\partial_u\chi^u)*\varphi_{\varepsilon,\delta}]\chi_{\varepsilon,\delta}^v\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^{n+1}}\chi_{\varepsilon,\delta}^v\Big[\int_{\mathbf{R}^n\times\mathbf{R}^+}f_{ix_i}(\xi,t,x)\psi_\delta(\xi-u(s,y))\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\\&\quad-\int_{\mathbf{R}^n\times\mathbf{R}^+}(f_{iy_i}(u(s,y),s,y))\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-u(s,y))\,\mathrm{d}y\,\mathrm{d}s\Big]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&\to\sum_{i=1}^n\int_0^T\int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^n\times\mathbf{R}^+}\chi(u,v)*\varphi_\varepsilon(f_{ix_i}(u(s,y),t,x)-(f_{iy_i}(u(s,y),s,y)))\\&\quad\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\Big]\,\mathrm{d}x\,\mathrm{d}t,\quad(\delta\to 0\text{ 且 }\mathrm{R}_i\to+\infty(i=1,2,3)).\end{aligned}$$

当 $\varepsilon\to 0$ 时, 上式的极限为 0. R_{62}^f 可以类似的处理. 从而引理 3.4 得证.

下面计算 VII.

$$\begin{aligned}\mathrm{VII}&=\int_0^T\int_{\mathbf{R}^{n+1}}[\partial_\xi(m_{\varepsilon,\delta}+n_{\varepsilon,\delta})\chi_{\varepsilon,\delta}^v+\partial_\xi(p_{\varepsilon,\delta}+q_{\varepsilon,\delta})\chi_{\varepsilon,\delta}^u]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\mathrm{R}_1^M+\mathrm{R}_2^M+\mathrm{R}_3^M,\end{aligned}$$

其中

$$\mathrm{R}_1^M=-\int_0^T\int_{\mathbf{R}^{n+1}}[(m_{\varepsilon,\delta}+n_{\varepsilon,\delta})\varphi_\delta+(p_{\varepsilon,\delta}+q_{\varepsilon,\delta})\varphi_\delta]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_2^M=\int_0^T\int_{\mathbf{R}^{n+1}}[(m_{\varepsilon,\delta}+n_{\varepsilon,\delta})\delta_{\varepsilon,\delta}(\xi-v)+(p_{\varepsilon,\delta}+q_{\varepsilon,\delta})\delta_{\varepsilon,\delta}(\xi-u)]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}_3^M=-\frac{1}{\mathrm{R}_3}\int_0^T\int_{\mathbf{R}^{n+1}}[(m_{\varepsilon,\delta}+n_{\varepsilon,\delta})\chi_{\varepsilon,\delta}^v+(p_{\varepsilon,\delta}+q_{\varepsilon,\delta})\chi_{\varepsilon,\delta}^u]\rho'\Big(\frac{\xi}{\mathrm{R}_3}\Big)\rho(t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

当 $\delta\to 0$ 且 $\mathrm{R}_i\to+\infty$ 时, 得

$$\mathrm{R}_3^M\to 0,\quad \mathrm{R}_1^M\to-\int_0^T\int_{\mathbf{R}^n}[(m_\varepsilon+n_\varepsilon)+(p_\varepsilon+q_\varepsilon)](0,t,x)\,\mathrm{d}x\,\mathrm{d}t.\tag{3.10}$$

