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A Nontrivial Homotopy Element of Order p^2 Detected by the Classical Adams Spectral Sequence*

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Abstract Let p be an odd prime. The authors detect a nontrivial element $\widetilde{a}_p$ of order p^2 in the stable homotopy groups of spheres by the classical Adams spectral sequence. It is represented by $a_0^{p-2}h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-1,pq+p-2}(\mathbb{Z}/p,\mathbb{Z}/p)$ in the E_2-term of the ASS and meanwhile $p\cdot\widetilde{a}_p$ is the first periodic element α_p.

Keywords Stable homotopy groups of sphere, Adams spectral sequence, May spectral sequence, Massey product
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1 Introduction

Let p be an odd prime. Let $\mathcal{A}$ be the mod p Steenrod algebra and S be the sphere spectrum localized at p. Throughout the paper we fix $q = 2(p-1)$. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to approach it is the classical Adams spectral sequence (ASS) whose E_2-term is given by $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$ which is the cohomology of $\mathcal{A}$. The Adams differential is given by

$$\mathrm{d}_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

From [6], we know that $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $a_0 \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{Z}/p,\mathbb{Z}/p)$, $h_i \in \mathrm{Ext}_{\mathcal{A}}^{1,p^iq}(\mathbb{Z}/p,\mathbb{Z}/p)$ for all $i \geq 0$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $\widetilde{\alpha}_2$, a_0^2, a_0h_i $(i>0)$, g_i $(i\geq 0)$, k_i $(i \geq 0)$, b_i $(i\geq 0)$ and h_ih_j $(j \geq i+2,\ i\geq 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $(p^{i+1}+2p^i)q$, $(2p^{i+1}+p^i)q$, $p^{i+1}q$ and $(p^i+p^j)q$, respectively. Aikawa [1] obtained all the generators of $\mathrm{Ext}_{\mathcal{A}}^{3,*}(\mathbb{Z}/p,\mathbb{Z}/p)$. Here we do not list out these generators as it being complicated. By now only partial structure of $\mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ is known for $s \geq 4$.

If a family of generators $x_i \in E_2^{s,*}$ converges nontrivially in the ASS, then we obtain a family of homotopy elements f_i in π_*S and we say that f_i has filtration s and is represented by $x_i \in E_2^{s,*}$ in the ASS. In order to compute π_*S by the ASS, it is critical to determine which element in the E_2-terms survives to E_∞. But so far not so much has been known about this. In known that only a_0 and h_0 in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ survive to π_*S. They survive to the degree p map and the mod p Hopf invariant one element, respectively. By the Thom map (see [9]), it is shown that a_0^2, $\widetilde{\alpha}_2$, k_0, a_0h_1 (for $p=3$), b_i $(i\geq 0)$ and h_0h_{n+2} $(n \geq 0)$ can survive to

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π_*S. Among them a_0^2, $\widetilde{\alpha}_2$, b_0 and k_0 survive to the degree p^2 map, the first periodic element α_2, the secondary periodic elements β_1 and β_2, respectively. In [2], R. Cohen showed that $h_0 b_{n-1} \in \mathrm{Ext}_{\mathcal{A}}^{3,(p^n+1)q}(\mathbb{Z}/p, \mathbb{Z}/p)$ survives to an element ζ_n in π_*S.

In [13], it was shown that the periodic elements detected by the Adams-Novikov spectral sequence are represented in the classical Adams spectral sequence as follows:

(1) For $s \not\equiv 0 \mod p$, the first periodic elements α_s is represented by

$$\widetilde{\alpha}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

which is represented by $s a_1^{s-1} h_{1,0}$ in the May spectral sequence.

(2) For $s \not\equiv 0, 1 \mod p$, the second periodic elements β_s is represented by

$$\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

which is represented by $s(s-1) a_2^{s-2} h_{2,0} h_{1,1}$ in the May spectral sequence.

(3) For $s \not\equiv 0, 1, 2 \mod p$, the third periodic elements γ_s is represented by

$$\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

which is represented by $s(s-1)(s-2) a_3^{s-3} h_{3,0} h_{2,1} h_{1,2}$ in the May spectral sequence.

