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Hartogs 域上延拓算子的性质*

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提要 主要研究 Roper-Suffridge 延拓算子在推广的 Hartogs 域上的性质. 借助双全纯映照的偏差定理, 得到延拓算子在 Ω_N 上保持强 α 次殆 β 型螺形映照、α 次殆 β 型螺形映照和 α 次 β 型螺形映照的性质, 进而得到 B^n 上相应的结论. 所得结论包含已有的结果并为研究 $\mathbb{C}^n$ 中的双全纯映照提供了新的途径.

关键词 星形映照, 螺形映照, 延拓算子, Hartogs 域
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1 引 言

多复变几何函数论源于单复变理论, 然而其本质却不同. Roper-Suffridge 算子

$$\phi_n(f)(z) = (f(z_1), \sqrt{f'(z_1)}z_0)'$$

(其中 $z = (z_1, z_0) \in B^n$, $z_1 \in D$, $z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$, $f(z_1) \in H(D)$, $\sqrt{f'(0)} = 1$) 是单复变与多复变之间的桥梁. 该算子由 Roper 和 Suffridge 在文 [1] 中引入, 在文 [1–2] 中被证明在 B^n 上保持凸性、星形性和 Bloch 映照的性质. 因此, 通过 Roper-Suffridge 算子我们可以由单复变中有特殊几何性质的双全纯函数来构造多复变中相应的映照. 因此许多学者开始研究该算子的性质. 随着不同的域和星形映照及凸映照子族的涌现, 对于 Roper-Suffridge 算子的研究已有了许多优美的结论[3–5].

2005 年, Muir[6]引进 Roper-Suffridge 延拓算子:

$$F(z) = (f(z_1) + f'(z_1)P(z_0), \sqrt{f'(z_1)}z_0)',$$

其中 f 是单位圆盘 D 上正规化双全纯函数, $z = (z_1, z_0)' \in B^n$, $z_1 \in D$, $z_0 = (z_2, \cdots, z_n)' \in \mathbb{C}^{n-1}$. 幂函数取分支, 使得 $\sqrt{f'(0)} = 1$. $P : \mathbb{C}^{n-1} \to \mathbb{C}$ 是 2 阶齐次多项式. Muir 和 Suffridge 证明了该算子在 $\|P\| \leqslant \frac{1}{4}$ 和 $\|P\| \leqslant \frac{1}{2}$ 时分别保持星形性和凸性. Kohr 在文 [7] 中借助 Loewner 链讨论了推广的算子. 王建飞和刘太顺[8]得到了算子

$$F(z) = (f(z_1) + f'(z_1)P(z_0), [f'(z_1)]^{\frac{1}{m}} z_0)'$$

保持 α 次殆星形性和 α 次星形性, 其中 $[f'(0)]^{\frac{1}{m}} = 1$, $P : \mathbb{C}^{n-1} \to \mathbb{C}$ 是 $m(m \geqslant 2)$ 阶齐次多项式. 2008 年, Muir[9]在复 Banach 空间单位球上引进了延拓算子

$$[\Phi_{G,\gamma}(f)](z) = (f(z_1) + G([f'(z_1)]^\gamma z_0), [f'(z_1)]^\gamma z_0)', \tag{1.1}$$

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其中 G 是 $\mathbb{C}^{n-1}$ 中的全纯函数, $G(0)=0$, $DG(0)=0$, $\gamma\geqslant 0$, $[f'(z_1)]^{\gamma}|_{z_1=0}=1$. $G(z)$ 的齐次展开式是 $\sum\limits_{j=2}^{\infty}Q_j(z)$, $Q_j(z)$ 是 j 阶齐次多项式. Muir 证明了 $[\Phi_{G,\gamma}(f)](z)$ 是一个 Loewner 链并保持一些几何性质不变. 2016 年唐言言[10]在 Bergman-Hartogs 域:
$\Omega_{p,q}^{B^n}=\{(w,z)\in\mathbb{C}^m\times B^n:\|w\|^{2p}<K_{B^n}(z,z)^{-q}\},\quad w=(w_{(1)},\cdots,w_{(m)}),\ z=(z_1,\cdots,z_0)$
上推广了 Roper-Suffridge 算子, 潘利双和王安在文 [11] 中对域进行了讨论. 唐言言证明了推广的算子在 $\Omega_{p,q}^{B^n}$ 上保持 α 次殆 β 型螺形性、α 次 β 型螺形性和强 β 型螺形性.

由文 [12–13] 中的 Hartogs 域, 这里将延拓算子推广为

$$\begin{aligned}F(\xi,z,w)=&((f'(z_1))^{\delta_1}(f'(w_1))^{\gamma_1}\xi_{(1)},\cdots,(f'(z_1))^{\delta_r}(f'(w_1))^{\gamma_r}\xi_{(r)},f(z_1)+G_1[(f'(z_1))^{\sigma_1}z_0],\\&(f'(z_1))^{\sigma_1}z_0,f(w_1)+G_2[(f'(w_1))^{\sigma_2}w_0],(f'(w_1))^{\sigma_2}w_0)',\quad (\xi,z,w)\in\Omega_N,\end{aligned}\tag{1.2}$$

其中幂函数选取主值支,

$$G_1[(f'(z_1))^{\sigma_1}z_0]=\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j}P_j(z_0),\quad G_2[(f'(w_1))^{\sigma_2}w_0]=\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j}Q_j(w_0),$$

P_j 和 Q_j 分别是 z_0 和 w_0 的 j $(j\geqslant 2)$ 次齐次多项式, 且

$$\begin{aligned}\Omega_N=\Big\{&(\xi_{(1)},\cdots,\xi_{(r)},z,w)\in\mathbb{C}^{m_1}\times\cdots\times\mathbb{C}^{m_r}\times B^{N_1}(0,1)\times B^{N_2}(0,1):\\&\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}<(1-\|z\|^2)^l(1-\|w\|^2)^t,\quad s_i>0(i=1,\cdots,r),\ l\geqslant 0,\ t\geqslant 0\Big\}.\end{aligned}$$

在 (1.2) 中若不考虑含有 ξ 及 w 的项, 则有以下算子

$$F(z)=(f(z_1)+G_1[(f'(z_1))^{\sigma_1}z_0],(f'(z_1))^{\sigma_1}z_0)'.\tag{1.3}$$

本文主要研究算子 (1.2) 所保持的几何性质. 第 3–5 节讨论 (1.2) 在 Ω_N 上在不同的条件下分别保持强 α 次殆 β 型螺形性、α 次殆 β 型螺形性和 α 次 β 型螺形性. 由此得到 (1.2) 在 Ω_N 上保持强 α 次殆星形性、强 β 型螺形性、α 次殆星形性和 β 型螺形性及 (1.1) 在 B^n 上的相应性质.

2 定义及引理

以下 D 表示 $\mathbb{C}$ 中的单位圆盘, B^n 表示 $\mathbb{C}^n$ 中的单位球. $DF(z)$ 表示 F 在 z 的 Fréchet 导数. $I[\frac{a}{b}]$ 表示 $\frac{a}{b}$ 的整数部分. 为得到主要结论, 我们需要以下定义及引理.

定义 2.1 [14] 设 Ω 是 $\mathbb{C}^n$ 中的有界星形圆型域, 其 Minkowski 泛函 $\rho(z)$ 在除去一个低微流行外是 C^1 的. 设 $f(z)$ 是 Ω 上正规化的局部双全纯映照. 若

$$\Big|\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}+\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{2}{\rho(z)}\frac{\partial\rho(z)}{\partial z}(Df(z))^{-1}f(z)-\frac{1+c^2}{1-c^2}\Big|<\frac{2c}{1-c^2},$$

则称 $f(z)$ 是 Ω 上的强 α 次殆 β 型螺形映照 $\big(\alpha\in[0,1),\ \beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)\big)$.

在定义 2.1 中分别令 $\alpha=0$, $\beta=0$, $\alpha=\beta=0$, 则得到强 α 次殆星形映照、强 β 型螺形映照和强星形映照的定义.

定义 2.2 [15] 设 Ω 和 $f(z)$ 同定义 2.1, 若

$$\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(z)}\frac{\partial\rho(z)}{\partial z}(Df(z))^{-1}f(z)\Big]\geqslant\alpha\cos\beta,\quad z\in\Omega\setminus\{0\},$$

其中 $\alpha\in[0,1)$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则称 $f(z)$ 是 Ω 上的 α 次殆 β 型螺形映照.

在定义 2.2 中分别令 $\alpha=0$, $\beta=0$, 则得到 β 型螺形映照和 α 次殆星形映照的定义.

定义 2.3[16] 设 Ω 和 $f(z)$ 同定义 2.1, 若

$$\Big|(1-\mathrm{i}\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho(z)}{\partial z}[Df(z)]^{-1}f(z)-\frac{1}{2\alpha}+\mathrm{i}\tan\beta\Big|<\frac{1}{2\alpha},\quad z\in\Omega\setminus\{0\},$$

其中 $\alpha\in(0,1)$, $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$, 则称 f 是 Ω 上的 α 次 β 型螺形映照.

在定义 2.3 中令 $\beta=0$ 和 $\Omega=B^n$, 则得到 B^n 上 α 次星形映照[17]的定义.

