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两参数奇异摄动非线性双曲型微分系统的过渡冲击层广义解*

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提要 研究了一类两参数双曲型微分系统奇异摄动初始边值问题. 首先, 利用奇异摄动理论和方法, 注意到两个小参数, 构造了问题的外部解. 其次, 利用多重尺度变量和伸长变量, 分别得到了原问题解的过渡冲击层、边界层和初始层校正项. 最后, 得到了原问题解的渐近展开式, 并利用泛函分析不动点理论, 证明了渐近解的一致有效性. 由本方法求得的原问题的渐近解, 它还可以进行微分, 积分等解析运算, 从而能了解相应过渡冲击层解的更进一步的性态. 因此本方法具有良好的应用前景.

关键词 微分系统, 过渡层, 小参数
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1 引 言

奇异摄动理论和方法, 已被广泛地应用在数学物理、弹性力学、流体力学和生物化学等自然科学中, 它在应用数学、系统科学理论中具有重要的地位. 许多学者做了很多的研究 [1−13]. 莫嘉琪、韩祥临等人也在这方面做了一些工作 [14−26]. 本文利用特殊而有效的理论和方法, 讨论了一类具有两参数的非线性奇异摄动双曲型微分系统的初始边值问题, 得到了相应微分系统的过渡冲击层渐近解.

今讨论如下一类广义双曲型微分系统的初始边值问题:

$$\mu^2(\psi, D_0^2u_i) - \varepsilon^2 B_1[\psi, u_i] = (\psi, f_i(t,x,u)),$$
$$0 < t \leqslant T_0,\ x \in \Omega,\ \forall\psi \in C_0^\infty(\Omega), \quad i = 1,2,\cdots,m, \tag{1.1}$$

$$(\psi, u_i) = (\psi, g_i), \quad x \in \partial\Omega,\ \forall\psi \in C_0^\infty(\Omega), \quad i = 1,2,\cdots,m, \tag{1.2}$$

$$(\psi, u_i) = (\psi, h_{1i}), \quad t = 0,\ \forall\psi \in C_0^\infty(\Omega), \quad i = 1,2,\cdots,m, \tag{1.3}$$

$$(\psi, \mu D_0 u_i) = (\psi, h_{2i}), \quad t = 0,\ \forall\psi \in C_0^\infty(\Omega), \quad i = 1,2,\cdots,m, \tag{1.4}$$

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其中 ε,μ 是正的小参数, $u(t,x)\equiv(u_1(t,x),u_2(t,x),\cdots,u_m(t,x))\in\Omega$, Ω 是 R^n 中的有界区域, $\partial\Omega$ 为 Ω 的 $C^{1+\alpha}$ 类的边界, α 为 Hölder 指数, T_0 为正常数,

$$B_1[\psi,u_i]\equiv\sum_{0\leqslant\mu,\sigma\leqslant1}(D^\mu\psi,(a^{\mu\sigma}(x)D^\sigma u_i)=(\psi,L[u_i]),\quad i=1,2,\cdots,m,$$

$$L_i=\sum_{0\leqslant\mu,\sigma\leqslant1}(-1)^\mu(D^\mu(a^{\mu\sigma}(x)D^\sigma)),$$

$$D_0=\frac{\partial}{\partial t},\quad D_j=\frac{\partial}{\partial x_j},\quad j=1,2,\cdots,n,\quad D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n},\quad \alpha=\sum_{j=1}^n\alpha_j,$$

系数 $a^{\mu\sigma}(x)$ 假设为在 $C^\infty(\Omega)$ 中的实值函数, L 为一致椭圆型算子, f,g,h_{ji} ($i=1,2,\cdots,m$, $j=1,2$) 为关于其变量为充分光滑的函数, $C_0^\infty(\Omega)$ 为由 Ω 中具有紧支函数并为 $C^\infty(\Omega)$ 的子集, 表示式 $B[v,u]$ 为双线性形式, 该形式是 u,v 在包含原点 (0,0) 的区域 Ω 中并被定义在 Sobolev 空间 $H^1(\Omega)$, 且具有有限模:

$$\|\xi\|_j=\Big\{\sum_{\alpha\leqslant j}\int_\Omega|D^\alpha\xi(x)|^2\Big\}^{\frac12},\quad \forall\xi\in C^1(\Omega),\ j=0,1,$$

同时 (u,v) 是被定义在 $H^1(\Omega)$ 中的内积.

假设 H1 模型 (1.1)–(1.4) 的广义退化微分系统

$$(\psi,f_i(t,x,u))=0,\quad 0<t\leqslant T_0,\ x\in\Omega,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m \tag{1.5}$$

有解 $U_{00}(t,x)=(U_{100},U_{200},\cdots,U_{m00})$, 其中 $U_{i00}\in H^1([0,T_0\times\Omega])$, 而

$$U_{i00}(t,x)=\begin{cases}U_{Lj}(t,x), & x_1\leqslant0,\ i=1,2,\cdots,m,\\ U_{Rj}(t,x), & x_1>0,\ i=1,2,\cdots,m.\end{cases}$$

假设 H2 $(\psi,f_u(t,x,u))\leqslant-c_1<0,\ 0<t\leqslant T_0,\ x\in\Omega,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m$, 其中 c_1 为正常数.

