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Exact Boundary Controllability on a Tree-Like Network of Nonlinear Planar Timoshenko Beams*

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Abstract This paper concerns a system of equations describing the vibrations of a planar network of nonlinear Timoshenko beams. The authors derive the equations and appropriate nodal conditions, determine equilibrium solutions and, using the methods of quasilinear hyperbolic systems, prove that for tree-like networks the natural initial-boundary value problem admits semi-global classical solutions in the sense of Li [Li, T. T., Controllability and Observability for Quasilinear Hyperbolic Systems, AIMS Ser. Appl. Math., vol 3, American Institute of Mathematical Sciences and Higher Education Press, 2010] existing in a neighborhood of the equilibrium solution. The authors then prove the local exact controllability of such networks near such equilibrium configurations in a certain specified time interval depending on the speed of propagation in the individual beams.

Keywords Nonlinear Timoshenko beams, Tree-like networks, Exact boundary controllability, Semi-global classical solutions
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1 Introduction

We consider a planar network of initially straight nonlinear Timoshenko beams under control at some external boundary nodes. The corresponding linear system has been modeled and analyzed with respect to wellposedness and controllability, observability and stabilizability, optimal control and domain decomposition methods by Lagenese, Leugering and Schmidt [5–6] and Lagnese and Leugering [8]. Nonlinear Timoshenko beams in three spatial dimensions including thermal effects have been introduced by the same authors in [7]. Modeling and well-posedness for nonlinear Timoshenko beams and in particular for networks of such beams, to the best knowledge of the authors, have not been studied by many authors. A planar couple-stress modeling has been recently described by Asghari et al. [1]. Other planar models have been given by Zhong and Guo [17] and Racke and Riviera [12], the latter with thermal effects but without longitudinal displacement. We will show below that these models are included in the approach described in [7]. Global wellposedness for nonlinear Timoshenko beams in a general framework does not seem to be available within the literature so far. Problems of exact controllability,

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observability and even stabilizability for single nonlinear Timoshenko beams has, to the best knowledge of the authors, not been considered in the literature, and this is particularly so for networks of nonlinear Timoshenko beams. This paper provides a first attempt to solve these problems. We concentrate here our attention to planar networks of such kind. Networks in three-spatial dimensions are subject to a forthcoming publication.

The plan of the paper is as follows. We first briefly describe the modeling procedure outlined in [7]. The focus here, however, is on planar, initially straight and isothermal shearable beams. These assumptions drastically reduce the complexity of the modeling procedure and, therefore, the description is of independent interest. We then formulate the corresponding initial-boundary value problem for a single beam under gravity. In the next step, we look for equilibria under a given set of boundary conditions. In order to discuss well-posedness, we rewrite the system in quasilinear form followed by a representation as a first order system. We notice that in doing this, an artificial zero eigenvalue appears related to the shear angle. Then, the first order format allows the application of the concept of semi-global classical solutions in the sense of Li Tatsien [13]. Having established semi-global existence for the solution of the problem and the problem resulting in interchanging the space and time variable a transformation that has to be verified in due course the exact controllability can be shown as in [13]. The remaining part of the paper is then devoted to extend the method to networks of such nonlinear Timoshenko beams. Indeed, as is well-known that this procedure applies to tree-like networks using the so-called peeling method.

2 Modeling of Nonlinear Beams

Let Ω be the domain of the undeformed planar beam:

$$\Omega := \{\mathbf{r}(x) := \mathbf{r}_0(x_1) + x_3\mathbf{e}_3 \mid x_3\mathbf{e}_3 \in A(x_1),\ x_1 \in [0, L]\},$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ constitute an orthonormal basis in $\mathbb{R}^3$, L and $A(x_1)$ denote the length of the beam and its cross section at x_1, respectively. As we consider a planar and initially straight reference configuration, $\mathbf{e}_1, \mathbf{e}_3$ describe the plane of deformation, while $\mathbf{e}_2$, pointing into the plane, is the axis of rotation. We may, in fact, assume that $A(x_1) = A$ is constant along the beam. It is clear that more general cases can also be dealt with. The reference configuration Ω is subject to deformation, and thus we consider $\mathbf{R}(x)$ as the vector pointing into the deformed configuration. Consequently, $\mathbf{V}(x) := \mathbf{R}(x) - \mathbf{r}(x)$ is the displacement. As in all beam theories, one finally wants to express everything in terms of variables related to the center line $x_3 = 0$ of the beam. Thus, the displacement of the center line is introduced as $\mathbf{W}(x_1) := \mathbf{V}(x_1, 0)$, where we suppress the variable x_2, which is zero in the planar case. The tangents are given by $\mathbf{G}_i := \mathbf{R}_{,i} := \frac{\partial}{\partial x_i}\mathbf{R}$ and $\mathbf{E}_i(x_1) := \mathbf{G}_i(x_1, 0),\ i = 1, 3$, respectively. We then can write down the strains

$$\epsilon_{ij} := \frac{1}{2}(\mathbf{G}_i \cdot \mathbf{G}_j - \delta_{ij}) \tag{2.1}$$

and

$$\bar{\epsilon}_{ij} := \frac{1}{2}(\mathbf{E}_i \cdot \mathbf{E}_j - \delta_{ij}). \tag{2.2}$$

Using standard arguments, see [7, 16], for a shearable beam one can establish the following representation of the potential energy of the beam under deformation:

$$U = \frac{1}{2}\int_0^L (EA\bar{\epsilon}_{11}^2 + GA\bar{\epsilon}_{13}^2 + EI\bar{\kappa}^2)\mathrm{d}x, \tag{2.3}$$

where $\overline{\kappa}$ is the curvature due to bending and E, A, I, G are Young's modulus, the area of cross section, the inertial moment around the $\mathbf{e}_2$ axis, and the shear modulus, respectively. In order to derive the equations governing the motion of the beam, we need to express the strains (2.2) and the curvature $\overline{\kappa}$ in terms of primitive variables such as displacements and rotation. We consider the deformation process as being composed of the mappings, a rotation with the angle Θ about the $\mathbf{e}_2$ axis carrying the orthonormal system $\mathbf{e}_1, \mathbf{e}_3$ into $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_3$ followed by a deformation of $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_3$ into the non-orthogonal system $\mathbf{E}_1, \mathbf{E}_3$, described by the strains $\overline{\epsilon}_{ij}$. The first smallness assumption concerns the rotation angle Θ such that the rotation takes the form

$$\widehat{\mathbf{e}}_1 = \mathbf{e}_1 - \Theta \mathbf{e}_3, \quad \widehat{\mathbf{e}}_3 = \Theta \mathbf{e}_1 + \mathbf{e}_3. \tag{2.4}$$

By definition, the curvature is given by

$$\overline{\kappa} := \widehat{\mathbf{e}}_3 \cdot \widehat{\mathbf{e}}_{1,1} = -\widehat{\mathbf{e}}_1 \cdot \widehat{\mathbf{e}}_{3,1} = -\Theta_{,1}. \tag{2.5}$$

It is obvious from the definition (2.2) that $\mathbf{E}_1, \mathbf{E}_3$ take the form

$$\mathbf{E}_1 = \widehat{\mathbf{e}}_1 + \overline{\epsilon}_{11}\mathbf{e}_1, \quad \mathbf{E}_3 = \widehat{\mathbf{e}}_3 + 2\overline{\epsilon}_{31}\mathbf{e}_1. \tag{2.6}$$

Using (2.4) we find

$$\begin{cases} \mathbf{E}_1 = (1+\overline{\epsilon}_{11})(\mathbf{e}_1 - \Theta \mathbf{e}_3) = \mathbf{e}_1 + \mathbf{W}_{,1}, \\ \mathbf{E}_3 = \mathbf{e}_3 + (\Theta + 2\overline{\epsilon}_{31})\mathbf{e}_1. \end{cases} \tag{2.7}$$

(2.7) describes the overall deformation from $\mathbf{e}_1, \mathbf{e}_3$ into $\mathbf{E}_1, \mathbf{E}_3$ and the second line suggests the introduction of a total rotation angle $\vartheta := \Theta + 2\overline{\epsilon}_{31}$, which accounts for rotation due to both bending and shear. We are now going to express $\overline{\epsilon}$ in terms of a linear symmetric part and a nonlinear part based on anti-symmetric quantities. Let

$$\begin{cases} \overline{e}_{ij} := \dfrac{1}{2}(\mathbf{e}_i \cdot \mathbf{E}_j + \mathbf{e}_j \cdot \mathbf{E}_i) - \delta_{ij}, \\ \overline{\omega}_{ij} := \dfrac{1}{2}(\mathbf{e}_i \cdot \mathbf{E}_j - \mathbf{e}_j \cdot \mathbf{E}_i). \end{cases} \tag{2.8}$$

With this notation, we can express $\overline{\epsilon}$, given by (2.2), as

$$\overline{\epsilon}_{ij} = \overline{e}_{ij} + \frac{1}{2}\sum_{p=1,3}(\overline{e}_{pi} + \overline{\omega}_{pi})(\overline{e}_{pj} + \overline{\omega}_{pj}). \tag{2.9}$$

At this point, we introduce the second hypothesis on the smallness, namely, we assume that the strains $\overline{e}_{ij}$ are small with respect to the rotations $\overline{\omega}_{ij}$, in other words, we set

$$\overline{\epsilon}_{ij} = \overline{e}_{ij} + \frac{1}{2}\sum_{p=1,3}(\overline{\omega}_{pi})(\overline{\omega}_{pj}). \tag{2.10}$$

We have $\mathbf{W}_{,1} \cdot \mathbf{e}_3 = (\mathbf{E}_1 - \mathbf{e}_1) \cdot \mathbf{e}_3 = -\Theta = W_{3,1}$ and, therefore,

$$\overline{e}_{11} = W_{1,1}, \quad \overline{e}_{13} = \frac{1}{2}(\vartheta - \Theta), \quad \overline{\omega}_{13} = \frac{1}{2}(\vartheta + \Theta). \tag{2.11}$$

With (2.11) we can now express the strains $\overline{\epsilon}_{ij}$ as follows:

$$\overline{\epsilon}_{11} = W_{1,1} + \frac{1}{8}(\vartheta + \Theta)^2, \quad \overline{\epsilon}_{13} = \frac{1}{2}(\vartheta - \Theta). \tag{2.12}$$

It depends now on how one handles Θ versus ϑ in connection with $W_{3,1}$ in order to obtain different models of Timoshenko beams and Euler-Bernoulli beams. If there is no shear strain, then upon $\Theta = \vartheta = -W_{3,1}$ one obtains

$$\bar{\epsilon}_{11} = W_{1,1} + \frac{1}{2}W_{3,1}^2, \quad \bar{\epsilon}_{13} = 0. \tag{2.13}$$

This leads to a nonlinear Euler-Bernoulli beam as discussed in [7]. If one keeps the shear strain, one arises at what has come to be known as von Karman relation, namely,

$$\bar{\epsilon}_{11} = W_{1,1} + \frac{1}{2}W_{3,1}^2, \quad \bar{\epsilon}_{13} = \frac{1}{2}(\vartheta + W_{3,1}). \tag{2.14}$$

We now express the quantities W_1, W_3, ϑ by $u, w, -\psi$, respectively, where the minus sign is introduced only for easier comparison with the traditional notation for linear models. We introduce then the following potential energy:

$$U = \frac{1}{2}\int_0^L \Big\{EA\Big(\frac{\partial}{\partial x}u + \frac{1}{2}\Big(\frac{\partial}{\partial x}w\Big)^2\Big)^2 + GA\Big(\frac{\partial}{\partial x}w - \psi\Big)^2 + EI\Big(\frac{\partial}{\partial x}\psi\Big)^2\Big\}\mathrm{d}x. \tag{2.15}$$

If we consider the deformation of the beam under its own weight, we have to add to the potential energy the corresponding gravitational effect:

$$U_g = \int_0^L \rho A g(\mathbf{e}\cdot\mathbf{e}_1 u + \mathbf{e}\cdot\mathbf{e}_3 w)\mathrm{d}x. \tag{2.16}$$

Obviously, axial forces no longer couple to rotation (shear), and we then derive the following nonlinear Timoshenko beam model:

$$\begin{cases} \rho A\dfrac{\partial^2}{\partial t^2}u = EA\dfrac{\partial^2}{\partial x^2}u + \dfrac{EA}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w\Big)^2 - \rho g A\mathbf{e}\cdot\mathbf{e}_1, \\ \rho A\dfrac{\partial^2}{\partial t^2}w = GA\Big(\dfrac{\partial^2}{\partial x^2}w - \dfrac{\partial}{\partial x}\psi\Big) + \dfrac{\partial}{\partial x}\Big(EA\Big(\dfrac{\partial}{\partial x}u + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w\Big)^2\Big)\dfrac{\partial}{\partial x}w\Big) - \rho g A\mathbf{e}\cdot\mathbf{e}_3, \\ \rho I\dfrac{\partial^2}{\partial t^2}\psi = EI\dfrac{\partial^2}{\partial x^2}\psi + GA\Big(\dfrac{\partial}{\partial x}w - \psi\Big). \end{cases} \tag{2.17}$$

Remark 2.1 Such a system (including higher order terms) has been investigated by Asghari et al. [1]. If one considers very thin beams, then their model reduces to (2.17). It should also be mentioned that upon neglecting the longitudinal displacement in (2.17), one derives a model that can be written in the format of Riviera and Racke [12], where the thermal coupling is also present. These models are included in the framework of Lagnese, Leugering and Schmidt [7]. While in [12] a wellposedness result is derived for the thermoelastic Timoshenko beam, a global in time existence and uniqueness result does not seem to be known in the literature as regards systems (2.17). Finally, if one assumes no shear, i.e., $\psi = \frac{\partial}{\partial x}w$, then one obtains from the Hamilton principle, after suitably adjusting the potential and kinetic energies, a nonlinear Euler-Bernoulli-beam equation coupled to longitudinal motion. If the latter, in turn, is neglected one arrives at a 1-d version of the von Karman-system. See Langese and Leugering [4] and Horn and Leugering [3] for stabilizability results for the latter beam equations.

We use the model (2.17) in this article. We have the following boundary conditions.

