Chin. Ann. Math.
38B(3), 2017, 787–794
DOI: 10.1007/s11401-017-1095-4

Chinese Annals of Mathematics, Series B

Global Well-Posedness of Incompressible Navier-Stokes Equations with Two Slow Variables*

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Abstract In this paper, the global well-posedness of the three-dimensional incompressible Navier-Stokes equations with a linear damping for a class of large initial data slowly varying in two directions are proved by means of a simpler approach.

Keywords Global well-posedness, Incompressible Navier-Stokes equations, Slow variables
2000 MR Subject Classification 35Q30, 76D03, 76D05

1 Introduction

The classical three-dimensional incompressible Navier-Stokes equations are given by

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \triangle u, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where

$$u(x,t) = (u_1,u_2,u_3)(x,t)$$

and $p(x,t)$ stand for the velocity vector and the pressure function of the flow at the point $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$, respectively, and for simplicity, the kinematic viscosity of the fluid is taken to be equal to one. The initial data u_0 should also be a divergence-free vector field.

The study on Navier-Stokes equations dates back long time ago. The first important result was obtained by Leray [12]. Despite much effort by many mathematicians and physicists, however, our understanding of Navier-Stokes equations remains minimal (see [7]). We can explain the main difficulty by means of the scaling property of the incompressible Navier-Stokes equations from a purely mathematical viewpoint. If $(u(x,t),p(x,t))$ solves the Navier-Stokes

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*This work was supported by the National Natural Science Foundation of China (Nos. 11471215, 11031001, 11121101, 11626156), Shanghai Leading Academic Discipline Project (No. XTKX2012), the Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), the Ministry of Education of China, Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University and 111 Program of MOE, China (No. B08018).

equations on the time interval $[0, T]$, then we can form a new solution $(u_\lambda(x,t), p_\lambda(x,t))$ to the Navier-Stokes equations on the time interval $[0, \lambda^{-2}T]$, by the formula

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) \tag{1.2}$$

with the initial data

$$u_\lambda(x,0) = \lambda u_0(\lambda x). \tag{1.3}$$

It can be easily checked that the unique conserved quantity — the energy $E(u)(t)$ is

$$E(u)(t) = \frac{1}{2}\int u^2(x,t)\mathrm{d}x + \int_0^t \int |\nabla u(x,\tau)|^2 \mathrm{d}x\mathrm{d}\tau. \tag{1.4}$$

If the dimension d is equal to 3, then

$$E(u_\lambda)(t) = \frac{1}{\lambda}E(u)(\lambda^2 t). \tag{1.5}$$

Thus, the energy is "super-critical", which makes that the energy becomes increasingly useless for controlling the solution as one moves to finer and finer scales.

For the problem with small initial data, we can see that the initial data should belong to scale-invariant spaces in the following sense: There exists a constant C such that for any given positive λ, we have

$$C^{-1}\|u\|_X \le \|u_\lambda\|_X \le C\|u\|_X. \tag{1.6}$$

The corresponding result in $\dot{H}^{\frac{1}{2}}$ is due to Fujita and Kato [8] (see also Leray [12], where the smallness of the initial data is measured by $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$). The study of Navier-Stokes equations in critical spaces was done by many authors, for example, Weissler [13], Kato [10], Giga and Miyakawa [9], Cannone, Meyer and Planchon [2], in particular, Koch and Tataru [11] proved the global well-posedness of the Navier-Stokes equations with small initial data in the space BMO^{-1}. For more information about the classical results, the reader can consult the book by Cannone [1] and the references therein.

There are some results for large initial data, for example, in a series of recent papers [3–6], Chemin et al. constructed some classes of large anisotropic initial data for the Navier-Stokes system. We will not describe them in details, but we will just mention their work related to our paper. In [4], Chemin, Gallagher and Paicu considered the well-poedness of the three-dimensional Navier-Stokes equations with initial data slowly varying in one direction:

$$\Big(v_1(x_1,x_2,\epsilon x_3), v_2(x_1,x_2,\epsilon x_3), \frac{1}{\epsilon}v_3(x_1,x_2,\epsilon x_3)\Big), \tag{1.7}$$

where $\epsilon > 0$ is a small parameter, (x_2, x_2) belongs to the torus $\mathbb{T}^2$ and x_3 belongs to $\mathbb{R}$. After a change of scale in the vertical variable, the system is not uniformly elliptic, which leads to lose the control on one derivative in the vertical variable. To compensate the loss of derivative, by working in the class of analytical functions and using some kind of the global Cauchy-Kowalewski result, the authors in [4] got the global well-posedness without small assumption on the norm of initial data for the three-dimensional incompressible Navier-Stokes equations.