2 广义双曲型微分系统外部解

现构造模型 (1.1)–(1.4) 的外部解 $U=(U_1,U_2,\cdots,U_m,)$. 设

$$U_i=\sum_{r,s=0}^\infty U_{irs}\varepsilon^r\mu^s,\quad i=1,2,\cdots,m. \tag{2.1}$$

将 (2.1) 式代入广义双曲型微分系统 (1.1), 按 ε 和 μ 的幂展开非线性扰动函数项 f_i, 合并 $\varepsilon^r\mu^s$ 同次幂的系数, 分别令它们为零. 对于 $\varepsilon^0\mu^0$ 的系数为零, 就是广义退化微分系统 (1.5). 由假设, 它的解为 $U_{100}(t,x),U_{200}(t,x),\cdots,U_{m00}(t,x)$.

关于 $\varepsilon^r\mu^s$ ($r,s=0,1,\cdots,\ r+s\neq0$) 的系数为零, 可得广义双曲型系统

$$\Big(\psi,-\frac{\partial^2U_{ir(s-2)}}{\partial t^2}\Big)-(\psi,LU_{i(r-2)s})-(\psi,F_{irs})=0,\quad \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m, \tag{2.2}$$

其中

$$F_{irs}=\frac{1}{r!s!}\Big[\frac{\partial^{r+s}}{\partial\varepsilon^r\mu^s}f_i\Big(t,x,\sum_{r,s=0}^\infty U_{irs}\varepsilon^r\mu^s\Big)\Big]_{\varepsilon=\mu=0},\quad i=1,2,\cdots,m.$$

上面和以下的各式中, 假设负下标的项为零. 由广义双曲型微分系统 (2.2), 可以得到解 $U_{rs}(t,x) = (U_{1rs}(t,x), U_{2rs}(t,x), \cdots, U_{mrs}(t,x))$ $(r,s = 0,1,\cdots,\ r+s \neq 0)$, 将它们和 $U_{00}(t,x)$ 代入 (2.1) 式, 便得到了广义双曲型微分系统模型 (1.1)–(1.4) 的广义外部解 $U = (U_1, U_2, \cdots, U_m)$. 但它未必满足在区域 $x_1 = 0$ 和边界条件 (1.2) 以及初始条件 (1.3), (1.4). 因此尚需构造广义解的过渡冲击层校正项 $V = (V_1, V_2, \cdots, V_m)$, 边界层校正项 $W = (W_1, W_2, \cdots, W_m)$ 和初始层校正项 $Z = (Z_1, Z_2, \cdots, Z_m)$.

3 过渡冲击层

在区域 Ω 中 $x_1 = 0$ 的邻域上构造局部坐标系 (ρ, ϕ): 在 $x_1 = 0$ 的邻域中的每一点 P 的坐标 ρ $(\leqslant \rho_0)$ 是点 P 到 $x_1 = 0$ 的距离, 这里 ρ_0 是足够小的正常数, 而坐标 $\phi = (\phi_1, \phi_2, \cdots, \phi_{n-1})$ 是 $n-1$ 维流形 $x_1 = 0$ 的邻域上的一个非奇异坐标系, 并设点 P 的坐标 ϕ 就是通过点 P 的内法线与 $x_1 = 0$ 的邻域的交点 Q 的坐标 ϕ. 故在 $x_1 = 0$ 的邻域 $0 \leqslant \rho \leqslant \rho_0$ 中, 算子 L 为

$$L \equiv \sum_{0 \leqslant \mu, \sigma \leqslant 1} (-1)^{\mu} \overline{D}^{\mu} (\overline{a}_{(x)}^{\mu\sigma} \overline{D}^{\sigma}), \tag{3.1}$$

其中

$$\overline{D}_n = \frac{\partial}{\partial \rho}, \quad \overline{D}_j = \frac{\partial}{\partial \phi_j}, \quad j = 1, 2, \cdots, n-1, \quad \overline{D}^{\alpha} = \overline{D}_1^{\overline{\alpha}_1} \overline{D}_2^{\overline{\alpha}_2} \cdots \overline{D}_n^{\overline{\alpha}_n}, \quad \overline{\alpha} = \sum_{j=1}^{n} \overline{\alpha}_j.$$

在 $x_1 = 0$ 的邻域 $0 \leqslant \rho \leqslant \rho_0$ 上, 引入多重尺度变量 [1−3]

$$\sigma = \frac{\xi(\rho, \phi)}{\varepsilon^2}, \quad \overline{\rho} = \rho, \quad \overline{\phi} = \phi,$$