(i) Dirichlet conditions at $x = 0$:

$$u(0,t) = v_D^1(t), \quad w(0,t) = v_D^2(t), \quad \psi(0,t) = v_D^3(t), \quad t \in [0,T]. \tag{2.18}$$

(ii) Neumann conditons at $x = L$:

$$EA\frac{\partial}{\partial x}u(L,t) + \frac{EA}{2}\Big(\frac{\partial}{\partial x}w(L,t)\Big)^2 = v_N^1(t), \quad t \in [0,T], \tag{2.19}$$

$$\begin{aligned} &GA\Big(\frac{\partial}{\partial x}w(L,t) - \psi(L,t)\Big) \\ &\quad + EA\Big[\Big(\frac{\partial}{\partial x}u(L,t) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w(L,t)\Big)^2\Big)\frac{\partial}{\partial x}w(L,t)\Big] = v_N^2(t), \quad t \in [0,T], \end{aligned} \tag{2.20}$$

$$EI\frac{\partial}{\partial x}\psi(L,t) = v_N^3(t), \quad t \in [0,T]. \tag{2.21}$$

Remark 2.2 It is clear that with homogeneous boundary data, $u = w = \psi = 0,\ x \in [0,L]$ is an equilibrium. The determination of all non-zero constant and also nonconstant equilibria is beyond the scope of the article. This will be considered in a forthcoming publication. Here we restrict ourselves only with some examples.

We are now in the position to formulate the initial-boundary value problem for a planar nonlinear Timoshenko beam.

$$\begin{cases} \rho A\dfrac{\partial^2}{\partial t^2}u(x,t) = EA\dfrac{\partial^2}{\partial x^2}u(x,t) + \dfrac{EA}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w(x,t)\Big)^2, \\ \rho A\dfrac{\partial^2}{\partial t^2}w(x,t) = GA\Big(\dfrac{\partial^2}{\partial x^2}w(x,t) - \dfrac{\partial}{\partial x}\psi(x,t)\Big) \\ \qquad\qquad + \dfrac{\partial}{\partial x}\Big(EA\Big(\dfrac{\partial}{\partial x}u(x,t) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w(x,t)\Big)^2\Big)\dfrac{\partial}{\partial x}w(x,t)\Big) - \rho gA, \\ \rho I\dfrac{\partial^2}{\partial t^2}\psi(x,t) = EI\dfrac{\partial^2}{\partial x^2}\psi(x,t) + GA\Big(\dfrac{\partial}{\partial x}w(x,t) - \psi(x,t)\Big), \\ \qquad\qquad (x,t) \in [0,L]\times[0,T], \end{cases} \tag{2.22}$$

$$\begin{cases} u(0,t) = v_D^1(t), \quad w(0,t) = v_D^2(t), \quad \psi(0,t) = v_D^3(t), \quad t \in [0,T], \\ EA\dfrac{\partial}{\partial x}u(L,t) + \dfrac{EA}{2}\Big(\dfrac{\partial}{\partial x}w(L,t)\Big)^2 = v_N^1(t), \\ GA\Big(\dfrac{\partial}{\partial x}w(L,t) - \psi(L,t)\Big) \\ \quad + \Big(EA\Big(\dfrac{\partial}{\partial x}u(L,t) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w(L,t)\Big)^2\Big)\dfrac{\partial}{\partial x}w(L,t)\Big) = v_N^2(t), \\ EI\dfrac{\partial}{\partial x}\psi(L,t) = v_N^3(t), \quad t \in [0,T], \end{cases} \tag{2.23}$$

$$\begin{cases} u(x,0) = u_0(x), \quad \dfrac{\partial}{\partial t}u(x,0) = u_1(x), \\ w(x,0) = w_0(x), \quad \dfrac{\partial}{\partial t}w(x,0) = w_1(x), \\ \psi(x,0) = \psi_0(x), \quad \dfrac{\partial}{\partial t}\psi(x,0) = \psi_1(x), \quad x \in [0,L]. \end{cases} \tag{2.24}$$

Here (2.22), (2.23) and (2.24) represent the state equations, the boundary conditions and the initial conditions, respectively.

2.1 Equilibrium solutions

We now consider equilibrium solutions of (2.22). Clearly, for the homogeneous system, the zero-state is an equilibrium. However, in the context of mechanics, we always have to deal with gravitational forces.

Example 2.1 The first case concerns a horizontal beam that is clamped at $x=0$ and free at the other end, i.e., $\mathbf{e}_1=(1,0)^{\mathrm{T}}$, $\mathbf{e}_3=(0,1)^{\mathrm{T}}=\mathbf{e}$. Therefore, looking for an equilibrium solution in the context of gravitation leads to the following ordinary differential system:

$$\begin{cases} 0=EA\dfrac{\partial^2}{\partial x^2}u(x)+\dfrac{EA}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w(x)\Big)^2, \\ \rho gA=GA\Big(\dfrac{\partial^2}{\partial x^2}w(x)-\dfrac{\partial}{\partial x}\psi(x)\Big) \\ \qquad\qquad +\dfrac{\partial}{\partial x}\Big(EA\Big(\dfrac{\partial}{\partial x}u(x)+\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w(x)\Big)^2\Big)\dfrac{\partial}{\partial x}w(x)\Big), \\ 0=EI\dfrac{\partial^2}{\partial x^2}\psi(x)+GA\Big(\dfrac{\partial}{\partial x}w(x)-\psi(x)\Big), \quad x\in[0,L], \end{cases} \tag{2.25}$$

$$\begin{cases} u(0)=0, \quad w(0)=0, \quad \psi(0)=0, \\ EA\dfrac{\partial}{\partial x}u(L)+\dfrac{EA}{2}\Big(\dfrac{\partial}{\partial x}w(L)\Big)^2=0, \\ GA\Big(\dfrac{\partial}{\partial x}w(L)-\psi(L)\Big)+\Big(EA\Big(\dfrac{\partial}{\partial x}u(L)+\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w(L)\Big)^2\Big)\dfrac{\partial}{\partial x}w(L)\Big)=0, \\ EI\dfrac{\partial}{\partial x}\psi(L)=0. \end{cases} \tag{2.26}$$

The solution of (2.25) and (2.26) is given by

$$\begin{aligned} \widehat{u}(x)=&-\frac{1}{2}\Big(\frac{\rho gA}{EI}\Big)^2\Big\{-\frac{1}{252}((L-x)^7-L^7)+\frac{1}{15}((L-x)^5-L^5))\frac{EI}{GA} \\ &+\frac{1}{72}L^3((L-x)^4-L^4)-\frac{1}{3}((L-x)^3-L^3)\Big(\frac{EI}{GA}\Big)^2 \\ &-\frac{L^3}{6}((L-x)^2-L^2)\frac{EI}{GA}+\frac{L^6}{36}x\Big\}, \end{aligned} \tag{2.27}$$

$$\widehat{w}(x)=\frac{\rho gA}{2EI}\Big(-\frac{1}{12}((L-x)^4-L^4)-\frac{1}{3}L^3x+((L-x)^2-L^2)\frac{EI}{GA}\Big), \tag{2.28}$$

$$\widehat{\psi}(x)=\frac{\rho gA}{6EI}((L-x)^3-L^3). \tag{2.29}$$

Example 2.2 In the case of a hanging beam, where the top end $x=0$ is clamped and the end $x=L$ is free, we have $\mathbf{e}_1=-(0,1)^{\mathrm{T}}=-\mathbf{e}$, $\mathbf{e}_3=(1,0)^{\mathrm{T}}$. Thus,

$$\begin{cases} -\rho gA=EA\dfrac{\partial^2}{\partial x^2}u(x)+\dfrac{EA}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w(x)\Big)^2, \\ 0=GA\Big(\dfrac{\partial^2}{\partial x^2}w(x)-\dfrac{\partial}{\partial x}\psi(x)\Big) \\ \qquad\qquad +\dfrac{\partial}{\partial x}\Big(EA\Big(\dfrac{\partial}{\partial x}u(x)+\dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w(x)\Big)^2\Big)\dfrac{\partial}{\partial x}w(x)\Big), \\ 0=EI\dfrac{\partial^2}{\partial x^2}\psi(x)+GA\Big(\dfrac{\partial}{\partial x}w(x)-\psi(x)\Big), \quad x\in[0,L]. \end{cases} \tag{2.30}$$

The boundary conditions are the same as in (2.26). In this case the first equation gives a tangential load

$$\rho g A(L-x) = EA\Big(\frac{\partial}{\partial x}u(x) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w(x)\Big)^2\Big), \tag{2.31}$$

$$0 = GA\Big(\frac{\partial^2}{\partial x^2}w(x) - \frac{\partial}{\partial x}\psi(x)\Big) + \rho g A\frac{\partial}{\partial x}\Big((L-x)\frac{\partial}{\partial x}w(x)\Big). \tag{2.32}$$

Obviously, the unique solution is the one, where $w(x) = \psi(x) = 0,\ \forall x \in [0, L]$, while

$$u(x) = \frac{\rho g}{2E}(L^2 - (L-x)^2),$$

which clearly shows the stretching due to gravitation.

2.2 Quasilinear form

We proceed to derive the quasilinear form of the system (2.17) or of the initial-boundary value problem (2.22)–(2.24).

$$\begin{cases} \rho A\dfrac{\partial^2}{\partial t^2}u = EA\dfrac{\partial^2}{\partial x^2}u + EA\dfrac{\partial}{\partial x}w\dfrac{\partial^2}{\partial x^2}w, \\ \rho A\dfrac{\partial^2}{\partial t^2}w = GA\Big(\dfrac{\partial^2}{\partial x^2}w - \dfrac{\partial}{\partial x}\psi\Big) \\ \qquad\qquad + EA\dfrac{\partial}{\partial x}w\dfrac{\partial^2}{\partial x^2}u + EA\Big(\dfrac{\partial}{\partial x}u + \dfrac{3}{2}\Big(\dfrac{\partial}{\partial x}w\Big)^2\Big)\dfrac{\partial^2}{\partial x^2}w - \rho g A, \\ \rho I\dfrac{\partial^2}{\partial t^2}\psi = EI\dfrac{\partial^2}{\partial x^2}\psi + GA\Big(\dfrac{\partial}{\partial x}w - \psi\Big). \end{cases} \tag{2.33}$$

We rewrite (2.33) as a system of second order equations in vectorial form as follows. We introduce the vectorial state as $\quad := (u, w, \psi)^{\mathrm{T}}$ and define

$$\begin{cases} M := \begin{pmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho I \end{pmatrix}, \\ G\Big(\ ,\dfrac{\partial}{\partial x}\ \Big) := \begin{pmatrix} EA & EA\dfrac{\partial}{\partial x}w & 0 \\ EA\dfrac{\partial}{\partial x}w & GA + EA\Big(\dfrac{\partial}{\partial x}u + \dfrac{3}{2}\Big(\dfrac{\partial}{\partial x}w\Big)^2\Big) & 0 \\ 0 & 0 & EI \end{pmatrix}, \\ F\Big(\ ,\dfrac{\partial}{\partial x}\ \Big) := \begin{pmatrix} 0 \\ -GA\dfrac{\partial}{\partial x}\psi - \rho g A \\ GA\Big(\dfrac{\partial}{\partial x}w - \psi\Big) \end{pmatrix}. \end{cases} \tag{2.34}$$

The system (2.33) can be written as

$$M\frac{\partial^2}{\partial t^2}\quad = G\Big(\ ,\frac{\partial}{\partial x}\ \Big)\frac{\partial^2}{\partial x^2}\quad + F\Big(\ ,\frac{\partial}{\partial x}\ \Big). \tag{2.35}$$

System (2.35) is a quasilinear system of second order in space and time. Now, given an equilibrium $\widehat{\ } := (\widehat{u}, \widehat{w}, \widehat{\psi})^{\mathrm{T}}$, we look for states $\quad = \widehat{\ } + \widetilde{\ }$ for possibly small $\widetilde{\ }$. Clearly, we can reformulate the quasilinear system in terms of the perturbation $\widetilde{\ }$. To this end, we define

$$\begin{cases} \mathcal{G}\Big(\widetilde{\ }, \dfrac{\partial}{\partial x}\widetilde{\ }\Big) := G\Big(\widehat{\ } + \widetilde{\ }, \dfrac{\partial}{\partial x}\widehat{\ } + \dfrac{\partial}{\partial x}\widetilde{\ }\Big), \\ \mathcal{F}\Big(\widetilde{\ }, \dfrac{\partial}{\partial x}\widetilde{\ }\Big) := \mathcal{G}\Big(\widetilde{\ }, \dfrac{\partial}{\partial x}\widetilde{\ }\Big)\dfrac{\partial^2}{\partial x^2}\widehat{\ } + F\Big(\widehat{\ } + \widetilde{\ }, \dfrac{\partial}{\partial x}\widehat{\ } + \dfrac{\partial}{\partial x}\widetilde{\ }\Big). \end{cases} \tag{2.36}$$

Because the $\widehat{\ }$ is an equilibrium solution, we have

$$\mathcal{F}(0,0)=0. \tag{2.37}$$

The system (2.33) takes now the form

$$M\frac{\partial^2}{\partial t^2}\widetilde{\ }=\mathcal{G}\Big(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big)\frac{\partial^2}{\partial x^2}\widetilde{\ }+\mathcal{F}\Big(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big). \tag{2.38}$$

Now,

$$M^{-1}\mathcal{G}\Big(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big)=\frac{E}{\rho}\begin{pmatrix} 1 & \frac{\partial}{\partial x}(\widehat{w}+w) & 0\\ \frac{\partial}{\partial x}(\widehat{w}+w) & \frac{G}{E}+\frac{\partial}{\partial x}(\widehat{u}+u)+\frac{3}{2}\Big(\frac{\partial}{\partial x}(\widehat{w}+w)\Big)^2 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.39}$$

is a symmetric matrix. Hyperbolicity is then a matter of showing that the eigenvalues of $M^{-1}\mathcal{G}$ are uniformly positive in a neighborhood of the equilibrium solution. To this end, we introduce

$$q=q\Big(\widehat{\ },\widetilde{\ },\frac{\partial}{\partial x}\widehat{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big):=1+\frac{G}{E}\frac{\partial}{\partial x}(\widehat{u}+u)+\frac{3}{2}\Big(\frac{\partial}{\partial x}(\widehat{w}+w)\Big)^2, \tag{2.40}$$

$$r=r\Big(\widehat{\ },\widetilde{\ },\frac{\partial}{\partial x}\widehat{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big):=\frac{G}{E}+\frac{\partial}{\partial x}(\widehat{u}+u)+\frac{1}{2}\Big(\frac{\partial}{\partial x}(\widehat{w}+w)\Big)^2. \tag{2.41}$$

Then the eigenvalues of $M^{-1}\mathcal{G}$ are $\mu_1:=\frac{E}{\rho},\ \mu_2,\ \mu_3$ with

$$\mu_{2,3}\Big(\widehat{\ },\widetilde{\ },\frac{\partial}{\partial x}\widehat{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big)=\frac{\mu_1}{2}\Big(q\pm\sqrt{q^2-4r}\Big). \tag{2.42}$$

As the general discussion on the hyperbolicity of the system depending on the magnitudes of E and G is a bit involved, for the sake of simplicity, in this article we resort to equilibria such that the eigenvalues μ_i are uniformly positive and smooth in a sufficiently small neighborhood of the equilibrium solution $\widehat{\ }$.