Based on the representation of the periodic maps in ASS and R. Cohen's elements ζ_n several non-trivial order p homotopy elements are detected (see [4–5, 13]). But except for a_0^r, all the non-trivial elements detected by the classical Adams spectral sequence are of order p. In this paper we will detect an order p^2 element $\widetilde{a}_p$ by the classical ASS.

Theorem 1.1 *Let p be an odd prime. Then there exists a nontrivial element $\widetilde{a}_p \in \pi_*S$ of order p^2, which is represented by $a_0^{p-2} h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-1, pq+p-2}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the Adams spectral sequence and $p \cdot \widetilde{a}_p$ is the first periodic element α_p.*

Remark 1.1 As shown in [9], when $p = 3$, the element $a_0 h_1$ converges to an order p^2 homotopy element $\alpha_{\frac{3}{2}}$ of π_*S. Thus our result generalizes the case $p = 3$ in [9] to the case of any odd prime.

This paper is organized as follows. In Section 2, we compute the generators of the E_1-term of the May spectral sequence (MSS) which converge to the E_2-term of the ASS. In Section 3, we give some higher May differentials related to Massey products. Then in Section 4 we give the proof of Theorem 1.1.

2 Computation via May Spectral Sequences

In this section we will recall the construction of cobar construction which connects the E_2-terms of the ASS and the MSS. Let $\mathcal{A}_*$ denote the dual algebra of the mod p Steenrod algebra $\mathcal{A}$. J. Milnor [10] showed that, as a Hopf algebra

$$\mathcal{A}_* = P[\xi_1, \xi_2, \cdots] \otimes E[\tau_0, \tau_1, \cdots],$$

where $P[\ ]$ is the polynomial algebra and $E[\ ]$ is the exterior algebra. The secondary degrees of ξ_i and τ_i are $2(p^i - 1)$ and $2(p^i - 1) + 1$, respectively. The coproduct $\Delta\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by

$$\Delta(\xi_n) = \xi_n \otimes 1 + 1 \otimes \xi_n + \sum_{i=1}^{n-1} \xi_{n-i}^{p^i} \otimes \xi_i,$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + 1 \otimes \tau_n + \sum_{i=0}^{n-1} \xi_{n-i}^{p^i} \otimes \tau_i.$$

Let $\varepsilon\colon \mathcal{A}_* \to \mathbb{Z}/p$ be the argumentation homomorphism and let $\overline{\mathcal{A}}_*$ =Ker ε which is called the argumentation ideal of $\mathcal{A}_*$. It then follows a bigraded cochain complex $(C^{*,*}(H^*S), \mathrm{d}) = (C^{*,*}(\mathbb{Z}/p), \mathrm{d})$, where $C^{*,*}(\mathbb{Z}/p)$ is the cobar construction with s-filtration

$$C^{s,*}(\mathbb{Z}/p) = \underbrace{\overline{\mathcal{A}}_* \otimes \cdots \otimes \overline{\mathcal{A}}_*}_{s},$$

and the differential $\mathrm{d}\colon C^{s,t}(\mathbb{Z}/p) \to C^{s+1,t}(\mathbb{Z}/p)$ is given by

$$\mathrm{d}(\alpha_1 \otimes \cdots \otimes \alpha_s) = \sum_{i=1}^{s} (-1)^{\lambda(i)+1} \alpha_1 \otimes \cdots \otimes (\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i) \otimes \cdots \otimes \alpha_s, \tag{2.1}$$

where $\lambda(i)$ is the total degree of $\alpha_1 \otimes \cdots \otimes \alpha_i'$ if $\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i = \Sigma \alpha_i' \otimes \alpha_i''$ (see [7]). For example, we have differentials $\mathrm{d}(\xi_2^{p^i}) = \xi_1^{p^{i+1}} \otimes \xi_1^{p^i}$ and $\mathrm{d}(\xi_1^{2p^i}) = 2\xi_1^{p^i} \otimes \xi_1^{p^i}$.

