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The Iteration Formulae of the Maslov-Type Index Theory in Weak Symplectic Hilbert Space*

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Abstract The authors prove a splitting formula for the Maslov-type indices of symplectic paths induced by the splitting of the nullity in weak symplectic Hilbert space. Then a direct proof of the iteration formulae for the Maslov-type indices of symplectic paths is given.

Keywords Maslov-type index, Positive path, Iteration formula
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1 Introduction

Since the pioneer work [3] on iteration of closed geodesics, the theory was generalized to convex Hamiltonian case (see [5]), to symplectic paths (see [13]), and to brake orbits (see [11]). The iteration theory for symplectic paths turns out to be extremely useful in the study of the periodical solutions of the finite dimensional Hamiltonian systems [5, 7–8, 11–12, 14–17].

All the known proofs of the iteration formulae follow the following line. Firstly, they give the relation between the (relative) Morse index for a certain differential operator and the Maslov-type index of its fundamental solution. Then they show that such an operator is in diagonal form under the iteration with respect to a certain orthogonal decomposition of the subspaces. The iteration formulae then follow from the above argument.

In this paper, we shall give a direct proof of the iteration formulae. The idea is that each splitting formula of the nullity will induce a splitting formula of the Maslov-type indices. The iteration formula then follows from the homotopy invariance of the Maslov-type indices.

Our method has three advantages. Firstly, we do not require that the symplectic path starts from the identity. So we can treat the iteration theory for symplectic paths in symplectic Hilbert spaces. It will be used in the study of the periodical solutions of the infinite dimensional Hamiltonian systems. Secondly, we do not need the orthogonality of the decomposition. So we can get more general formulae in some cases. It can be applied to non-contractible periodical solutions of Hamiltonian systems. Finally, our results apply to symplectic paths on weak symplectic Hilbert space.

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2 A Splitting Formula of the Maslov-Type Index Induced by the Splitting of the Nullity

In this paper, we denote by $\dim V$ the complex dimension of a complex vector space V. A pair of complex linear subspaces (λ, μ) of V is called Fredholm if $\dim(\lambda \cap \mu)$ and $\dim V/(\lambda + \mu)$ are finite. The index of the pair (λ, μ) in V is defined as

$$\operatorname{index}(\lambda, \mu) = \dim(\lambda \cap \mu) - \dim V/(\lambda + \mu). \tag{2.1}$$

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with a bounded injective map J such that $J^* = -J$. Then we have the symplectic structure ω on H defined by $\omega(x, y) := \langle Jx, y \rangle$ for each $x, y \in H$. Since J^* is injective, $\operatorname{im} J$ is dense in H. The operator J^{-1} on H is a closed operator with dense domain $\operatorname{im} J$. The annihilator of a linear subspace λ of H is defined by

$$\lambda^\omega := \{y \in X,\ \omega(x, y) = 0 \text{ for all } x \in \lambda\}.$$

The subspace λ is called a Lagrangian subspace of H if $\lambda = \lambda^\omega$. We denote by $\mathcal{L}(H)$ the set of the Lagrangian subspaces of H. We define

$$\mathcal{FL}(H) := \{(\lambda, \mu) \in (\mathcal{L}(H))^2,\ (\lambda, \mu) \text{ is Fredholm}\}, \tag{2.2}$$

$$\mathcal{FL}_k(H) := \{(\lambda, \mu) \in \mathcal{FL}(H),\ \operatorname{index}(\lambda, \mu) = k\}. \tag{2.3}$$

Denote by $\mathrm{GL}(H)$, $\mathcal{B}(H)$, $\mathrm{S}(H)$, $\mathrm{P}(H)$ and $\mathrm{U}(H)$ the set of bounded invertible operators, bounded operators, bounded self-adjoint operators, positive definite operators and unitary operators on H respectively. The symplectic group $\mathrm{Sp}(H)$ and the symplectic Lie algebra $\mathrm{sp}(H)$ are defined as