对 R_2^M, 由于 m 和 p 非负, n 和 q 满足 (2.12), 得

$$\mathrm{R}_2^M\geqslant\int_0^T\int_{\mathbf{R}^{n+1}}[n_{\varepsilon,\delta}\delta(\xi-v)*\varphi_{\varepsilon,\delta}+q_{\varepsilon,\delta}\delta(\xi-u)*\varphi_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^T \int_{\mathbf{R}^{n+1}} \sum_{k=1}^K \Big[\int_{\mathbf{R}^+\times\mathbf{R}^n} \int_{\mathbf{R}^+\times\mathbf{R}^n} \Big(\Big(\sum_{i=1}^n \Big(\partial_{y_i} \kappa_{ik}^{\sqrt{\psi_\delta(\xi-u(s,y))}} - \kappa_{iky_i}^{\sqrt{\psi_\delta(\xi-u(s,y))}} \Big) \Big)^2 \\
&\quad \cdot \psi_\delta(\xi - v(\widetilde{s},\widetilde{y})) + \Big(\sum_{j=1}^n \Big(\partial_{\widetilde{y}_j} \kappa_{jk}^{\sqrt{\psi_\delta(\xi-v(\widetilde{s},\widetilde{y}))}} - \kappa_{jk\widetilde{y}_j}^{\sqrt{\psi_\delta(\xi-v(\widetilde{s},\widetilde{y}))}} \Big) \Big)^2 \psi_\delta(\xi - u(s,y)) \Big) \\
&\quad \cdot \varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}\widetilde{y} \,\mathrm{d}\widetilde{s} \Big] \rho(\xi,t,x) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&\geqslant 2 \int_0^T \int_{\mathbf{R}^{n+1}} \sum_{k=1}^K \sum_{i,j=1}^n \int_{\mathbf{R}^+\times\mathbf{R}^n} \int_{\mathbf{R}^+\times\mathbf{R}^n} \Big[\Big(\partial_{y_i} \kappa_{ik}^{\sqrt{\psi_\delta(\xi-u(s,y))}} - \kappa_{iky_i}^{\sqrt{\psi_\delta(\xi-u(s,y))}} \Big) \\
&\quad \cdot \Big(\partial_{\widetilde{y}_j} \kappa_{jk}^{\sqrt{\psi_\delta(\xi-v(\widetilde{s},\widetilde{y}))}} - \kappa_{jk\widetilde{y}_j}^{\sqrt{\psi_\delta(\xi-v(\widetilde{s},\widetilde{y}))}} \Big) \sqrt{\psi_\delta(\xi - v(\widetilde{s},\widetilde{y}))} \sqrt{\psi_\delta(\xi - u(s,y))} \\
&\quad \cdot \varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}\widetilde{y} \,\mathrm{d}\widetilde{s} \Big] \rho(\xi,t,x) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&= 2 \int_0^T \int_{\mathbf{R}^{n+1}} \Big(\int_{\mathbf{R}^n\times\mathbf{R}^+} \int_{\mathbf{R}^n\times\mathbf{R}^+} \sum_{k=1}^K \Big[\Big(\sum_{i=1}^n (\partial_{y_i} \kappa_{ik}^{\psi_\delta(\xi-u(s,y))} - \kappa_{iky_i}^{\psi_\delta(\xi-u(s,y))}) \Big) \\
&\quad \cdot \Big(\sum_{j=1}^n (\partial_{\widetilde{y}_j} \kappa_{jk}^{\psi_\delta(\xi-v(\widetilde{y},\widetilde{s}))} - \kappa_{jk\widetilde{y}_j}^{\psi_\delta(\xi-v(\widetilde{y},\widetilde{s}))}) \Big) \Big] \\
&\quad \cdot \varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}\widetilde{y} \,\mathrm{d}\widetilde{s} \Big) \rho(\xi,t,x) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

由于

$$
\begin{aligned}
&\partial_{y_i} \kappa_{ik}^{\psi_\delta(\xi-u(s,y))} - \kappa_{iky_i}^{\psi_\delta(\xi-u(s,y))} \\
&= \partial_{y_i} \int_{\mathbf{R}} \sigma_{ik}(\eta,s,y)\chi(\eta,u)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta - \int_{\mathbf{R}} \sigma_{iky_i}(\eta,s,y)\chi(\eta,u)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

从而

$$
\mathrm{R}_2^M \geqslant 2 \int_0^T \int_{\mathbf{R}^{n+1}} \rho(\xi,t,x) \Big(\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+} \int_{\mathbf{R}^{n+1}\times\mathbf{R}^+} \Phi_{ijk} \mathrm{d}\eta \,\mathrm{d}\widetilde{\eta} \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}\widetilde{y} \,\mathrm{d}\widetilde{s} \Big) \mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t, \tag{3.11}
$$

