

A Boundary Schwarz Lemma for Holomorphic Mappings on the Polydisc*

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Abstract The authors prove a general Schwarz lemma at the boundary for holomorphic mappings from the polydisc to the unit ball in any dimensions. For the special case of one complex variable, the obtained results give the classic boundary Schwarz lemma.

Keywords Boundary Schwarz lemma, Holomorphic mapping, Polydisc, Unit ball

2000 MR Subject Classification 32H02, 30C80

1 Introduction

The Schwarz lemma is known as one of the most important results in complex analysis. The Schwarz-Pick lemma is a variant of the Schwarz lemma, which essentially states that a holomorphic self-map of the unit disk decreases the distance of points in the Poincaré metric. It has been generalized to the derivatives of arbitrary order in one complex variable (see [1–2]). Concerning several complex variables, Rudin [3] gave a first derivative estimate for the bounded holomorphic functions on the polydisc, which is really a precursor to Schwarz-Pick estimate in high dimensions. Knese [4] studied the extremal problem for the holomorphic functions from the polydisc to the disk. Later, Liu [5] generalized the result of [1–2] to the holomorphic mappings on the unit ball and polydisc in \mathbb{C}^n .

The Schwarz lemma at the boundary is another active topic in complex analysis (see [6]), which was applied to geometric function theory of one complex variable and several complex variables [7–10] as well as Hilbert space (see [11]). The following result is the classic boundary version of Schwarz lemma in one complex variable.

Theorem 1.1 (see [6]) *Let D be the unit disk in \mathbb{C} , and let f be the self-holomorphic mapping of D . If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$.*

It is a well-known result that there are no biholomorphic mappings between D^n and B^n . Therefore, holomorphic mappings between D^n and B^n are interesting in several complex variables. There have been many results proved about such mappings since the 1970s (see [5, 12–13]). Before giving the proof of the boundary version of Schwarz lemma for holomorphic mappings between D^n and B^N , we give some notations and definitions first.

Manuscript received February 18, 2016. Revised June 1, 2016.

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*This work was supported by the National Science Foundation of China (Nos. 11671361, 11571256).

For any $z = (z_1, \dots, z_n)^T$, $w = (w_1, \dots, w_n)^T \in \mathbb{C}^n$, the inner product and the corresponding norm are given by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$, $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$. In addition, denote $\|z\|_\infty = \max_{1 \leq j \leq n} |z_j|$. Let $D^n \subset \mathbb{C}^n$ be the polydisc in the n -dimensional complex space, which is described by $D^n = \{z \in \mathbb{C}^n \mid \|z\|_\infty < 1\}$. $\partial D^n = \{z \in \mathbb{C}^n \mid \|z\|_\infty = 1\}$ denotes the topological boundary of D^n , and the distinguished boundary of D^n is given by $T^n = \{z \in \mathbb{C}^n \mid |z_j| = 1, 1 \leq j \leq n\}$ (see [3]). Denote by $H(D^n, B^N)$ the set of all holomorphic mappings from D^n to B^N . For any $f \in H(D^n, B^N)$, we denote it by $f = (f_1, \dots, f_N)^T$ and the Jacobian matrix of f at $z \in D^n$ is given by

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j} \right)_{N \times n}.$$

For a bounded domain $V \in \mathbb{C}^n$, $C^\alpha(V)$ for $0 < \alpha < 1$ is the set of all functions f on V for which

$$\sup \left\{ \frac{|f(z) - f(z')|}{|z - z'|^\alpha} \mid z, z' \in \overline{V} \right\}$$

is finite. $C^{k+\alpha}(V)$ is the set of all functions f on V whose k^{th} order partial derivatives exist and belong to $C^\alpha(V)$ for an integer $k \geq 0$.