Inspired by their work, this paper is devoted to the study of the following system with a linear damping:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = -u + \triangle u \quad \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot u = 0 \end{cases} \tag{1.8}$$

on the domain

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{T} = [-\pi, \pi]\} \tag{1.9}$$

with the initial data slowing varying in two directions

$$t = 0 : (u_1, u_2, u_3) = \Big(\frac{1}{\epsilon} U_1(\epsilon x_1, \epsilon x_2, x_3), \frac{1}{\epsilon} U_2(\epsilon x_1, \epsilon x_2, x_3), U_3(\epsilon x_1, \epsilon x_2, x_3)\Big), \tag{1.10}$$

where $\epsilon > 0$ is a small parameter.

For $U_j(x_1, x_2, x_3)$ $(j = 1, 2, 3)$, we take the Fourier transformation with respect to $x_1, x_2 \in \mathbb{R}^2$, and also take the coefficients in the Fourier expansion with respect to $x_3 \in [-\pi, \pi]$. More precisely, let

$$\widehat{U}_{j,n}(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}(x_1\xi_1 + x_2\xi_2)} \mathrm{e}^{-\mathrm{i}nx_3} U_j(x_1, x_2, x_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3, \quad j = 1, 2, 3. \tag{1.11}$$

We give the following assumptions.

(1) $U_j(x_1, x_2, x_3)$ $(j = 1, 2, 3)$ are analytic functions of x_1, x_2, $U_j(x_1, x_2, x_3)$ $(j = 1, 2)$ are even functions with respect to x_3, and $U_3(x_1, x_2, x_3)$ is an odd function with respect to x_3.

(2) For $\widehat{U}_{j,n}(\xi_1, \xi_2)$ $(j = 1, 2)$, the following inequality holds:

$$\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{a(|\xi_1| + |\xi_2|)} |\widehat{U}_{j,n}(\xi_1, \xi_2)| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \le \delta, \tag{1.12}$$

where a is a positive number and $\delta > 0$ is a small constant.

(3) For $\widehat{U}_{3,n}(\xi_1, \xi_2)$, the following inequality holds:

$$\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{a(|\xi_1| + |\xi_2|)} |\widehat{U}_{3,n}(\xi_1, \xi_2)| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \le M, \tag{1.13}$$

where M is a bounded constant.

Our result for system (1.8) is the following.

Theorem 1.1 *Under assumptions* (1)–(3)*, there exists* $\delta = \delta(a) > 0$ *so small that if* $\epsilon M \le \delta$*, then the three-dimensional Navier-Stokes problem with damping* (1.8) *and with the initial data given by* (1.10) *generates a unique global solution for any given small* $\epsilon > 0$.

Remark 1.1 The linear damping term is put in system (1.8) to deal with the zero Fourier mode. Comparing with [4], we give a much simpler proof at the cost of all initial data having some analytical condition.

2 Proof of Theorem 1.1

Noting that the divergence free condition recovers p from u through the following formula:

$$-\triangle p = \nabla \cdot (u \cdot \nabla u) = \nabla \cdot [\nabla \cdot (u \otimes u)], \tag{2.1}$$

we can put the system (1.8) in the following formula:

$$\begin{cases} u_t + (I - \triangle^{-1}(\nabla \otimes \nabla))(u \cdot \nabla u) + (1 - \triangle)u = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot u = 0. \end{cases} \tag{2.2}$$

Let

$$u_\epsilon(x,t) = \frac{1}{\epsilon} u\Big(\frac{x}{\epsilon}, \frac{t}{\epsilon^2}\Big).$$

The rescaled Navier-Stokes equations can be obtained from (1.8) as follows (to simplify the notation, we will drop ϵ in the rest of this section):

$$\begin{cases} u_t + (I - \triangle^{-1}(\nabla \otimes \nabla))(u \cdot \nabla u) + \Big(\frac{1}{\epsilon^2} - \triangle\Big)u = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ \nabla \cdot u = 0 \end{cases} \tag{2.3}$$

on the domain

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathbb{R}^2, x_3 \in \mathbb{T}_\epsilon = [-\epsilon\pi, \epsilon\pi]\} \tag{2.4}$$

with the initial data

$$\Big(\frac{1}{\epsilon^2} U_1\Big(x_1, x_2, \frac{x_3}{\epsilon}\Big), \frac{1}{\epsilon^2} U_2\Big(x_1, x_2, \frac{x_3}{\epsilon}\Big), \frac{1}{\epsilon} U_3\Big(x_1, x_2, \frac{x_3}{\epsilon}\Big)\Big). \tag{2.5}$$