For the analysis of (2.35), in particular for its controllability, it is important to consider the invertibility of the matrix $\mathcal{G}(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ })$ in a neighborhood of $\widehat{\ }$. Indeed, we formally have

$$\begin{aligned}&\mathcal{G}\Big(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big)^{-1}\\ &=\frac{1}{d\Big(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }\Big)}\begin{pmatrix}\frac{\partial}{\partial x}(\widehat{u}+\widetilde{u})+\frac{3}{2}\Big(\frac{\partial}{\partial x}(\widehat{w}+\widetilde{w})\Big)^2+\frac{G}{E} & -\frac{\partial}{\partial x}(\widehat{w}+\widetilde{w}) & 0\\ -\frac{\partial}{\partial x}(\widehat{w}+\widetilde{w}) & 1 & 0\\ 0 & 0 & \frac{E}{I}\Big(\frac{\partial}{\partial x}(\widehat{u}+\widetilde{u})+\frac{1}{2}\Big(\frac{\partial}{\partial x}\widetilde{(\widehat{w}+w)}\Big)^2+\frac{G}{E}\Big)\end{pmatrix},\end{aligned}$$

where $d(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ }):=EA\big(\frac{G}{E}+\frac{\partial}{\partial x}\big(\widehat{u}+\widetilde{u}+\frac{1}{2}\big(\frac{\partial}{\partial x}(\widehat{w}+\widetilde{w})\big)^2\big)$. It is obvious that the matrix $\mathcal{G}(\widetilde{\ },\frac{\partial}{\partial x}\widetilde{\ })$ is indeed invertible in a neighborhood of the equilibrium state $\widehat{\ }$ that satisfies the requirements above. Under these conditions, one may then multiply the system (2.38) by $\mathcal{G}^{-1}$ and obtain the second order derivative in the spatial variable with the identity matrix as coefficient. Then, one may interchange x and t. Moreover, in this case, the transformed system is again a hyperbolic system of second order. Formally, the original system and the one obtained after interchanging x and t are not of the same type, as first order spatial derivatives in the coefficients of the $x-t$ version are now time derivatives in the $t-x$ version of the problem. In order to fully symmetrize the situation, one can use the format discussed by Wang [14].

3 First Order System

In order to analyze the well-posedness of system (2.33), we transform the second order equations into a quasilinear hyperbolic system of first order. There are a number of equivalent ways to do that. In order to avoid the $\widetilde{\ }$-notation, we write the perturbations without the $\widetilde{\ }$-sign. To this end, we introduce the following variables $U=(u_1,\cdots,u_7)$:

$$\begin{cases} u_1 := \dfrac{\partial}{\partial x}u, \quad u_2 := \dfrac{\partial}{\partial t}u, \quad u_3 := \dfrac{\partial}{\partial x}w, \quad u_4 := \dfrac{\partial}{\partial t}w, \\ u_5 := \dfrac{\partial}{\partial x}\psi, \quad u_6 := \dfrac{\partial}{\partial t}\psi, \quad u_7 := \psi \end{cases} \tag{3.1}$$

and write (2.17) in the following form:

$$\begin{cases} \dfrac{\partial}{\partial t}u_1 = \dfrac{\partial}{\partial x}u_2, \\ \dfrac{\partial}{\partial t}u_2 = \dfrac{E}{\rho}\dfrac{\partial}{\partial x}u_1 + \dfrac{E}{\rho}(\widehat{u_3}+u_3)\dfrac{\partial}{\partial x}u_3, \\ \dfrac{\partial}{\partial t}u_3 = \dfrac{\partial}{\partial x}u_4, \\ \dfrac{\partial}{\partial t}u_4 = \dfrac{E}{\rho}(\widehat{u_3}+u_3)\dfrac{\partial}{\partial x}u_1 + \dfrac{G+E(\widehat{u_1}+u_1+\frac{3}{2}(\widehat{u_3}+u_3)^2)}{\rho}\dfrac{\partial}{\partial x}u_3 - \dfrac{G}{\rho}(\widehat{u_5}+u_5) - g, \\ \dfrac{\partial}{\partial t}u_5 = \dfrac{\partial}{\partial x}u_6, \\ \dfrac{\partial}{\partial t}u_6 = \dfrac{E}{\rho}\dfrac{\partial}{\partial x}u_5 + \dfrac{GA}{\rho I}(\widehat{u_3}+u_3-\widehat{u_7}-u_7), \\ \dfrac{\partial}{\partial t}u_7 = u_6. \end{cases} \tag{3.2}$$

We rewrite (3.2) into matrix format and introduce

$$A(U) := -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{E}{\rho} & 0 & \dfrac{E}{\rho}(\widehat{u_3}+u_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \dfrac{E}{\rho}(\widehat{u_3}+u_3) & 0 & \dfrac{G+E((\widehat{u_1}+u_1+\dfrac{3}{2}(\widehat{u_3}+u_3)^2))}{\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{E}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

$$B(U) := -A(U)\frac{\partial}{\partial x}\widehat{U} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{G}{\rho}(\widehat{u_5}+u_5)-g \\ 0 \\ \dfrac{GA}{\rho I}(\widehat{u_3}+u_3-\widehat{u_7}-u_7) \\ u_6 \end{pmatrix} \quad \text{with } B(0)=0. \tag{3.4}$$

With (3.3) and (3.4), system (3.2) can be written in standard form as follows:

$$\frac{\partial}{\partial t}U + A(U)\frac{\partial}{\partial x}U = B(U). \tag{3.5}$$

In order to verify the hyperbolicity of (3.5), we need the eigenvalues and eigenvectors of $A(U)$. We define

$$\begin{aligned} q &:= 2E\Big(\widehat{u_1}+u_1+\frac{3}{2}(\widehat{u_3}+u_3)^2+\Big(1+\frac{G}{E}\Big)\Big), \\ r &:= 16E^2\Big(\widehat{u_1}+u_1+\frac{1}{2}(\widehat{u_3}+u_3)^2+\frac{G}{E}\Big). \end{aligned}$$

There are three cases to deal with:

(1) $G<E$,

(2) $G=E$,

(3) $G>E$.

We first consider the case $E>G$. Then the eigenvalues are given in increasing order by

$$\begin{cases} \lambda_1 = -\sqrt{\dfrac{E}{\rho}}, \quad \lambda_2 = -\dfrac{1}{2\sqrt{\rho}}\sqrt{q+\sqrt{q^2-r}}, \quad \lambda_3 = -\dfrac{1}{2\sqrt{\rho}}\sqrt{q-\sqrt{q^2-r}}, \\ \lambda_4 = 0, \\ \lambda_5 = \dfrac{1}{2\sqrt{\rho}}\sqrt{q-\sqrt{q^2-r}}, \quad \lambda_6 = \dfrac{1}{2\sqrt{\rho}}\sqrt{q+\sqrt{q^2-r}}, \quad \lambda_7 = \sqrt{\dfrac{E}{\rho}}. \end{cases} \tag{3.6}$$

It is clear that the eigenvalues $\lambda_j (j=1,2,3,5,6,7)$ correspond to $\pm\sqrt{\mu_i} (i=1,2,3)$ with (2.42) of the second order system. In order to establish the relation of the nonlinear model under gravity with linear Timoshenko model without gravity, we set for the moment $g=0$, i.e., we look at the case $\widehat{U}=0$. In this case, we have as $(u_1,u_3)\to(0,0)$,

$$\lambda_2 \to \begin{cases} -\sqrt{\dfrac{E}{\rho}}, & E>G, \\ -\sqrt{\dfrac{G}{\rho}}, & E<G, \end{cases} \qquad \lambda_3 \to \begin{cases} -\sqrt{\dfrac{G}{\rho}}, & E>G, \\ -\sqrt{\dfrac{E}{\rho}}, & E<G, \end{cases} \tag{3.7}$$

$$\lambda_5 \to \begin{cases} \sqrt{\dfrac{G}{\rho}}, & E>G, \\ \sqrt{\dfrac{E}{\rho}}, & E<G, \end{cases} \qquad \lambda_6 \to \begin{cases} \sqrt{\dfrac{E}{\rho}}, & E>G, \\ \sqrt{\dfrac{G}{\rho}}, & E<G. \end{cases} \tag{3.8}$$

Remark 3.1 It should be noted that for $\widehat{U} = U = 0$ the eigenvalues $\lambda_i (i = 1, \cdots, 7)$ coincide with those of the linear Timoshenko beam system, namely,

$$\mu_1 = -\sqrt{\frac{E}{\rho}} = \mu_2, \quad \mu_3 = -\sqrt{\frac{G}{\rho}}, \quad \mu_4 = 0, \quad \mu_5 = \sqrt{\frac{G}{\rho}}, \quad \mu_6 = \sqrt{\frac{E}{\rho}} = \mu_7,$$

ordered by magnitude for $E > G$. Notice that for $\widehat{U} = U = 0$ the eigenvalues $\pm\sqrt{\frac{E}{\rho}}$ have double multiplicity, while the system is strictly hyperbolic for $\widehat{U} \neq 0$ and U such that $(u_1, u_3) \neq (\frac{E-G}{E}, 0)$. The zero eigenvalue is an artificial one which can be avoided by directly considering the original system (2.35) of second order. It is, however, necessary to use the first order format in order to utilize the concept of semi-global classical solutions in the sense of Li [13].

For the case $E > G$ we find the following right-eigenvectors:

$$\begin{cases} \mathbf{v}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\left(\frac{\rho}{E}\right)^{\frac{1}{2}} & 1 & 0 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\lambda_2} & 1 & \frac{1}{\lambda_2}\Big(\left(\frac{\lambda_2}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & \Big(\left(\frac{\lambda_2}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_3 = \begin{pmatrix} \frac{\widehat{u_3} + u_3}{\lambda_3} \frac{1}{(\frac{\lambda_3}{\lambda_1})^2 - 1} & \frac{\widehat{u_3} + u_3}{(\frac{\lambda_3}{\lambda_1})^2 - 1} & \frac{1}{\lambda_3} & 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_5 = \begin{pmatrix} \frac{\widehat{u_3} + u_3}{\lambda_5} \frac{1}{(\frac{\lambda_5}{\lambda_1})^2 - 1} & \frac{\widehat{u_3} + u_3}{(\frac{\lambda_5}{\lambda_1})^2 - 1} & \frac{1}{\lambda_5} & 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_6 = \begin{pmatrix} \frac{1}{\lambda_6} & 1 & \frac{1}{\lambda_6}\Big(\left(\frac{\lambda_6}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & \Big(\left(\frac{\lambda_6}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \\ \mathbf{v}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & \left(\frac{\rho}{E}\right)^{\frac{1}{2}} & 1 & 0 \end{pmatrix}^{\mathrm{T}}, \end{cases} \tag{3.9}$$

while for the case $G > E$, we have to interchange the role of $\mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_5, \mathbf{v}_6$, according to the change in the magnitude of G and E. $E = G$ is a degenerate case with the additional property that now, in the limit, we have three double eigenvalues. Nevertheless, even in this case the system is strictly hyperbolic in a neighborhood of the origin. We need to evaluate the left-eigenvectors. These, for the case $G < E$, are given by

$$\begin{cases} \mathrm{l}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \left(\frac{E}{\rho}\right)^{\frac{1}{2}} & 1 & 0 \end{pmatrix}, \\ \mathrm{l}_2 = \begin{pmatrix} \lambda_2 & 1 & \lambda_2\Big((\frac{\lambda_2}{\lambda_1})^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & \Big(\left(\frac{\lambda_2}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & 0 & 0 & 0 \end{pmatrix}, \\ \mathrm{l}_3 = \begin{pmatrix} \frac{\lambda_3}{(\frac{\lambda_3}{\lambda_1})^2 - 1}(\widehat{u_3} + u_3) & \frac{1}{(\frac{\lambda_3}{\lambda_1})^2 - 1}(\widehat{u_3} + u_3) & \lambda_3 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ \mathrm{l}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathrm{l}_5 = \begin{pmatrix} \frac{\lambda_5}{(\frac{\lambda_5}{\lambda_1})^2 - 1}(\widehat{u_3} + u_3) & \frac{1}{(\frac{\lambda_5}{\lambda_1})^2 - 1}(\widehat{u_3} + u_3) & \lambda_5 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ \mathrm{l}_6 = \begin{pmatrix} \lambda_6 & 1 & \lambda_6\Big(\left(\frac{\lambda_6}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & \Big(\left(\frac{\lambda_2}{\lambda_1}\right)^2 - 1\Big)\frac{1}{\widehat{u_3} + u_3} & 0 & 0 & 0 \end{pmatrix}, \\ \mathrm{l}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\left(\frac{E}{\rho}\right)^{\frac{1}{2}} & 1 & 0 \end{pmatrix}. \end{cases} \tag{3.10}$$

We suppress the analogous cases for $G > E$ and $E = G$.