引理 2.1[18] 设 Ω 同定义 2.1, 则

$$2\frac{\partial\rho(z)}{\partial z}z=\rho(z),\quad \frac{\partial\rho}{\partial z}(\lambda z)=\frac{\partial\rho(z)}{\partial z},\quad \lambda\geqslant0,\quad \frac{\partial\rho}{\partial z}(\mathrm{e}^{\mathrm{i}\theta}z)=\mathrm{e}^{-\mathrm{i}\theta}\frac{\partial\rho(z)}{\partial z},\quad \theta\in R.$$

引理 2.2 设 $\rho(\xi,z,w)=\rho$ 是 Ω_N 的 Minkowski 泛函, 则

$$\begin{aligned}\frac{\partial\rho}{\partial\xi_{ij}}=&\frac{s_i}{2}\|\xi_{(i)}\|^{2(s_i-1)}\overline{\xi}_{ij}\rho^{3-2s_i}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\&+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)\\&+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1},\end{aligned}\tag{2.1}$$

$$\begin{aligned}\frac{\partial\rho}{\partial z_{j_1}}=&\frac{\rho l}{2}\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t}\overline{z}_{j_1}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\&+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)\\&+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1},\end{aligned}\tag{2.2}$$

$$\begin{aligned}\frac{\partial\rho}{\partial w_{j_2}}=&\frac{\rho t}{2}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l}\overline{w}_{j_2}\Big\{\sum_{k=1}^{r}\|\xi_{(k)}\|^{2s_k}s_k\rho^{2-2s_k}\\&+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)\\&+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big]\Big\}^{-1}.\end{aligned}\tag{2.3}$$

若 $(\xi,z,w)\in\partial\Omega_N$, 则有 $\rho=1$,

$$\begin{cases}\dfrac{\partial\rho}{\partial\xi_{ij}}=\dfrac{s_i\|\xi_{(i)}\|^{2(s_i-1)}\overline{\xi}_{ij}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)},\\ \dfrac{\partial\rho}{\partial z_{j_1}}=\dfrac{\rho\triangle_2\overline{z}_{j_1}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)},\\ \frac{\partial\rho}{\partial w_{j_2}}=\frac{\rho\triangle_3\overline{w}_{j_2}}{2(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)},\end{cases}\tag{2.4}$$

其中 $i=1,\cdots,r,\quad j=1,\cdots,m_i,\quad j_1=1,\cdots,N_1,\quad j_2=1,\cdots,N_2,$

$$\triangle_1=\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}s_i,\quad \triangle_2=l(1-\|z\|^2)^{l-1}(1-\|w\|^2)^{t},\quad \triangle_3=t(1-\|w\|^2)^{t-1}(1-\|z\|^2)^{l}.$$

证 由于 $\rho(\xi,z,w)$ 是 Ω_N 的 Minkowski 泛函, 则 $\frac{(\xi,z,w)}{\rho}\in\partial\Omega_N$, 于是

$$\Big\|\frac{\xi_{(1)}}{\rho}\Big\|^{2s_1}+\cdots+\Big\|\frac{\xi_{(r)}}{\rho}\Big\|^{2s_r}=\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^l\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^t. \tag{2.5}$$

(2.5) 式两端关于 ξ_{ij} 求偏导并由 (2.1) 式, 得

$$\begin{aligned}&s_i\|\xi_{(i)}\|^{2(s_i-1)}\overline{\xi}_{ij}\rho^{-2s_i}+\sum_{k=1}^r\|\xi_{(k)}\|^{2s_k}(-2s_k)\rho^{-2s_k-1}\frac{\partial\rho}{\partial\xi_{ij}}\\ =&\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}2\rho^{-3}\frac{\partial\rho}{\partial\xi_{ij}}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big].\end{aligned}$$

(2.5) 式两端关于 z_{j_1} 求偏导并由 (2.2) 式, 得

$$\begin{aligned}&\sum_{k=1}^r\|\xi_{(k)}\|^{2s_k}(-2s_k)\rho^{-2s_k-1}\frac{\partial\rho}{\partial z_{j_1}}\\ =&-l\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\overline{z}_{j_1}\rho^{-2}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^t\\ &+\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)^{l-1}\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)^{t-1}2\rho^{-3}\frac{\partial\rho}{\partial z_{j_1}}\Big[l\|z\|^2\Big(1-\Big\|\frac{w}{\rho}\Big\|^2\Big)+t\|w\|^2\Big(1-\Big\|\frac{z}{\rho}\Big\|^2\Big)\Big].\end{aligned}$$

类似地有 (2.3) 式. 当 $(\xi,z,w)\in\partial\Omega_N$ 时, 有 $\rho=1$ 和 (2.4) 式成立, 这里

$$\triangle_1=\sum_{i=1}^r\|\xi_{(i)}\|^{2s_i}s_i,\quad \triangle_2=l(1-\|z\|^2)^{l-1}(1-\|w\|^2)^t,\quad \triangle_3=t(1-\|w\|^2)^{t-1}(1-\|z\|^2)^l.$$

引理 2.3 设 $F(\xi,z,w)$ 是由 (1.2) 式所定义的函数, $(\xi,z,w)\in\partial\Omega_N$, 则

$$\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)=\frac{H}{\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3},$$

其中

$$\begin{aligned}H=&\triangle_1\Big\{1-\delta_i\frac{f''(z_1)}{f'(z_1)}\Big[\frac{f(z_1)}{f'(z_1)}+\sum_{j=2}^\infty(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]\\ &-\gamma_i\frac{f''(w_1)}{f'(w_1)}\Big[\frac{f(w_1)}{f'(w_1)}+\sum_{j=2}^\infty(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\Big\}\\ &+\triangle_2\Big\{|z_1|^2\Big[\frac{f(z_1)}{z_1f'(z_1)}+\frac{1}{z_1}\sum_{j=2}^\infty(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]\\ &+\|z_0\|^2\Big[1-\sigma_1\frac{f''(z_1)}{f'(z_1)}\Big(\frac{f(z_1)}{f'(z_1)}+\sum_{j=2}^\infty(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big)\Big]\Big\}\\ &+\triangle_3\Big\{|w_1|^2\Big[\frac{f(w_1)}{w_1f'(w_1)}+\frac{1}{w_1}\sum_{j=2}^\infty(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\\ &+\|w_0\|^2\Big[1-\sigma_2\frac{f''(w_1)}{f'(w_1)}\Big(\frac{f(w_1)}{f'(w_1)}+\sum_{j=2}^\infty(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big)\Big]\Big\}.\end{aligned}$$

证 由 (1.2) 式得

$$DF=\begin{pmatrix} \lambda_1 & \cdots & 0_{m_1\times m_r} & v_1 & 0_{m_1\times(N_1-1)} & \sigma_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{m_r\times m_1} & \cdots & \lambda_r & v_r & 0_{m_r\times(N_1-1)} & \sigma_r \\ 0_{1\times m_1} & \cdots & 0_{1\times m_r} & u_1 & \sum\limits_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j}DP_j(z_0) & 0_{1\times N_2} \\ 0_{(N_1-1)\times m_1} & \cdots & 0_{(N_1-1)\times m_r} & u_2 & (f'(z_1))^{\sigma_1}I_{N_1-1} & 0_{(N_1-1)\times N_2} \\ 0_{N_2\times m_1} & \cdots & 0_{N_2\times m_r} & 0_{N_2\times 1} & 0_{N_2\times(N_1-1)} & J \end{pmatrix},$$

其中

$$\lambda_i=(f'(z_1))^{\delta_i}(f'(w_1))^{\gamma_i}I_{m_i},\quad v_i=\delta_i(f'(z_1))^{\delta_i-1}f''(z_1)(f'(w_1))^{\gamma_i}\xi'_{(i)},\quad i=1,\cdots,r,$$

$$\sigma_i=\Big(\gamma_i(f'(w_1))^{\gamma_i-1}f''(w_1)(f'(z_1))^{\delta_i}\xi'_{(i)},0_{m_i\times(N_2-1)}\Big),\quad i=1,\cdots,r,$$

$$u_1=f'(z_1)+\sum_{j=2}^{\infty}\sigma_1 j(f'(z_1))^{\sigma_1 j-1}f''(z_1)P_j(z_0),\quad u_2=\sigma_1(f'(z_1))^{\sigma_1-1}f''(z_1)z_0',$$

$$J=\begin{pmatrix} f'(w_1)+\sum\limits_{j=2}^{\infty}\sigma_2 j(f'(w_1))^{\sigma_2 j-1}f''(w_1)Q_j(w_0) & \sum\limits_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j}DQ_j(w_0) \\ \sigma_2(f'(w_1))^{\sigma_2-1}f''(w_1)w_0' & (f'(w_1))^{\sigma_2}I_{N_2-1} \end{pmatrix}.$$

经简单计算得 $(DF(\xi,z,w))^{-1}F(\xi,z,w)=(h_1,\cdots,h_r,h_{(2)},h_{(3)},h_{(4)},h_{(5)})'$, 其中

$$\begin{cases} h_i=\xi_{(i)}\Big\{1-\delta_i\dfrac{f''(z_1)}{f'(z_1)}\Big[\dfrac{f(z_1)}{f'(z_1)}+\sum\limits_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big] \\ \qquad -\gamma_i\dfrac{f''(w_1)}{f'(w_1)}\Big[\dfrac{f(w_1)}{f'(w_1)}+\sum\limits_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\Big\},\quad i=1,\cdots,r, \\ h_{(2)}=\dfrac{f(z_1)}{f'(z_1)}+\sum\limits_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0), \\ h_{(3)}=z_0\Big\{1-\sigma_1\dfrac{f''(z_1)}{f'(z_1)}\Big[\dfrac{f(z_1)}{f'(z_1)}+\sum\limits_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]\Big\}, \\ h_{(4)}=\dfrac{f(w_1)}{f'(w_1)}+\sum\limits_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0), \\ h_{(5)}=w_0\Big\{1-\sigma_2\dfrac{f''(w_1)}{f'(w_1)}\Big[\dfrac{f(w_1)}{f'(w_1)}+\sum\limits_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\Big\}. \end{cases}$$

于是由引理 2.2 知结论成立.