Remark 1.1 From the definition of the norm on D^n , the boundary points could be classified into several kinds. Let $z_0 = (z_1, \dots, z_n)^T \in \partial D^n$. If there are only r components of z_0 whose norm equals 1, then we denote the set of all this kind of boundary points by E_r , $1 \leq r \leq n$. It is easy to see that there are n different E_r for ∂D^n , i.e.,

$$\bigcup_{1 \leq r \leq n} E_r = \partial D^n.$$

Especially, $E_n = T^n$ which is the distinguished boundary of D^n .

In this paper, we study the mapping $f \in H(D^n, B^N)$ for any $n, N \geq 1$. Our main results are listed as follows.

Theorem 1.2 *Let $f \in H(D^n, B^N)$ for any $n, N \geq 1$. Given $z_0 \in \partial D^n$, assume $z_0 \in E_r$ with the first r components at the boundary of D for some $1 \leq r \leq n$. If f is $C^{1+\alpha}$ at z_0 and $f(z_0) = w_0 \in \partial B^N$, then there exist a sequence of nonnegative real numbers $\gamma_1, \dots, \gamma_r$ satisfying $\sum_{j=1}^r \gamma_j \geq 1$ and $\lambda \in \mathbb{R}$ such that*

$$\overline{J_f(z_0)}^T w_0 = \lambda \text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0) z_0, \quad (1.1)$$

where $\lambda = \frac{|1 - \overline{a}^T w_0|^2}{1 - \|a\|^2} > 0$, $a = f(0)$ and diag represents the diagonal matrix.

Remark 1.2 Especially, when $r = n$, $z_0 \in T^n$. Denote by e_i^n the i -th column of identity matrix I_n with degree n . If $z_0 \in E_r$ is given by other expression such as $z_0 = \sum_{j=1}^r e_{i_j}^n$ where i_j are different from each other and $i_j \in \{1, \dots, n\}$ for $1 \leq j \leq r$. Then from the proof of the theorem, $\text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0)$ should be replaced by the diagonal matrix M with the i_j -th row and i_j -th column element $M(i_j, i_j) = \gamma_{i_j}$ ($j = 1, \dots, r$), otherwise being 0.

For $r = 1$, we have the following corollary.

Corollary 1.1 *Let $f \in H(D^n, B^N)$ for any $n \geq 1$. Given $z_0 \in E_1 \subset \partial D^n$, if f is $C^{1+\alpha}$ at z_0 and $f(z_0) = w_0 \in \partial B^N$, then there exists a real number $\lambda \in \mathbb{R}$ such that*

$$\overline{J_f(z_0)}^T w_0 = \lambda z_0, \quad (1.2)$$

where $\lambda \geq \frac{|1-\bar{a}^T w_0|^2}{1-\|a\|^2} > 0$ and $a = f(0)$.

For $N = 1$, the following result is obtained.

Corollary 1.2 *Let $f \in H(D^n, D)$ for any $n \geq 1$. Given $z_0 \in \partial D^n$, assume $z_0 \in E_r$ with the first r components in the boundary of D for some $1 \leq r \leq n$. If f is $C^{1+\alpha}$ at z_0 and $f(z_0) = e^{i\theta} \in \partial D$ for $0 \leq \theta \leq 2\pi$. Then there exist a sequence of nonnegative real numbers $\gamma_1, \dots, \gamma_r$ satisfying $\sum_{j=1}^r \gamma_j \geq 1$ and $\lambda \in \mathbb{R}$ such that*

$$\overline{J_f(z_0)}^T e^{i\theta} = \lambda \text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0) z_0, \quad (1.3)$$

where $\lambda = \frac{|1-\bar{a}e^{i\theta}|^2}{1-|a|^2} > 0$ and $a = f(0)$.

Remark 1.3 This theorem is a general Schwarz lemma at the boundary for holomorphic mappings from the polydisc to the unit ball in any dimensions. For $n = N = 1$, Corollary 1.1 gives Theorem 1.1 in [6]. The smooth condition of f is $C^{1+\alpha}$ at z_0 here.