$$\mathrm{Sp}(H) := \{M \in \mathrm{GL}(H),\ M^*JM = J\}, \tag{2.4}$$

$$\mathrm{sp}(H) := \{M \in \mathcal{B}(H),\ M^*J + JM = 0\}, \tag{2.5}$$

respectively. Then we have $J \in \mathrm{sp}(H)$ and $\mathrm{e}^{Js} \in \mathrm{Sp}(H)$ for $s \in \mathbb{R}$. The path e^{Js}, $s \in \mathbb{R}$ is crucial in our main result, Theorem 2.2 below. So our method does not work in the general symplect Banach space case. For each $M \in \mathcal{B}(H)$ such that $M^*JM = J$, the operator M is injective. If $\dim H$ is infinite and J is invertible, there exists an $M \in \mathcal{B}(H) \setminus \mathrm{GL}(H)$ such that $M^*JM = J$. For each $M \in \mathrm{Sp}(H)$, we have $M^{-1} = J^{-1}M^*J$ and $\operatorname{im}(M^*J) \subset \operatorname{im} J$. Let $z \in \mathbb{C}$ and $|z| = 1$. Then we have $z\mathrm{I}_H \in \mathrm{Sp}(H)$, where I_H denotes the identity map on H. The fact is used in Definition 2.1 below. So we use the concept of complex symplectic Hilbert space instead of the real one.

Let $\lambda(s)$, $s \in (-\varepsilon, \varepsilon)$ be a path in $\mathcal{L}(H)$ differentiable at $s = 0$ and $\lambda(0)$ be an L-complemented Lagrangian subspace, i.e., there exists a $\mu \in \mathcal{L}(H)$ such that $H = \lambda(0) \oplus \mu$. If J is an isomorphism, each Lagrangian subspace of H is an L-complemented Lagrangian subspace. Define the form $q(\lambda, 0)$ on $\lambda(0)$ by

$$q(\lambda, 0)(x, y) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \omega(x, y(s)), \tag{2.6}$$

where $x, y \in \lambda(0)$, $y(s) \in \lambda(s)$ and $y(s) - y \in \mu$. Then the form $q(\lambda, 0)$ is independent of the choices of $\mu \in \mathcal{L}(H)$ with $H = \lambda(0) \oplus \mu$. The path λ is said to be (semi-)positive (see [1, Definition 3.2.8]) if $q(\lambda, s)$ is (semi-)positive definite for each $s \in (-\varepsilon, \varepsilon)$.

Lemma 2.1 *Let $\gamma(s)$, $s \in (-\varepsilon, \varepsilon)$ be a path in* $\mathrm{Sp}(H)$ *differentiable at $s = 0$ with $\gamma(0) = \mathrm{I}_H$ and $\lambda(0)$ be an L-complemented Lagrangian subspace. Set $\lambda(s) := \gamma(s)\lambda(0)$ for each $s \in (-\varepsilon, \varepsilon)$. Then we have*

$$q(\lambda, 0)(x, y) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \omega(x, \gamma(s)y), \tag{2.7}$$

where $x, y \in \lambda(0)$.

Proof Let $\mu \in \mathcal{L}(H)$ be such that $H = \lambda(0) \oplus \mu$. Let $\gamma(s)$ be of the form

$$\gamma(s) = \begin{pmatrix} A(s) & B(s) \\ C(s) & D(s) \end{pmatrix}.$$

For $x, y \in \lambda(0)$, let $y(s) \in \lambda(s)$ be such that $y(s) - y \in \mu$. Then we have $y(s) = y + C(s)A(s)^{-1}y$, and $\omega(x, y(s) - \gamma(s)y) = \omega(x, C(s)(A(s)^{-1} - \mathrm{I}_{\lambda(0)})y)$. By (2.6), our result (2.7) holds.

Set $X := H \times H$. We define the symplectic structure of X by

$$\widetilde{\omega}(v, w) = \langle \widetilde{J}v, w \rangle, \quad \forall v, w \in X, \tag{2.8}$$

where $\widetilde{J} = (-J) \oplus J$. For any $M \in \mathrm{Sp}(H)$, we have $X = \mathrm{Gr}(M) \oplus \mathrm{Gr}(-M)$, where

$$\mathrm{Gr}(M) := \{(x, Mx),\ x \in H\}.$$

By [2, Lemma 4], $\mathrm{Gr}(M)$ is an L-complemented Lagrangian subspace of X.