其中 $\xi(\rho, \phi)$ 为待定函数, 它将在下面决定. 为方便起见, 以下仍以 (ρ, ϕ) 来表示 $(\overline{\rho}, \overline{\phi})$. 由 (3.1) 式, 有

$$L = \frac{1}{\varepsilon^2} K_0 + \frac{1}{\varepsilon} K_1 + K_2,$$

其中

$$K_0 = \overline{a}^{nn} \xi_{\rho}^2 \frac{\partial^2}{\partial \sigma^2},$$

$$K_1 = 2\overline{a}^{nn} \xi_{\rho} \frac{\partial^2}{\partial \sigma \partial \rho} + \sum_{i=1}^{n-1} \overline{a}^{ni} \xi_{\rho} \frac{\partial^2}{\partial \sigma \partial \phi_1} + (\overline{a}^{nn} \xi_{\rho} + \overline{a} \xi_{\rho}) \frac{\partial}{\partial \sigma},$$

$$K_2 = \overline{a}^{nn} \frac{\partial^2}{\partial \rho^2} + \sum_{i=1}^{n-1} \overline{a}^{ni} \frac{\partial^2}{\partial \sigma \partial \phi_i} + \sum_{i,j=1}^{n-1} \overline{a}^{i} \frac{\partial^2}{\partial \phi_i \partial \phi_j} + \overline{a}^{n} \frac{\partial}{\partial \rho} + \sum_{i=1}^{n-1} \overline{a} \frac{\partial}{\partial \phi_i}.$$

令 $\xi_{\rho} = \sqrt{\frac{1}{\overline{a}^{nn}}}$, 且设广义双曲型微分系统 (1.1)–(1.4) 的解为 $u = (u_1, u_2, \cdots, u_m)$, 其中

$$u_i = U_i(t,x) + V(\sigma, \rho, \phi), \quad i = 1, 2, \cdots, m, \tag{3.2}$$

而

$$V=\sum_{r,s=0}^{m} v_{irs}(\sigma,\rho,\phi)\varepsilon^{r}\mu^{s},\quad i=1,2,\cdots,m. \tag{3.3}$$

将 (3.2) 式、(3.3) 式代入微分系统 (1.1), 按 σ,μ 展开非线性的项, 合并 $\sigma^{j}\mu^{k}$ $(j,k=0,1,\cdots)$ 同次幂的系数, 可得

$$(\psi,K_0 v_{i00})=0,\quad 0\leqslant\rho\leqslant\rho_0,\quad \forall\psi\in C_0^{\infty}(\Omega),\ i=1,2,\cdots,m, \tag{3.4}$$

$$(\psi,v_{i00})=(\psi,-u_{i00}(t,\phi)),\quad \rho=0,\ \forall\psi\in C_0^{\infty}(\Omega),\ i=1,2,\cdots,m, \tag{3.5}$$

$$\begin{aligned}&(\psi,K_0 v_{irs})=(\psi,G_{irs}),\\&0\leqslant\rho\leqslant\rho_0,\forall\psi\in C_0^{\infty}(\Omega),\quad r,s=0,1,\cdots,r+s\neq0,\ i=1,2,\cdots,m,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&(\psi,v_{irs})=(\psi,-u_{irs}(t,\phi)),\\&\qquad\rho=0,\ \forall\psi\in C_0^{\infty}(\Omega),\ r,s=0,1,\cdots,\ r+s\neq0,\ i=1,2,\cdots,m,\end{aligned} \tag{3.7}$$

其中 G_{irs} $(r,s=0,1,\cdots,\ r+s\neq0)$ 为逐次已知函数, 其结构也从略. 于是由 (3.4)–(3.5) 可得到解 v_{i00}. 再由 (3.6)–(3.7) 式, 可以依次得到解 v_{irs} $(r,s=0,1,\cdots;\ r+s\neq0)$.

由假设, 不难看出: v_{irs} $(r,s=0,1,\cdots)$ 具有过渡冲击层性态[1−2]:

$$v_{irs}=O\Big(\exp\Big(-\delta_{irs}\frac{\xi(\rho,\phi)}{\varepsilon}\Big)\Big),\quad r,s=0,1,\cdots,\ i=1,2,\cdots,m,\ 0<\varepsilon\ll1, \tag{3.8}$$

其中 $\delta_{irs}>0$ $(r,s=0,1,\cdots,\ i=1,2,\cdots,m)$ 为常数.

令 $\overline{v}_{irs}=\zeta(\rho)v_{irs}$, 其中 $\zeta(\rho)$ 为在 $0\leqslant\rho\leqslant\rho_0$ 内的充分光滑的分隔函数, 并满足

$$\zeta(\rho)=\begin{cases}1, & 0\leqslant\rho\leqslant\dfrac{1}{3}\rho_0,\\[2mm] 0, & \rho\geqslant\dfrac{2}{3}\rho_0.\end{cases}$$

为了方便, 以下仍然用 v_{irs} 代替 $\overline{v}_{irs}$. 再由 (3.3) 式, 便得到在 $x_1=0$ 的邻域 $(0\leqslant\rho\leqslant\rho_0)$ 内的过渡冲击层校正项 $V=(V_1,V_2,\cdots,V_m)$.