According to the above statements, the cohomology of $C^{*,*}(\mathbb{Z}/p)$ is

$$H^{s,t}(C^{*,*}(\mathbb{Z}/p), \mathrm{d}) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$$

which is the E_2-term of the ASS. From [6], we know that

$$\begin{aligned}
a_0 &= \{\tau_0\}, \quad h_i = \{\xi_1^{p^i}\}, \\
\widetilde{\alpha}_2 &= \{2\xi_1 \otimes \tau_1 + \xi_1^2 \otimes \tau_0\}, \\
g_i &= \Big\{\xi_2^{p^i} \otimes \xi_1^{p^i} + \frac{1}{2}\xi_1^{p^{i+1}} \otimes \xi_1^{2p^i}\Big\}, \\
k_i &= \Big\{\xi_1^{p^{i+1}} \otimes \xi_2^{p^i} + \frac{1}{2}\xi_1^{2p^{i+1}} \otimes \xi_1^{p^i}\Big\}, \\
b_i &= \Big\{\sum_{j=1}^{p-1} \binom{p}{j} \Big/ p(\xi_1^{p^i(p-j)} \otimes \xi_1^{p^i j})\Big\}
\end{aligned}$$

are generators of $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Based on [11, Theorem 3.2.5], we set a May filtration on $\mathcal{A}_*$ by $M(\tau_{i-1}) = M(\xi_i^{p^j}) = 2i - 1$. It induces a corresponding filtration

$$F^0 \subseteq F^1 \subseteq \cdots \subseteq F^{M-1} \subseteq F^M \subseteq \cdots \subseteq \mathcal{A}_*. \tag{2.2}$$

This shows that for the associated bigraded Hopf algebra

$$E^0\mathcal{A}_* = \oplus(F^M/F^{M-1}),$$

there is an isomorphism

$$E^0\mathcal{A}_* \cong E[\tau_i \mid i \geqslant 0] \otimes T[\xi_{i,j} \mid i > 0,\ j \geqslant 0],$$

where $T[\,]$ denotes the truncated polynomial algebra of height p on the indicated generators, τ_i and $\xi_{i,j}$ are the projections of τ_i and $\xi_i^{p^i}$ respectively. Applying the filtration (2.2) to the cobar construction $C^{*,*}(\mathbb{Z}/p)$, we obtain a filtration

$$F^{*,*,0} \subseteq F^{*,*,1} \subseteq \cdots \subseteq F^{*,*,M-1} \subseteq F^{*,*,M} \subseteq \cdots \subseteq C^{*,*}(\mathbb{Z}/p). \tag{2.3}$$

Then we have a tri-graded exact couple which induces the so-called May spectral sequence (MSS)

$$\{E_r^{s,t,M}, \mathrm{d}_r\} \Longrightarrow \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

where $\mathrm{d}_r\colon E_r^{s,t,M} \to E_r^{s+1,t,M-r}$ is the r-th differential of the MSS. Since the MSS converges to the E_2-term of the ASS, to show the nontriviality of the elements in the Ext-group $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ is equivalent to showing that its representation in the MSS is an infinite cycle. This is the base of proving our main theorem.

The E_0-term of the MSS is $C^{*,*}(E^0\mathcal{A}_*) = \oplus(F^{*,*,M}/F^{*,*,M})$ and the E_1-term $E_1 = H^*(E_0\mathcal{A}_*, \mathrm{d}_0)$ is isomorphic to

$$E[h_{i,j} \mid i>0, j\geqslant 0]\otimes P[b_{i,j} \mid i>0, j\geqslant 0]\otimes P[a_i \mid i\geqslant 0],$$

where

$$h_{i,j}\in E_1^{1,2(p^i-1)p^j,2i-1},\quad b_{i,j}\in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)},\quad a_i\in E_1^{1,2(p^i-1)+1,2i+1}.$$

In the filtrated cobar complexes, $h_{i,j}$, $b_{i,j}$ and a_i are represented by

$$\xi_i^{p^j},\quad \sum_{k=1}^{p-1}\binom{p}{k}\Big/ p\,\xi_i^{kp^j}\otimes\xi_i^{(p-k)p^j}\quad\text{and}\quad \tau_i$$

respectively. It is known that the generators $h_{1,i}$, $b_{1,i}$ and a_0 converge to h_i, b_i, $a_0\in\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ respectively.