Clearly, for $E > G$ ($E < G$), the eigenvalues and the eigenvectors are C^1-smooth as functions of u_1, u_3 in a neighborhood of $(\widehat{u_1}, \widehat{u_3})$. However, if we consider the case $(\widehat{u_1}, \widehat{u_3}) = (0,0)$, the case without gravity, then we only get directional differentiability along lines $t(u_1, u_3),\ u_1 \neq 0,\ t > 0$ when $t \to 0$. Clearly, on $t(u_1, 0),\ t > 0$ the expressions are not defined. For this reason, that situation is not considered in this article further. We introduce

$$\mathbf{r}_i := \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}, \quad i = 1, 2, 3.$$

We have

$$\mathbf{l}_i \cdot \mathbf{r}_i = \delta_{ij}, \quad \mathbf{r}_i \cdot \mathbf{r}_i^{\mathrm{T}} = 1, \quad i, j = 1, \cdots, 7.$$

We can also express the boundary conditions at $x = 0$ and $x = L$ in terms of U. To this end, we introduce the matrices

$$R_D(U) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.11}$$

and

$$R_N(U) := \begin{pmatrix} EA & 0 & \dfrac{EA}{2} u_3 & 0 & 0 & 0 & 0 \\ EA u_3 & 0 & \dfrac{EA}{2} u_3^2 + GA & 0 & 0 & 0 & -GA \\ 0 & 0 & 0 & 0 & EI & 0 & 0 \end{pmatrix}. \tag{3.12}$$

With (3.11) and (3.12), we can express nonhomogeneous Dirichlet conditions at $x = 0$ and Neumann conditions at $x = L$ as

$$\begin{aligned} R_D(U(0,t))U(0,t) &= V_D(t), \\ R_N(U(L,t))U(L,t) &= V_N(t), \quad t \in [0, T]. \end{aligned} \tag{3.13}$$

We are now in the position to formulate the initial-boundary value problem (2.22)–(2.24) as a first order hyperbolic system of equations in the classical format:

$$\begin{cases} \dfrac{\partial}{\partial t} U(x,t) + A(U(x,t)) \dfrac{\partial}{\partial x} U(x,t) = B(U(x,t))U(x,t), \quad (x,t) \in [0, L] \times [0, T], \\ R_D(U(0,t))U(0,t) = V_D(t), \quad R_N(U(L,t))U(L,t) = V_N(t), \quad t \in [0, T], \\ U(x, 0) = U_0(x), \quad x \in [0, L]. \end{cases} \tag{3.14}$$

In order to apply the theory for semi-global classical solutions of Li [13], we need to express the boundary conditions in terms of the variables

$$v_i := \mathbf{l}_i(U) \cdot U, \quad i = 1, \cdots, 7.$$

Namely,

$$\begin{cases} x = 0 : (v_5, v_6, v_7) = G_0(t, v_1, v_2, v_3, v_4) + V_0(t), \\ x = L : (v_1, v_2, v_3) = G_L(t, v_4, v_5, v_6, v_7) + V_L(t), \end{cases} \tag{3.15}$$

together with

$$G_0(t, 0) = G_L(t, 0) = 0.$$

We analyze the situation for (3.10); hence for $G \le E$, we leave the case to the reader. Here we obtain

$$\begin{cases} v_1 + v_7 = 2u_6, \\ v_2 + v_6 = 2u_2 + 2u_4\Big(\Big(\dfrac{\lambda_6}{\lambda_1}\Big)^2 - 1\Big)\dfrac{1}{\widehat{u_3} + u_3}, \\ v_3 + v_5 = 2u_4 + 2u_2\dfrac{\widehat{u_3} + u_3}{(\frac{\lambda_5}{\lambda_1})^2 - 1}, \\ v_7 - v_1 = 2\lambda_1 u_5, \\ v_6 - v_2 = 2\lambda_6 u_1 + 2u_3\lambda_6\Big(\Big(\dfrac{\lambda_6}{\lambda_1}\Big)^2 - 1\Big)\dfrac{1}{\widehat{u_3} + u_3}, \\ v_5 - v_3 = 2\lambda_5\dfrac{\widehat{u_3} + u_3}{(\frac{\lambda_5}{\lambda_1})^2 - 1}u_1 + 2\lambda_5 u_3. \end{cases} \tag{3.16}$$

This can be written in matrix form as follows:

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} v_7 \\ v_6 \\ v_5 \end{pmatrix} &= 2\begin{pmatrix} 0 & 0 & 1 \\ 1 & \Big(\Big(\dfrac{\lambda_6}{\lambda_1}\Big)^2 - 1\Big)\dfrac{1}{\widehat{u_3} + u_3} & 0 \\ \dfrac{\widehat{u_3} + u_3}{(\frac{\lambda_5}{\lambda_1})^2 - 1} & 1 & 0 \end{pmatrix}\begin{pmatrix} u_2 \\ u_4 \\ u_6 \end{pmatrix} \\ &=: M(u_1, u_3)\begin{pmatrix} u_2 \\ u_4 \\ u_6 \end{pmatrix} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \begin{pmatrix} v_7 \\ v_6 \\ v_5 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= 2\begin{pmatrix} 0 & 0 & \lambda_1 \\ \lambda_6 & \lambda_6\Big(\Big(\dfrac{\lambda_6}{\lambda_1}\Big)^2 - 1\Big)\dfrac{1}{\widehat{u_3} + u_3} & 0 \\ \lambda_5\dfrac{\widehat{u_3} + u_3}{(\frac{\lambda_5}{\lambda_1})^2 - 1} & \lambda_5 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_3 \\ u_5 \end{pmatrix} \\ &= \mathrm{diag}(\lambda_1, \lambda_6, \lambda_5)M(u_1, u_3)\begin{pmatrix} u_1 \\ u_3 \\ u_5 \end{pmatrix} =: Q(u_1, u_3)\begin{pmatrix} u_1 \\ u_3 \\ u_5 \end{pmatrix}. \end{aligned} \tag{3.18}$$

In order to simplify the notation, we order the system variables in a different way. Namely, the vector U is now ordered according to $\widehat{U}^{\mathrm{T}} = (u_1, u_3, u_5, u_2, u_4, u_6, u_7) =: (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$, where $\mathbf{w}_1 = (u_1, u_3, u_5)$, $\mathbf{w}_2 = (u_2, u_4, u_6)$, $\mathbf{w}_3 = u_7$. We also introduce $\xi_+ := (v_7, v_6, v_5)^{\mathrm{T}}$ and $\xi_- := (v_1, v_2, v_3)^{\mathrm{T}}$. With this notation, we can reformulate (3.17) and (3.18) as follows:

$$\xi_+ + \xi_- = M(\mathbf{w}_1)\mathbf{w}_2, \quad \xi_+ - \xi_- = Q(\mathbf{w}_1)\mathbf{w}_1. \tag{3.19}$$

The derivative

$$D_{(u_1,u_3)}\left(Q(u_1, u_3)\begin{pmatrix} u_1 \\ u_3 \\ u_5 \end{pmatrix}\right)\Bigg|_{(u_1,u_3,u_5)=0_3} = Q(0, 0)$$

is invertible and, therefore, we can apply the implicit function theorem and conclude that there exists a function Θ such that

$$\mathbf{w}_1 = \Theta(\xi_+ - \xi_-).$$

Inserting this into the first equation in (3.19), we obtain

$$\xi_+ + \xi_- = M(\Theta(\xi_+ - \xi_-))\mathbf{w}_2.$$

We then define

$$(\xi_+, \xi_-, \mathbf{w}_2) := \xi_+ + \xi_- - M(\Theta(\xi_+ - \xi_-))\mathbf{w}_2.$$

Obviously, we have

$$(0,0,0) = 0, \quad D_{\xi_+} \quad (0,0,0) = I.$$

Thus, we can apply the implicit function theorem again in order to solve for ξ_+. There exists a vector function $G_{D,0}(\cdot,\cdot)$ such that

$$\xi_+ = G_{D,0}(\xi_-, \mathbf{w_2}), \quad G_{D,0}(0,0) = 0. \tag{3.20}$$

Upon defining Dirichlet controls at $x = 0$ as

$$\mathbf{w}_2(0,t) = (u_2, u_4, u_6)^{\mathrm{T}}(0,t) = \Big(\frac{\partial}{\partial t}u(0,t), \frac{\partial}{\partial t}w(0,t), \frac{\partial}{\partial t}\psi(0,t)\Big)^{\mathrm{T}} = (h_{01}(t), h_{02}(t), h_{03}(t))^{\mathrm{T}},$$

we rewrite (3.20) with

$$G_{D,0}(\xi_-(0,t);t) := G_{0,D}(\xi_-(0,t), \mathbf{h}_0(t)) - G_{D,0}(0, \mathbf{h}_0(t))$$

and $V_D(t) = G_{D,0}(0, \mathbf{h}_0(t))$ as

$$\xi_+(0,t) = G_{D,0}(\xi_-;t) + V_D(t), \tag{3.21}$$

where now $G_{D,0}(0;t) = 0$. We go back to the previous notation and conclude

$$(v_7, v_6, v_5)(0,t) = G_{D,0}((v_1, v_2, v_3, v_4)(0,t);t) + V_D(t). \tag{3.22}$$

This is the format required in (3.15). As for the Neumann boundary conditions (2.19)–(2.21), we have, after dividing through the constants in each condition and renaming the controls v_N^i $(i = 1,2,3)$

$$\begin{cases} u_1(L,t) + \dfrac{1}{2}u_3^2(L,t) = v_N^1(t), \\ u_3(L,t) - u_7(L,t) + \dfrac{E}{G}(u_1(L,t) + \dfrac{1}{2}u_3(L,t)^2)u_3(L,t) = v_N^2(t), \\ u_5(L,t) = v_N^3(t), \quad t \in [0,T]. \end{cases} \tag{3.23}$$

Using the first equation of (3.23) in the second condition, we arrive at

$$\begin{cases} u_1(L,t) + \dfrac{1}{2}u_3^2(L,t) = v_N^1(t), \\ \Big(1 + \dfrac{E}{G}v_N^1(t)\Big)u_3(L,t) - u_7(L,t) = v_N^2(t), \\ u_5(L,t) = v_N^3(t), \quad t \in [0,T]. \end{cases} \tag{3.24}$$

We are going to use (3.17)–(3.18). We notice that $v_4 = u_7$. According to the second equation in (3.24), $u_3(L,t)$ can be expressed in terms of $u_7(L,T)$ and, hence, in terms of $v_4(L,T)$. By the first equation in (3.24), this is true also for $u_1(L,T)$. Thus,

$$u_3(L,t) = \Big(1 + \frac{E}{G}v_N^1(t)\Big)^{-1}v_N^2(t) + \Big(1 + \frac{E}{G}v_N^1(t)\Big)^{-1}v_4$$

and

$$u_1(L,T) = v_N^1(t) - \frac{1}{2}\Big(\Big(1+\frac{E}{G}v_N^1(t)\Big)^{-1} v_N^2(t) + \Big(1+\frac{E}{G}v_N^1(t)\Big)^{-1} v_4\Big)^2.$$

As $u_5(L,T) = \frac{1}{2\lambda_1}(v_7 - v_1) = v_N^3(t)$, we can express the third condition in (3.23) as $v_1(L,t) = v_7(L,t) - 2\lambda_1 v_N^3(t)$. Now, the last two equations of (3.16) contain on their right-hand sides terms in the variables u_1, u_3, only. As seen above, these can be expressed in terms of v_4. Thus, v_2, v_6 can be expressed in terms of v_4 and terms involving the controls $v_N^i(t)$ $(i = 1, 2, 3)$. This shows that we can express (3.23) in the form

$$(v_1, v_2, v_3)(L,t) = G_{N,L}((v_4, v_5, v_6, v_7)(L,T); t) + V_N(t). \tag{3.25}$$

Similarly, Dirichlet boundary conditions at $x = 0$ and Neumann conditions at $x = L$ can be shown to satisfy

$$(v_1, v_2, v_3)^{\mathrm{T}}(L,t) = G_{D,L}((v_4, v_5, v_6, v_7)(L,t); t) + V_D(t), \tag{3.26}$$

$$(v_7, v_6, v_5)^{\mathrm{T}}(0,t) = G_{N,0}((v_1, v_2, v_3, v_4)(0,t); t) + V_N(t). \tag{3.27}$$

Remark 3.2 It should be remarked that the system matrices with $\widehat{U}^{\mathrm{T}} =: (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$, where $\mathbf{w}_1 = (u_1, u_3, u_5)$, $\mathbf{w}_2 = (u_2, u_4, u_6)$, $\mathbf{w}_3 = u_7$, have now the following forms:

$$\widehat{A}(\mathbf{w}) := -\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dfrac{E}{\rho} & \dfrac{E}{\rho}(\widehat{u_3} + u_3) & 0 & 0 & 0 & 0 & 0 \\ \dfrac{E}{\rho}(\widehat{u_3} + u_3) & \dfrac{G + E(\widehat{u_1} + u_1 + \frac{3}{2}((\widehat{u_3} + u_3)^2)}{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{E}{\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

$$\widehat{B}(\mathbf{w}) := \widehat{A}(\mathbf{w})\frac{\partial}{\partial x}\widehat{\mathbf{w}} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{G}{\rho}(\widehat{u_5} + u_5) - g \\ \dfrac{GA}{\rho I}(\widehat{u_3} + u_3 - (\widehat{u_7} + u_7) \\ (\widehat{u_6} + u_6) \end{pmatrix} \quad \text{with} \quad \widehat{B}(0) = 0. \tag{3.29}$$

With (3.28) and (3.29), system (3.2) can be written in the following equivalent form:

$$\frac{\partial}{\partial t}\mathbf{w} + \widehat{A}(\mathbf{w})\frac{\partial}{\partial x}\mathbf{w} = \widehat{B}(\mathbf{w}). \tag{3.30}$$

The advantage of this form is that it reveals the typical block structure for wave equations.

4 Existence of Solutions

In order to study the well-posedness of (2.22)–(2.24) in the framework of semi-global classical solutions, we need to assume regular initial and boundary data, as well as compatibility conditions.

De nition 4.1 *We say that the initial conditions and boundary conditions satisfy C^1-compatibility conditions for the Dirichlet case at $x=0$ and for the Neumann case at $x=L$, if the following conditions hold:*

$$\begin{cases} v_D^1(0) = u_0(0), \quad v_D^2(0) = w_0(0), \quad v_D^3(0) = \psi_0(0), \\ (v_D^1)'(0) = u_1(0), \quad (v_D^2)' = w_1(0), \quad (v_D^3)'(0) = \psi_1(0), \\ (v_D^1)''(0) = \dfrac{E}{\rho}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}u_0 + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w_0\Big)^2\Big)(0), \\ (v_D^2)''(0) = \dfrac{G}{\rho}\Big(\dfrac{\partial^2}{\partial x^2}w_0 - \dfrac{\partial}{\partial x}\psi_0\Big)(0) \\ \qquad\qquad + \dfrac{E}{\rho}\Big(\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}u_0\dfrac{\partial}{\partial x}w_0\Big) + \dfrac{3}{2}\Big(\dfrac{\partial}{\partial x}w_0\Big)^2\dfrac{\partial^2}{\partial x^2}w_0\Big)(0), \\ (v_D^3)''(0) = \psi_0''(0) - \dfrac{GA}{\rho I}\Big(\dfrac{\partial}{\partial x}w_0 - \dfrac{\partial}{\partial x}\psi_0\Big)(0) \end{cases} \tag{4.1}$$

as well as

$$\begin{cases} EA\dfrac{\partial}{\partial x}u_0(L) + \dfrac{EA}{2}\Big(\dfrac{\partial}{\partial x}w_0(L)\Big)^2 = v_N^1(0), \\ GA\Big(\dfrac{\partial}{\partial x}w_0(L) - \psi_0(L)\Big) \\ \quad + \Big(EA\Big(\dfrac{\partial}{\partial x}u_0(L) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w_0(L)\Big)^2\Big)\dfrac{\partial}{\partial x}w_0(L)\Big) = v_N^2(0), \\ EI\dfrac{\partial}{\partial x}\psi_0(L) = v_N^3(0) \end{cases} \tag{4.2}$$

and

$$\begin{cases} EA\dfrac{\partial}{\partial x}u_1(L) + EA\dfrac{\partial}{\partial x}w_0(L)\dfrac{\partial}{\partial x}w_1(L) = (v_N^1)'(0), \\ GA\Big(\dfrac{\partial}{\partial x}w_1(L) - \psi_1(L)\Big) \\ \quad + EA\Big(\dfrac{\partial}{\partial x}u_0(L)\dfrac{\partial}{\partial x}w_1(L) + \dfrac{\partial}{\partial x}u_1(L)\dfrac{\partial}{\partial x}w_0(L) \\ \quad + \dfrac{3}{2}\Big(\dfrac{\partial}{\partial x}w_0(L)\Big)^2\dfrac{\partial}{\partial x}w_1(L)\Big) = (v_N^2)'(0), \\ EI\dfrac{\partial}{\partial x}\psi_1(L) = (v_N^3)'(0). \end{cases} \tag{4.3}$$

We say that the initial and boundary conditions satisfy C^2-compatibility conditions if they satisfy the C^1-compatibility conditions and, in addition, the partial differential equations (2.33) hold at $x = 0, L$ on time $t = 0$, where the second order in time derivatives are replaced with the second order in time derivatives of the controls at $x = 0$ and $x = L$, respectively.