For any given point

$$x = (x_1, x_2, x_3) \in \Omega,$$

we denote its horizontal coordinates by

$$x_h = (x_1, x_2).$$

Similarly, the horizontal components of any given vector field

$$u = (u_1, u_2, u_3)$$

will be denoted by

$$v = (u_1, u_2),$$

and $\xi=(\xi_1,\xi_2)$ will be the frequency variable with respect to

$$x_h=(x_1,x_2).$$

Let

$$\widehat{u}_n(\xi,t)=\frac{1}{2\epsilon\pi}\int_{-\epsilon\pi}^{\epsilon\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}x_h\cdot\xi}\mathrm{e}^{-\mathrm{i}n\frac{x_3}{\epsilon}}u(x_h,x_3,t)\mathrm{d}x_h\mathrm{d}x_3. \tag{2.6}$$

Thanks to the relationship between Fourier transformation, Fourier series and the derivative with respect to x, the first equation in (2.3) becomes

$$\begin{aligned}&\partial_t\widehat{u}_n(\xi,t)+\Big(|\xi|^2+\frac{1+n^2}{\epsilon^2}\Big)\widehat{u}_n(\xi,t)\\&=-\Big(1-\frac{(\xi,\frac{n}{\epsilon})\otimes(\xi,\frac{n}{\epsilon})}{|\xi|^2+\frac{n^2}{\epsilon^2}}\Big)\sum_{n_1+n_2=n}\Big(\int_{\mathbb{R}^2}\widehat{v}_{n_1}(\xi-\eta,t)\cdot\eta\widehat{u}_{n_2}(\eta,t)\mathrm{d}\eta\\&\quad+\int_{\mathbb{R}^2}\widehat{u}_{3,n_1}(\xi-\eta,t)\frac{n_2}{\epsilon}\widehat{u}_{n_2}(\eta,t)\mathrm{d}\eta\Big).\end{aligned} \tag{2.7}$$

The incompressible condition in (2.3) turns to

$$\xi\cdot\widehat{v}_n(\xi,t)+\frac{n}{\epsilon}\widehat{u}_{3,n}(\xi,t)=0. \tag{2.8}$$

Moreover, for $j=1,2$ the initial data (2.5) becomes

$$\begin{aligned}&\widehat{u}_{j,n}(\xi,0)\\&=\frac{1}{2\epsilon\pi}\int_{-\epsilon\pi}^{\epsilon\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x_1\xi_1+x_2\xi_2)}\mathrm{e}^{-\mathrm{i}n\frac{x_3}{\epsilon}}\frac{1}{\epsilon^2}U_j\Big(x_1,x_2,\frac{x_3}{\epsilon}\Big)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\&=\frac{1}{2\pi}\int_{\pi}^{\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x_1\xi_1+x_2\xi_2)}\mathrm{e}^{-\mathrm{i}nx_3}\frac{1}{\epsilon^2}U_j(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\&=\frac{1}{\epsilon^2}\widehat{U}_{j,n}(\xi_1,\xi_2),\end{aligned} \tag{2.9}$$

while, for $j=3$ we have

$$\begin{aligned}&\widehat{u}_{3,n}(\xi,0)\\&=\frac{1}{2\epsilon\pi}\int_{-\epsilon\pi}^{\epsilon\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x_1\xi_1+x_2\xi_2)}\mathrm{e}^{-\mathrm{i}n\frac{x_3}{\epsilon}}\frac{1}{\epsilon}U_3\Big(x_1,x_2,\frac{x_3}{\epsilon}\Big)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\&=\frac{1}{2\pi}\int_{\pi}^{\pi}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}(x_1\xi_1+x_2\xi_2)}\mathrm{e}^{-\mathrm{i}nx_3}\frac{1}{\epsilon}U_3(x_1,x_2,x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\&=\frac{1}{\epsilon}\widehat{U}_{3,n}(\xi_1,\xi_2).\end{aligned} \tag{2.10}$$

Multiplying $\frac{\widehat{u}_n(\xi,t)}{|\widehat{u}_n(\xi,t)|}$ on both sides of (2.7), we get

$$\begin{aligned}&\partial_t|\widehat{u}_n(\xi,t)|+\Big(|\xi|^2+\frac{1+n^2}{\epsilon^2}\Big)|\widehat{u}_n(\xi,t)|\\&\le C\sum_{n_1+n_2=n}\int_{\mathbb{R}^2}|\widehat{v}_{n_1}(\xi-\eta,t)||\eta||\widehat{u}_{n_2}(\eta,t)|\mathrm{d}\eta\\&\quad+\frac{|\widehat{u}_{3,n_1}(\xi-\eta,t)|}{\epsilon}|n_2||\widehat{u}_{n_2}(\eta,t)|\mathrm{d}\eta.\end{aligned} \tag{2.11}$$