引理 2.4 [19] 设 $f(z)$ 是 D 上的全纯函数, $|f(z)|<1$, 则 $|f'(z)|\leqslant\frac{1-|f(z)|^2}{1-|z|^2}$, $\quad z\in D$.

引理 2.5 [10] 设 $f(z_1)$ 是 D 上正规化的双全纯函数, $a,b\in\mathbb{R}^+$, $a+b\leqslant\frac{1}{2}$, $z=(z_1,z_0)\in B^n$, 则

$$\Big|\frac{f''(z_1)}{f'(z_1)}[a(1-|z_1|^2)+b\|z_0\|^2]-\overline{z}_1\Big|<2(a+b)+1.$$

引理 2.6 设 $z=(z_1,\cdots,z_n)\in B^n$, $d,\sigma_1\in\big[0,\frac{1}{2}\big]$, 则

$$M=|z_1|^2+[d(1-\|z\|^2)+\sigma_1\|z_0\|^2]\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\geqslant 0.$$

证 当 $|z_1|\in[0,\sqrt{2}-1)$ 时, $1-2|z_1|-|z_1|^2\geqslant 0$, 从而 $M\geqslant 0$; 当 $|z_1|\in(\sqrt{2}-1,1]$ 时, 有

$$\begin{aligned}M&=|z_1|^2+[d(1-|z_1|^2)+(\sigma_1-d)\|z_0\|^2]\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\\&=|z_1|^2+d(1-2|z_1|-|z_1|^2)+(\sigma_1-d)\|z_0\|^2\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}.\end{aligned}$$

(i) 若 $\sigma_1\leqslant d$, 则 $(\sigma_1-d)\|z_0\|^2\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\geqslant 0$. 另外, 当 $d\in[0,\frac{1}{2}]$ 时,

$$|z_1|^2+d(1-2|z_1|-|z_1|^2)=(1-d)\Big(|z_1|-\frac{d}{1-d}\Big)^2+\frac{d(1-2d)}{1-d}\geqslant 0,$$

于是 $M\geqslant 0$.

(ii) 若 $\sigma_1>d$, 由于 $\sigma_1\in[0,\frac{1}{2}]$, 则

$$\begin{aligned}M&=|z_1|^2+[d(1-|z_1|^2)+(\sigma_1-d)\|z_0\|^2]\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\\&\geqslant|z_1|^2+\sigma_1(1-2|z_1|-|z_1|^2)=(1-\sigma_1)\Big(|z_1|-\frac{\sigma_1}{1-\sigma_1}\Big)^2+\frac{\sigma_1(1-2\sigma_1)}{1-\sigma_1}\geqslant 0.\end{aligned}$$

引理 2.7 [20] 设 $f(z)$ 是 D 上正规化的双全纯函数, 则 $\frac{1-|z|}{(1+|z|)^3}\leqslant|f'(z)|\leqslant\frac{1+|z|}{(1-|z|)^3}$.

引理 2.8 [21] 设 $p(z)$ 是 D 上的全纯函数, $p(0)=1$, $\Re p(z)>0$, 则 $|p'(z)|\leqslant\frac{2\Re p(z)}{1-|z|^2}$.

引理 2.9 [22] 设 $f:D^n\to\mathbb{C}^n$ 是 α 次殆 β 型螺形映照, 则对于 $\forall z\in D^n\backslash\{0\}$, 存在单位向量 $\xi(z)$, 使得

$$\|Df(z)\xi(z)\|\leqslant\frac{1+\|z\|}{\cos\beta[1-(1-2\alpha)\|z\|]^{\frac{2(\alpha-1)}{2\alpha-1}+1}},\quad\|z\|=\max|z_j|,\quad 1\leqslant j\leqslant n.$$

引理 2.10 [23] 设 $f(z)$ 是 D 上的 α 次 β 型螺形函数, $\alpha\in(0,1)$, $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$. 则

$$|f'(z)|\leqslant\frac{1+(1-2\alpha)|z|}{\cos\beta(1-|z|)^{1+2(1-\alpha)}},\quad z\in D.$$

3 强 α 次殆 β 型螺形映照的不变性

定理 3.1 设 $f(z_1)$ 是 D 上的强 α 次殆 β 型螺形函数, $\alpha\in[0,1)$, $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$, $c\in(0,1)$. 令 $F(\xi,z,w)$ 是由 (1.2) 式所定义的函数, $\delta_i,\gamma_i,\delta_i+\gamma_i\in[0,1]$, $\frac{s_i\delta_i}{l},\frac{s_i\gamma_i}{t}\in\big[0,\frac{1}{2}\big](i=1,\cdots,r)$, $\sigma_1,\sigma_2\in\big[0,\frac{1}{2}\big]$. 令 $P_j=0\big(j>\frac{4}{6\sigma_1-1}\big),Q_j=0\big(j>\frac{4}{6\sigma_2-1}\big)$, $\sigma_1,\sigma_2\in\big(\frac{1}{6},\frac{1}{2}\big]$. 若

$$\begin{cases}\displaystyle\sum_{j=2}^{I[\frac{4}{2\sigma_1+1}]}(j-1)\|P_j\|+\sum_{j=I[\frac{4}{2\sigma_1+1}]+1}^{\infty}(j-1)\|P_j\|2^{\frac{2\sigma_1+1}{2}j-2}\leqslant\frac{c(1-\sigma_1)(1-\alpha)\cos\beta}{1+c},\\ \displaystyle\sum_{j=2}^{I[\frac{4}{2\sigma_2+1}]}(j-1)\|Q_j\|+\sum_{j=I[\frac{4}{2\sigma_2+1}]+1}^{\infty}(j-1)\|Q_j\|2^{\frac{2\sigma_2+1}{2}j-2}\leqslant\frac{c(1-\sigma_2)(1-\alpha)\cos\beta}{1+c},\end{cases}$$

则 $F(\xi,z,w)$ 是 Ω_N 上的强 α 次殆 β 型螺形映照.

证 由定义 2.1, 需证

$$\Big|\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{2}{\rho}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF)^{-1}F+\frac{1-c^2}{2c}\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}-\frac{1+c^2}{2c}\Big|<1. \tag{3.1}$$

令 $(\xi,z,w)=\zeta(\eta,\lambda,\mu)=|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu)$, 其中 $(\eta,\lambda,\mu)\in\partial\Omega$, $\zeta\in\overline{D}\setminus\{0\}$, 则由引理 2.1, 有

$$\begin{aligned}&\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)\\ &=\frac{2}{\rho(|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu))}\frac{\partial\rho}{\partial(\xi,z,w)}(|\zeta|\mathrm{e}^{\mathrm{i}\theta}(\eta,\lambda,\mu))(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)\\ &=\frac{2\partial\rho(\eta,\lambda,\mu)}{\partial(\xi,z,w)}\frac{(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)}{\zeta}.\end{aligned}$$

显然 $\frac{2\partial\rho(\eta,\lambda,\mu)}{\partial(\xi,z,w)}\frac{(DF(\zeta\eta,\zeta\lambda,\zeta\mu))^{-1}F(\zeta\eta,\zeta\lambda,\zeta\mu)}{\zeta}$ 关于 ζ 全纯. 由全纯函数的最大模原理, 只需证 $(\xi,z,w)\in\partial\Omega_N$ 时 (3.1) 式成立, 此时 $\rho(\xi,z,w)=1$. 令

$$q(z_1)=\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)}{z_1f'(z_1)}+\frac{1-c^2}{2c}\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}-\frac{1+c^2}{2c}. \tag{3.2}$$

则 $|q(z_1)|<1$,

$$\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}=-c-q(z_1)-z_1q'(z_1). \tag{3.3}$$

又由 $(\xi,z,w)\in\partial\Omega$ 得 $\sum\limits_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}=(1-\|z\|^2)^l(1-\|w\|^2)^t$, 则

$$\triangle_2=\frac{l}{1-\|z\|^2}\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i},\quad \triangle_3=\frac{t}{1-\|w\|^2}\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}. \tag{3.4}$$

由引理 2.3, (3.2)–(3.4) 式, 得

$$\begin{aligned}&\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)+\frac{1-c^2}{2c}\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}-\frac{1+c^2}{2c}\\ &=\frac{E}{\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3},\end{aligned}$$