The Kobayashi distances for the polydisc in \mathbb{C}^n is given as follows (see [14]):

$$K_{D^n}(z, w) = \frac{1}{2} \log \frac{1 + \|\varphi_z(w)\|_\infty}{1 - \|\varphi_z(w)\|_\infty}, \quad z, w \in D^n, \quad (1.4)$$

where $\varphi_z(w)$ is the automorphism of D^n given by

$$\varphi_z(w) = \left(\frac{w_1 - z_1}{1 - \bar{z}_1 w_1}, \dots, \frac{w_n - z_n}{1 - \bar{z}_n w_n} \right)^T.$$

Meanwhile, the Kobayashi distances for the unit ball in \mathbb{C}^N could be expressed by

$$K_{B^N}(z, w) = \frac{1}{2} \log \frac{1 + \|\phi_z(w)\|}{1 - \|\phi_z(w)\|}, \quad z, w \in B^N, \quad (1.5)$$

where $\phi_z(w)$ is the automorphism of B^N , and

$$\phi_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - \langle w, z \rangle}, \quad z, w \in B^N$$

with P_z being the orthogonal projection of \mathbb{C}^n by

$$P_z(w) = \frac{\langle w, z \rangle}{\|z\|^2} z, \quad \text{if } z \neq 0,$$

and $Q_z(w) = w - P_z(w)$, as well as $s_z = \sqrt{1 - \|z\|^2}$ (see [15]). It is found that $\phi_z(0) = z$, $\phi_z(z) = 0$ and $\phi_z = \phi_z^{-1}$.

It is the fact that the Kobayashi distance non-increases under holomorphic mappings (see [16]). Consider the mapping $f \in H(D^n, B^N)$ and $f(0) = 0$. Then from (1.4)–(1.5), $K_{B^N}(f(0), f(w)) \leq K_{D^n}(0, w)$, i.e.,

$$\frac{1}{2} \log \frac{1 + \|f(w)\|}{1 - \|f(w)\|} \leq \frac{1}{2} \log \frac{1 + \|w\|_\infty}{1 - \|w\|_\infty}.$$

Since $t \rightarrow \frac{1}{2} \log \frac{1+t}{1-t}$ is an increasing function for $t \in [0, 1)$, we have $\|f(w)\| \leq \|w\|_\infty$. Therefore, the following lemma is obtained which would play an important role in the proof of main results.

Lemma 1.1 Assume $f \in H(D^n, B^N)$ for any $n, N \geq 1$. If $f(0) = 0$, then

$$\|f(w)\| \leq \|w\|_\infty, \quad w \in D^n.$$

2 Proof of Theorem 1.2

We will prove the theorem in the following five steps.

Step 1 Let $z_0 \in E_r \subset \partial D^n$. Without loss of generality, we assume $z_0 = \sum_{i=1}^r e_i^n$. f is $C^{1+\alpha}$ in a neighborhood V of z_0 . Moreover, we assume $f(0) = 0$ and $f(z_0) = w_0 = e_1^N \in \partial B^N$.

Let $p = z_0$, $q_j = -\sum_{i=1}^r e_i^n + ike_j^n$ for $1 \leq j \leq r$. Then $p + tq_j = (1-t)z_0 + ikte_j^n$ for $t \in \mathbb{R}$. $\|p + tq_j\|_\infty < 1 \Leftrightarrow |1-t+ikt| < 1$ and $|1-t| < 1 \Leftrightarrow 0 < t < \frac{2}{1+k^2}$, which means that for a given $k \in \mathbb{R}$ when $t \rightarrow 0^+$, $p + tq_j \in D^n \cap V$. For such t , taking the Taylor expansion of $f(p + tq_j)$ at $t = 0$, we have

$$f(p + tq_j) = w_0 + J_f(z_0)q_j t + O(t^{1+\alpha}).$$

By Lemma 1.1,

$$\|f(p + tq_j)\|^2 = \|w_0 + J_f(z_0)q_j t + O(t^{1+\alpha})\|^2 \leq \|p + tq_j\|_\infty^2,$$

i.e.,

$$1 + 2\operatorname{Re}(\overline{w_0}^T J_f(z_0)q_j t) + O(t^{1+\alpha}) \leq 1 - 2t + O(t^2).$$