其中

$$
\begin{aligned}
\Phi_{ijk} = \sum_{k=1}^K \sum_{i,j=1}^n & [(\sigma_{ik}(\eta,s,y)\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\partial_{y_i}\varphi_\varepsilon(t-s,x-y)\partial_{\widetilde{y}_j}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \\
& + (\sigma_{ik}(\eta,s,y)\sigma_{jk\widetilde{y}_j}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\partial_{y_i}\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \\
& + (\sigma_{iky_i}(\eta,s,y)\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\varphi_\varepsilon(t-s,x-y)\partial_{\widetilde{y}_j}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y}) \\
& + (\sigma_{iky_i}(\eta,s,y)\sigma_{jk\widetilde{y}_j}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})] \\
& \cdot \chi(\eta,u)\chi(\widetilde{\eta},v)\psi_\delta(\xi-\eta)\psi_\delta(\xi-\widetilde{\eta}).
\end{aligned}
$$

对于含 a_{ij} 的项,

$$
\begin{aligned}
&\mathrm{IV} + \mathrm{V} \\
&= \sum_{i,j=1}^n \int_0^T \int_{\mathbf{R}^{n+1}} (-\partial_{x_i x_j}^2 (a_{ij}\chi^u) * \varphi_{\varepsilon,\delta}\chi_{\varepsilon,\delta}^v - \partial_{x_i x_j}^2 (a_{ij}\chi^v) * \varphi_{\varepsilon,\delta}\chi_{\varepsilon,\delta}^u) \rho(\xi,t,x) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

$$+\sum_{i,j=1}^{n}\int_0^T\int_{\mathbf{R}^{n+1}}(\partial_{x_i}(a_{ijx_j}\chi^u)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+\partial_{x_i}(a_{ijx_j}\chi^v)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta})\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=\mathrm{R}^a_1+\mathrm{R}^a_2+\mathrm{R}^a_3+\mathrm{R}^a_4,$$

其中

$$\mathrm{R}^a_1=\sum_{i,j=1}^{n}\int_0^T\int_{\mathbf{R}^{n+1}}(\partial_{x_j}(a_{ij}\chi^u)_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+\partial_{x_j}(a_{ij}\chi^v)_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta})\partial_{x_i}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}^a_2=-\sum_{i,j=1}^{n}\int_0^T\int_{\mathbf{R}^{n+1}}((a_{ijx_j}\chi^u)_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+(a_{ijx_j}\chi^v)_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta})\partial_{x_i}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}^a_3=\sum_{i,j=1}^{n}\int_0^T\int_{\mathbf{R}^{n+1}}(\partial_{x_j}(a_{ij}\chi^u)_{\varepsilon,\delta}\partial_{x_i}\chi^v_{\varepsilon,\delta}+\partial_{x_j}(a_{ij}\chi^v)_{\varepsilon,\delta}\partial_{x_i}\chi^u_{\varepsilon,\delta})\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{R}^a_4=-\sum_{i,j=1}^{n}\int_0^T\int_{\mathbf{R}^{n+1}}((a_{ijx_j}\chi^u)_{\varepsilon,\delta}\partial_{x_i}\chi^v_{\varepsilon,\delta}+(a_{ijx_j}\chi^v)_{\varepsilon,\delta}\partial_{x_i}\chi^u_{\varepsilon,\delta})\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

与前面的讨论一样, 当 $\delta\to 0$ 且 $\mathrm{R}_i\to+\infty(i=1,2,3)$ 时可得 R^a_1 与 R^a_2 都趋于 0. 经过计算, 可得

$$\begin{aligned}\mathrm{R}^a_3=\int_0^T\int_{\mathbf{R}^{n+1}}\Big[&\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(\sigma_{ik}(\eta,y,s)\sigma_{jk}(\eta,y,s)\\&+\sigma_{ik}(\widetilde{\eta},\widetilde{y},\widetilde{s})\sigma_{jk}(\widetilde{\eta},\widetilde{y},\widetilde{s}))\chi(\eta,u)\chi(\widetilde{\eta},v)\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\\&\cdot\psi_\delta(\xi-\eta)\psi_\delta(\xi-\widetilde{\eta})\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.12}$$