4 边界层

在区域 Ω 的边界 $\partial\Omega$ 的邻域上构造局部坐标系 $(\widetilde{\rho},\widetilde{\phi})$: 在区域 Ω 的边界 $\partial\Omega$ 的邻域中的每一点 M 的坐标 $\widetilde{\rho}(\leqslant\widetilde{\rho}_0)$ 是点 M 到 $\partial\Omega$ 的距离, 这里 $\widetilde{\rho}_0$ 是足够小的正常数, 使得在边界 $\partial\Omega$ 上的每一点的内法线在 $\partial\Omega$ 的邻域 $0\leqslant\widetilde{\rho}\leqslant\widetilde{\rho}_0$ 中相互不相交. 而坐标 $\widetilde{\phi}=(\widetilde{\phi}_1,\widetilde{\phi}_2,\cdots,\widetilde{\phi}_{n-1})$ 是 $n-1$ 维流形 $\partial\Omega$ 上的一个非奇异坐标系, 设点 M 的坐标 $\widetilde{\phi}$ 是通过点 M 的内法线与边界的交点 N 的坐标 $\widetilde{\phi}$, 故在 $\partial\Omega$ 的邻域 $0\leqslant\widetilde{\rho}\leqslant\widetilde{\rho}_0$ 内, 算子 $\overline{L}$ 为

$$\overline{L}\equiv\sum_{0\leqslant\mu,\sigma\leqslant1}(-1)^{\mu}\widetilde{D}^{\mu}(\widetilde{a}^{\mu\sigma}(x)\widetilde{D}^{\sigma}), \tag{4.1}$$

其中

$$\widetilde{D}_n=\frac{\partial}{\partial\widetilde{\rho}},\quad \widetilde{D}_j=\frac{\partial}{\partial\widetilde{\phi}_j},\quad j=1,2,\cdots,n-1,\quad \widetilde{D}^{\widetilde{\alpha}}=\widetilde{D}_1^{\widetilde{\alpha}_1}\widetilde{D}_2^{\widetilde{\alpha}_2}\cdots\widetilde{D}_n^{\widetilde{\alpha}_n},\quad \widetilde{\alpha}=\sum_{j=1}^{n}\widetilde{\alpha}_j.$$

在 $\partial\Omega$ 的邻域 $0 \leqslant \widetilde{\rho} \leqslant \widetilde{\rho}_0$ 上, 引入多重尺度变量 [1−3]

$$\widetilde{\sigma} = \frac{\widetilde{\xi}(\widetilde{\rho}, \widetilde{\phi})}{\varepsilon^2}, \quad \widehat{\rho} = \widetilde{\rho}, \quad \widehat{\phi} = \widetilde{\phi},$$

其中 $\widetilde{\xi}$ 为待定函数, 它将在下面决定. 为方便起见, 以下仍以 $(\widetilde{\rho}, \widetilde{\phi})$ 来表示 $(\widehat{\rho}, \widehat{\rho})$. 由 (4.1) 式, 有

$$\overline{L} = \frac{1}{\mu^2}\overline{K}_0 + \frac{1}{\mu}\overline{K}_1 + \overline{K}_2, \tag{4.2}$$

其中

$$\overline{K}_0 = \widetilde{a}^{nn}\widetilde{\xi}_{\widetilde{\rho}}^2\frac{\partial^2}{\partial\widetilde{\sigma}^2},$$

$$\overline{K}_1 = 2\widetilde{a}^{nn}\widetilde{\xi}_{\widetilde{\rho}}\frac{\partial^2}{\partial\widetilde{\sigma}\partial\widetilde{\rho}} + \sum_{i=1}^{n-1}\widetilde{a}^{ni}\widetilde{\xi}_{\widetilde{\rho}}\frac{\partial^2}{\partial\widetilde{\sigma}\partial\widetilde{\phi}_1} + (\widetilde{a}^{nn}\widetilde{\xi}_{\widetilde{\rho}} + \widetilde{a}\widetilde{\xi}_{\widetilde{\rho}})\frac{\partial}{\partial\widetilde{\sigma}}.$$

$$\overline{K}_2 = \widetilde{a}^{nn}\frac{\partial^2}{\partial\widetilde{\rho}^2} + \sum_{i=1}^{n-1}\widetilde{a}^{ni}\frac{\partial^2}{\partial\widetilde{\sigma}\partial\widetilde{\phi}_i} + \sum_{i,j=1}^{n-1}\widetilde{a}^{i}\frac{\partial^2}{\partial\widetilde{\phi}_i\partial\widetilde{\phi}_j} + \widetilde{a}^{n}\frac{\partial}{\partial\widetilde{\rho}} + \sum_{i=1}^{n-1}\widetilde{a}\frac{\partial}{\partial\widetilde{\phi}_i}.$$