Here and hereafter C denotes a positive constant. Since u_3 is an odd function with respect to x_3, we have

$$\widehat{u}_{3,0}(\xi,t)\equiv 0. \tag{2.12}$$

The combination of (2.8) with (2.12) leads to

$$\frac{|\widehat{u}_{3,n}(\xi,t)|}{\epsilon}\le |\xi||\widehat{v}_n(\xi,t)|,\quad \forall n. \tag{2.13}$$

Plugging (2.13) into (2.11) gives

$$\begin{aligned}&\partial_t|\widehat{u}_n(\xi,t)|+\Big(|\xi|^2+\frac{1+n^2}{\epsilon^2}\Big)|\widehat{u}_n(\xi,t)|\\ &\le C\sum_{n_1+n_2=n}\int_{\mathbb{R}^2}|\widehat{v}_{n_1}(\xi-\eta,t)||\eta||\widehat{u}_{n_2}(\eta,t)|\mathrm{d}\eta\\ &\quad+|\xi-\eta||\widehat{v}_{n_1}(\xi-\eta,t)||n_2||\widehat{u}_{n_2}(\eta,t)|\mathrm{d}\eta\\ &\le C\sum_{n_1+n_2=n}\int_{\mathbb{R}^2}(|n_1||\widehat{u}_{n_1}(\xi-\eta,t)|+|\widehat{u}_{n_1}(\xi-\eta,t)|)|\eta||\widehat{u}_{n_2}(\eta,t)|\mathrm{d}\eta.\end{aligned} \tag{2.14}$$

Suppose that there exists a C^1 function $\theta(t)$ satisfying

$$\theta(t)\le\frac{a}{2},\quad \theta(0)=0. \tag{2.15}$$

Multiplying $\mathrm{e}^{(a-\theta(t))|\xi|}$ on both sides of (2.14), and using the triangle inequality, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_n\int_{\mathbb{R}^2}\mathrm{e}^{(a-\theta(t))|\xi|}|\widehat{u}_n(\xi,t)|\mathrm{d}\xi+\sum_n\int_{\mathbb{R}^2}\dot{\theta}(t)|\xi||\widehat{u}_n(\xi,t)|\mathrm{e}^{(a-\theta(t))|\xi|}\mathrm{d}\xi\\ &+\sum_n\int_{\mathbb{R}^2}\Big(|\xi|^2+\frac{1+n^2}{\epsilon^2}\Big)|\widehat{u}_n(\xi,t)|\mathrm{e}^{(a-\theta(t))|\xi|}\mathrm{d}\xi\\ &\le C\Big(\sum_n\int_{\mathbb{R}^2}(|n|+1)|\widehat{u}_n(\xi,t)|\mathrm{e}^{(a-\theta(t))|\xi|}\mathrm{d}\xi\Big)\Big(\sum_n\int_{\mathbb{R}^2}|\eta||\widehat{u}_n(\eta,t)|\mathrm{e}^{(b-\theta(t))|\eta|}\mathrm{d}\eta\Big).\end{aligned} \tag{2.16}$$

Integrating both sides of (2.16) with respect to t, we get

$$\begin{aligned}&\sum_n\int\mathrm{e}^{(a-\theta(t))|\xi|}|\widehat{u}_n(\xi,t)|\mathrm{d}\xi\\ &+\int_0^t\sum_n\int\dot{\theta}(\tau)|\xi||\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\mathrm{d}\tau\\ &-\sum_n\int\mathrm{e}^{a|\xi|}|\widehat{u}_n(\xi,0)|\mathrm{d}\xi+\int_0^t\sum_n\int\Big(|\xi|^2+\frac{1+n^2}{\epsilon^2}\Big)|\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\mathrm{d}\tau\\ &\le C\int_0^t\Big(\sum_n\int(|n|+1)|\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\Big)\Big(\sum_n|\eta||\widehat{u}_n(\eta,\tau)|\mathrm{e}^{(a-\theta(\tau))|\eta|}\mathrm{d}\eta\Big)\mathrm{d}\tau.\end{aligned} \tag{2.17}$$