其中

$$\begin{aligned}E&=\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}H+\Big(\frac{1-c^2}{2c}\frac{-\alpha+\mathrm{i}\tan\beta}{1-\alpha}-\frac{1+c^2}{2c}\Big)(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\ &=\triangle_1\Big\{c(\delta_i+\gamma_i-1)+\delta_i[z_1q'(z_1)+q(z_1)]+\gamma_i[w_1q'(w_1)+q(w_1)]+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\\ &\quad\times\Big[\delta_i\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1j-1}(j-1)P_j(z_0)+\gamma_i\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2j-1}(j-1)Q_j(w_0)\Big]\Big\}\\ &\quad+\triangle_2\Big\{|z_1|^2\Big[q(z_1)+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\frac{1}{z_1}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1j-1}(1-j)P_j(z_0)\Big]+\|z_0\|^2\Big[c(\sigma_1-1)\\ &\quad+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\sigma_1\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1j-1}(j-1)P_j(z_0)+\sigma_1(z_1q'(z_1)+q(z_1))\Big]\Big\}\\ &\quad+\triangle_3\Big\{|w_1|^2\Big[q(w_1)+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{(1-\alpha)w_1}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2j-1}(1-j)Q_j(w_0)\Big]+\|w_0\|^2\Big[c(\sigma_2-1)\end{aligned}$$

$$+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\sigma_2\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)+\sigma_2(w_1q'(w_1)+q(w_1))\Big]\Big\}$$

$$=\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{c\Big[s_i(\delta_i+\gamma_i-1)+\frac{l\|z_0\|^2}{1-\|z\|^2}(\sigma_1-1)+\frac{t\|w_0\|^2}{1-\|w\|^2}(\sigma_2-1)\Big]$$

$$+\frac{l}{1-\|z\|^2}\Big[|z_1|^2q(z_1)+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big(\frac{f''(z_1)}{f'(z_1)}\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)-\overline{z}_1\Big)$$

$$\times\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)+\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)(z_1q'(z_1)+q(z_1))\Big]$$

$$+\frac{t}{1-\|w\|^2}\Big[|w_1|^2q(w_1)+\frac{1-c^2}{2c}\frac{1-\mathrm{i}\tan\beta}{1-\alpha}\Big(\frac{f''(w_1)}{f'(w_1)}\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)-\overline{w}_1\Big)$$

$$\times\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)+\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)(w_1q'(w_1)+q(w_1))\Big]\Big\}.$$

由于 $\delta_i+\gamma_i\in[0,1]$, $\frac{s_i\delta_i}{l},\frac{s_i\gamma_i}{t}\in\big[0,\frac{1}{2}\big]$, $\sigma_1,\sigma_2\in\big[0,\frac{1}{2}\big]$, 由引理 2.4–2.7, 得

$$|E|-(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)<\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}A,$$

其中

$$A=c\Big[s_i(1-\delta_i-\gamma_i)+\frac{l\|z_0\|^2}{1-\|z\|^2}(1-\sigma_1)+\frac{t\|w_0\|^2}{1-\|w\|^2}(1-\sigma_2)\Big]$$

$$+\frac{l}{1-\|z\|^2}\Big\{|z_1|^2|q(z_1)|+\frac{1-c^2}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)|P_j(z_0)|$$

$$+\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big]\Big[\frac{2|z_1|(1-|q(z_1)|)}{1-|z_1|^2}+|q(z_1)|\Big]\Big\}$$

$$+\frac{t}{1-\|w\|^2}\Big\{|w_1|^2|q(w_1)|+\frac{1-c^2}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)|Q_j(w_0)|$$

$$+\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big]\Big[\frac{2|w_1|(1-|q(w_1)|)}{1-|w_1|^2}+|q(w_1)|\Big]\Big\}$$

$$-\Big(s_i+\frac{l\|z\|^2}{1-\|z\|^2}+\frac{t\|w\|^2}{1-\|w\|^2}\Big)$$

$$=c\Big[s_i(1-\delta_i-\gamma_i)+\frac{l\|z_0\|^2}{1-\|z\|^2}(1-\sigma_1)+\frac{t\|w_0\|^2}{1-\|w\|^2}(1-\sigma_2)\Big]-s_i$$

$$+\frac{l}{1-\|z\|^2}\Big\{(|q(z_1)|-1)\Big[|z_1|^2+\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)\Big]$$

$$+\frac{1-c^2}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)|P_j(z_0)|+\frac{s_i\delta_i}{l}(1-\|z\|^2)+(\sigma_1-1)\|z_0\|^2\Big\}$$

$$+\frac{t}{1-\|w\|^2}\Big\{(|q(w_1)|-1)\Big[|w_1|^2+\frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2}\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)\Big]$$

$$+\frac{1-c^2}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)|Q_j(w_0)|+\frac{s_i\gamma_i}{t}(1-\|w\|^2)+(\sigma_2-1)\|w_0\|^2\Big\}$$

$$
\begin{aligned}
&= (c-1)s_i(1-\delta_i-\gamma_i)\\
&\quad + \frac{l}{1-\|z\|^2}\Big\{(|q(z_1)|-1)\Big[|z_1|^2+\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)\Big]\\
&\quad + (c-1)\|z_0\|^2\Big[(1-\sigma_1)-\frac{1+c}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)\|P_j\|\|z_0\|^{j-2}\Big]\Big\}\\
&\quad + \frac{t}{1-\|w\|^2}\Big\{(|q(w_1)|-1)\Big[|w_1|^2+\frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2}\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)\Big]\\
&\quad + (c-1)\|w_0\|^2\Big[(1-\sigma_2)-\frac{1+c}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)\|Q_j\|\|w_0\|^{j-2}\Big]\Big\}\\
&< \frac{l(c-1)\|z_0\|^2}{1-\|z\|^2}\Big[(1-\sigma_1)-\frac{1+c}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}(j-1)\|P_j\|(1+|z_1|)^{\frac{2\sigma_1+1}{2}j-2}\\
&\quad \times(1-|z_1|)^{\frac{1-6\sigma_1}{2}j+2}\Big]+\frac{t(c-1)\|w_0\|^2}{1-\|w\|^2}\Big[(1-\sigma_2)-\frac{1+c}{c(1-\alpha)\cos\beta}\sum_{j=2}^{\infty}(j-1)\|Q_j\|\\
&\quad \times(1+|w_1|)^{\frac{2\sigma_2+1}{2}j-2}(1-|w_1|)^{\frac{1-6\sigma_2}{2}j+2}\Big].
\end{aligned}
$$

当 $\sigma_1\in[0,\frac{1}{6}]$ 时, $\frac{1-6\sigma_2}{2}j+2\geqslant 0$. 当 $\sigma_1\in(\frac{1}{6},\frac{1}{2}]$ 时, 若 $j\leqslant\frac{4}{6\sigma_1-1}$, 则 $\frac{1-6\sigma_2}{2}j+2\geqslant 0$. 因此当 $P_j=0(j>\frac{4}{6\sigma_1-1})$ 时, $(1-|z_1|)^{\frac{1-6\sigma_1}{2}j+2}$ 关于 $|z_1|$(对于所有的 $\sigma_1\in[0,\frac{1}{2}]$) 单调递减. 另外, 对于 $\sigma_1\in[0,\frac{1}{2}]$, $(1+|z_1|)^{\frac{2\sigma_1+1}{2}j-2}$ 在 $j\geqslant\frac{4}{2\sigma_1+1}$ 时关于 $|z_1|$ 单调递增, 反之递减, 则

$$
\begin{aligned}
&\sum_{j=2}^{\infty}(j-1)\|P_j\|(1+|z_1|)^{\frac{2\sigma_1+1}{2}j-2}(1-|z_1|)^{\frac{1-6\sigma_1}{2}j+2}\\
&\leqslant \sum_{j=2}^{I[\frac{4}{2\sigma_1+1}]}(j-1)\|P_j\|+\sum_{j=I[\frac{4}{2\sigma_1+1}]+1}^{\infty}(j-1)\|P_j\|2^{\frac{2\sigma_1+1}{2}j-2}.
\end{aligned}
$$

于是, 如果 $P_j=0\big(j>\frac{4}{6\sigma_1-1}\big)$, $Q_j=0\big(j>\frac{4}{6\sigma_2-1}\big)$ $\big(\sigma_1,\sigma_2\in\big(\frac{1}{6},\frac{1}{2}\big]\big)$ 且

$$
\begin{cases}
\displaystyle\sum_{j=2}^{I[\frac{4}{2\sigma_1+1}]}(j-1)\|P_j\|+\sum_{j=I[\frac{4}{2\sigma_1+1}]+1}^{\infty}(j-1)\|P_j\|2^{\frac{2\sigma_1+1}{2}j-2}\leqslant\frac{c(1-\sigma_1)(1-\alpha)\cos\beta}{1+c},\\
\displaystyle\sum_{j=2}^{I[\frac{4}{2\sigma_2+1}]}(j-1)\|Q_j\|+\sum_{j=I[\frac{4}{2\sigma_2+1}]+1}^{\infty}(j-1)\|Q_j\|2^{\frac{2\sigma_2+1}{2}j-2}\leqslant\frac{c(1-\sigma_2)(1-\alpha)\cos\beta}{1+c},
\end{cases}
$$

则有 $A<0$. 所以 $|E|<(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)$, 从而 (3.1) 式成立, 则定理得证.