令 $\widetilde{\xi}_\rho = \sqrt{\frac{1}{\widetilde{a}^{nn}}}$, 且设广义双曲型微分系统 (1.1)–(1.4) 的解为 $u = (u_1, u_2, \cdots, u_m)$, 其中

$$u_i = U_i(t,x) + V(\sigma,\rho,\phi) + W(\widetilde{\sigma},\widetilde{\rho},\widetilde{\phi}), \quad i = 1,2,\cdots,m. \tag{4.3}$$

将 (4.3) 代入微分系统 (1.1), 按 $\widetilde{\sigma}$ 和 μ 展开非线性项, 合并 $\widetilde{\sigma}^j$, μ^k $(j,k = 0,1,\cdots)$ 同次幂的系数, 可得

$$(\psi, \mu^2 W_{itt}) - (\psi, \mu^2 L W_i) = (\psi, f_i(\widetilde{\rho}, \widetilde{\phi}, U+V+W)) - (\psi, f_i(\rho,\phi,U+V)),$$
$$\widetilde{\rho} = 0,\ \forall\psi \in C_0^\infty(\Omega), \tag{4.4}$$

$$(\psi, W) = (\psi, g_i(t,0,\widetilde{\phi}) - (\psi, U(t,0,\widetilde{\phi} + V(t,0,\widetilde{\phi})),$$
$$\widetilde{\rho} = 0,\ \forall\psi \in C_0^\infty(\Omega),\ i = 1,2,\cdots,m. \tag{4.5}$$

设

$$W_i = \sum_{r,s=0}^{\infty} w_{irs}(\widetilde{\sigma},\widetilde{\rho},\widetilde{\phi})\widetilde{\sigma}\mu^s. \tag{4.6}$$

将 (4.3) 式、(4.6) 式代入 (4.4) 式、(4.5) 式, 按 $\widetilde{\sigma}$ 和 $\widetilde{\mu}$ 的幂展开非线性函数的项, 合并 $\widetilde{\sigma}^r$, $\widetilde{\mu}^s$ 的同次幂项的系数, 有

$$(\psi, \overline{K}_0 w_{i00}) = 0, \quad 0 \leqslant \widetilde{\rho} \leqslant \widetilde{\rho}_0,\ \forall\psi \in C_0^\infty(\Omega),\ i = 1,2,\cdots,m, \tag{4.7}$$

$$(\psi, w_{i00}) = (\psi, -u_{i00}(t,\widetilde{\phi}) - v_{i00}(t,\widetilde{\phi})), \quad \widetilde{\rho} = 0,\ \forall\psi \in C_0^\infty(\Omega),\ i = 1,2,\cdots,m, \tag{4.8}$$

$$(\psi, \overrightarrow{K}_0 w_{irs}) = (\psi, \overline{G}_{irs}),$$
$$0 \leqslant \rho \leqslant \rho_0,\ \forall\psi \in C_0^\infty(\Omega),\ r,s = 0,1,\cdots,\ r+s \neq 0,\ i = 1,2,\cdots,m, \tag{4.9}$$

$$(\psi, w_{irs}) = (\psi, -u_{irs}(t,\widetilde{\phi}) - v_{irs}(t,\widetilde{\phi})),$$
$$\rho = 0,\ \forall\psi \in C_0^\infty(\Omega),\ r,s = 0,1,\cdots,\ r+s \neq 0,\ i = 1,2,\cdots,m, \tag{4.10}$$

其中 $\overline{G}_{irs}$ $(r,s=0,1,\cdots,\ r+s\neq 0)$, $i=1,2,\cdots,m$ 为逐次已知函数, 其结构也从略. 于是, 由 (4.7) 式、(4.8) 式可得到解 w_{i00}. 再由 (4.9), (4.10) 式, 可以依次得到解 w_{irs} $(r,s=0,1,\cdots,\ r+s\neq 0)$, $i=1,2,\cdots,m$.

由假设不难看出: w_{irs} $(r,s=0,1,\cdots)$ 具有边界层性态:

$$w_{irs}=O\Big(\exp\Big(-\widetilde{\delta}_{irs}\frac{\widetilde{\xi}(\widetilde{\rho},\widetilde{\phi})}{\varepsilon}\Big)\Big),\quad r,s=0,1,\cdots,\ i=1,2,\cdots,m,\ 0<\varepsilon\ll 1,\tag{4.11}$$

其中 $\widetilde{\delta}_{irs}>0$ $(r,s=0,1,\cdots,\ i=1,2,\cdots,m)$ 为常数.

令 $\overline{w}_{irs}=\widetilde{\zeta}(\widetilde{\rho})w_{irs}$, 其中 $\widetilde{\zeta}(\widetilde{\rho})$ 为在 $0\leqslant\widetilde{\rho}\leqslant\widetilde{\rho}_0$ 内的充分光滑的分隔函数, 并满足

$$\widetilde{\zeta}(\widetilde{\rho})=\begin{cases}1, & 0\leqslant\widetilde{\rho}\leqslant\dfrac{1}{3}\widetilde{\rho}_0,\\[2ex] 0, & \widetilde{\rho}\geqslant\dfrac{2}{3}\widetilde{\rho}_0.\end{cases}$$

为了方便, 以下仍然用 w_{irs} 代替 $\widetilde{w}_{irs}$. 再由 (4.6) 式, 便得到在 $\partial\Omega$ 的邻域 $(0\leqslant\widetilde{\rho}\leqslant\widetilde{\rho}_0)$ 内的过渡冲击层校正项 $W=(W_1,W_2,\cdots,W_m)$.