In the May spectral sequence, one has

$$\mathrm{d}_r(xy)=\mathrm{d}_r(x)y+(-1)^{s+t}x\mathrm{d}_r(y)$$

for $x\in E_r^{s,t,*}$. The first May differential d_1 is given by

$$\mathrm{d}_1(h_{i,j})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j},\quad \mathrm{d}_1(a_i)=\sum_{0\leq k<i}h_{i-k,k}a_k,\quad \mathrm{d}_1(b_{i,j})=0.$$

Given an element $x\in E_1^{s,t,M}$, we define $\dim(x)=s$, $\deg(x)=t$ and $M(x)=M$, which are the homological dimension, inner degree and May filtration of x, respectively. Then we have

$$\begin{aligned}
\dim(h_{i,j}) &= \dim(a_i)=1,\\
\dim(b_{i,j}) &= 2,\\
M(h_{i,j}) &= M(a_{i-1})=2i-1,\\
M(b_{i,j}) &= (2i-1)p,\\
\deg(h_{i,j}) &= 2(p^i-1)p^j=(p^j+\cdots+p^{i+j-1})q,\\
\deg(b_{i,j}) &= 2(p^i-1)p^{j+1}=(p^{j+1}+\cdots+p^{i+j})q,\\
\deg(a_i) &= 2p^i-1=(1+\cdots+p^{i-1})q+1,\\
\deg(a_0) &= 1,
\end{aligned}$$

where $i\geqslant 1$ and $j\geqslant 0$.

Now we consider the convergence of the element $a_0^{p-2}h_1\in\mathrm{Ext}_{\mathcal{A}}^{p-1,pq+p-2}(\mathbb{Z}/p,\mathbb{Z}/p)$. Since a_0 converges to the degree p map in π_*S, in order to show that $a_0^{p-2}h_1$ converges to a nontrivial element of order p^2 in π_*S, we plan to do it in two steps: (i) To show that $a_0^{p-2}h_1$ and $a_0^{p-1}h_1$ converge to two nontrivial elements in π_*S; (ii) To show that $a_0^p h_1$ vanishes in the Adams spectral sequence. We will complete these two steps via the May spectral sequence. In what follows, we give some computation results which are needed to prove Theorem 1.1.

Theorem 2.1 *Let $r \geq 2$ and $k = 1$ or 2. Then we have*
(1) $E_1^{p-k+r+1,pq+p-k+r-1,*} = \mathbb{Z}/p\{a_0^{p-k+r-1}b_{1,0}\}$;
(2) $E_1^{p-k-r+1,pq+p-k-r+1,*} = 0$;
(3) $E_1^{p-1,pq+p-1,*} = E_1^{p-2,pq+p-2,*} = 0$;
(4) $E_1^{p-1,(q+1)(p-1),*} = \mathbb{Z}/p\{a_1^{p-1}\}$.

Proof Considering the degree of each above $E_1^{s,t,*}$-term, the possible generators must have the form

$$a_0^{x_1}a_1^{x_2}h_{1,0}^{x_3}h_{1,1}^{x_4}b_{1,0}^{x_5},$$

where $a_0 \in E_1^{1,1,1}$, $a_1 \in E_1^{1,q+1,3}$, $h_{1,0} \in E_1^{1,q,1}$, $h_{1,1} \in E_1^{1,pq,1}$, $b_0 \in E_1^{2,pq,p}$. For the total degree $t-s$, we have

$$qx_2 + (q-1)x_3 + (pq-1)x_4 + (pq-2)x_5 = t-s,$$

i.e.,

$$(x_2 + x_3 + px_4 + px_5)q - (x_3 + x_4 + 2x_5) = t-s.$$

(1) In this case, we have $t - s = pq - 2$. Then there is the following group of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + 2x_5 = p-k+r+1, \\ x_2 + x_3 + px_4 + px_5 = p, \\ x_3 + x_4 + 2x_5 = 2. \end{cases}$$

We get the solutions as

$$x_1 = p-k+r-1, \quad x_2 = x_3 = x_4 = 0, \quad x_5 = 1$$

or

$$x_1 = r-k+1, \quad x_2 = p-2, \quad x_3 = 2, \quad x_4 = x_5 = 0.$$

One gets the generators $a_0^{p-k+r-1}b_{1,0}$ and $a_0^{r-k+1}a_1^{p-2}h_{1,0}^2$. The second one is zero due to $h_{1,0}^2 = 0$.