Definition 2.1 (see [19, Definition 4.6]) *Let (H, ω) and $(X, \widetilde{\omega})$ be symplectic Hilbert spaces as defined above. Let V be a Lagrangian subspace of X. Let $\gamma(t)$, $t \in [a, b]$ be a path in* $\mathrm{Sp}(H)$. *Assume that $(\mathrm{Gr}(\gamma(t)), V)$ is a Fredholm pair for each $t \in [a, b]$. Denote by* $\mathrm{Mas}\{\cdot, \cdot\}$ *the Maslov index for the path of Fredholm pairs of Lagrangian subspaces in X with index* 0 *defined by* [1, *Definition* 3.1.4]. *The Maslov-type index $i_V(\gamma)$ is defined by*

$$i_V(\gamma) = \mathrm{Mas}\{\mathrm{Gr}(\gamma), V\}. \tag{2.9}$$

For each $M \in \mathrm{Sp}(H)$, we define the nullity $\nu_V(M)$ and co-nullity $\widetilde{\nu}_V(M)$ by

$$\nu_V(M) = \dim(\mathrm{Gr}(M) \cap V), \quad \widetilde{\nu}_V(M) = \dim(X/(\mathrm{Gr}(M) + V)). \tag{2.10}$$

If $N \in \mathrm{Sp}(H)$ and $V = \mathrm{Gr}(N)$, we define $i_N(\gamma) := i_V(\gamma)$, $\nu_N(M) := \nu_V(M)$ and $\widetilde{\nu}_N(M) := \widetilde{\nu}_V(M)$. If $z \in \mathbb{C}$, $|z| = 1$ and $V = \mathrm{Gr}(z\mathrm{I}_H)$, we define $i_z(\gamma) := i_V(\gamma)$, $\nu_z(M) := \nu_V(M)$ and $\widetilde{\nu}_z(M) := \widetilde{\nu}_V(M)$.

By [19, Lemma 4.4], for each $N \in \mathrm{Sp}(H)$ and $\gamma \in C([a, b], \mathrm{Sp}(H))$, we have

$$i_N(\gamma) = i_1(\gamma N^{-1}) = i_1(N^{-1}\gamma). \tag{2.11}$$

By [2, Proposition 1], for each $V \in \mathcal{L}(X)$ and $M \in \mathrm{Sp}(H)$, we have

$$\nu_V(M) \le \widetilde{\nu}_V(M). \tag{2.12}$$

Definition 2.2 (see [3, Definition 3.1]) *Let (H, ω) be a symplectic Hilbert space as defined above. A C^1 path $\gamma : [a, b] \to \mathrm{Sp}(H)$ is called a positive path if $\dot{\gamma}(t)\gamma(t)^{-1} \in K$ for each $t \in [a, b]$, where we denote by $\dot{\gamma} := \frac{\mathrm{d}\gamma}{\mathrm{d}t}$, and the cone K is defined by*

$$K = \{A \in \mathcal{B}(H),\ -JA = A^*J > 0\}.$$

By [4, Lemma 3.1], the path γ is positive if and only if $\mathrm{Gr}(\gamma)$ is positive.

The following lemma is our key observation.

Lemma 2.2 *Let (H,ω) be a symplectic Hilbert space as defined above. Let $\gamma_1(t)$, $\gamma_2(t)$ with $t\in[a,b]$ be two positive paths in $\mathrm{Sp}(H)$. Then $\gamma(t):=\gamma_1(t)\gamma_2(t),\ t\in[a,b]$ is a positive path.*

Proof We have

$$\begin{aligned}-J\dot\gamma\gamma^{-1}&=-J(\dot\gamma_1\gamma_2+\gamma_1\dot\gamma_2)(\gamma_1\gamma_2)^{-1}\\&=-J\dot\gamma_1\gamma_1^{-1}+(\gamma_1^*)^{-1}(-J\dot\gamma_2\gamma_2^{-1})(\gamma_1)^{-1}.\end{aligned}$$

Since γ_1, γ_2 are both positive paths, $-J\dot\gamma_1\gamma_1^{-1}$ and $(\gamma_1^*)^{-1}(-J\dot\gamma_2\gamma_2^{-1})(\gamma_1)^{-1}$ are both positive definite. So $-J\dot\gamma\gamma^{-1}$ is positive definite and the lemma follows.