Substitute $w_0 = e_1^N$ and let $t \rightarrow 0^+$, we have

$$\operatorname{Re}\left(\overline{e_1^N}^T J_f(z_0)\left(-\sum_{i=1}^r e_i^n + ike_j^n\right)\right) \leq -1,$$

i.e.,

$$\operatorname{Re}\left(-\sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} + ik \frac{\partial f_1(z_0)}{\partial z_j}\right) \leq -1.$$

Let $\frac{\partial f_1(z_0)}{\partial z_j} = \operatorname{Re} \frac{\partial f_1(z_0)}{\partial z_j} + i \operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_j}$. Then from the above inequality, one gets

$$-\operatorname{Re} \sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} - k \operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_j} \leq -1,$$

i.e.,

$$-k \operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_j} \leq \operatorname{Re} \sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} - 1. \quad (2.1)$$

Since (2.1) is valid for any $k \in \mathbb{R}$, so that

$$\operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_j} = 0, \quad 1 \leq j \leq r,$$

which gives

$$\operatorname{Re} \sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} \geq 1 \quad (2.2)$$

and

$$\frac{\partial f_1(z_0)}{\partial z_j} = \operatorname{Re} \frac{\partial f_1(z_0)}{\partial z_j}, \quad 1 \leq j \leq r. \quad (2.3)$$

Step 2 Let $p = z_0$, $q_j = -\sum_{i=1}^r e_i^n + ke_j^n$ for $1 \leq j \leq r$ and $k \leq 0$. Then $p + tq_j = (1-t)z_0 + kte_j^n$ for $t \in \mathbb{R}$. $\|p + tq_j\|_\infty < 1 \Leftrightarrow |1-t+kt| < 1$ and $|1-t| < 1 \Leftrightarrow 0 < t < \frac{2}{1-k}$, which means that for any given $k \leq 0$, there are $t \rightarrow 0^+$ such that $p + tq_j \in D^n \cap V$. Taking the Taylor expansion of $f(p + tq_j)$ at $t = 0$, we have

$$\|f(p + tq_j)\|^2 = \|w_0 + J_f(z_0)q_j t + O(t^{1+\alpha})\|^2 \leq \|p + tq_j\|_\infty^2,$$

i.e.,

$$1 + 2\operatorname{Re}(\overline{w_0}^T J_f(z_0)q_j t) + O(t^{1+\alpha}) \leq 1 - 2t + t^2.$$

Substituting $w_0 = e_1^N$ and letting $t \rightarrow 0^+$, we have

$$\operatorname{Re}\left(\overline{e_1^N}^T J_f(z_0)\left(-\sum_{i=1}^r e_i^n + ke_j^n\right)\right) \leq -1,$$

i.e.,

$$\operatorname{Re}\left(-\sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} + k \frac{\partial f_1(z_0)}{\partial z_j}\right) \leq -1.$$

From (2.3), one gets

$$-\sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} + k \frac{\partial f_1(z_0)}{\partial z_j} \leq -1,$$

which is equal to

$$k \frac{\partial f_1(z_0)}{\partial z_j} \leq \sum_{i=1}^r \frac{\partial f_1(z_0)}{\partial z_i} - 1.$$

The right-hand side of the above inequality is nonnegative from (2.2)–(2.3). Since $k \leq 0$ is arbitrary, it is obtained that

$$\frac{\partial f_1(z_0)}{\partial z_j} \geq 0, \quad 1 \leq j \leq r. \quad (2.4)$$