For (2) and (3), we can obtain a similar group of equations as (1) and find out that there are no solutions. For (4), the corresponding group of equations has the solution

$$x_1 = 0, \quad x_2 = p-1, \quad x_3 = x_4 = x_5 = 0.$$

Then the generator a_1^{p-1} follows.

3 Higher May Differentials Related to Massey Products

Let us first recall the definition of Massey products.

Recalling from [8, Section 1], let R be a differential graded algebra. We assume that the differentials have degree +1 and $\mathrm{d}(x \cdot y) = \mathrm{d}(x) \cdot y + (-1)^{\deg x} x \cdot \mathrm{d}(y)$, where $\deg x$ denotes the total degree of x. If $V \in R$, we define

$$\overline{V} = (-1)^{1+\deg V}V.$$

Let V and W be two elements in R or $H(R)$. we have the following relations:

$$\mathrm{d}(\overline{V}) = -\overline{\mathrm{d}(V)}, \quad \overline{V \cdot W} = -\overline{V} \cdot \overline{W} \quad \text{and} \quad \mathrm{d}(V \cdot W) = \mathrm{d}(V) \cdot W - \overline{V} \cdot \mathrm{d}(W). \tag{3.1}$$

Let $V_1, V_2, \cdots, V_n$ be a sequence of homology classes in $H(R)$. Let $A_{i-1,i} \in R$ be a representative cycle for V_i, abbreviated $\{A_{i-1,i}\} = V_i$. Suppose that $\overline{V}_i V_{i+1} = 0$. Then there exists $A_{i-1,i+1} \in R$ such that $\mathrm{d}(A_{i-1,i+1}) = \overline{A}_{i-1,i} A_{i,i+1}$ and

$$\overline{A}_{i-1,i} A_{i,i+2} + \overline{A}_{i-1,i+1} A_{i+1,i+2}$$

is a cycle in R, and we say that its homology class belongs to $\langle V_i, V_{i+1}, V_{i+2} \rangle$.

Inductively suppose that we get $A_{i,j} \in R$ for $0 \leqslant i < j \leqslant n$ and $(i,j) \neq (0,n)$ such that

$$\{A_{i-1,i}\} = V_i, \quad \mathrm{d}(A_{i,j}) = wA_{i,j} = \sum_{k=i+1}^{j-1} \overline{A}_{i,k} A_{k,j}. \tag{3.2}$$

Then

$$\widetilde{A}_{0,n} = \sum_{k=1}^{n-1} \overline{A}_{0,k} A_{k,n} \tag{3.3}$$

is a cycle in R. We say that $\{\widetilde{A}_{0,n}\} \in \langle V_1, V_2 \cdots, V_n \rangle$.

We say that the Massey product $\langle V_1, V_2, \cdots, V_n \rangle$ is defined if there exist $A_{i,j} \in R$, $0 \leqslant i < j \leqslant n$ and $(i,j) \neq (0,n)$, which satisfies (3.2). The set of elements $\{A_{i,j}\}$ is said to be a defining system for $\langle V_1, V_2, \cdots, V_n \rangle$. We say that $\langle V_1, V_2, \cdots, V_n \rangle$ is strictly defined if each $\langle V_i, \cdots, V_j \rangle$, $1 \leqslant j-i \leqslant n-2$ is defined and contains only the zero matrix. In particular, every defined triple product is strictly defined.

Consider the Massey product in the E_1-term of the May spectral spectral sequence where $E_1^{s,t,*}$ is a differential graded algebra. One has the following result.

Theorem 3.1 *In the May spectral sequence, up to nonzero scalar there is a nontrivial May differential* $\mathrm{d}_1 \colon E_1^{p-1,(q+1)(p-1),*} \to E_1^{p,(q+1)(p-1),*}$ *given by*

$$\mathrm{d}_1\Big(\frac{1}{(p-1)!} a_1^{p-1}\Big) = \langle a_0^{p-1}, h_{1,0}, h_{1,0}, \cdots, h_{1,0} \rangle.$$