Remark 4.1 The representation as a first order system, say, in the format described in Remark 3.2, is useful. In particular, $\mathbf{w}_1, \mathbf{w}_2$, at $t = 0$, are related to (the spatial derivatives of) the initial displacements u_0, w_0, ψ_0 and the initial velocities u_1, w_1, ψ_1, respectively.

Theorem 4.1 *Let $T > 0$ be given. Let the boundary controls $V_D := (v_D^i)_{i=1}^3 \in C^2([0,T])^3$, $V_N := (v_N^i)_{i=1}^3 \in C^1([0,T])^3$ and the initial data $\ _0 = (u_0, w_0, \psi_0)^{\mathrm{T}} \in C^2([0,L])^3$, $\ _1 := (u_1, w_1, \psi_1)^{\mathrm{T}} \in C^1([0,T])$ satisfy the C^2 compatibility conditions of Definition 4.1, such that $\|(\ _0, \ _1)\|_{C^2([0,L])^3 \times C^1([0,L])^3}$ and $\|(V_D, V_N)\|_{C^2([0,T])^3 \times C^1([0,L])^3}$ are sufficiently small. Then* *there exists a unique semi-global C^2-solution* $(x,t) = (u(x,t), w(x,t), \psi(x,t))$ *with small C^2-norm on $R(T)$ defined as*

$$R(T) := \{(x,t) \mid 0 \le x \le L,\ 0 \le t \le T\}. \tag{4.4}$$

Proof The proof is similar to the one given in [10] and follows the lines of [13].

5 Exact Controllability

We are now in the position to formulate the problem of local one-sided exact boundary controllability around an equilibrium solution $\widehat{\ } = (\widehat{u}, \widehat{w}, \widehat{\psi})$. We denote the perturbations of $\widetilde{\ }$ by . Let us recall the eigenvalues μ_i $(i = 1,2,3)$ given by (2.42). For the sake of brevity, we write $\mu_i(x, \ )$ $(i = 1,2,3)$ in order to indicate the dependence of μ_i on the spatial variable and the perturbation. Thus, $\mu_i(x,0)$ $(i = 1,2,3)$ signifies the eigenvalues in (2.42) at the spatial point x at equilibrium. Both, the systems at equilibrium and at the actual position, define speeds of propagation. The maximum travel time in the beam can be estimated as follows:

$$T_0 := \max_{i=1,2,3} \max_{x\in[0,L]} \frac{L}{\sqrt{\mu_i(x,0)}}. \tag{5.1}$$

For a given ϵ-neighborhood $B_\epsilon(\ )$ of , we can bound the travel time by

$$T_1 := \max_{\|\mathbf{\Phi}\|<\epsilon} \max_{i=1,2,3} \max_{x\in[0,L]} \frac{L}{\sqrt{\mu_i(x,\ )}}. \tag{5.2}$$

Indeed, by our assumptions, we find such an $\epsilon_0 > 0$ such that $T > 2T_0$ implies $T > 2T_1$. Thus, we can bound the time that a signal needs to travel from the boundary, where controls that apply to the clamped origin and back can be estimated by the corresponding time for the system seen at equilibrium.

De nition 5.1 *Let a control time $T > 0$, initial and final data $(\ _0,\ _1)$, $(\ _0,\ _1)$ where* $_0 = (u_0, w_0, \psi_0)^{\mathrm T} \in C^2([0,L])^3$, $_1 := (u_1, w_1, \psi_1)^{\mathrm T} \in C^1([0,L])^3$, $_0 = (u_0^{\mathrm T}, w_0^{\mathrm T}, \psi_0^{\mathrm T})^{\mathrm T} \in C^2([0,L])^3$, $_1 := (u_1^{\mathrm T}, w_1^{\mathrm T}, \psi_1^{\mathrm T})^{\mathrm T} \in C^1([0,L]^3)$ *be given. We say that the problem* (2.22)–(2.24) *is exactly controllable in time T with one-sdided controls, if there exist boundary controls V_D with $V_N = 0$ or V_N with $V_D = 0$, satifying the compatibility conditions* (4.1)–(4.3)*, such that the corresponding solution, satisfying the conditions of Theorem* 4.1*, admits the final values*

$$\begin{cases} u(x,T) = u_0^{\mathrm T}(x), \quad w(x,T) = w_0^{\mathrm T}(x), \quad \psi(x,T) = \psi_0^{\mathrm T}(x), \\ \dfrac{\partial}{\partial t}u(x,T) = u_1^{\mathrm T}(x), \quad \dfrac{\partial}{\partial t}w(x,T) = w_1^{\mathrm T}(x), \quad \dfrac{\partial}{\partial t}\psi(x,T) = \psi_1^{Z}(x), \quad x \in [0,L]. \end{cases} \tag{5.3}$$

Theorem 5.1 *Let T be given by $T > 2T_0$. There exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of $(\widehat{\ }, 0)$ such that for each pair of initial states in $\mathcal{U}_0$ and final states in $\mathcal{U}_1$ satisfying the regularity and compatibility conditions given in Theorem* 4.1*, there exist $C^2(0,T;\mathbb{R}^3)$-controls V_D or V_N such that the solutions of* (2.38) *satisfy the conditions in Definition* 5.1.

Proof The remaining part of the proof consists of applying method described in [13]. For the sake of brevity, we refer to the proof of Theorem 7.1 (Section 7 below) for $n = 1$.

Remark 5.1 We remark that we do not consider the exact controllability problem on the level of the first order system (3.5). Indeed, the exact controllability of the full state via boundary controls is generally impossible, due to the appearance of the zero eigenvalue. This eigenvalue would make it necessary to involve a distributed control (see [13] and a more recent discussion in Hu [2]). The controllability result is a local one. It can be extended to a local-global result if one considers two distinct equilibria connected by a path of equilibria (see [10] for the analogous case in the context of nonlinear strings).

6 Networks of Nonlinear Timoshenko Beams

We now consider networks of planar initially straight nonlinear Timoshenko beams according to (2.17). We introduce some notation in order to describe the network. We suppose that there are n beams indexed by $i \in \mathcal{I} = \{1, \cdots, n\}$. We let the i-th beam be parametrized by its rest arc length x with $x \in [0, L_i]$, L_i the length of that beam. The position and s-hear at time t of the point corresponding to the parameter x will be denoted by the vector $((u, w, \psi)^i(x,t))^{\mathrm{T}}$. The positions and shear at the endpoints, which we refer to as nodes, are given by functions $((v_D^1, v_D^2, v_D^3)^j)^{\mathrm{T}}(t)$ with $j \in \mathcal{J} = \{1, \cdots, m\}$. A similar statement holds for the simple nodes where Neumann conditions are applied, see below. At multiple nodes which we denote by $\mathcal{J}^M$, where several beams meet there is a common location $\mathbf{N}^j$. Simple nodes are those corresponding to the endpoints of only one beam. This set is split into nodes $\mathcal{J}^{S_D}$, where Dirchilet conditions are satisfied, and $\mathcal{J}^{S_N}$ where Neumann conditions hold. We let $\mathcal{I}^j = \{i \in \mathcal{I} : \mathbf{N}^j$ is an end point of the i-th beam$\}$, $\mathcal{J}^M$ be the subset of $\mathcal{J}$ corresponding to multiple nodes, while $\mathcal{J}^S$ contains the indices of simple nodes. We assume that there are simple nodes so that $\mathcal{J}^S$ is not empty. For $j \in \mathcal{J}^S$ we have $\mathcal{I}^j = \{i_j\}$. For $i \in \mathcal{I}^j$ we let $x_{ij} = 0$ or $x_{ij} = L_i$ depending on whether the beam begins or ends at the simple node, respectively. For purposes of integration by parts we also introduce ϵ_{ij} to equal 1 or -1 depending on whether x_{ij} is equal to L_i or 0.

We only consider rigid joints such that the positions and, hence, the displacements as well as the angles between to adjacent beams before and after deformation coincide. Pinned joints will be treated elsewhere. The continuity of displacements is expressed as

$$\mathbf{W}^i(x_{ij}) = \mathbf{W}^k(x_{kj}), \quad \forall i, k \in \mathcal{I}^j,\ j \in \mathcal{J}^M. \tag{6.1}$$

Reflecting the meaning of $W = u\mathbf{e}_1 + w\mathbf{e}_3$, we can rewrite (6.1) as

$$u^i(x_{ij})\mathbf{e}_1^i + w^i(x_{ij})\mathbf{e}_3^i = u^k(x_{ij})\mathbf{e}_1^k + w^k(x_{ij})\mathbf{e}_3^k, \quad \forall i, k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,$$

where we notice that because the undeformed and the deformed configurations $\mathbf{r}^i, \mathbf{R}^i$ satisfy the same conditions, these conditions are also valid for the displacements at the centerline. As for the total angles ψ^i, we obtain at a rigid joint

$$\psi^i(x_{ij}) = \psi^k(x_{kj}), \quad \forall i, k \in \mathcal{I}^j,\ j \in \mathcal{J}^M. \tag{6.2}$$

We can now derive the following conditions on balance of forces and moment at a multiple node:

$$\begin{aligned}&\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big\{E_iA_i\Big(\frac{\partial}{\partial x}u^i(x_{ij}) + \frac{1}{2}\frac{\partial}{\partial x}w^i(x_{i_jj})^2\Big)\mathbf{e}_1^i \\ &+ G_iA_i\Big(\frac{\partial}{\partial x}w^i(x_{ij}) - \psi^i(x_{ij})\Big)\mathbf{e}_3^i\Big\} = F_j, \end{aligned}\tag{6.3}$$

$$\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big(E_iI_i\frac{\partial}{\partial x}\psi^i(x_{ij})\Big) = M_j. \tag{6.4}$$

Remark 6.1 We notice that the conditions (6.3)–(6.4) at multiple node are vectorial conditions coupling longitudinal motion and shearing, while the rotation around $\mathbf{e}_2$ does not couple to other primitive variables. This is in contrast to scalar beam models which represent out-of-the-plane dsiplacements, only. The fact that the multiple node conditions are vectorial implies that the network depends on the angels between the beams, thus, on the topology. In scalar networks, the angles do not matter.

Let $\mathbf{e}$ be the upright unit vector. The full initial-, boundary-, nodal-value network problem for initially straight planar nonlinear Timoshenko beams reads as follows:

$$\begin{cases} \rho_i A_i \dfrac{\partial^2}{\partial t^2} u^i(x,t) = E_i A_i \dfrac{\partial^2}{\partial x^2} u^i(x,t) + \dfrac{E_i A_i}{2} \dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x} w^i(x,t)\Big)^2 - \rho_i A_i g \mathbf{e}\cdot \mathbf{e}_1^i, \\ \rho_i A_i \dfrac{\partial^2}{\partial t^2} w^i(x,t) = G_i A_i \Big(\dfrac{\partial^2}{\partial x^2} w^i(x,t) - \dfrac{\partial}{\partial x}\psi^i(x,t)\Big) \\ \qquad\qquad + \dfrac{\partial}{\partial x}\Big(E_i A_i \Big(\dfrac{\partial}{\partial x} u^i(x,t) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x} w^i(x,t)\Big)^2\Big) \dfrac{\partial}{\partial x} w^i(x,t)\Big) \\ \qquad\qquad -\rho_i A_i \mathbf{e}\cdot \mathbf{e}_3^i, \\ \rho_i I_i \dfrac{\partial^2}{\partial t^2}\psi^i(x,t) = E_i I_i \dfrac{\partial^2}{\partial x^2}\psi^i(x,t) + G_i A_i\Big(\dfrac{\partial}{\partial x} w^i(x,t) - \psi^i(x,t)\Big), \\ \qquad\qquad (x,t)\in[0,L_i]\times[0,T], \quad i\in\mathcal{I}, \end{cases} \tag{6.5}$$

$$\begin{cases} u^i(x_{i_j j},t)) = v_{D1}^i(t), \quad w^i(x_{i_j j},t) = v_{D2}^i(t), \quad \psi^i(x_{i_j j},t) = v_{D3}^i(t), \\ \quad i\in\mathcal{I}^j,\ j\in\mathcal{J}^{S_D},\ t\in[0,T], \\ E_i A_i \dfrac{\partial}{\partial x} u^i(x_{i_j j},t) + \dfrac{E_i A_i}{2}\Big(\dfrac{\partial}{\partial x} w^i(x_{i_j j},t)\Big)^2 = v_{N1}^i(t), \\ G_i A_i\Big(\dfrac{\partial}{\partial x} w^i(x_{i_j j},t) - \psi^i(x_{i_j j},t)\Big) \\ \quad +\Big(E_i A_i\Big(\dfrac{\partial}{\partial x} u^i(x,t) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x} w^i(x,t)\Big)^2\Big)\dfrac{\partial}{\partial x} w^i(x,t)\Big) = v_{N2}^i(t), \\ E_i I_i \dfrac{\partial}{\partial x}\psi^i(x_{i_j j},t) = v_{N3}^i(t), \quad i\in\mathcal{I}^j,\ j\in\mathcal{J}^{S_N},\ t\in[0,T], \end{cases} \tag{6.6}$$

$$\begin{cases} \mathbf{W}^i(x_{ij},t) = \mathbf{W}^k(x_{kj},t), \quad \forall i,k\in\mathcal{I}^j,\ j\in\mathcal{J}^M,\ t\in[0,T], \\ \psi^i(x_{ij}) = \psi^k(x_{kj}), \quad \forall i,k\in\mathcal{I}^j\ j\in\mathcal{J}^M,\ t\in[0,T], \\ \displaystyle\sum_{i\in\mathcal{I}^j}\epsilon_{ij}\Big\{E_i A_i\Big(\dfrac{\partial}{\partial x} u^i(x_{ij}) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x} w^i(x_{i_j j})\Big)^2\Big)\mathbf{e}_1^i \\ \quad + G_i A_i\Big(\dfrac{\partial}{\partial x} w^i(x_{ij}) - \psi^i(x_{ij})\Big)\mathbf{e}_3^i\Big\} = F_j, \quad t\in[0,T], \\ \displaystyle\sum_{i\in\mathcal{I}^j}\epsilon_{ij}\Big(E_i I_i \dfrac{\partial}{\partial x}\psi^i(x_{ij})\Big) = M_j, \quad t\in[0,T], \end{cases} \tag{6.7}$$

$$\begin{cases} u^i(x,0) = u_0^i(x), \quad \dfrac{\partial}{\partial t} u^i(x,0) = u_1^i(x), \quad x\in[0,L_i], \\ w^i(x,0) = w_0^i(x), \quad \dfrac{\partial}{\partial t} w^i(x,0) = w_1^i(x), \quad x\in[0,L_i], \\ \psi^i(x,0) = \psi_0^i(x), \quad \dfrac{\partial}{\partial t}\psi^i(x,0) = \psi_1^i(x), \quad x\in[0,L_i]. \end{cases} \tag{6.8}$$