Step 3 Let $q_l = -\sum_{j=1}^r e_j^n + ike_l^n$ for $r+1 \leq l \leq n$ and $k \neq 0 \in \mathbb{R}$. Then $p + tq_l = (1-t)\sum_{j=1}^r e_j^n + ikte_l^n$ for $t \in \mathbb{R}$. $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1-t| < 1$ and $|ikt|^2 < 1 \Leftrightarrow 0 < t < \min\{\frac{1}{|k|}, 2\}$. Therefore, given a $k(\neq 0) \in \mathbb{R}$, there exist $t \rightarrow 0^+$ such that $p + tq \in D^n \cap V$. Similarly, taking the Taylor expansion of $f(p + tq_l)$ at $t = 0$, we have

$$f(p + tq_l) = w_0 + J_f(z_0)q_l t + O(t^{1+\alpha}).$$

By Lemma 1.1,

$$\|f(p + tq_l)\|^2 = \|w_0 + J_f(z_0)q_l t + O(t^{1+\alpha})\|^2 \leq \|p + tq_l\|_\infty^2,$$

i.e.,

$$1 + 2\operatorname{Re}\left(\overline{w_0}^\top J_f(z_0)\left(-\sum_{j=1}^r e_j^n + ik e_l^n\right)t\right) + O(t^{1+\alpha}) \leq 1 - 2t + O(t^2).$$

Substituting $w_0 = e_1^N$ and letting $t \rightarrow 0^+$, we have

$$\operatorname{Re}\left(\overline{e_1^N}^\top J_f(z_0)\left(-\sum_{j=1}^r e_j^n + ik e_l^n\right)\right) \leq -1,$$

i.e.,

$$\operatorname{Re}\left(-\sum_{j=1}^r \frac{\partial f_1(z_0)}{\partial z_j} + ik \frac{\partial f_1(z_0)}{\partial z_l}\right) \leq -1.$$

From the above inequality as well as inequality (2.3), one has

$$-k \operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_l} \leq \sum_{j=1}^r \frac{\partial f_1(z_0)}{\partial z_j} - 1.$$

Since the right-hand side of the above inequality is a nonnegative scalar, with the similar argument to Step 1, we have

$$\operatorname{Im} \frac{\partial f_1(z_0)}{\partial z_l} = 0, \quad r+1 \leq l \leq n.$$

Meanwhile, if we assume $p = z_0$, $q = -\sum_{j=1}^r e_j^n + k e_l^n$ for $r+1 \leq l \leq n$ and any $k \neq 0 \in \mathbb{R}$. It is easy to find

$$\operatorname{Re} \frac{\partial f_1(z_0)}{\partial z_l} = 0, \quad r+1 \leq l \leq n$$

as well. Therefore,

$$\frac{\partial f_1(z_0)}{\partial z_l} = 0, \quad r+1 \leq l \leq n. \quad (2.5)$$

As a result of (2.2) and (2.4)–(2.5), we have

$$\overline{J_f(z_0)}^\top w_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) z_0 \quad (2.6)$$

for $w_0 = e_1^N$, $z_0 = \sum_{j=1}^r e_j^n$ and $\lambda_j = \frac{\partial f_1(z_0)}{\partial z_j} \geq 0$ with $\sum_{j=1}^r \lambda_j \geq 1$ and $1 \leq j \leq r$.

Step 4 Now let $z_0 \in E_r$ be any given point at ∂D^n with the first r components in the boundary of D , i.e., z_0 is not necessary $\sum_{j=1}^r e_j^n$. Then there exists a special kind of diagonal unitary matrix U_{z_0} such that $U_{z_0}(z_0) = \sum_{j=1}^r e_j^n$. Assume $f(0) = 0$, $f(z_0) = w_0$ and w_0 is not necessary e_1^N at ∂B^N . Similarly, there is a U_{w_0} such that $U_{w_0}(w_0) = e_1^N$. Denote

$$g(z) = U_{w_0} \circ f \circ \overline{U_{z_0}}^\top.$$

Then $g(0) = 0$, $g\left(\sum_{j=1}^r e_j^n\right) = e_1^N$. Moreover,

$$J_g(z) = U_{w_0} J_f(\overline{U_{z_0}}^\top z) \overline{U_{z_0}}^\top. \quad (2.7)$$