由定理 3.1 可得以下 B^n 上的结论.

推论 3.1 设 $f(z_1)$ 是 D 上的强 α 次殆 β 型螺形函数, $\alpha\in[0,1)$, $\beta\in\big(-\frac{\pi}{2},\frac{\pi}{2}\big)$, $c\in(0,1)$. $F(z)$ 是 (1.3) 式所定义的函数, $\sigma_1\in\big[0,\frac{1}{2}\big]$. 令 $P_j=0\big(j>\frac{4}{6\sigma_1-1}$, $\sigma_1\in\big(\frac{1}{6},\frac{1}{2}\big]\big)$. 若

$$
\sum_{j=2}^{I[\frac{4}{2\sigma_1+1}]}(j-1)\|P_j\|+\sum_{j=I[\frac{4}{2\sigma_1+1}]+1}^{\infty}(j-1)\|P_j\|2^{\frac{2\sigma_1+1}{2}j-2}\leqslant\frac{c(1-\sigma_1)(1-\alpha)\cos\beta}{1+c},
$$

则 $F(z)$ 是 B^n 上的强 α 次殆 β 型螺形映照.

注 3.1 在定理 3.1 及推论 3.1 中分别令 $\alpha=0$ 和 $\beta=0$, 则得到相应的关于强 β 型螺形映照及强 α 次殆星形映照的结论.

4 α 次殆 β 型螺形映照的不变性

定理 4.1 设 $f(z_1)$ 是 D 上的 α 次殆 β 型螺形函数, $\alpha\in[0,1)/\{\frac{1}{2}\}$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$. $F(\xi,z,w)$ 是 (1.2) 式所定义的函数, $\delta_i,\gamma_i,\delta_i+\gamma_i\in[0,1]$, $\frac{s_i\delta_i}{l},\frac{s_i\gamma_i}{t}\in[0,\frac{1}{2}](i=1,\cdots,r)$, $\sigma_1,\sigma_2\in[0,\frac{1}{2}]$. 若

(i) 当 $\alpha\in([0,\frac{1}{2})\bigcup(\frac{1}{2},\frac{3}{4}])$ 时,

$$\begin{cases}\sum\limits_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}+\sum\limits_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|(\frac{2}{\cos\beta})^{\sigma_1 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}\leqslant c_1,\\ \sum\limits_{j=2}^{I[\frac{1}{\sigma_2}]}(j-1)\|Q_j\|(\cos\beta)^{1-\sigma_2 j}+\sum\limits_{j=I[\frac{1}{\sigma_2}]+1}^{\infty}(j-1)\|Q_j\|(\frac{2}{\cos\beta})^{\sigma_2 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_2 j)}\leqslant c_2;\end{cases}$$

(ii) 当 $\alpha\in(\frac{3}{4},1)$ 时,

$$\begin{cases}\sum\limits_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}+\sum\limits_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|(\frac{2}{\cos\beta})^{\sigma_1 j-1}\leqslant c_1,\\ \sum\limits_{j=2}^{I[\frac{1}{\sigma_2}]}(j-1)\|Q_j\|(\cos\beta)^{1-\sigma_2 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_2 j)}+\sum\limits_{j=I[\frac{1}{\sigma_2}]+1}^{\infty}(j-1)\|Q_j\|(\frac{2}{\cos\beta})^{\sigma_2 j-1}\leqslant c_2,\end{cases}$$

则 $F(\xi,z,w)$ 是 Ω_N 上的 α 次殆 β 型螺形映照, 这里 $c_k=\frac{(1-\alpha)\cos\beta(1-\sigma_i)}{2}$, $k=1,2$.

证 由定义 2.2 及调和函数的最小值原理, 类似于定理 3.1, 需证当 $(\xi,z,w)\in\partial\Omega_N$ 时,

$$\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big]\geqslant 0,\tag{4.1}$$

此时 $\rho(\xi,z,w)=1$. 令

$$q(z_1)=\frac{\mathrm{e}^{-\mathrm{i}\beta}\frac{f}{z_1f'}-\alpha\cos\beta+\mathrm{i}\sin\beta}{(1-\alpha)\cos\beta}.\tag{4.2}$$

由于 $f(z_1)$ 是 D 上的 α 次殆 β 型螺形函数, 则 $\Re q(z_1)\geqslant 0, q(0)=1$. 于是 $|q'(z_1)|\leqslant\frac{2\Re q(z_1)}{1-|z_1|^2}$,

$$\mathrm{e}^{-\mathrm{i}\beta}\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}=(1-\alpha)\cos\beta(1-q(z_1)-z_1q'(z_1)).\tag{4.3}$$

由引理 2.3, (4.2) 式, (4.3) 式得

$$\begin{aligned}&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big]\\&=\mathrm{e}^{-\mathrm{i}\beta}H-\alpha\cos\beta(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\&=\triangle_1\Big\{(1-\alpha)\cos\beta[\delta_i(z_1q'(z_1)+q(z_1)-1)+\gamma_i(w_1q'(w_1)+q(w_1)-1)]\\&\quad+\mathrm{e}^{-\mathrm{i}\beta}\Big[1+\delta_i\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)\end{aligned}$$

$$
\begin{aligned}
&+\gamma_i\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\Big]\Big\}\\
&+\triangle_2\Big\{|z_1|^2\Big[(1-\alpha)\cos\beta q(z_1)+\alpha\cos\beta-\mathrm{i}\sin\beta+\frac{\mathrm{e}^{-\mathrm{i}\beta}}{z_1}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]\\
&+\|z_0\|^2\Big[\mathrm{e}^{-\mathrm{i}\beta}\Big(1+\sigma_1\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)\Big)\\
&+\sigma_1(1-\alpha)\cos\beta(z_1q'(z_1)+q(z_1)-1)\Big]\Big\}\\
&+\triangle_3\Big\{|w_1|^2\Big[(1-\alpha)\cos\beta q(w_1)+\alpha\cos\beta-\mathrm{i}\sin\beta+\frac{\mathrm{e}^{-\mathrm{i}\beta}}{w_1}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\\
&+\|w_0\|^2\Big[\mathrm{e}^{-\mathrm{i}\beta}\Big(1+\sigma_2\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\Big)\\
&+\sigma_2(1-\alpha)\cos\beta(w_1q'(w_1)+q(w_1)-1)\Big]\Big\}-\alpha\cos\beta(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\
=&\,\triangle_1\Big\{(1-\alpha)\cos\beta[\delta_i(z_1q'(z_1)+q(z_1)-1)+\gamma_i(w_1q'(w_1)+q(w_1)-1)]+(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)\\
&+\mathrm{e}^{-\mathrm{i}\beta}\Big[\delta_i\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)+\gamma_i\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\Big]\Big\}\\
&+\triangle_2\Big\{|z_1|^2\Big[(1-\alpha)\cos\beta q(z_1)-\mathrm{i}\sin\beta+\frac{\mathrm{e}^{-\mathrm{i}\beta}}{z_1}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]\\
&+\|z_0\|^2\Big[(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta}\sigma_1\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)\\
&+\sigma_1(1-\alpha)\cos\beta(z_1q'(z_1)+q(z_1)-1)\Big]\Big\}\\
&+\triangle_3\Big\{|w_1|^2\Big[(1-\alpha)\cos\beta q(w_1)-\mathrm{i}\sin\beta+\frac{\mathrm{e}^{-\mathrm{i}\beta}}{w_1}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]\\
&+\|w_0\|^2\Big[(\mathrm{e}^{-\mathrm{i}\beta}-\alpha\cos\beta)+\mathrm{e}^{-\mathrm{i}\beta}\sigma_2\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\\
&+\sigma_2(1-\alpha)\cos\beta(w_1q'(w_1)+q(w_1)-1)\Big]\Big\}.
\end{aligned}
$$

由于 $\delta_i+\gamma_i\in[0,1]$, 由 (3.4) 式和引理 2.5–2.6、引理 2.8–2.9, 得

$$
\begin{aligned}
&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-\alpha\cos\beta\Big]\\
=&\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}M,
\end{aligned}
$$