Here (6.5) includes the equations governing the motion of the beams along the individual edges, (6.6) describes the boundary conditions at simple Dirichlet- and Neumann-nodes, (6.7) provides the conditions of continuity and force/moment balance at multiple nodes and (6.8) denotes the initial conditions. As for the single element case, we introduce the mass-matrices and stiffness operators for each individual beam: For each $i \in \mathcal{I}$, we define

$$\begin{cases} M_i := \begin{pmatrix} \rho A_i & 0 & 0 \\ 0 & \rho_i A_i & 0 \\ 0 & 0 & \rho_i I_i \end{pmatrix}, \\ G_i\Big(\ _i, \dfrac{\partial}{\partial x} \ _i\Big) := \begin{pmatrix} E_i A_i & E_i A_i \dfrac{\partial}{\partial x} w_i & 0 \\ E_i A_i \dfrac{\partial}{\partial x} w_i & G_i A_i + E_i A_i \Big(\dfrac{\partial}{\partial x} u_i + \dfrac{3}{2}\Big(\dfrac{\partial}{\partial x} w_i\Big)^2\Big) & 0 \\ 0 & 0 & E_i I_i \end{pmatrix}, \\ F_i\Big(\ _i, \dfrac{\partial}{\partial x} \ _i\Big) := \begin{pmatrix} 0 \\ -G_i A_i \dfrac{\partial}{\partial x} \psi_i - \rho_i g A_i \\ G_i A_i \Big(\dfrac{\partial}{\partial x} w_i - \psi_i\Big) \end{pmatrix}. \end{cases} \tag{6.9}$$

Then each Timoshenko-beam system can be written as

$$M_i \frac{\partial^2}{\partial t^2} \ _i = G_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big) \frac{\partial^2}{\partial x^2} \ _i + F_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big). \tag{6.10}$$

Now, given an equilibrium configuration such that on the edge i we have $\widehat{\ }_i := (\widehat{u}_i, \widehat{w}_i, \widehat{\psi}_i)^{\mathrm{T}}$, we look for states $\ _i = \widehat{\ }_i + \ _i$ for possibly small $\ _i$. Notice that we omit the tilde for the perturbations right away. We can reformulate the quasilinear system in terms of the perturbation $\ _i$:

$$\mathcal{G}_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big) := G_i\Big(\widehat{\ }_i + \ _i, \frac{\partial}{\partial x} \widehat{\ }_i + \frac{\partial}{\partial x} \ _i\Big), \tag{6.11}$$

$$\mathcal{F}_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big) := \mathcal{G}_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big) \frac{\partial^2}{\partial x^2} \widehat{\ }_i + F_i\Big(\widehat{\ }_i + \ _i, \frac{\partial}{\partial x} \widehat{\ }_i + \frac{\partial}{\partial x} \ _i\Big). \tag{6.12}$$

Because the $\widehat{\ }_i$ correspond to an equilibrium solution we have

$$\mathcal{F}_i(0,0) = 0. \tag{6.13}$$

We thus have

$$M_i \frac{\partial^2}{\partial t^2} \ _i = \mathcal{G}_i\Big(\ _i, \frac{\partial}{\partial x} \ _i\Big) \frac{\partial^2}{\partial x^2} \ _i + \mathcal{F}\Big(\ _i, \frac{\partial}{\partial x} \tilde{\ }_i\Big). \tag{6.14}$$

Under precisely the same conditions as in the single link case, we may invert $\mathcal{G}_i(\ _i, \frac{\partial}{\partial x} \ _i)$, as this matrix is uniformly positive definite in a sufficiently small neighborhood of the equilibrium. This fact will be important for the proof of controllability. In order to proceed with existence in the sense of [13], we need compatibility conditions both at the simple nodes and at the multiple nodes. At a simple node $j \in \mathcal{J}^{S_D, S_N}$, there is only one edge incident and the location is denoted as above by x_{i_j}. It is straightforward to reformulate the compatibility conditions (4.1)–(4.3) for such simple node conditions. We refrain from displaying the corresponding conditions. However, the new multiple node conditions (6.7) require the following new compatibility conditions:

$$\begin{cases} u_0^i(x_{ij})\mathbf{e}_1^i + w_0^i(x_{ij})\mathbf{e}_3^i = u_0^k(x_{kj})\mathbf{e}_1^k + w_0^k(x_{kj})\mathbf{e}_3^i, \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T], \\ u_1^i(x_{ij})\mathbf{e}_1^i + w_1^i(x_{ij})\mathbf{e}_3^i = u_1^k(x_{kj})\mathbf{e}_1^k + w_1^k(x_{kj})\mathbf{e}_3^i, \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T], \\ \psi_0^i(x_{ij}) = \psi_0^k(x_{kj}), \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T], \\ \psi_1^i(x_{ij}) = \psi_1^k(x_{kj}), \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T], \end{cases} \tag{6.15}$$

$$\begin{cases} \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big\{E_iA_i\Big(\frac{\partial}{\partial x}u_0^i(x_{ij}) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w_0^i(x_{i_jj})\Big)^2\Big)\mathbf{e}_1^i \\ \qquad +G_iA_i\Big(\frac{\partial}{\partial x}w_0^i(x_{ij}) - \psi_0^i(x_{ij})\Big)\mathbf{e}_3^i\Big\} = F_j(0), \quad t\in[0,T], \\ \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big(E_iI_i\frac{\partial}{\partial x}\psi_0^i(x_{ij})\Big) = M_j(0), \quad t\in[0,T], \\ \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big\{E_iA_i\Big(\frac{\partial}{\partial x}u_1^i(x_{ij}) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w_0^i(x_{i_jj})\Big)\frac{\partial}{\partial x}w_1^i(x_{ij})\Big)\mathbf{e}_1^i, \\ \qquad +G_iA_i\Big(\frac{\partial}{\partial x}w_1^i(x_{ij}) - \psi_1^i(x_{ij})\Big)\mathbf{e}_3^i\Big\} = F_j'(0), \quad t\in[0,T], \\ \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big(E_iI_i\frac{\partial}{\partial x}\psi_1^i(x_{ij})\Big) = M_j'(0), \quad t\in[0,T]. \end{cases} \tag{6.16}$$

6.1 Equilibrium solutions for the network

We assume from now on that we have an equilibrium solution of the entire tree-like network. That is a solution $\widehat{u}^i(x), \widehat{w}^i(x), \widehat{\psi}^i(x)$ of the following steady state problem:

$$\begin{cases} \rho_i g A_i \mathbf{e}\cdot\mathbf{e}_1^i = E_iA_i\frac{\partial^2}{\partial x^2}u^i(x) + \frac{E_iA_i}{2}\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x}w^i(x)\Big)^2, \\ \rho_i g A_i \mathbf{e}\cdot\mathbf{e}_3^i = G_iA_i\Big(\frac{\partial^2}{\partial x^2}w^i(x) - \frac{\partial}{\partial x}\psi^i(x)\Big) \\ \qquad\qquad +\frac{\partial}{\partial x}\Big(E_iA_i\Big(\frac{\partial}{\partial x}u^i(x) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w^i(x)\Big)^2\Big)\frac{\partial}{\partial x}w^i(x)\Big), \\ 0 = E_iI_i\frac{\partial^2}{\partial x^2}\psi^i(x) + G_iA_i\Big(\frac{\partial}{\partial x}w^i(x) - \psi^i(x)\Big), \\ \quad x\in[0,L_i],\ i\in\mathcal{I}, \end{cases} \tag{6.17}$$

$$\begin{cases} u^i(x_{i_jj}) = 0, \quad w^i(x_{i_jj}) = 0, \quad \psi^i(x_{i_jj}) = 0, \quad i\in\mathcal{I}^j,\ j\in\mathcal{J}^{S_D}, \\ E_iA_i\frac{\partial}{\partial x}u^i(x_{i_jj}) + \frac{E_iA_i}{2}\Big(\frac{\partial}{\partial x}w^i(x_{i_jj})\Big)^2 = 0, \\ G_iA_i\Big(\frac{\partial}{\partial x}w^i(x_{i_jj}) - \psi^i(x_{i_jj})\Big) \\ \quad +\Big(E_iA_i\Big(\frac{\partial}{\partial x}u^i(x) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w^i(x)\Big)^2\Big)\frac{\partial}{\partial x}w^i(x)\Big) = 0, \\ E_iI_i\frac{\partial}{\partial x}\psi^i(x_{i_jj}) = 0, \quad i\in\mathcal{I}^j,\ j\in\mathcal{J}^{S_N}, \end{cases} \tag{6.18}$$

$$\begin{cases} \mathbf{W}^i(x_{ij}) = \mathbf{W}^k(x_{kj}), \quad \forall i,k\in\mathcal{I}^j,\ j\in\mathcal{J}^M, \\ \psi^i(x_{ij}) = \psi^k(x_{kj}), \quad \forall i,k\in\mathcal{I}^j,\ j\in\mathcal{J}^M, \\ \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big\{E_iA_i\Big(\frac{\partial}{\partial x}u^i(x_{ij}) + \frac{1}{2}\Big(\frac{\partial}{\partial x}w^i(x_{i_jj})\Big)^2\Big)\mathbf{e}_1^i \\ \qquad +G_iA_i\Big(\frac{\partial}{\partial x}w^i(x_{ij}) - \psi^i(x_{ij})\Big)\mathbf{e}_3^i\Big\} = 0, \\ \displaystyle\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\Big(E_iI_i\frac{\partial}{\partial x}\psi^i(x_{ij})\Big) = 0. \end{cases} \tag{6.19}$$

It is clear that for the local equations (6.17) a similar analysis as for the single-link case (2.25) can be performed. The full analysis of the equlibirum problem is beyond the scope of this article.

We provide, however, a simple example that makes the vectorial nodal conditions evident.

Example 6.1 In order to elucidate the network setup, we give an example of a "carpenter square". Here, we have a horizontal beam (labelled 1) and a second hanging beam (labelled 2) mounted at the end of the first beam. The constellation of local bases is as follows: $\mathbf{e}_1^1 = (1,0)$, $\mathbf{e}_3^1 = (0,1)$ and $\mathbf{e}_1^2 = (-1,0)$, $\mathbf{e}_3^2 = (1,0)$.

$$\begin{cases} 0 = E_1A_1\dfrac{\partial^2}{\partial x^2}u^1(x) + \dfrac{E_1A_1}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w^1(x)\Big)^2, \\ -\rho_1 gA_1 = G_1A_2\Big(\dfrac{\partial^2}{\partial x^2}w^1(x) - \dfrac{\partial}{\partial x}\psi^1(x,t)\Big) \\ \qquad +\dfrac{\partial}{\partial x}\Big(E_1A_1\Big(\dfrac{\partial}{\partial x}u^1(x) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w^1(x)\Big)^2\Big)\dfrac{\partial}{\partial x}w^1(x)\Big), \\ 0 = E_1I_1\dfrac{\partial^2}{\partial x^2}\psi^1(x) + G_1A_1\Big(\dfrac{\partial}{\partial x}w^1(x) - \psi^1(x)\Big), \\ \qquad x \in [0, L_1], \end{cases} \tag{6.20}$$

$$\begin{cases} \rho_2 gA_2 = E_2A_2\dfrac{\partial^2}{\partial x^2}u^2(x,t) + \dfrac{E_2A_2}{2}\dfrac{\partial}{\partial x}\Big(\dfrac{\partial}{\partial x}w^2(x)\Big)^2, \\ 0 = G_2A_2\Big(\dfrac{\partial^2}{\partial x^2}w^2(x) - \dfrac{\partial}{\partial x}\psi^2(x,t)\Big) \\ \qquad +\dfrac{\partial}{\partial x}\Big(E_2A_2\Big(\dfrac{\partial}{\partial x}u^2(x) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w^2(x)\Big)^2\Big)\dfrac{\partial}{\partial x}w^2(x)\Big), \\ 0 = E_2I_2\dfrac{\partial^2}{\partial x^2}\psi^2(x) + G_2A_2\Big(\dfrac{\partial}{\partial x}w^2(x) - \psi^2(x)\Big), \\ \qquad x \in [0, L_2], \end{cases} \tag{6.21}$$

The boundary conditions at the simple nodes are

$$\begin{cases} u^1(0) = 0, \quad w^1(0) = 0, \quad \psi^1(0) = 0, \\ E_2A_2\dfrac{\partial}{\partial x}u^2(L_2) + \dfrac{E_2A_2}{2}\Big(\dfrac{\partial}{\partial x}w^2(L_2)\Big)^2 = 0, \\ G_2A_2\Big(\dfrac{\partial}{\partial x}w^2(L_2) - \psi^2(L_2)\Big) \\ \quad +\Big(E_2A_2\Big(\dfrac{\partial}{\partial x}u^2(x) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w^2(L_2)\Big)^2\Big)\dfrac{\partial}{\partial x}w^2(L_2)\Big) = 0, \\ E_2I_2\dfrac{\partial}{\partial x}\psi^2(0) = 0, \end{cases} \tag{6.22}$$

while the transmission conditions at the multiple node read as follows:

$$\begin{cases} u^1(L_1) = w^2(0), \quad w^1(L_1) = -u^2(0), \quad \psi^1(L_1) = \psi^2(0), \\ E_1A_1\Big(\dfrac{\partial}{\partial x}u^1(L_1) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w^1(L_1)\Big)^2\Big) = G_2A_2\Big(\dfrac{\partial}{\partial x}w^2(0) - \psi^2(0)\Big), \\ E_2A_2\Big(\dfrac{\partial}{\partial x}u^2(0) + \dfrac{1}{2}\Big(\dfrac{\partial}{\partial x}w^2(0)\Big)^2\Big) = G_1A_1\Big(\dfrac{\partial}{\partial x}w^1(L_1) - \psi^1(L_1)\Big), \\ E_1I_1\dfrac{\partial}{\partial x}\psi^1(L_1) = E_2I_2\dfrac{\partial}{\partial x}\psi^2(0). \end{cases} \tag{6.23}$$

The conditions at the multiple node are intuitive, as they clearly show that the longitudinal displacement of the horizontal beam converts to transversal displacements of the second beam and vice versa. The analogous observation is evident for the balance of forces. It can be shown that this system of ordinary differential equations has a unique solution. However,

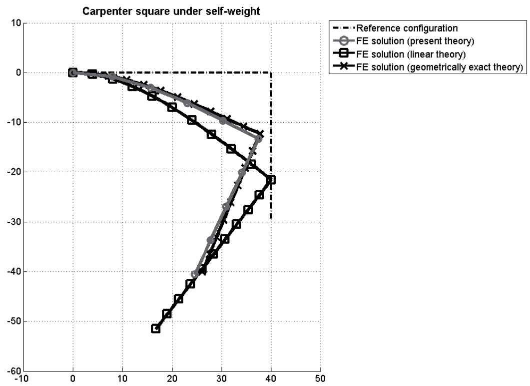

Figure 1 A carpenter's square: Current, geometric exact and linear model.

the analytical solution cannot be provided due to space limitations. See Figure 1, however, for a numerical comparison between the current beam model, the geometric exact model and the linear Timoshenko beam model. It is clearly seen that the two nonlinear models are in good agreement, while the classical linear model deviates significantly. In the linear theory the downwards movement of the tip of the horizontal beam follows the vertical line, which is clearly non-intuitive for large displacements.