From Steps 2 and 3, we have

$$\overline{J_g \left(\sum_{j=1}^r e_j^n \right)}^T e_1^N = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \sum_{j=1}^r e_j^n,$$

where $\lambda_j = \frac{\partial g_1(z_0)}{\partial z_j} \geq 0$ with $\sum_{j=1}^r \lambda_j \geq 1$ and $1 \leq j \leq r$, which is equal to

$$\overline{U_{w_0} J_f \left(\overline{U_{z_0}}^T \sum_{j=1}^r e_j^n \right) \overline{U_{z_0}}^T}^T e_1^N = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \sum_{j=1}^r e_j^n,$$

i.e.,

$$\overline{U_{z_0} J_f(z_0)}^T \overline{U_{w_0}}^T e_1^N = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \sum_{j=1}^r e_j^n.$$

Multiplying $\overline{U_{z_0}}^T$ at both sides of the above equation gives

$$\overline{U_{z_0}}^T \overline{U_{z_0} J_f(z_0)}^T \overline{U_{w_0}}^T e_1^N = \overline{U_{z_0}}^T \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \sum_{j=1}^r e_j^n.$$

Since $\overline{U_{z_0}}^T$ is also a diagonal matrix, we have

$$\overline{U_{z_0}}^T \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \overline{U_{z_0}}^T,$$

and therefore,

$$\overline{J_f(z_0)}^T w_0 = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \overline{U_{z_0}}^T \sum_{j=1}^r e_j^n = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0) z_0, \quad (2.8)$$

where $\lambda_j = \frac{\partial g_1(z_0)}{\partial z_j} \geq 0$ with $\sum_{j=1}^r \lambda_j \geq 1$ and $1 \leq j \leq r$.

Step 5 Let $f(z_0) = w_0$ with $z_0 \in \partial D^n$, $w_0 \in \partial B^N$. If $f(0) = a \neq 0$, then we use the automorphism of B^N to get the result. Assume that $\phi_a(w)$ is an automorphism of B^N such that $\phi_a(a) = 0$. Then $\phi_a(w_0) \in \partial B^N$ as well. With similar analysis to Step 3, there exists a U_{ϕ_a} such that $U_{\phi_a}(\phi_a(w_0)) = w_0$. Let

$$h = U_{\phi_a} \circ \phi_a \circ f.$$

Then $h(0) = 0$, $h(z_0) = w_0$. As a result from Step 4, there is a sequence of real numbers $\gamma_j \geq 0$ and $\sum_{j=1}^r \gamma_j \geq 1$ such that

$$\overline{J_h(z_0)}^T w_0 = \text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0) z_0.$$

According to the expression of h , it is obtained that

$$\overline{J_h(z_0)}^T w_0 = \overline{U_{\phi_a} J_{\phi_a}(w_0) J_f(z_0)}^T w_0 = \overline{J_f(z_0)}^T \overline{J_{\phi_a}(w_0)}^T \overline{U_{\phi_a}}^T w_0. \quad (2.9)$$

Since $U_{\phi_a}(\phi_a(w_0)) = w_0$, we have $\overline{U_{\phi_a}}^T w_0 = \phi_a(w_0)$. From the expression of the automorphism ϕ_a given by [15], we have the following equality:

$$\overline{J_{\phi_a}(w_0)}^T \overline{U_{\phi_a}}^T w_0 = \overline{J_{\phi_a}(w_0)}^T \phi_a(w_0) = \frac{1 - \|a\|^2}{|1 - \bar{a}^T w_0|^2} w_0.$$

Therefore, combining with (2.9) we get

$$\overline{J_f(z_0)}^T \frac{1 - \|a\|^2}{|1 - \bar{a}^T w_0|^2} w_0 = \text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0) z_0.$$

Consequently,

$$\overline{J_f(z_0)}^T w_0 = \lambda \text{diag}(\gamma_1, \dots, \gamma_r, 0, \dots, 0) z_0, \quad (2.10)$$

where $\lambda = \frac{|1 - \bar{a}^T w_0|^2}{1 - \|a\|^2} > 0$, $\sum_{j=1}^r \gamma_j \geq 1$ and $a = f(0)$.

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