5 初始层

设

$$u=U+V+W+Z,\tag{5.1}$$

其中 $Z=(Z_1,Z_2,\cdots,Z_m)$ 为双曲型微分系统的初始层校正项, 而

$$Z(\tau,x)=\sum_{r,s=0}^{\infty}z_{irs}\varepsilon^r\mu^s,\quad i=1,2,\cdots,m,\tag{5.2}$$

其中 $\tau=\frac{t}{\varepsilon}$ 为伸长变量 [1−3].

将 (5.1) 式、(5.2) 式代入广义双曲型微分系统的初始边值问题 (1.1)–(1.4), 按 ε,μ 展开非线性项, 再合并 $\varepsilon^r\mu^s$ 同次幂的系数, 可得

$$\Big(\psi,\frac{\partial^2 z_{i00}}{\partial\tau^2}\Big)=(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00}+z_{i00}))-(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00})),$$
$$i=1,2,\cdots,m,\quad\forall\psi\in C_0^\infty(\Omega),\tag{5.3}$$

$$(\psi,z_{i00})=(\psi,h_{1i}(x))-(\psi,U_{i00}+v_{i00}+w_{i00}),$$
$$\tau=0,\ i=1,2,\cdots,m,\quad\forall\psi\in C_0^\infty(\Omega),\tag{5.4}$$

$$\Big(\psi,\frac{\partial z_{i00}}{\partial\tau}\Big)=(\psi,h_{2i}(x))-\Big(\psi,\frac{\partial(U_{i00}+v_{i00}+w_{i00})}{\partial\tau}\Big),$$
$$\tau=0,\quad i=1,2,\cdots,m,\ \forall\psi\in C_0^\infty(\Omega),\tag{5.5}$$

$$\Big(\psi,\frac{\partial^2 z_{irs}}{\partial\tau^2}\Big)=(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00}+z_{i00}))-(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00})),$$
$$i=1,2,\cdots,m,\ \forall\psi\in C_0^\infty(\Omega),\tag{5.6}$$

$$(\psi,z_{irs})=-(\psi,U_{i00}+v_{i00}+w_{i00}),\quad\tau=0,\ i=1,2,\cdots,m,\ \forall\psi\in C_0^\infty(\Omega),\tag{5.7}$$

$$\Big(\psi,\frac{\partial z_{irs}}{\partial\tau}\Big)=-\Big(\psi,\frac{\partial(U_{i00}+v_{i00}+w_{i00})}{\partial\tau}\Big),$$

$$\tau=0,\ \forall\psi\in C_0^\infty,\ i=1,2,\cdots,m,\ \forall\psi\in C_0^\infty(\Omega),\quad(5.8)$$

其中 $\widetilde{F}_{irs}$ $(r,s=0,1,\cdots,\ r+s\neq0)$, $i=1,2,\cdots,m$ 为逐次已知函数, 其结构也从略. 于是由 (5.3)–(5.5) 式, 可得到解 w_{i00}. 再由 (5.6)–(5.8) 式, 可以依次得到解 z_{irs} $(r,s=0,1,\cdots,\ r+s\neq0)$, $i=1,2,\cdots,m$.

由假设, z_{irs} $(r,s=0,1,\cdots,\ r+s\neq0)$, $i=1,2,\cdots,m$ 具有初始层性态:

$$(\psi,z_{irs})=O\Big(\psi,\exp\Big(-\widetilde{k}_{irs}\frac{t}{\mu}\Big)\Big),\quad r,s=0,1,\cdots,\ i=1,2,\cdots,m,\ 0<\varepsilon,\mu\ll1,\quad(5.9)$$

其中 $\widetilde{k}_{irs}>0$ $(r,s=0,1,\cdots,\ i=1,2,\cdots,m)$ 为适当小的常数. 设

$$\overline{z}_{irs}=\eta(t)z_{irs},$$

其中 $\eta(t)$ 为在充分光滑的分隔函数, 且满足

$$\eta(t)=\begin{cases}1,& 0\leqslant t\leqslant\frac{1}{3}t_0,\\ 0,& t\geqslant\frac{2}{3}t_0.\end{cases}$$

同样, 为了方便起见, 以下我们仍然用 z_{irs} 代替 $\widetilde{z}_{irs}$. 将得到的 z_{irs} 代入 (5.2) 式, 由此, 可构造两参数非线性广义双曲型微分方程初始边值问题 (1.1)–(1.4) 的广义解, 可知有如下形式的渐近展开式:

$$u\sim\sum_{r,s=0}^{M}(U_{rs}+V_{rs}+W_{rs}+Z_{rs})\varepsilon^r\mu^s+O(\lambda),\quad(5.10)$$

其中 M 为正整数, $\lambda=\max(\varepsilon^{M+1}\mu^M,\varepsilon^M\mu^{M+1})$.