其中

$$
M=(1-\alpha)\cos\beta\Big(s_i+\frac{l\|z_0\|^2}{1-\|z\|^2}+\frac{t\|w_0\|^2}{1-\|w\|^2}\Big)+\frac{l}{1-\|z\|^2}\Big\{|z_1|^2(1-\alpha)\cos\beta\Re q(z_1)
$$

$$
\begin{aligned}
&+\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)\Big(\frac{f''(z_1)}{f'(z_1)}\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)-\overline{z}_1\Big)\Big]\\
&+\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big](1-\alpha)\cos\beta\Re[z_1q'(z_1)+q(z_1)-1]\Big\}\\
&+\frac{t}{1-\|w\|^2}\Big\{|w_1|^2(1-\alpha)\cos\beta\Re q(w_1)\\
&+\Re\Big[\mathrm{e}^{-\mathrm{i}\beta}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\Big(\frac{f''(w_1)}{f'(w_1)}\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)-\overline{w}_1\Big)\Big]\\
&+\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big](1-\alpha)\cos\beta\Re[w_1q'(w_1)+q(w_1)-1]\Big\}\\
\geqslant{}&(1-\alpha)\cos\beta\Big(s_i+\frac{l\|z_0\|^2}{1-\|z\|^2}+\frac{t\|w_0\|^2}{1-\|w\|^2}\Big)\\
&+\frac{l}{1-\|z\|^2}\Big\{|z_1|^2(1-\alpha)\cos\beta\Re q(z_1)-2\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)|P_j(z_0)|\\
&+\Big[\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big](1-\alpha)\cos\beta\Big[\Re q(z_1)-\frac{2|z_1|}{1-|z_1|^2}\Re q(z_1)-1\Big]\Big\}\\
&+\frac{t}{1-\|w\|^2}\Big\{|w_1|^2(1-\alpha)\cos\beta\Re q(w_1)-2\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)|Q_j(w_0)|\\
&+\Big[\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big](1-\alpha)\cos\beta\Big[\Re q(w_1)-\frac{2|w_1|}{1-|w_1|^2}\Re q(w_1)-1\Big]\Big\}\\
={}&(1-\alpha)\cos\beta s_i(1-\delta_i-\gamma_i)+\frac{l}{1-\|z\|^2}\Big\{(1-\alpha)\cos\beta\Re q(z_1)\Big[|z_1|^2+\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)\\
&+\sigma_1\|z_0\|^2\Big)\frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2}\Big]+(1-\alpha)\cos\beta(1-\sigma_1)\|z_0\|^2-2\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)|P_j(z_0)|\Big\}\\
&+\frac{t}{1-\|w\|^2}\Big\{(1-\alpha)\cos\beta\Re q(w_1)\Big[|w_1|^2+\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)\frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2}\Big]\\
&+(1-\alpha)\cos\beta(1-\sigma_2)\|w_0\|^2-2\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)|Q_j(w_0)|\Big\}\\
\geqslant{}&\frac{l\|z_0\|^2}{1-\|z\|^2}\Big\{(1-\alpha)\cos\beta(1-\sigma_1)-2\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)\|P_j\|\|z_0\|^{j-2}\Big\}\\
&+\frac{t\|w_0\|^2}{1-\|w\|^2}\Big\{(1-\alpha)\cos\beta(1-\sigma_2)-2\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)\|Q_j\|\|w_0\|^{j-2}\Big\}\\
\geqslant{}&\frac{l\|z_0\|^2}{1-\|z\|^2}\Big\{(1-\alpha)\cos\beta(1-\sigma_1)-2\sum_{j=2}^{\infty}\Big[\frac{1+|z_1|}{\cos\beta(1-(1-2\alpha)|z_1|)^{\frac{2(\alpha-1)}{2\alpha-1}+1}}\Big]^{\sigma_1 j-1}(j-1)\|P_j\|\\
&\times\|z_0\|^{j-2}\Big\}+\frac{t\|w_0\|^2}{1-\|w\|^2}\Big\{(1-\alpha)\cos\beta(1-\sigma_2)-2\sum_{j=2}^{\infty}\Big[\frac{1+|z_1|}{\cos\beta(1-(1-2\alpha)|z_1|)^{\frac{2(\alpha-1)}{2\alpha-1}+1}}\Big]^{\sigma_2 j-1}\\
&\times(j-1)\|Q_j\|\|w_0\|^{j-2}\Big\}.
\end{aligned}
$$

令 $c_i=\frac{(1-\alpha)\cos\beta(1-\sigma_i)}{2}$, $i=1,2$ 及

$$A=(1-\alpha)\cos\beta(1-\sigma_1)-2\sum_{j=2}^{\infty}\Big[\frac{1+|z_1|}{\cos\beta(1-(1-2\alpha)|z_1|)^{\frac{2(\alpha-1)}{2\alpha-1}+1}}\Big]^{\sigma_1 j-1}(j-1)\|P_j\|\|z_0\|^{j-2}.$$

(i) 当 $\alpha\in([0,\frac{1}{2})\bigcup(\frac{1}{2},\frac{3}{4}])$ 时, 显然有 $(1+|z_1|)^{\sigma_1 j-1}$ 及 $[1-(1-2\alpha)|z_1|]^{(1+\frac{2(\alpha-1)}{2\alpha-1})(1-\sigma_1 j)}$ 关于 $|z_1|$ 在 $\sigma_1 j\geqslant 1$ 时单调递增, 在 $\sigma_1 j<1$ 时单调递减, 于是

$$A\geqslant(1-\alpha)\cos\beta(1-\sigma_1)-2\Big[\sum_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}+\sum_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|\Big(\frac{2}{\cos\beta}\Big)^{\sigma_1 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}\Big]\geqslant 0.$$

故 $M\geqslant 0$, 这里

$$\begin{cases}\sum\limits_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}+\sum\limits_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|(\frac{2}{\cos\beta})^{\sigma_1 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}\leqslant c_1,\\ \sum\limits_{j=2}^{I[\frac{1}{\sigma_2}]}(j-1)\|Q_j\|(\cos\beta)^{1-\sigma_2 j}+\sum\limits_{j=I[\frac{1}{\sigma_2}]+1}^{\infty}(j-1)\|Q_j\|(\frac{2}{\cos\beta})^{\sigma_2 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_2 j)}\leqslant c_2.\end{cases}$$

(ii) 当 $\alpha\in(\frac{3}{4},1)$ 时, 显然 $[1-(1-2\alpha)|z_1|]^{(1+\frac{2(\alpha-1)}{2\alpha-1})(1-\sigma_1 j)}$ 关于 $|z_1|$ 在 $\sigma_1 j\geqslant 1$ 时单调递减, 在 $\sigma_1 j<1$ 时单调递增, 而 $(1+|z_1|)^{\sigma_1 j-1}$ 则反之, 于是

$$A\geqslant(1-\alpha)\cos\beta(1-\sigma_1)-2\Big[\sum_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}+\sum_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|\Big(\frac{2}{\cos\beta}\Big)^{\sigma_1 j-1}\Big]\geqslant 0,$$

故 $M\geqslant 0$, 这里

$$\begin{cases}\sum\limits_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}+\sum\limits_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|(\frac{2}{\cos\beta})^{\sigma_1 j-1}\leqslant c_1,\\ \sum\limits_{j=2}^{I[\frac{1}{\sigma_2}]}(j-1)\|Q_j\|(\cos\beta)^{1-\sigma_2 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_2 j)}+\sum\limits_{j=I[\frac{1}{\sigma_2}]+1}^{\infty}(j-1)\|Q_j\|(\frac{2}{\cos\beta})^{\sigma_2 j-1}\leqslant c_2.\end{cases}$$

于是 (4.1) 式成立, 则定理得证.

在定理 4.1 中令 $\xi_{(i)}=0, w_j=0$, 则有以下结论.

推论 4.1 设 $f(z_1)$ 是 D 上的 α 次殆 β 型螺形函数, $\alpha\in[0,1)/\{\frac{1}{2}\}$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$. $F(z)$ 是 (1.3) 式所定义的函数, $\sigma_1\in[0,\frac{1}{2}]$. 令 $c=\frac{(1-\alpha)\cos\beta(1-\sigma_1)}{2}$. 若

(i) 当 $\alpha\in[0,\frac{1}{2})\bigcup(\frac{1}{2},\frac{3}{4}]$ 时,

$$\sum_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}+\sum_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|\Big(\frac{2}{\cos\beta}\Big)^{\sigma_1 j-1}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}\leqslant c;$$

(ii) 当 $\alpha\in(\frac{3}{4},1)$ 时,

$$\sum_{j=2}^{I[\frac{1}{\sigma_1}]}(j-1)\|P_j\|(\cos\beta)^{1-\sigma_1 j}(2\alpha)^{\frac{4\alpha-3}{2\alpha-1}(1-\sigma_1 j)}+\sum_{j=I[\frac{1}{\sigma_1}]+1}^{\infty}(j-1)\|P_j\|\Big(\frac{2}{\cos\beta}\Big)^{\sigma_1 j-1}\leqslant c,$$

则 $F(z)$ 是 B^n 上的 α 次殆 β 型螺形映照.

注 4.1 在定理 4.1 及推论 4.1 中, 分别令 $\alpha=0$ 和 $\beta=0$, 则得到相应的关于 β 型螺形映照及 α 次殆星形映照的结论.

5 α 次 β 型螺形映照的不变性

定理 5.1 设 $f(z_1)$ 是 D 上的 α 次 β 型螺形函数, $\alpha\in(0,1)$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$. $F(\xi,z,w)$ 是 (1.2) 式所定义的函数, $\delta_i,\gamma_i,\delta_i+\gamma_i\in[0,1]$, $\frac{s_i\delta_i}{l}$, $\frac{s_i\gamma_i}{t}\in[0,\frac{1}{2}](i=1,\cdots,r)$, $\sigma_1,\sigma_2\in[0,\frac{1}{2}]$. 令 $P_j=0(j>\frac{1}{\sigma_1})$, $Q_j=0(j>\frac{1}{\sigma_2})$. 若

(i) 当 $\alpha\in\big(0,\frac{1}{2}\big]$ 时,

$$\sum_{j=2}^{\infty}(j-1)\|P_j\|(\cos\beta)^{-\sigma_1 j}\leqslant\frac{1-\sigma_1}{2},\quad \sum_{j=2}^{\infty}(j-1)\|Q_j\|(\cos\beta)^{-\sigma_2 j}\leqslant\frac{1-\sigma_2}{2};$$

(ii) 当 $\alpha\in\big(\frac{1}{2},1\big)$ 时,

$$\sum_{j=2}^{\infty}(j-1)\|P_j\|(2-2\alpha)^{\sigma_1 j-2}(\cos\beta)^{-\sigma_1 j}\leqslant\frac{1-\sigma_1}{4\alpha},$$

$$\sum_{j=2}^{\infty}(j-1)\|Q_j\|(2-2\alpha)^{\sigma_2 j-2}(\cos\beta)^{-\sigma_2 j}\leqslant\frac{1-\sigma_2}{4\alpha},$$

则 $F(\xi,z,w)$ 是 Ω_N 上的 α 次 β 型螺形映照.