及

$$\begin{aligned}\mathrm{R}^a_4=\int_0^T\int_{\mathbf{R}^{n+1}}\sum_{k=1}^{K}\sum_{i,j=1}^{n}\Big(&\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\partial_{y_j}(\sigma_{ik}(\eta,s,y)\sigma_{jk}(\eta,s,y))\chi(\eta,u)\varphi_\varepsilon(t-s,x-y)\\&\cdot\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\widetilde{\eta},v)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\psi_\delta(\xi-\widetilde{\eta})\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\\&+\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\partial_{\widetilde{y}_i}(\sigma_{ik}(\widetilde{\eta},\widetilde{s},\widetilde{y})\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\chi(\widetilde{\eta},v)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\psi_\delta(\xi-\widetilde{\eta})\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\\&\cdot\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\eta,u)\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big)\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.13}$$

由 (3.11)–(3.13), 可得

$$\mathrm{R}^a_3+\mathrm{R}^a_4-\mathrm{R}^M_2\leqslant\mathrm{R}^a_5+\mathrm{R}^a_6+\mathrm{R}^a_7.$$

其中的 R^a_5, R^a_6, R^a_7 在下面将依次给出并计算它们.

$$\begin{aligned}\mathrm{R}^a_5=\int_0^T\int_{\mathbf{R}^{n+1}}\Big[&\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}(\chi(\eta,u)\chi(\widetilde{\eta},v)\psi_\delta(\xi-\eta)\psi_\delta(\xi-\widetilde{\eta})\\&\cdot\sigma_{ik}(\eta,s,y)\sigma_{jk}(\eta,s,y)-2\sigma_{ik}(\eta,s,y)\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y})+\sigma_{ik}(\widetilde{\eta},\widetilde{s},\widetilde{y})\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\\&\cdot\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\to \int_0^T\int_{\mathbf{R}^{n+1}}\Big[\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\chi(\xi,u)\chi(\xi,v)$$

$$\cdot(\sigma_{ik}(\xi,s,y)\sigma_{jk}(\xi,s,y)-2\sigma_{jk}(\xi,s,y)\sigma_{ik}(\xi,\widetilde{s},\widetilde{y})+\sigma_{ik}(\xi,\widetilde{s},\widetilde{y})\sigma_{jk}(\xi,\widetilde{s},\widetilde{y}))$$

$$\cdot\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=-\int_0^T\int_{\mathbf{R}^{n+1}}\Big[\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}(-2\sigma_{jk}(\xi,s,y)\sigma_{ik\widetilde{y}_i}(\xi,\widetilde{s},\widetilde{y})$$

$$+\partial_{\widetilde{y}_i}(\sigma_{ik}(\xi,\widetilde{s},\widetilde{y})\sigma_{jk}(\xi,\widetilde{s},\widetilde{y})))$$

$$\cdot\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^T\int_{\mathbf{R}^n}\Big[\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\chi(v,u)v_{\widetilde{y}_i}(\sigma_{ik}(v,s,y)\sigma_{jk}(v,s,y)$$

$$-2\sigma_{ik}(v,s,y)\sigma_{jk}(v,\widetilde{s},\widetilde{y})+\sigma_{ik}(v,\widetilde{s},\widetilde{y})\sigma_{jk}(v,\widetilde{s},\widetilde{y}))$$

$$\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=\mathrm{L}_5^a+\mathrm{L}_8^a.$$

而

$$\mathrm{R}_6^a=\int_0^T\int_{\mathbf{R}^{n+1}}\Big(\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\eta,u)\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)$$

$$(\partial_{y_j}(\sigma_{ik}(\eta,s,y)\sigma_{jk}(\eta,s,y))-2\sigma_{jky_j}(\eta,s,y)\sigma_{ik}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s$$

$$\cdot\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\widetilde{\eta},v)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\psi_\delta(\xi-\widetilde{\eta})\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}$$

$$+\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\widetilde{\eta},v)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\psi_\delta(\xi-\widetilde{\eta})$$

$$\cdot(\partial_{\widetilde{y}_i}(\sigma_{ik}(\widetilde{\eta},\widetilde{s},\widetilde{y})\sigma_{jk}(\widetilde{\eta},\widetilde{s},\widetilde{y}))-2\sigma_{jk}(\eta,s,y)\sigma_{ik\widetilde{y}_i}(\widetilde{\eta},\widetilde{s},\widetilde{y}))\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}$$