6 广义解的一致有效性

现有如下定理.

定理 6.1 在假设 H1, H2 下, 奇异摄动非线性广义双曲型微分系统初始边值问题 (1.1)–(1.4) 存在一个广义解 $u=(u_1,u_2,\cdots,u_m)$, 在 $(t,x)\in[0,T_0]\times\overline{\Omega}])$ 上关于 ε,μ 成立一致有效的渐近展开式 (5.10).

证 现用泛函分析不动点理论来估计双曲型微分系统模型 (1.1)–(1.4) 广义渐近解 (5.10) 的余项 R_i $(i=1,2,\cdots,m)$. 设

$$u_i=\overline{u}_i+R_i,\quad i=1,2,\cdots,m,\quad(6.1)$$

其中

$$\overline{u}_i=\sum_{r,s=0}^{M}(U_{irs}+v_{irs}+w_{irs}+z_{irs})\varepsilon^r\mu^s,\quad i=1,2,\cdots,m.$$

利用 (3.8) 式, (4.11) 式, (5.9) 式和 (6.1) 式, 有

$$\mu^2\Big(\psi,\frac{\partial^2 R_i}{\partial t^2}\Big)-\varepsilon^2 B_1[\psi,R_i]-(\psi,f_i(t,x,\overline{u}_i+R_i))+(\psi,f_i(t,x,\overline{u}_i))$$
$$=-(\psi,f_i(t,x,U_{00})+\sum_{r,s=0,r+s\neq 0}^{M}\Big[\Big(\psi,-\frac{\partial^2 U_{ir(s-2)}}{\partial t^2}\Big)-(\psi,LU_{i(r-2)s})-(\psi,F_{irs})\Big]\varepsilon^r\mu^s$$
$$+(\psi,K_0v_{i00})+\sum_{r,s=0,r+s\neq 0}^{M}[(\psi,K_0v_{irs})-(\psi,G_{irs})]\varepsilon^r\mu^s$$
$$+(\psi,\overline{K}_0w_{i00})+\sum_{r,s=0,r+s\neq 0}^{M}[(\psi,\overline{K}_0w_{irs})-(\psi,\overline{G}_{irs})]\varepsilon^r\mu^s$$
$$+\Big(\psi,\frac{\partial^2 z_{i00}}{\partial \tau^2}\Big)-(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00}+z_{i00}))+(\psi,f_i(0,x,U_{i00}+v_{i00}+w_{i00}))$$
$$+\sum_{r,s=0,r+s\neq 0}^{M}\Big[\Big(\psi,\frac{\partial^2 z_{irs}}{\partial \tau^2}\Big)-(\psi,f_{u_i}(0,x,U_{i00}+v_{i00}+w_{i00})w_{irs})-(\psi,\widetilde{F}_{irs})\Big]\varepsilon^r\mu^s+O(\lambda)$$
$$=O(\lambda),\quad (t,x)\in[0,T_0]\times\Omega,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m,$$

$$(\psi,R_i)$$
$$=(\psi,g_i)-(\psi,-U_{i00}(t,\widetilde{\phi})-v_{i00}(t,\widetilde{\phi}))+\sum_{r,s=0,r+s\neq 0}^{M}(\psi,-U_{irs}(t,\widetilde{\phi})-v_{irs}(t,\widetilde{\phi})\varepsilon^r\mu^s+O(\lambda)$$
$$=O(\lambda),\quad x\in\partial\Omega,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m,$$

$$(\psi,R_i)$$
$$=(\psi,h_{1i})-(\psi,U_{i00}+v_{i00}+w_{i00})-\sum_{r,s=0,r+s\neq 0}^{M}(\psi,U_{irs}+v_{irs}+w_{irs})\varepsilon^r\mu^s+O(\lambda)$$
$$=O(\lambda),\quad t=0,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m,$$

$$\Big(\psi,\frac{\partial R_i}{\partial t}\Big)$$
$$=(\psi,h_{2i})-\Big(\psi,\frac{\partial(U_{i00}+v_{i00}+w_{i00})}{\partial\tau}\Big)-\sum_{r,s=0,r+s\neq 0}^{M}\Big(\psi,\frac{\partial(U_{irs}+v_{irs}+w_{irs})}{\partial\tau}\Big)\varepsilon^r\mu^s+O(\lambda)$$
$$=O(\lambda),\quad t=0,\ \forall\psi\in C_0^\infty(\Omega),\ i=1,2,\cdots,m.$$

设线性化的微分算子 $\overline{L}$ 满足

$$(\psi,\overline{L}[p_i])=\mu^2\Big(\psi,\frac{\partial^2 p_i}{\partial t^2}\Big)+\varepsilon^2B_1[\psi,p_i]-(\psi,f_{iu}(t,x,\overline{u})p_i),$$
$$t=0,\ \forall\psi\in C_0^\infty,\ i=1,2,\cdots,m.$$