证 由定义 2.3, 与定理 3.1 同理只需证 $(\xi,z,w)\in\partial\Omega$ 时,

$$\Big|2\alpha(1-\mathrm{i}\tan\beta)\frac{2}{\rho(\xi,z,w)}\frac{\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF(\xi,z,w))^{-1}F(\xi,z,w)-1+\mathrm{i}2\alpha\tan\beta\Big|<1,\quad(5.1)$$

此时 $\rho(\xi,z,w)=1$. 令

$$q(z_1)=2\alpha(1-\mathrm{i}\tan\beta)\frac{f(z_1)}{z_1f'(z_1)}-1+\mathrm{i}2\alpha\tan\beta,\tag{5.2}$$

则 $|q(z_1)|<1$ 且

$$2\alpha(1-\mathrm{i}\tan\beta)\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}=2\alpha-1-q(z_1)-z_1q'(z_1).\tag{5.3}$$

由引理 2.3、(5.2)–(3.4) 式, 得

$$\begin{aligned}&(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[2\alpha(1-\mathrm{i}\tan\beta)\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF)^{-1}F(\xi,z,w)-1+\mathrm{i}2\alpha\tan\beta\Big]\\&=2\alpha(1-\mathrm{i}\tan\beta)H+(\mathrm{i}2\alpha\tan\beta-1)(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\\&=\triangle_1\Big\{(2\alpha-1)(1-\delta_i-\gamma_i)+\delta_i[z_1q'(z_1)+q(z_1)]+\gamma_i[w_1q'(w_1)+q(w_1)]\end{aligned}$$

$$+2\alpha(1-\mathrm{i}\tan\beta)\Big[\delta_i\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)+\gamma_i\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}$$

$$\times(j-1)Q_j(w_0)\Big]\Big\}+\triangle_2\Big\{|z_1|^2\Big[q(z_1)+\frac{2\alpha(1-\mathrm{i}\tan\beta)}{z_1}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(1-j)P_j(z_0)\Big]$$

$$+\|z_0\|^2\Big[(2\alpha-1)(1-\sigma_1)+2\alpha(1-\mathrm{i}\tan\beta)\sigma_1\frac{f''(z_1)}{f'(z_1)}\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)$$

$$+\sigma_1(z_1q'(z_1)+q(z_1))\Big]\Big\}+\triangle_3\Big\{|w_1|^2\Big[q(w_1)+\frac{2\alpha(1-\mathrm{i}\tan\beta)}{w_1}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(1-j)Q_j(w_0)\Big]$$

$$+\|w_0\|^2\Big[(2\alpha-1)(1-\sigma_2)+2\alpha(1-\mathrm{i}\tan\beta)\sigma_2\frac{f''(w_1)}{f'(w_1)}\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)$$

$$+\sigma_2(w_1q'(w_1)+q(w_1))\Big]\Big\}$$

$$=\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{(2\alpha-1)\Big[s_i(1-\delta_i-\gamma_i)+\frac{l\|z_0\|^2}{1-\|z\|^2}(1-\sigma_1)+\frac{t\|w_0\|^2}{1-\|w\|^2}(1-\sigma_2)\Big]$$

$$+\frac{l}{1-\|z\|^2}\Big[|z_1|^2q(z_1)+2\alpha(1-\mathrm{i}\tan\beta)\sum_{j=2}^{\infty}(f'(z_1))^{\sigma_1 j-1}(j-1)P_j(z_0)$$

$$\times\Big(\frac{f''(z_1)}{f'(z_1)}\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)-\overline{z}_1\Big)+\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)(z_1q'(z_1)+q(z_1))\Big]$$

$$+\frac{t}{1-\|w\|^2}\Big[|w_1|^2q(w_1)+2\alpha(1-\mathrm{i}\tan\beta)\sum_{j=2}^{\infty}(f'(w_1))^{\sigma_2 j-1}(j-1)Q_j(w_0)\Big(\frac{f''(w_1)}{f'(w_1)}$$

$$\times\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)-\overline{w}_1\Big)+\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)(w_1q'(w_1)+q(w_1))\Big]\Big\}.$$

由于 $\delta_i+\gamma_i\in[0,1]$, 由引理 2.4–2.6 及引理 2.10, 得

$$(\triangle_1+\|z\|^2\triangle_2+\|w\|^2\triangle_3)\Big[\Big|2\alpha(1-\mathrm{i}\tan\beta)\frac{2\partial\rho(\xi,z,w)}{\partial(\xi,z,w)}(DF)^{-1}F(\xi,z,w)-1+\mathrm{i}2\alpha\tan\beta\Big|-1\Big]$$

$$<\sum_{i=1}^{r}\|\xi_{(i)}\|^{2s_i}\Big\{|2\alpha-1|\Big[s_i(1-\delta_i-\gamma_i)+\frac{l\|z_0\|^2}{1-\|z\|^2}(1-\sigma_1)+\frac{t\|w_0\|^2}{1-\|w\|^2}(1-\sigma_2)\Big]$$

$$+\frac{l}{1-\|z\|^2}\Big[|z_1|^2|q(z_1)|+\frac{4\alpha}{\cos\beta}\sum_{j=2}^{\infty}|f'(z_1)|^{\sigma_1 j-1}(j-1)|P_j(z_0)|$$

$$+\Big(\frac{s_i\delta_i}{l}(1-\|z\|^2)+\sigma_1\|z_0\|^2\Big)\Big(\frac{2|z_1|}{1-|z_1|^2}(1-|q(z_1)|)+|q(z_1)|\Big)\Big]$$

$$+\frac{t}{1-\|w\|^2}\Big[|w_1|^2|q(w_1)|+\frac{4\alpha}{\cos\beta}\sum_{j=2}^{\infty}|f'(w_1)|^{\sigma_2 j-1}(j-1)|Q_j(w_0)|$$

$$+\Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2)+\sigma_2\|w_0\|^2\Big)\Big(\frac{2|w_1|}{1-|w_1|^2}(1-|q(w_1)|)+|q(w_1)|\Big)\Big]$$

$$-\Big(s_i+\frac{l\|z\|^2}{1-\|z\|^2}+\frac{t\|w\|^2}{1-\|w\|^2}\Big)\Big\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \|\xi_{(i)}\|^{2s_i} \Big\{ |2\alpha-1| \Big[s_i(1-\delta_i-\gamma_i) + (1-\sigma_1)\frac{l\|z_0\|^2}{1-\|z\|^2} + (1-\sigma_2)\frac{t\|w_0\|^2}{1-\|w\|^2} \Big] - s_i \\
&\quad + \frac{l}{1-\|z\|^2} \Big[(|q(z_1)|-1) \Big(|z_1|^2 + \Big(\frac{s_i\delta_i}{l}(1-\|z\|^2) + \sigma_1\|z_0\|^2 \Big) \frac{1-2|z_1|-|z_1|^2}{1-|z_1|^2} \Big) \\
&\quad + \frac{s_i\delta_i}{l}(1-\|z\|^2) + (\sigma_1-1)\|z_0\|^2 + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} |f'(z_1)|^{\sigma_1 j-1} (j-1)|P_j(z_0)| \Big] \\
&\quad + \frac{t}{1-\|w\|^2} \Big[(|q(w_1)|-1) \Big(|w_1|^2 + \Big(\frac{s_i\gamma_i}{t}(1-\|w\|^2) + \sigma_2\|w_0\|^2 \Big) \frac{1-2|w_1|-|w_1|^2}{1-|w_1|^2} \Big) \\
&\quad + \frac{s_i\gamma_i}{t}(1-\|w\|^2) + (\sigma_2-1)\|w_0\|^2 + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} |f'(w_1)|^{\sigma_2 j-1} (j-1)|Q_j(w_0)| \Big] \Big\} \\
&\leqslant \sum_{i=1}^{r} \|\xi_{(i)}\|^{2s_i} \Big\{ \frac{l\|z_0\|^2}{1-\|z\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_1) \\
&\quad + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} \Big(\frac{1+(1-2\alpha)|z_1|}{\cos\beta(1-|z_1|)^{1+2(1-\alpha)}} \Big)^{\sigma_1 j-1} (j-1)\|P_j\|\|z_0\|^{j-2} \Big] \\
&\quad + \frac{t\|w_0\|^2}{1-\|w\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_2) + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} \Big(\frac{1+(1-2\alpha)|w_1|}{\cos\beta(1-|w_1|)^{1+2(1-\alpha)}} \Big)^{\sigma_2 j-1} \\
&\quad \times (j-1)\|Q_j\|\|w_0\|^{j-2} \Big] \Big\} = \sum_{i=1}^{r} \|\xi_{(i)}\|^{2s_i} A.
\end{aligned}
$$