Proof According to the defining system of the Massey product $\langle a_0^{p-1}, h_{1,0}, h_{1,0}, \cdots, h_{1,0} \rangle$ in the E_1-term of the May spectral sequence (see [7–8]), we obtain $\langle a_0^{p-1}, h_{1,0}, h_{1,0} \rangle = a_0^{p-2} a_1 h_{1,0}$ from $\mathrm{d}_1(a_1) = a_0 h_{1,0}$ and $h_{1,0} \cdot h_{1,0} = 0$ in the E_1-term of the May spectral sequence. Thus in the May spectral sequence, one has

$$\mathrm{d}_1\Big(\frac{1}{2} a_0^{p-3} a_1^2\Big) = a_0^{p-2} a_1 h_{1,0} = \langle a_0^{p-1}, h_{1,0}, h_{1,0} \rangle.$$

By induction one obtains

$$\mathrm{d}_1\Big(\frac{1}{(p-1)!} a_1^{p-1}\Big) = \frac{1}{(p-2)!} a_0 a_1^{p-2} h_{1,0} = \langle a_0^{p-1}, h_{1,0}, h_{1,0}, \cdots, h_{1,0} \rangle$$

from the fact that the k-fold Massey product $\langle h_{1,0}, h_{1,0}, \cdots, h_{1,0} \rangle = 0$ for $k < p$.

4 Proof of Theorem 1.1

Before giving the proof of our main result, we first need two lemmas.

Lemma 4.1 *Let $k = 1$ or 2. Then the element $a_0^{p-k} h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-k+1,pq+p-i}(\mathbb{Z}/p, \mathbb{Z}/p)$ is nonzero.*

Proof First we know that the element $a_0^{p-k}h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-k+1,pq+p-k}(\mathbb{Z}/p,\mathbb{Z}/p)$ is represented by $a_0^{p-k}h_{1,1} \in E_1^{p-k+1,pq+p-k,*}$ in the E_1-term of the May spectral sequence. According to Theorem 2.1(3), one has $E_1^{p-k,pq+p-k,*} = 0$. It follows that $E_r^{p-k,pq+p-k,*} = 0$ for $r \geq 1$. Thus $a_0^{p-k}h_{1,1}$ can not be hit by any May differential. Then it can converge nontrivially to $a_0^{p-k}h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-k+1,pq+p-k}(\mathbb{Z}/p,\mathbb{Z}/p)$.

Lemma 4.2 *Let $k = 1$ or 2. Then* $\mathrm{Ext}_{\mathcal{A}}^{p-k+r+1,pq+p-k+r-1}(\mathbb{Z}/p,\mathbb{Z}/p) = 0$.

Proof According to Theorem 3.1, there is a May differential

$$\mathrm{d}_1\Big(\frac{1}{(p-1)!}a_1^{p-1}\Big) = \langle a_0^{p-1}, h_{1,0}, \cdots, h_{1,0}\rangle,$$

from which follows that $\langle a_0^{p-1}, h_{1,0}, \cdots, h_{1,0}\rangle = 0 \in E_p^{p,q(p-1)+p-1,*}$. Since $\langle a_0^{p-1}, h_{1,0}, \cdots, h_{1,0}\rangle$ represents $\langle a_0^{p-1}, h_0, \cdots, h_0\rangle \in \mathrm{Ext}_{\mathcal{A}}^{p,q(p-1)+p-1}(\mathbb{Z}/p,\mathbb{Z}/p)$ in the ASS, we have

$$\langle a_0^{p-1}, h_0, \cdots, h_0\rangle = 0 \in \mathrm{Ext}_{\mathcal{A}}^{p,q(p-1)+p-1}(\mathbb{Z}/p,\mathbb{Z}/p).$$

Then it follows in Ext-group of the ASS that

$$0 = \langle a_0^{p-1}, h_0, \cdots, h_0\rangle h_0 = a_0^{p-1}\langle h_0, \cdots, h_0, h_0\rangle = a_0^{p-1}b_0,$$

where $b_0 = \langle h_0, \cdots, h_0, h_0\rangle$ is the p-fold Massey product (see [11]).

Since $r \geq k$, we also have $a_0^{p-k+r-1}b_0 = 0 \in \mathrm{Ext}_{\mathcal{A}}^{p-k+r+1,pq+p-k+r-1}(\mathbb{Z}/p,\mathbb{Z}/p)$. The desired result then follows from Theorem 2.1(1).