$$\cdot\int_{\mathbf{R}^{n+1}\times\mathbf{R}^+}\chi(\eta,u)\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\psi_\delta(\xi-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}s\Big)\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$\to\int_0^T\int_{\mathbf{R}^{n+1}}\Big(\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}(\partial_{y_j}(\sigma_{ik}(\xi,s,y)\sigma_{jk}(\xi,s,y))-2\sigma_{jky_j}(\xi,s,y)\sigma_{ik}(\xi,\widetilde{s},\widetilde{y}))$$

$$\cdot\chi(\xi,u)\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\int_{\mathbf{R}^n\times\mathbf{R}^+}\chi(\xi,v)\partial_{\widetilde{y}_i}\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}$$

$$+\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}(\partial_{\widetilde{y}_i}(\sigma_{ik}(\xi,\widetilde{s},\widetilde{y})\sigma_{jk}(\xi,\widetilde{s},\widetilde{y}))-2\sigma_{jk}(\xi,s,y)\sigma_{ik\widetilde{y}_i}(\xi,\widetilde{s},\widetilde{y}))$$

$$\cdot\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\chi(\xi,v)\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\int_{\mathbf{R}^n\times\mathbf{R}^+}\chi(\xi,u)\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\Big)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

$$=\int_0^T\int_{\mathbf{R}^{n+1}}\Big(\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}2\sum_{k=1}^{K}\sum_{i,j=1}^{n}(\sigma_{jky_j}(\xi,s,y)\sigma_{ik\widetilde{y}_i}(\xi,\widetilde{s},\widetilde{y}))\chi(\xi,u)\chi(\xi,v)$$

$$
\begin{aligned}
&\cdot\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\\
&+\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}(\partial_{\widetilde{y}_i}(\sigma_{ik}(\xi,\widetilde{s},\widetilde{y})\sigma_{jk}(\xi,\widetilde{s},\widetilde{y}))-2\sigma_{jk}(\xi,s,y)\sigma_{ik\widetilde{y}_i}(\xi,\widetilde{s},\widetilde{y}))\\
&\cdot\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\chi(\widetilde{\xi},v)\chi(\xi,u)\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_{\mathbf{R}^n}\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}(\partial_{y_j}(\sigma_{ik}(v,s,y)\sigma_{jk}(v,s,y))\\
&-2\sigma_{jky_j}(v,s,y)\sigma_{ik}(v,\widetilde{s},\widetilde{y}))\\
&\cdot\chi(v,u)v_{\widetilde{y}_i}\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
=&\ \mathrm{L}_6^a+\mathrm{L}_9^a.
\end{aligned}
$$

上两式都取了 $\delta\to 0,\mathrm{R}_i\to+\infty(i=1,2,3)$ 时的极限. 同时

$$
\mathrm{R}_7^a=-2\int_0^T\int_{\mathbf{R}^{n+1}}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^2}\sum_{k=1}^{K}\sum_{i,j=1}^{n}\Psi_{ijk}\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\mathrm{d}\eta\,\mathrm{d}\widetilde{\eta}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s},
$$

这里

$$
\begin{aligned}
\Psi_{ijk}=&\ \sigma_{iky_i}(\eta,s,y)\sigma_{jk\widetilde{y}_j}(\widetilde{\eta},\widetilde{s},\widetilde{y})\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\\
&\cdot\chi(\eta,u)\chi(\widetilde{\eta},v)\psi_\delta(\xi-\eta)\psi_\delta(\xi-\widetilde{\eta})\rho(\xi,t,x).
\end{aligned}
$$

易知当 $\delta\to 0,\mathrm{R}_i\to+\infty$, R_7^a 收敛到函数 L_7^a, 其中

$$
\begin{aligned}
\mathrm{L}_7^a=&-2\int_0^T\int_{\mathbf{R}^{n+1}}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\Big(\sum_{k=1}^{K}\sum_{i,j=1}^{n}\sigma_{jky_j}(\xi,s,y)\sigma_{ik\widetilde{y}_i}(\xi,\widetilde{s},\widetilde{y})\Big)\chi(\xi,u)\chi(\xi,v)\\
&\cdot\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{3.14}
$$