(i) 当 $\alpha \in \big(0, \frac{1}{2}\big]$ 时, 显然有 $[1+(1-2\alpha)|z_1|]^{\sigma_1 j-1}$ 关于 $|z_1|$ 在 $\sigma_1 j \leqslant 1$ 时单调递减, $[1+(1-2\alpha)|w_1|]^{\sigma_2 j-1}$ 关于 $|w_1|$ 在 $\sigma_2 j \leqslant 1$ 时单调递减. 由于 $P_j = 0(j > \frac{1}{\sigma_1})$, $Q_j = 0(j > \frac{1}{\sigma_2})$, 若 $\sum\limits_{j=2}^{\infty}(j-1)\|P_j\|(\cos\beta)^{-\sigma_1 j} \leqslant \frac{1-\sigma_1}{2}$, $\sum\limits_{j=2}^{\infty}(j-1)\|Q_j\|(\cos\beta)^{-\sigma_2 j} \leqslant \frac{1-\sigma_2}{2}$, 则有

$$
\begin{aligned}
A &\leqslant \frac{l\|z_0\|^2}{1-\|z\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_1) + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} (\cos\beta)^{1-\sigma_1 j}(j-1)\|P_j\| \Big] \\
&\quad + \frac{t\|w_0\|^2}{1-\|w\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_2) + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} (\cos\beta)^{1-\sigma_2 j}(j-1)\|Q_j\| \Big] \leqslant 0.
\end{aligned}
$$

(ii) 当 $\alpha \in (\frac{1}{2}, 1)$ 时, $[1+(1-2\alpha)|z_1|]^{\sigma_1 j-1}$ 关于 $|z_1|$ 在 $\sigma_1 j \leqslant 1$ 时单调递增, $[1+(1-2\alpha)|w_1|]^{\sigma_2 j-1}$ 关于 $|w_1|$ 在 $\sigma_2 j \leqslant 1$ 时单调递增. 由于 $P_j = 0\big(j > \frac{1}{\sigma_1}\big)$, $Q_j = 0\big(j > \frac{1}{\sigma_2}\big)$, 则

$$
\begin{aligned}
A &\leqslant \frac{l\|z_0\|^2}{1-\|z\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_1) + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} (2-2\alpha)^{\sigma_1 j-1}(\cos\beta)^{1-\sigma_1 j}(j-1)\|P_j\| \Big] \\
&\quad + \frac{t\|w_0\|^2}{1-\|w\|^2} \Big[(|2\alpha-1|-1)(1-\sigma_2) + \frac{4\alpha}{\cos\beta} \sum_{j=2}^{\infty} (2-2\alpha)^{\sigma_2 j-1}(\cos\beta)^{1-\sigma_2 j}(j-1)\|Q_j\| \Big] \\
&\leqslant 0,
\end{aligned}
$$

这里 $\sum\limits_{j=2}^{\infty}(j-1)\|P_j\|(2-2\alpha)^{\sigma_1 j-2}(\cos\beta)^{-\sigma_1 j} \leqslant \frac{1-\sigma_1}{4\alpha}$, $\sum\limits_{j=2}^{\infty}(j-1)\|Q_j\|(2-2\alpha)^{\sigma_2 j-2}(\cos\beta)^{-\sigma_2 j} \leqslant \frac{1-\sigma_2}{4\alpha}$. 于是 (5.1) 式成立, 则定理得证.

在定理 5.1 中若不考虑含有 ξ 及 w 的项, 则有以下结论.

推论 5.1　设 $f(z_1)$ 是 D 上的 α 次 β 型螺形函数, $\alpha\in(0,1)$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$. $F(z)$ 是 (1.3) 式所定义的函数, $\sigma_1\in\big[0,\frac{1}{2}\big]$. 令 $P_j=0\big(j>\frac{1}{\sigma_1}\big)$. 若

(i) $\alpha\in\big(0,\frac{1}{2}\big]$ 时, $\sum\limits_{j=2}^{\infty}(j-1)\|P_j\|(\cos\beta)^{-\sigma_1 j}\leqslant\frac{1-\sigma_1}{2}$;

(ii) $\alpha\in\big(\frac{1}{2},1\big)$ 时, $\sum\limits_{j=2}^{\infty}(j-1)\|P_j\|(2-2\alpha)^{\sigma_1 j-2}(\cos\beta)^{-\sigma_1 j}\leqslant\frac{1-\sigma_1}{4\alpha}$,

则 $F(z)$ 是 B^n 上的 α 次 β 型螺形映照.

注 5.1　在定理 5.1 及推论 5.1 中令 $\beta=0$, 则得到相应的关于 α 次星形映照的结论.

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参　考　文　献

[1] Roper K A, Suffridge T J. Convex mappings on the unit ball of $\mathbb{C}^n$ [J]. *J Anal Math*, 1995, 65:333–347.

[2] Graham I, Kohr G. Univalent mappings associated with the Roper-Suffridge extension operator [J]. *J Analyse Math*, 2000, 81:331–342.

[3] 刘小松, 刘太顺. 关于 α 次的 β 型螺形映照推广的 Roper-Suffridge 算子 [J]. 数学年刊 A 辑, 2006, 27(6):789–798.

[4] 刘小松, 冯淑霞. 关于 α 次的 β 型螺形映射推广的 Roper-Suffridge 算子的一个注记 [J]. 数学季刊, 2009, 24(2):310–316.

[5] 刘浩, 夏红川. Reinhardt 域上一类推广的 Roper-Suffridge 算子 [J]. 数学学报, 2016, 59(2):253–266.

[6] Muir J R. A modification of the Roper-Suffridge extension operator [J]. *Comput Methods Funct Theory*, 2005, 5(1):237–251.

[7] Kohr G. Loewner chains and a modification of the Roper-Suffridge extension operator [J]. *Mathematica*, 2006, 71(1):41–48.

[8] 王建飞, 刘太顺. 全纯映射子族上改进的 Roper-Suffridge 算子 [J]. 数学年刊 A 辑, 2010, 31(4):487–496.

[9] Muir J R. A class of Loewner chain preserving extension operators [J]. *J Math Anal Appl*, 2008, 337(2):862–879.

[10] 唐言言. Bergman-Hartogs 型域上的 Roper-Suffridge 算子 [D]. 开封: 河南大学硕士论文, 2016.

[11] 潘利双, 王安. Bergman-Hartogs 型域的全纯自同构群 [J]. 中国科学, 2015, 45:31–42.

[12] 叶薇薇, 王安. 一类 Hartogs 域的 Einstein-Kähler 度量和 Kobayashi 度量的比较定理 [J]. 数学年刊 A 辑, 2012, 33(6):687–704.

[13] Wang A, Liu Y. Zeroes of the Bergman kernels on some new Hartogs domains [J]. *Chin Quart J of Math*, 2011, 26(3):325–334.

[14] 蔡荣华, 刘小松. 强螺形函数子族的第三项和第四项系数估计 [J]. 湛江师范学院学报, 2010, 31:38–43.

[15] Zhu Y C, Liu M S. The generalized Roper-Suffridge extension operator on Reinhardt domain D_p [J]. *Taiwanese Jour of Math*, 2010, 14(2):359–372.

[16] Feng S, Liu T. The generalized Roper-Suffridge extension operator [J]. *Acta Math Sci*, 2008, 28B(1):63–80.

[17] 冯淑霞, 刘太顺, 任广斌. 复 Banach 空间单位球上几类映射的增长掩盖定理 [J]. 数学年刊 A 辑, 2007, 28(2):215–230.

[18] Liu T S, Ren G B. The growth theorem for starlike mappings on bounded starlike circular domains [J]. *Chin Ann Math, Ser B*, 1998, 19(4):401–408.

[19] Ahlfors L V. Complex analysis [M]. 3rd, ed. New York: Mc Graw-Hill Book Co., 1979.

[20] Duren P L. Univalent functions [M]. New York: Springer-Verlag, 1983.

[21] Graham I, Kohr G. Geometric function theory in one and higher dimensions [M]. New York: Marcel Dekker, 2003.

[22] 张洁, 卢金. 单位多圆柱上 α 次的殆 β 型螺形映射的偏差定理 [J]. 湖州师范学院学报, 2011, 33(2):46–50.

[23] 王朝君, 崔艳艳, 刘浩. B^n 上推广的 Roper-Suffridge 算子的性质 [J]. 数学学报, 2016, 59(6):721–864.

Properties of the Perturbed Extension Operators on Hartogs Domains

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Abstract This paper mainly discuss the properties of the generalized Roper-Suffridge operators on the extended Hartogs domains. By using the distortion results of subclasses of biholomorphic mappings, the authors conclude that the generalized operators preserve the properties of strong and almost spirallike mappings of type β and order α, almost spirallike mappings of type β and order α, spirallike mappings of type β and order α on Ω_N under different conditions, respectively. Thus the authors get the corresponding results on B^n. These conclusions involve some known results and provide new approaches to research the biholomorphic mappings in $\mathbb{C}^n$.

Keywords Starlike mappings, Spirallike mappings, Extension operators, Hartogs domains

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