故有

$$(\psi,\Psi[p_i])\equiv(\psi,F[p_i]-\overline{L}[p_i])$$
$$=(\psi,f(t,x,\overline{u}_i)-(\psi,f(t,x,\overline{u}_i+p_i))+(\psi,f(t,x,\overline{u}_i+\theta_ip_i)p_i),$$

$$0<\theta_i<1,\ \forall\psi\in C_0^\infty,\ i=1,2,\cdots,m.$$

固定 ε,μ, 选择线性赋范空间 N:

$N=\{p_i\mid p_i\in C^2((0,T_0)]\times\Omega\}$,

$(\psi,p_i|_{\partial\Omega})=g_i,\quad (\psi,p_i|_{t=0})=(\psi,h_{1i}),\ \Big(\psi,\dfrac{\partial p_i}{\partial t}|_{t=0}\Big)=(\psi,h_{2i}),\quad \forall\psi\in C_0^\infty,\ i=1,2,\cdots,m,$

具有范数 $\|p_i\|=\sup\limits_{t\in(0,T_0],x\in\Omega}|p_i|$. 而设 Banach 空间 B 为

$$B=\{q\mid q\in C((0,T_0]\times\Omega)\},$$

且范数 $\|q\|=\sup\limits_{t\in(0,T_0],x\in\Omega}|q|$. 并由假设, 成立

$$\|(\psi,\overline{L}^{-1}[g_i])\|\leqslant l^{-1}\|(\psi,g)\|,\quad \forall g_i\in B,\ \forall\psi\in C_0^\infty(\Omega),$$

其中 l^{-1} 独立于 ε,μ, $\overline{L}^{-1}$ 为 $\overline{L}$ 的连续逆算子, 且满足 Lipschitz 条件

$$\begin{aligned}\|\Psi[p_2]-\Psi[p_1]\|&=\sup_{t\in(0,T_0],x\in\Omega}\Big(\psi,\Big|\frac{\partial^2 f_i}{\partial u^2}(t,x,\overline{u}+\theta_2p_2)p_2^2\Big|\Big)\\&=\sup_{t\in(0,T_0],x\in\Omega}\Big|\Big(\psi,\frac{\partial^2 f_i}{\partial u^2}(t,x,\overline{u}+\theta_2p_2)(p_2^2-p_1^2)\Big)\\&\quad+\Big(\psi,\Big(\frac{\partial^2 f_i}{\partial u^2}(t,x,\overline{u}+\theta_2p_2)\Big)\Big)\Big|\\&<C\|p_2-p_1\|,\quad \forall\psi\in C_0^\infty(\Omega),\end{aligned}$$

其中 C_1,C_2 和 C 为独立于 ε,μ 的常数, 并对任意的 p_1,p_2 在球 $K_N(r),\|r\|\leqslant1$ 中成立. 再由泛函分析不动点原理 [1−2], 奇异摄动非线性广义双曲型微分系统 (1.1)–(1.4) 广义解的渐近展开式 (6.1) 的余项 R, 满足

$$\sup_{t\in(0,T_0,x\in\Omega]}|(\psi,R_i(t,x))|=O(\lambda),\quad i=1,2,\cdots,m,$$

其中 $\lambda=\max(\varepsilon^{M+1}\mu^M,\varepsilon^M\mu^{M+1})$. 定理证毕.

7 结　论

非线性微分系统冲击波理论出自于一类复杂的自然现象. 因此需要简化它为基本模型. 利用近似方法求解这类模型是冲击波理论的重要方面. 本文就是利用奇异摄动理论, 用一个简单而有效的方法, 得到了非线性系统过渡冲击层渐近解. 由渐近方法求解模型的近似解, 不同于用单纯的数值模拟方法.

由于渐近解是具有解析形式的结构, 因此它还可以进一步进行微分、积分等解析运算, 从而能了解相应过渡冲击层解的更进一步的性态.

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The Generalized Solution for Transitional Shock Layer to Singularly Perturbed Nonlinear Hyperbolic Type Differential System with Two Parameters

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Abstract A class of nonlinear hyperbolic type differential system to singularly perturbed initial-boundary problem with two parameters is studied. Firstly, using singular perturbation theory and method, the outer solution for the problem is structured related two small parameters. Secondly, using the multi-scale and stretched variables, the transitional shock layer, boundary layer and initial layer corrective terms are obtained for the original problem respectively. Finally, the asymptotic expansion of solution for the original problem is given. And the uniform validity of its asymptotic solution is proved by using the theory of fixed point of functional analysis. Using this method to obtained asymptotic solution of original problem, it can also carry on analytical operation for the differential and integral and so on. It is known more behaviors for the transitional shock layer of solution. Thus this method possesses good applied foreground.

Keywords Differential system, Transitional layer, Small parameter

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