Now, setting

$$\dot{\theta}(t)=C\sum_n\int_{\mathbb{R}^2}(|n|+1)|\widehat{u}_n(\xi,t)|\mathrm{e}^{(a-\theta(t))|\xi|}\mathrm{d}\xi, \tag{2.18}$$

we have

$$\begin{aligned}&\int_0^t \sum_n \int_{\mathbb{R}^2} \Big(|\xi|^2 + \frac{1+n^2}{\epsilon^2}\Big)|\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\mathrm{d}\tau \\ &+\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{(a-\theta(t))|\xi|}|\widehat{u}_n(\xi,t)|\mathrm{d}\xi \\ \le{}& \sum_n \int_{\mathbb{R}^2} \mathrm{e}^{a|\xi|}|\widehat{u}_n(\xi,0)|\mathrm{d}\xi.\end{aligned} \tag{2.19}$$

According to the original assumptions (2)–(3), it is obvious that

$$\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{a|\xi|}|\widehat{u}_n(\xi,0)|\mathrm{d}\xi \le \frac{\delta}{\epsilon^2} + \frac{M}{\epsilon}, \tag{2.20}$$

then, owing to $\epsilon M \le \delta$, we have

$$\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{a|\xi|}|\widehat{u}_n(\xi,0)|\mathrm{d}\xi \le \frac{C\delta}{\epsilon^2}. \tag{2.21}$$

It follows from (2.19) that

$$\int_0^t \sum_n \int_{\mathbb{R}^2} \frac{1+n^2}{\epsilon^2}|\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\mathrm{d}\tau \le \frac{C\delta}{\epsilon^2}. \tag{2.22}$$

By the definition of $\theta(t)$, obviously we have

$$\theta(t) \le 2C\int_0^t \sum_n \int_{\mathbb{R}^2} (1+n^2)|\widehat{u}_n(\xi,\tau)|\mathrm{e}^{(a-\theta(\tau))|\xi|}\mathrm{d}\xi\mathrm{d}\tau \le \widetilde{C}\delta, \tag{2.23}$$

then we can choose δ so small that

$$\theta(t) \le \frac{a}{4}. \tag{2.24}$$

Recalling (2.15) and (2.18), the bootstrap argument ensures the existence of $\theta(t)$.

Thus, according to (2.19), it follows that

$$\sum_n \int_{\mathbb{R}^2} \mathrm{e}^{\frac{a}{2}|\xi|}|\widehat{u}_n(\xi,t)|\mathrm{d}\xi \le \frac{C\delta}{\epsilon^2} \tag{2.25}$$

for all time $t \ge 0$. This means the global existence of the solution.

Acknowledgement The authors would like to express their gratitude to Professor Ta-Tsien Li for his helpful advice.

References

[1] Cannone, M., Chapter 3 Harmonic analysis tools for solving the incompressible Navier-Stokes equations, Handbook of Mathematical Fluid Dynamics, Vol. III, North-Holland, Amsterdam, 2004, 161–244.

[2] Cannone, M., Meyer, Y. and Planchon, F., Solutions auto-similaries des équations de Navier-Stokes, Séminaire sur les Equations aux Dérivées Partielles, 1993–1994, Exp. No. VIII, 12 pages, Ecole Polytech., Palaiseau, 1994.

[3] Chemin, J. Y. and Gallagher, I., Large, global solutions to the Navier-Stokes equations, slowly varying in one direction, *Trans. Amer. Math. Soc.*, **362**(6), 2010, 2859–2873.

[4] Chemin, J. Y., Gallagher, I. and Paicu, M., Global regularity for some classes of large solutions to the Navier-Stokes equations, *Ann. of Math.*, **173**, 2011, 983–1012.

[5] Chemin, J. Y., Paicu, M. and Zhang, P., Global large solutions to 3-D inhomogeneous Navier-Stokes system with one slow variable, *J. Differential Equations*, **256**(1), 2014, 223–252.

[6] Chemin, J. Y. and Zhang, P., On the global well-posedness to the 3-D incompressible anisotropic Navier-Stokes equations, *Comm. Math. Phys.*, **272**(2), 2007, 529–566.

[7] Clay mathematics institute, Millennium prize problems, http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation

[8] Fujita, H. and Kato, T., On the Navier-Stokes initial value problem, I, *Arch. Ration. Mech. Anal.*, **16**, 1964, 269–315.

[9] Giga, Y. and Miyakama, T., Solutions in L^r of the Navier-Stokes initial value problem, *Arch. Ration. Mech. Anal.*, **89**(3), 1985, 267–281.

[10] Kato, T., Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, *Math. Z.*, **187**(4), 1984, 471–480.

[11] Koch, H. and Tataru, D., Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157**(1), 2001, 22–35.

[12] Leray, J., Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.*, **63**, 1934, 193–248.

[13] Weissler, F. B., The Navier-Stokes initial value problem in L^p, *Arch. Ration. Mech. Anal.*, **74**, 1980, 219–230.