6.2 Network equations in rst order format

By taking (3.2) into a matrix format, we denote the individual matrices for the i-th beam as follows:

$$A_i(U^i) := -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{E_i}{\rho_i} & 0 & \dfrac{E_i}{\rho_i}(\widehat{u_3^i}+u_3^i) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \dfrac{E_i}{\rho_i}(\widehat{u_3^i}+u_3^i) & 0 & \dfrac{G_i + E_i((\widehat{u_1^i}+u_1^i+\frac{3}{2}(\widehat{u_3^i}+u_3^i)^2))}{\rho_i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{E_i}{\rho_i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.24}$$

$$B_i(U^i) := -A_i(U^i)\frac{\partial}{\partial x}\widehat{U^i} + \begin{pmatrix} 0 \\ -g\mathbf{e}\cdot\mathbf{e}_3^i \\ 0 \\ -\dfrac{G_i}{\rho_i}(\widehat{u_5^i}+u_5^i) - g\mathbf{e}\cdot\mathbf{e}_1^i \\ 0 \\ \dfrac{G_iA_i}{\rho_iI_i}(\widehat{u_3^i}+u_3^i-\widehat{u_7^i}-u_7^i) \\ u_6^i \end{pmatrix} \quad \text{with} \quad B_i(0)=0. \tag{6.25}$$

Also for the multiple node, we introduce the representation of the local coordinate systems in terms of global coordinates in the 1-3-plane as follows: $\mathbf{e}_1^i = (e_{11}^i, e_{13}^i)$, $\mathbf{e}_3^i = (e_{31}^i, e_{33}^i)$. Moreover, the continuity conditions have to be differentiated with respect to time, so that instead of u^i, w^i, ψ^i, $\partial_t u^i, \partial_t w^i, \partial_t \psi^i$, hence, u_2^i, u_4^i, u_6^i are involed. Then

$$C^i := \begin{pmatrix} 0 & e_{11}^i & 0 & e_{31}^i & 0 & 0 & 0 \\ 0 & e_{13}^i & 0 & e_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.26}$$

and

$$F^i(U^i) := \begin{pmatrix} \frac{E_i}{\rho_i} e_{11}^i & 0 & \frac{G_i}{\rho_i} e_{31}^i + \frac{E_i}{\rho_i}\frac{1}{2}(\widehat{u}_3^i + u_3^i) e_{11}^i & 0 & 0 & 0 & -\frac{G_i}{\rho_i} e_{31}^i \\ \frac{E_i}{\rho_i} e_{13}^i & 0 & \frac{G_i}{\rho_i} e_{33}^i + \frac{E_i}{\rho_i}\frac{1}{2}(\widehat{u}_3^i + u_3^i) e_{13}^i & 0 & 0 & 0 & -\frac{G_i}{\rho_i} e_{33}^i \\ 0 & 0 & 0 & 0 & \frac{E_i I_i}{\rho_i} & 0 & 0 \end{pmatrix}. \tag{6.27}$$

With this notation, the general network problem can be expressed as follows:

$$\begin{cases} \frac{\partial}{\partial t} U^i(x,t) + A_i(U^i(x,t)) \frac{\partial}{\partial x} U^i(x,t) = B_i(U^i(x,t)) U^i(x,t), \\ \quad (x,t) \in [0, L_i] \times [0,T], \\ R_D(U^i(x_{i_j j}, t)) U^i(x_{i_j j}, t) = V_{iD}(t), \quad i \in \mathcal{I}^j,\ j \in \mathcal{J}^{S_D},\ t \in [0,T], \\ R_N(U^i(x_{i_j j}, t)) U^i(x_{i_j j}, t) = V_{iN}(t), \quad i \in \mathcal{I}^j,\ j \in \mathcal{J}^{S_N},\ t \in [0,T], \\ C^i U^i(x_{ij}, t) = C^k U^k(x_{kj}, t), \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T], \\ \sum_{i \in \mathcal{I}^j} \epsilon_{ij} F^i(U^i)(\widehat{U}^i + U^i)(x_{ij}, t) = 0, \quad i \in \mathcal{J}^M,\ t \in [0,T], \\ U^i(x,0) = U_0(x), \quad x \in [0, L_i],\ i \in \mathcal{I}. \end{cases} \tag{6.28}$$

An alternative formulation is as follows: We rewrite the system in terms of $\mathbf{w}$ according to $\widehat{U}^{\mathrm{T}} =: (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$, where $\mathbf{w}_1 = (u_1, u_3, u_5)$, $\mathbf{w}_2 = (u_2, u_4, u_6)$, $\mathbf{w}_3 = u_7$ has now the following form:

$$\widehat{A}_i(\mathbf{w}^i) := -\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{E_i}{\rho_i} & \frac{E_i}{\rho_i}(\widehat{u}_3^i + u_3^i) & 0 & 0 & 0 & 0 & 0 \\ \frac{E_i}{\rho_i}(\widehat{u}_3^i + u_3^i) & \frac{G_i + E_i(\widehat{u}_1^i + u_1^i + \frac{3}{2}((\widehat{u}_3^i + u_3^i)^2)}{\rho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{E_i}{\rho_i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.29}$$

$$\widehat{B}_i(\mathbf{w}^i) := \widehat{A}_i(\mathbf{w}^i) \frac{\partial}{\partial x} \widehat{\mathbf{w}}^i - \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g\mathbf{e} \cdot \mathbf{e}_3^i \\ -\frac{G_i}{\rho_i}(\widehat{u}_5^i + u_5^i) - g\mathbf{e} \cdot \mathbf{e}_1^i \\ \frac{G_i A_i}{\rho_i I_i}(\widehat{u}_3^i + u_3^i - (\widehat{u}_7^i + u_7) \\ (\widehat{u}_6^i + u_6) \end{pmatrix} \quad \text{with } \widehat{B}_i(0) = 0, \tag{6.30}$$

$$
\begin{cases}
\dfrac{\partial}{\partial t}\mathbf{w}^i(x,t) + \widehat{A}_i(\mathbf{w}^i(x,t))\dfrac{\partial}{\partial x}\mathbf{w}^i(x,t) = \widehat{B}_i(\mathbf{w}^i(x,t))\mathbf{w}^i(x,t), \\
\quad (x,t) \in [0,L_i] \times [0,T], \\
\mathbf{w}_2^i(x_{i_j j},t) = V_{iD}(t), \quad i \in \mathcal{I}^j,\ j \in \mathcal{J}^{S_D},\ t \in [0,T], \\
\widehat{R}_N(\mathbf{w}_1^i(x_{i_j j},t), \mathbf{w}_3^i(x_{i_j j},t)) = V_{iN}(t), \quad i \in \mathcal{I}^j,\ j \in \mathcal{J}^{S_N},\ t \in [0,T], \\
\mathbf{w}_s^i(x_{ij},t) = \mathbf{w}_s^k(x_{kj},t), \quad \forall i,k \in \mathcal{I}^j,\ j \in \mathcal{J}^M,\ t \in [0,T],\ s = 2,3, \\
\displaystyle\sum_{i \in \mathcal{I}^j} \epsilon_{ij}\widehat{F}^i(\mathbf{w}_1^i(x_{ij},t), \mathbf{w}_3^i(x_{ij},t)) = 0, \quad i \in \mathcal{J}^M,\ t \in [0,T], \\
\mathbf{w}^i(x,0) = \mathbf{w}_0(x), \quad x \in [0,L_i],\ i \in \mathcal{I},
\end{cases}
\tag{6.31}
$$

where we have used the obvious condensed forms of the operators R_N^i, F^i with respect to the partition of variables. We can now apply the same calculus as in the single-link case. This means that we may introduce, for each individual edge, a system of left and right eigenvalues of the matrices $A_i(U^i)$, $\mathbf{l}_j^i$ $(j = 1, \cdots, 7)$, $\mathbf{r}_j^i$ $(j = 1, \cdots, 7)$ or $\widehat{A}$, respectively. This makes it possible to rewrite the system (6.28) as a first order system in these new variables, analogous to the single-link case. For more details see [13]. We introduce

$$
\mathbf{V}_+^i := (v_1^i, v_2^i, v_3^i)^{\mathrm{T}}, \quad \mathbf{V}_-^i := (v_7^i, v_6^i, v_5^i)^{\mathrm{T}}, \quad \mathbf{V}_0^i := v_4^i = u_7^i. \tag{6.32}
$$

According to (3.17)–(3.18), we obtain

$$
\begin{cases}
\mathbf{V}_+^i + \mathbf{V}_-^i = M^i(\mathbf{w}_1^i)\mathbf{w}_2^i, \\
\mathbf{V}_+^i - \mathbf{V}_-^i = Q^i(\mathbf{w}_1^i)\mathbf{w}_1^i.
\end{cases}
\tag{6.33}
$$

We consider the case that $x_{ij} = 0$, the other case is completely analogous. In a tree, and this is the case we consider, one can always arrange the multiple nodes in such a way that all incident edges either start at the node, i.e., $x_{ij} = 0$, $\forall i \in \mathcal{I}^j$, or end there, i.e., $x_{ij} = L$, $\forall i \in \mathcal{I}^j$. For this, one has to work with a simple scaling in order to transform the lengths to a uniform quantity. Now, the second equation in (6.33), after applying the implicit function theorem at $\mathbf{w} = 0$, provides a function ϕ^i such that

$$
\mathbf{w}_1^i = \phi^i(\mathbf{V}_+^i, \mathbf{V}_-^i). \tag{6.34}
$$

We insert this in the first equation of (6.33) and get

$$
\mathbf{V}_+^i + \mathbf{V}_-^i = M^i(\phi^i(\mathbf{V}_+^i, \mathbf{V}_-^i))\mathbf{w}_2^i.
$$

In view of this, we can define the map

$$
{}^i := \mathbf{V}_+^i - \mathbf{V}_-^i - M^i(\phi^i(\mathbf{V}_+^i, \mathbf{V}_-^i))\mathbf{w}_2^i = 0
$$

together with

$$
{}^i(0,0,0) = -M^i(\phi^i(0,0))0 = 0, \quad D_{\mathbf{V}_+^i}\ {}^i(0,0,0) = I.
$$

We may, thus, apply the implicit function theorem again and obtain

$$
\mathbf{V}_+^i = G_0^i(\mathbf{V}_-^i, \mathbf{w}_2^i), \tag{6.35}
$$

so that at a controlled Dirichlet simple node we obtain

$$
\mathbf{V}_+^i(0,t) = G_0^i(\mathbf{V}_-^i(0,t), (\mathbf{V}_D^i)'(t)), \tag{6.36}
$$

where $\mathbf{V}^i_D(t) := (v^i_{D,1}, v^i_{D,2}, v_{D,3})(t)$ is the vector of Dirichlet controls at $x = 0$. This is the format required in [13]. Another way of writing (6.33) is

$$\begin{cases} \mathbf{V}^i_+ = \dfrac{1}{2}(M^i(\mathbf{w}^i_1)\mathbf{w}^i_2 + Q^i(\mathbf{w}^i_1)\mathbf{w}^i_1), \\ \mathbf{V}^i_- = \dfrac{1}{2}(M^i(\mathbf{w}^i_1)\mathbf{w}^i_2 - Q^i(\mathbf{w}^i_1)\mathbf{w}^i_1). \end{cases} \tag{6.37}$$

Now, (6.37) defines a map $\Theta^i(\mathbf{w}^i_1, \mathbf{w}^i_2) := (\mathbf{V}^i_+, \mathbf{V}^i_-)$, the Jacobian of which is given by

$$D_{\mathbf{w}^i_1}|_{0,0}\Theta = \frac{1}{2}\begin{pmatrix} Q^i_0(\mathbf{w}^i_1) & M_0(\mathbf{w}^i_1) \\ -Q_0(\mathbf{w}^i_1) & M^i_0(\mathbf{w}^i_1) \end{pmatrix},$$

which, in turn, is invertible. Notice that

$$Q^i = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} M^i.$$

Therefore, we may apply the implicit function theorem once again in order to find mappings ϕ^i, ψ^i with

$$\mathbf{w}^i_1 = \phi^i(\mathbf{V}^i_+, \mathbf{V}^i_-), \tag{6.38}$$
$$\mathbf{w}^i_2 = \psi^i(\mathbf{V}^i_+, \mathbf{V}^i_-). \tag{6.39}$$

With these expressions the Neumann node conditions at controlled simple nodes can be expressed in terms of $\mathbf{w}^i(0,t)$:

$$\widehat{R}_N(\mathbf{w}^i_1(0,t), \mathbf{w}^i_3(0,t)) = \widehat{R}(\phi(\mathbf{V}^i_+(0,t), \mathbf{V}^i_-(0,t)), \mathbf{w}^i_3(0,t)) = \mathbf{V}^i_N(t). \tag{6.40}$$

The continuity condition at a multiple node (taken at $x = 0$ for all incident edges) reads

$$\mathbf{w}^i_2(0,t) = \mathbf{w}^1_2(0,t), \quad i = 2, \cdots, n_j,$$

where $n_j = |\mathcal{I}^j|$ is the edge degree of the current multiple node $j \in \mathcal{J}^M$ at $x_{ij} = 0$. We can now use (6.38) and the lst equation to obtain

$$\mathbf{w}^i_1(0,t) = \phi^i(\mathbf{V}^i_+(0,t), \mathbf{V}^i_-(0,t)) = \phi^i(G^i_0(\mathbf{V}^i_+(0,t), \mathbf{w}^1_2(0,t)), \mathbf{V}^i_-(0,t)). \tag{6.41}$$

The final point is now to consider the transmission conditions involving the forces and moments. These conditions can be expressed as follows:

$$\sum_{i=1}^{n} \epsilon_{ij}\widehat{F}^i(\mathbf{w}^i_1(0,t), \mathbf{w}^i_3(0,t)) = 0, \tag{6.42}$$

which turns into

$$\sum_{i=1}^{n} \epsilon_{ij}\widehat{F}^i(\phi^i(G^i_0(\mathbf{V}^i_-(0,t), \mathbf{w}^1_2(0,t)), \mathbf{V}^i_-(0,t)), \mathbf{w}^i_3(0,t)) = 0. \tag{6.43}$$

This gives rise to the map

$$H_0(\mathbf{w}^1_2(0,t), \{\mathbf{w}^i_3(0,t)\}^n_{i=1}, \{\mathbf{V}^i_-(0,t)\}^n_{i=1})$$

$$= \sum_{i=1}^{n} \epsilon_{ij} \widehat{F}^i(\phi^i(G_0^i(\mathbf{V}_-^i(0,t), \mathbf{w}_2^1(0,t)), \mathbf{V}_-^i(0,t)), \mathbf{w}_3^i(0,t)) = 0. \tag{6.44}$$

Now, $H_0(0,0,0) = 0$ and $D_{\mathbf{w}_2^1} H_0(0,0,0)$ is invertible. This finally shows that $\mathbf{w}_2^1(0,t)$ is a function of $\{\mathbf{V}_-^i(0,t)\}_1^n, \{\mathbf{V}_0^i(0,t)\}_1^n$, i.e.,

$$\mathbf{V}_+^i(0,t) = G_0^i(\{\mathbf{V}_-^i(0,t)\}_{i=1}^n, \{\mathbf{V}_0^i(0,t)\}_{i=1}^n) \tag{6.45}$$

and this is the precisely format required in [13].