由于 $\mathrm{L}_5^a+\mathrm{L}_6^a+\mathrm{L}_7^a=0$, 从而

$$
\mathrm{R}_5^a+\mathrm{R}_6^a+\mathrm{R}_7^a\to\mathrm{L}_8^a+\mathrm{L}_9^a\quad(\delta\to 0,\mathrm{R}_i\to+\infty).
$$

而

$$
\begin{aligned}
|\mathrm{L}_8^a|\leqslant&\int_0^T\int_{\mathbf{R}^n}\Big[\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\Big|\chi(v,u)v_{\widetilde{y}_i}(\sigma_{ik}(v,s,y)\sigma_{jk}(v,s,y)\\
&-2\sigma_{ik}(v,s,y)\sigma_{jk}(v,\widetilde{s},\widetilde{y})+\sigma_{ik}(v,\widetilde{s},\widetilde{y})\sigma_{jk}(v,\widetilde{s},\widetilde{y}))\\
&\cdot\partial_{y_j}\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\Big|\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\Big]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
\leqslant&\ c\Big(\frac{\varepsilon_1}{\varepsilon_2}+\varepsilon_1+\varepsilon_2\Big)||v_0||_{BV((R^n)}\to 0,\quad\varepsilon\to 0.
\end{aligned}
\tag{3.15}
$$

及

$$
\begin{aligned}
|\mathrm{L}_9^a|\leqslant&\int_0^T\int_{\mathbf{R}^n}\sum_{k=1}^{K}\sum_{i,j=1}^{n}\int_{\mathbf{R}^n\times\mathbf{R}^+}\int_{\mathbf{R}^n\times\mathbf{R}^+}\Big|\chi(v,u)v_{\widetilde{y}_i}\\
&(\partial_{y_j}(\sigma_{ik}(v,s,y)\sigma_{jk}(v,s,y))-2\sigma_{jky_j}(v,s,y)\sigma_{ik}(v,\widetilde{s},\widetilde{y}))
\end{aligned}
$$

$$\varphi_\varepsilon(t-s,x-y)\varphi_\varepsilon(t-\widetilde{s},x-\widetilde{y})\Big|\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{s}\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant c(\varepsilon_1+\varepsilon_2)||v||_{BV((R^n)}\to 0,\quad \varepsilon\to 0. \tag{3.16}$$

最后, 考虑 VIII.

$$\begin{aligned}\mathrm{VIII}&=\int_0^T\int_{\mathbf{R}^{n+1}}[(g\partial_u\chi^u)*\varphi_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}+(g\partial_v\chi^v)*\varphi_{\varepsilon,\delta}\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\&=\mathrm{R}_1^g+\mathrm{R}_2^g+\mathrm{R}_3^g+\mathrm{R}_4^g,\end{aligned}$$

其中

$$\begin{aligned}\mathrm{R}_1^g&=\int_0^T\int_{\mathbf{R}^{n+1}}[g\varphi_\delta\chi^v_{\varepsilon,\delta}+g\varphi_\delta\chi^u_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{R}_2^g&=\int_0^T\int_{\mathbf{R}^{n+1}}[g'\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}]\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{R}_3^g&=\frac{1}{\mathrm{R}_3}\int_0^T\int_{\mathbf{R}^{n+1}}[g\chi^u_{\varepsilon,\delta}\chi^v_{\varepsilon,\delta}]\rho'\Big(\frac{\xi}{\mathrm{R}_3}\Big)\rho(t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t,\\
\mathrm{R}_4^g&=\int_0^T\int_{\mathbf{R}^{n+1}}[(g\partial_u\chi^u)*\varphi_{\varepsilon,\delta}-g\partial_u\chi^u_{\varepsilon,\delta}]\chi^v_{\varepsilon,\delta}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\mathbf{R}^{n+1}}[(g\partial_v\chi^v)*\varphi_{\varepsilon,\delta}-g\partial_v\chi^v_{\varepsilon,\delta}]\chi^u_{\varepsilon,\delta}\rho(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

引理 3.5　令 $\delta\to 0$, $\mathrm{R}_i\to+\infty$ $(i=1,2,3)$, 得

$$\mathrm{R}_1^g\to 0,\qquad \mathrm{R}_3^g\to 0,$$
$$\mathrm{R}_2^g\to\int_0^T\int_{\mathbf{R}^{n+1}}[g'\chi^u_\varepsilon\chi^v_\varepsilon]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

且 R_4^g 收敛到函数 L_4^g, 当 $\varepsilon\to 0$ 时, 它趋于 0.