We need a special case of [1, Theorem 3.2.12].

Theorem 2.1 *Let (H,ω) and $(\widetilde H,\widetilde\omega)$ be two symplectic Hilbert space. For $0\le a\le\delta$, $\delta>0$, we are given continuous two-parameter families*

$$\{(\lambda(s,a),\mu(s))\in(\mathcal{L}(H,\omega))^2\}\ \textit{and}\ \{(\widetilde\lambda(s,a),\widetilde\mu(s))\in(\mathcal{L}(\widetilde H,\widetilde\omega))^2\}. \tag{2.13}$$

We assume

$$(\lambda(s,0),\mu(s))\in\mathcal{FL}_0(H)\ \textit{and}\ (\widetilde\lambda(s,0),\widetilde\mu(s))\in\mathcal{FL}_0(\widetilde H), \tag{2.14}$$

$$\{\lambda(s,a)\}\ \textit{differentiable in } a \textit{ and semi-positive for fixed } s, \tag{2.15}$$

$$\{\widetilde\lambda(s,a)\}\ \textit{differentiable in } a \textit{ and positive for fixed } s, \tag{2.16}$$

$$\dim(\lambda(s,a)\cap\mu(s))-\dim(\widetilde\lambda(s,a)\cap\widetilde\mu(s))=c(s). \tag{2.17}$$

Then we have

$$\mathrm{Mas}\{\lambda(s,0),\mu(s);\omega\}=\mathrm{Mas}\{\widetilde\lambda(s,0),\widetilde\mu(s);\widetilde\omega\}. \tag{2.18}$$

The main result of the paper is as follows. Note that the path Me^{Js}, $s\in\mathbb{R}$ is positive for each $M\in\mathrm{Sp}(H)$.

Theorem 2.2 (Splitting of the Maslov-Type Index Induced by Splitting of the Nullity) *Let (H,ω) and $(X,\widetilde\omega)$ be symplectic Hilbert spaces as defined above. Let $f_j:\mathrm{Sp}(H)\to\mathrm{Sp}(H)$ $(j=1,\cdots,k)$ be a family of C^1 maps, where k is a positive integer. Assume that there are Lagrangian subspaces $\{V_j\}_{j=0,\cdots,k}$ of X such that the following hold for each $M\in\mathrm{Sp}(H)$:*

(i) *The pair $\big(\mathrm{Gr}\big(\prod_{1\le j\le k}f_k(M)\big),V_0\big)$ is a Fredholm pair with index 0 if and only if all pairs $(\mathrm{Gr}(f_j(M)),V_j)$ are Fredhlom pairs with index 0, and*

$$\nu_{V_0}\Big(\prod_{1\le j\le k}f_j(M)\Big)=\sum_{1\le j\le k}\nu_{V_j}(f_j(M)). \tag{2.19}$$

(ii) *$f_j(e^{Js}M)$, $s\in\mathbb{R}$ is positive for each $j=1,\cdots,k$.*

Let $\gamma\in C([a,b],H)$ such that the pair $\big(\mathrm{Gr}\big(\prod_{1\le j\le k}f_j(\gamma(t))\big),V_0\big)$ is a Fredholm pair with index 0 for each $t\in[0,1]$. Then we have

$$i_{V_0}\Big(\prod_{1\le j\le k}f_j(\gamma)\Big)=\sum_{1\le j\le k}i_{V_j}(f_j(\gamma)). \tag{2.20}$$

Proof Let $X := X$ and $\widetilde{X} := X^k$. For each $(s,t) \in \mathbb{R} \times [0,1]$, set

$$\lambda(t,s) := \mathrm{Gr}\Big(\prod_{1\le j\le k} f_j(\mathrm{e}^{Js}\gamma(t))\Big), \quad \mu(t) := V_0,$$
$$\widetilde{\lambda}(t,s) := \prod_{1\le j\le k} \mathrm{Gr}(f_j(\mathrm{e}^{Js}\gamma(t))), \quad \widetilde{\mu}(t) := \prod_{1\le j\le k} V_j.$$