Theorem 6.1 *Consider a tree-like network of Timosheko beams as described by* (6.5)–(6.8)*. For each $T > 0$, there exist constants c_0 and c_T such that for initial data ${}_0^i = (u_0^i, w_0^i, \psi_0^i) \in C^2([0,L_i])^3$, ${}_1^i = (u_1^i, w_1^i, \psi_1^i) \in C^1([0,L_i])^3$ and boundary data $\mathbf{V}_D^i = (v_{D,}^i, v_{D2}^i, v_{D3}^i) \in C^2([0,T])^3$, $\mathbf{V}_N^i = (v_{N1}^i, v_{N2}^i, v_{N3}^i) \in C^1([0.T])^3$ satisfying a uniform smallness condition, i.e.,*

$$\max\{\| {}_0^i\|_2, \| {}_1^i\|_1, \|\mathbf{V}_D^i\|_2, \|\mathbf{V}_N^i\|\}_{i\in\mathcal{I}^j, j\in\mathcal{J}^S} \le c_0, \tag{6.46}$$

and the compatibility conditions (4.1)–(4.3)*, extended to all simple nodes, and* (6.15)–(6.16) *at the multiple nodes. Then, there exists a unique piecewise twice continuously differentiable solution $u^i, w^i, \psi^i \in \prod_{i=1}^{n} C^2([0,L_i] \times [0,T])$ depending continuously on the data:*

$$\|(u^i, w^i, \psi^i)\|_2 \le \max\{\| {}_0^i\|_2, \| {}_1^i\|_1, \|\mathbf{V}_D^i\|_2, \|\mathbf{V}_N^i\|\}_{i\in\mathcal{I}^j, j\in\mathcal{J}^S} \le c_T. \tag{6.47}$$

7 Exact Controllability on Star-Like Networks and Trees

In this section, we assume that n beams meet at one node, such that for each beam the junction is at $x = 0$. Thus, $|\mathcal{J}^M| = |\mathcal{J}^1| = 1$ and $x_{i1} = 0$, $i \in \mathcal{I}^1$. We define the travel times

$$T_0 = \max_{j=1,2,3} \max_{x\in[0,L_1]} \frac{L_1}{\sqrt{\mu_j^1(x,0)}} + \max_{i=2,\cdots,n} \max_{j=1,2,3} \max_{x\in[0,L_i]} \frac{L_i}{\sqrt{\mu_j^i(x,0)}}, \tag{7.1}$$

$$T_1 = \max_{j=1,2,3} \max_{x\in[0,L_1]} \frac{L_1}{\sqrt{\mu_j^1(x, \ \ {}^1)}} + \max_{i=2,\cdots,n} \max_{j=1,2,3} \max_{x\in[0,L_i]} \frac{L_i}{\sqrt{\mu_j^i(x, \ \ {}^i)}}. \tag{7.2}$$

Theorem 7.1 *Let $\{ \ {}_i\}_{i\in\mathcal{I}}$ be an equilibrium solution of* (6.5)–(6.8) *and let $T > 2T_0$. Then there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of $(\{ \ {}_i\}_{i\in\mathcal{I}}, 0)$ such that given initial and final data*

$$\{(\ {}_i^0, \ {}_i^1)\}_{i\in\mathcal{I}} \in \mathcal{U}_0, \quad \{(\ {}_i^{0,T}, \ {}_i^{1,T})\}_{i\in\mathcal{I}} \in \mathcal{U}_1, \tag{7.3}$$

one can find Dirichlet-controls $v_i \in C^2(0,T;\mathbb{R}^3)$ such that the corresponding solutions of (6.5)–(6.8) *satisfy*

$$ {}_i(\cdot,T) = \ {}_1^{0,T}, \quad \frac{\partial}{\partial t} \ {}_i(\cdot,T) = \ {}_i^{1,T}, \quad i = 1,\cdots,n.$$

Proof We follow the spirit of the proof of Theorem 5.2 in [13]. The principal idea in exact boundary controllability of 1D-hyperbolic systems is to solve a forward problem with given initial data, a backward problem with given final data and a corresponding initial-boundary value problem from "the left or the right". In particular, for the latter it is convenient to interchange the spatial and time variables x and t, and then solve a mixed Cauchy-problem from the left or the right, once the corresponding boundary conditions have been reduced from the initial and final data. We assume equilibria for which $\mathcal{G}_i(0,0)$ and $\mathcal{G}_i(\ {}_i, \frac{\partial}{\partial x} \ {}_i)$ are positive definite uniformly with respect to (x,t) for a sufficiently small neighborhood of $\widehat{\ }_i$. If one has Cauchy-data $\left({}_i, \frac{\partial}{\partial x}\ {}_i\right)$ at a boundary point, say $x = L_i$ and "boundary data" ${}_i(x,0),\ \ {}_i(x,T),\ x \in [0,L_i]$ one can solve the wave-type equation "from $x = L_i$ to $x = 0$". For this procedure, it is important to understand that the Cauchy data at $t = 0$ and $t = T$ can be converted to the proper boundary conditions for the system when the role of x and t is reversed. As the existence and uniqueness results are obtained on the level of first order equations, one needs to first invert on the second order level and then rewrite the resulting second order system as a first order system such that the corresponding boundary conditions are of the standard type. Due to the zero eigenvalue for the first order system, the direct inversion on the level of the first order system is impossible. We describe the idea of the proof as follows: There are five steps. Step 1: In the first step we proceed forward from $t = 0$ to $t = T_1$. We solve the initial-boundary value problem with artificial controls at $x = L_i$. For each beam $i \in \mathcal{I}$, we define the set $R_I^i := \{(x,t) \in [0,L_i] \times [0,T_1]\}$ and for the network we set $R_I := \bigcup\limits_{i=1}^{n} R_I^i$. The first beam to be fixed at $x = L_1$. We may, for the sake of convenience, assume that we impose a homogenous Dirichlet condition there. This specifies the corresponding compatibility conditions. We impose artificial inhomogeneous Dirichlet conditions at $x = L_i$ $(i = 2, \cdots, n)$, i.e., ${}_i(L_i,t) = \mathsf{v}^i(t)$ $(i = 2, \cdots, n)$, where $\mathsf{v}^i(\cdot)$ are small in $C^2(0,T_1;\mathbb{R}^3)$. We also have sufficiently small initial data $\left({}_i(x,0), \frac{\partial}{\partial t}\ {}_i^{}(x,0)\right) = \left({}_i^0(x),\ {}_i^1(x)\right)$ for all beams. We apply the existence Theorem 6.1 and obtain a unique solution on R_I. We can now take traces of $\left({}_1(L_1,t), \frac{\partial}{\partial x}\ {}_1(L_1,t)\right) = (\mathsf{a}^1(t), \mathsf{a}^2(t))$ (here $\mathsf{a}^1(t) = 0$) at the boundary of the first beam along $\{L_1\} \times [0,T_1]$ and of $\left({}_i(0,t), \frac{\partial}{\partial x}\ {}_i(0,t)\right) = (\mathsf{b}_1^i(t), \mathsf{b}_2^i(t))$ for all beams at $\{0\} \times [0,T_1]$. It is clear that $(\mathsf{b}_1^i(t), \mathsf{b}_2^i(t))$ satisfy the nodal conditions at the common node. Moreover, all data is small in the appropriate spaces.

Step 2: We perform the same procedure, but now reversing the time and progressing from the final time T to $T - T_1$. More precisely, we introduce the individual domains $R_{II}^i := \{(x,t) \in [0,L_i] \times [T - T_1, T]\}$ $(i = 1, \cdots, n)$ and the global one $R_{II} = \bigcup\limits_{i=1}^{n} R_{II}^i$. By the same argument, a unique semi-global small solution $\left({}_i^{II}, \frac{\partial}{\partial x}\ {}_i^{II}\right)$ of the network problem exists, and we can take traces $\left({}_1^{II}(L_1,t), \frac{\partial}{\partial x}\ {}_1^{II}(L_1,t)\right) = (\overline{\mathsf{a}}^1(t), \overline{\mathsf{a}}^2(t))$ at $\{L_1\} \times [T - T_1, T]$ for the first beam (again $\overline{\mathsf{a}}^1(t) = 0$) and $\left({}_i^{II}(0,t), \frac{\partial}{\partial x}\ {}_i^{II}(0,t)\right) = (\overline{\mathsf{b}}_1^i(t), \overline{\mathsf{b}}_2^i(t))$ at $\{L_i\} \times [T - T_1, T]$ for the beams labelled $i = 2, \cdots, n$.

In order to prepare Step 3, we extend the Cauchy-data at $\{\{L_1\} \times [0,T_1]\} \cup \{\{L_1\} \times [T - T_1, T]\}$ in the C^2-sense to $\{L_1\} \times [0,T]$ as $(\widetilde{\mathsf{a}}^1(t), \widetilde{\mathsf{a}}^2(t))$. After that we can use these Cauchy-data along $\{L_1\} \times [0,T]$ as "initial conditions".

Step 3: We change the order of x and t as explained in the beginning of the proof. The Cauchy-data just constructed can be taken as "initial conditions" for the first beam "starting" at $x = L_1$ with 'boundary conditions' at $t = 0$ and $t = T$ taken from the original initial and final data. Applying the semi-global existence Theorem 6.1 to that situation, we can evaluate the solution $\left({}_1(x,t), \frac{\partial}{\partial x}\ {}_1(x,t)\right)$ at $\{0\} \times [0,T]$. On the set $\left\{(x,t) \in [0,L_1],\ 0 \le t \le T_2 + \frac{(T_1 - T_2)x}{L_1}\right\}$ this solution ${}_1$ is identical to ${}_1^I$. Therefore, at $t = 0$ we have

$$ {}_1(x,0) = {}_1^0(x), \quad \frac{\partial}{\partial t}\ {}_1(x,0) = {}_1^1(x), \quad x \in [0,L_1]. $$

At $x = 0$ we have

$$ {}_1(0,t) = \mathsf{b}_1^1(t), \quad \frac{\partial}{\partial x}\ {}_1(0,t) = \mathsf{b}_2^1(t), \quad t \in [0,T_2]. $$

The analogous uniqueness argument applies for the backward solution of Step 2, such that the final conditions are

$$ {}_1(x,T) = {}^{0,T}_1(x), \quad \frac{\partial}{\partial t} {}_1(x,T) = {}^{1,T}_1(x), \quad x \in [0, L_1], $$

while the evaluation at $x = 0$ provides the Cauchy-data

$$ {}_1(0,t) = \overline{\mathsf{b}}^1_1(t), \quad \frac{\partial}{\partial x} {}_1(0,t) = \overline{\mathsf{b}}^1_2(t), \quad t \in [T - T_2, T]. $$

Step 4: We now extend the Cauchy-data $(\mathsf{b}^1_1(t), \mathsf{b}^1_2(t))$, $t \in [0, T_2]$ together with $(\overline{\mathsf{b}}^1_2(t), \overline{\mathsf{b}}^1_2(t))$, $t \in [T - T_2, T]$ to Cauchy-data $(\widetilde{\mathsf{b}}^1_1(t), \widetilde{\mathsf{b}}^1_2(t))$, $t \in [0, T]$ such that corresponding solutions satisfy the nodal conditions.

Step 5: We now have Cauchy-data on $\{0\} \times [0, T]$ such that the nodal conditions are satisfied. Therefore, we can use these as compatible initial conditions for the beams labelled $i = 2, \cdots, n$ after interchanging x and t. Thus, on the domains $R^i_{IV} := \{(x,t) \in [0, L_i] \times [0, T]\}$ we solve the initial boundary value problems with Cauchy-data

$$ {}_i(0,t) = \widetilde{\mathsf{b}}^i_1, \quad \frac{\partial}{\partial x} {}_i(0,t) = \widetilde{\mathsf{b}}^i_2, \quad t \in [0, T] $$

and boundary conditions

$$ {}_i(x,0) = {}^0_i(x), \qquad {}_i(x,T) = {}^{1,T}_i(x), \quad x \in [0, L_i]. $$

By construction, the solutions are small in the sense described above. A similar uniqueness argument applies to the region $\{(x,t) \mid x \in [0, L_i],\ 0 \le t \le T_2(1 - \frac{x}{L_i})\}$ to the effect that

$$ {}_i(x,0) = {}^0_i(x), \quad \frac{\partial}{\partial t} {}_i(x,0) = {}^1_i(x), \quad x \in [0, L_i]. $$

The analogous argument on the “upper” domain leads to

$$ {}_i(x,T) = {}^{0,T}_i(x), \quad \frac{\partial}{\partial t} \Phi_i(x,T) = {}^{1,T}_i(x), \quad x \in [0, L_i]. $$

This gives the solution to the problem stated.

Remark 7.1 It is clear that a local-global controllability theorem can be proved, provided that we have two non-identical equilibrium solutions connected by a path of equilibria. However, as the analysis of general network equilibria is still open, we refrain from stating the theorem here. Also, the analogous controllability holds for tree-like networks as usual. This is proved by the so-called peeling method (see [13]). Moreover, it should be remarked that the observability does not directly follow as in the linear case (see the corresponding remarks in the monograph of Li [13]). Also feedback stabilization is an open problem that will be addressed in a forthcoming publication.

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