Proof of Theorem 1.1 For $k = 1$ or 2, we have

$$a_0^{p-k}h_1 \in \mathrm{Ext}_{\mathcal{A}}^{p-k+1,pq+p-k}(\mathbb{Z}/p,\mathbb{Z}/p),$$

which is nonzero due to Lemma 4.1. From Lemma 4.2, we see that the Adams differential

$$\mathrm{d}_r\colon E_r^{p-k+1,pq+p-k} \to E_r^{p-k+r+1,pq+p-k+r-1}$$

is trivial for $r \geq 2$. Thus $a_0^{p-k}h_1$ is a permanent cycle in the ASS. From Theorem 2.1(2), we know that $E_1^{p-k-r+1,pq+p-k-r+1,*} = 0$. It follows that $\mathrm{Ext}_{\mathcal{A}}^{p-k-r+1,pq+p-k-r+1}(\mathbb{Z}/p,\mathbb{Z}/p) = 0$ and then $E_r^{p-k-r+1,pq+p-k-r+1} = 0$ in the ASS for $r \geq 2$. Thus the Adams differential

$$\mathrm{d}_r\colon E_r^{p-k-r+1,pq+p-k-r+1} \to E_r^{p-k+1,pq+p-k}$$

is also trivial. Thus $a_0^{p-k}h_1$ can not be hit by any Adams differential. It then follows that $a_0^{p-k}h_1$ is a permanent element in the ASS.

Consider the Adams-Novikov spectral sequence with E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) \Longrightarrow \pi_*S$$

that converges to π_*S. By the method of infinite descent (see [3, 11–12]), we get that

$$\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) \quad \text{with } t-s = pq-1$$

is the $\mathbb{Z}/p^2$ module generated by $\alpha_{\frac{p}{2}}$ which converges to an order p^2 homotopy element $\widetilde{a}_p$ of π_*S. According to the above result, we see that in the ASS $a_0^{p-2}h_1$ exactly represents the element $\alpha_{\frac{p}{2}}$ in the ANSS. This implies that $a_0^{p-2}h_1$ represents an order p^2 homotopy element of $\pi_{pq-1}S$.

It is known that between the Brown-Peterson spectrum and the Eilenberg-MacLane spectrum, there is the Thom map $\Phi: BP \to K\mathbb{Z}/p$ which induces $\Phi\colon BP_*BP \to \mathcal{A}_*$ and the Thom map

$$\Phi\colon \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \to \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p)$$

between the Adams-Novikov spectral sequence and the classical Adams spectral sequence. The map $\Phi\colon BP_*BP \to \mathcal{A}_*$ sends t_i and v_i to ξ_i and zero, respectively. From the definition of the first periodic element, we know that

$$\alpha_p = \frac{\eta_R(v_1^p) - v_1^p}{p} = \frac{(v_1 + pt_1)^p - v_1^p}{p} = \sum \binom{p}{j} \cdot \frac{1}{p} \cdot v_1^j p^{p-j} t_1^{p-j} + p^{p-1} t_1^p.$$

Now $p^{p-1}t_1^p$ corresponds to $\tau_0^{\otimes p-1} \otimes \xi_1^p$ which represents the filtration p element $a_0^{p-1}h_1$ in the ASS, meanwhile $v_1^{p-1}pt_1$ corresponds to $a_1^{p-1}a_0h_0$ which is a filtration $p+1$ element. It follows that the filtration p element $a_0^{p-1}h_1$ converges to α_p. Since in the ASS $a_0^{p-1}h_1$ converges to $p \cdot \widetilde{a}_p$ as what we have shown, the desired result then follows.

As we think: Except for the Thom map, there is a correspondence between the ANSS and the classical ASS where $v_i \in BP_*$ corresponds to τ_i in the cobar complex of $\mathcal{A}$. Based on this, we give a conjecture which is a general version of our results.

Conjecture In the classical ASS, $a_0^{p^n-n-1}h_n \in \mathrm{Ext}_{\mathcal{A}}^{p^n-n,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ converges to the p^{n+1} order element $\alpha_{\frac{p^n}{n+1}}$, meanwhile $a_0^{p^n-1}h_n$ converges to the first periodic element $\alpha_{p^n} = p^n \alpha_{\frac{p^n}{n+1}}$.

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