这个引理的证明类似于引理 3.4, 从而略去细节. 利用引理 3.2–3.5 及 (3.10), 得

$$\begin{aligned}&\int_{\mathbf{R}^{n+1}}(\chi^u_\varepsilon\chi^v_\varepsilon(\xi,T,x)-\chi^u_\varepsilon\chi^u_\varepsilon(\xi,0,x))\,\mathrm{d}\xi\,\mathrm{d}x\\
&\geqslant\int_0^T\int_{\mathbf{R}^{n+1}}g'\chi^u_\varepsilon\chi^v_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t-\mathrm{L}_1^f-\mathrm{L}_2^f-\mathrm{L}_5^f+\mathrm{L}_6^f-\mathrm{L}_8^a-\mathrm{L}_9^a\\
&\quad+\mathrm{L}_4^g-\int_0^T\int_{\mathbf{R}^n}(m_\varepsilon+n_\varepsilon)(0,t,x)+(p_\varepsilon+q_\varepsilon)(0,t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{3.17}$$

将 (3.5)–(3.6) 和 (3.17) 合在一起, 得到

$$\begin{aligned}&\int_{\mathbf{R}^{n+1}}(|\chi^u_\varepsilon|+|\chi^v_\varepsilon|-2\chi^u_\varepsilon\chi^v_\varepsilon)(\xi,T,x)\,\mathrm{d}\xi\,\mathrm{d}x-\int_{\mathbf{R}^{n+1}}(|\chi^u_\varepsilon|+|\chi^v_\varepsilon|-2\chi^u_\varepsilon\chi^v_\varepsilon)(\xi,0,x)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\leqslant\int_0^T\int_{\mathbf{R}^{n+1}}(g'(|\chi^u_\varepsilon|+|\chi^v_\varepsilon|-2\chi^u_\varepsilon\chi^v_\varepsilon))(\xi,t,x)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+2\mathrm{L}_1^f+2\mathrm{L}_2^f+2\mathrm{L}_5^f-2\mathrm{L}_6^f+2\mathrm{L}_8^a+2\mathrm{L}_9^a-2\mathrm{L}_4^g+T^u_\varepsilon+T^v_\varepsilon.\end{aligned}$$

首先令 $\varepsilon_1\to 0$, 然后令 $\varepsilon_2\to 0$, 利用引理 3.1–3.5 及 (3.15)–(3.16), 得

$$\int_{\mathbf{R}^n}|u(T,x)-v(T,x)|\,\mathrm{d}x$$

$$\leqslant \int_{\mathbf{R}^n} |u(0,x)-v(0,x)|\,\mathrm{d}x + M\int_0^T\int_{\mathbf{R}^n} |u(t,x)-v(t,x)|\,\mathrm{d}x\,\mathrm{d}t.$$

通过 Gronwall 不等式, 最后得到

$$||u(T,\cdot)-v(T,\cdot)||_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant \mathrm{e}^{MT}||u_0(\cdot)-v_0(\cdot)||_{L^1(\mathbf{R}^n)},$$

这正是所要的结果.

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Uniqueness of Kinetic Solutions to Quasilinear Anisotropic Degenerate Parabolic-Hyperbolic Equation

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Abstract This paper deals with the uniqueness of the kinetic solutions to Cauchy problem of general anisotropic degenerate parabolic-hyperbolic equations. Kinetic formulation is extended to such general degenerate parabolic-hyperbolic equations with coefficients depending on time-spatial variables. Contraction property of kinetic solutions is established under appropriate conditions on diffusion and convection functions.

Keywords Degenerate parabolic-hyperbolic equation, Kinetic solutions, Entropy solutions, Uniqueness

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