By (i), for each $t \in [0,1]$, there hold that the pair $(\lambda(t,0),\mu(t))$ is a Fredholm pair of Lagrangian subspace in X, $(\widetilde{\lambda}(t,0),\widetilde{\mu}(t))$ is a Fredholm pair of Lagrangian subspace in $\widetilde{X}$, and $\dim(\lambda(t,s)\cap \mu(t)) = \dim(\widetilde{\lambda}(t,s)\cap\widetilde{\mu}(t))$. By (ii) and Lemma 2.2, the two paths $\lambda(t,\cdot)$ and $\widetilde{\lambda}(t,\cdot)$ are both positive for each fixed t. By [1, Theorem 2.2.1.c], the Maslov index is additive under direct sum. We have

$$\mathrm{Mas}\{\widetilde{\lambda}(\cdot,0),\widetilde{\mu};\widetilde{X}\} = \sum_{1\le j\le k}\mathrm{Mas}\{\mathrm{Gr}(f_j(\cdot,0)),V_j\}.$$

By Theorem 2.1, we have

$$\begin{aligned} i_{V_0}\Big(\prod_{1\le j\le k} f_j(\gamma)\Big) &= \mathrm{Mas}\{\lambda(\cdot,0),\mu;X\} = \mathrm{Mas}\{\widetilde{\lambda}(\cdot,0),\widetilde{\mu};\widetilde{X}\} \\ &= \sum_{1\le j\le k}\mathrm{Mas}\{\mathrm{Gr}(f_j(\cdot,0)),V_j\} = \sum_{1\le j\le k} i_{V_j}(f_j(\gamma)). \end{aligned}$$

Remark 2.1 Each pair of Lagrangian subspaces of a finite dimensional symplectic space is a Fredhlom pair with index 0. We shall choose suitable functions $\{f_i\}$ to get some classical iteration formulae for the symplectic paths in Section 4.

3 The Global Structure of Sp(H) When ω is Strong

Let (H,ω) be the symplectic Hilbert space defined in Section 2. Assume that ω is strong, i.e., J is an isomorphism. In this section we give the global structure of $\mathrm{Sp}(H)$.

Lemma 3.1 *Let (H,ω,J) be a strong symplectic Hilbert space. Set $J_1 := (-J^2)^{-\frac{1}{2}}J$ and $\omega_1(x,y) := \langle J_1x,y\rangle$ for each $x,y\in H$. Then the following hold:*

(a) *(H,ω_1,J_1) is a strong symplectic Hilbert space, $J_1^2 = -\mathrm{I}_H$, $J_1^* = -J_1$.*

(b) *$(-J^2)^{\frac{1}{4}} : (H,\omega)\to(H,\omega_1)$ is symplectic.*

(c) *We have a homeomorphism*

$$\varphi : \mathrm{Sp}(H,\omega)\to\mathrm{Sp}(H,\omega_1)$$

defined by $\varphi(M) = (-J^2)^{\frac{1}{4}}M(-J^2)^{\frac{1}{4}}$ for each $M\in\mathrm{Sp}(H,\omega)$.

Proof Direct computation. Note that $J(-J^2)^t = (-J^2)^tJ$ for each $t\in\mathbb{R}$.

By the lemma above, we can assume that $J^2 = -\mathrm{I}_H$. By Kuiper's theorem, $\mathrm{U}(H)$ is contractible if $\dim H = +\infty$. By the method similar to that in [14, Section 1.1, Section 2.2], we get the following Proposition 3.1. Here we give a short proof.

Proposition 3.1 *Let (H,ω) be a symplectic Hilbert space defined in Section 2. Assume that $J^2 = -\mathrm{I}_H$. Set $H^{\pm} := \dim\ker(J\mp\sqrt{-1})$. Then $H = H^+\oplus H^-$.*

(a) *Each* $M \in \mathrm{Sp}(H)$ *is uniquely represented by* $M = AU$*, where* $A \in \mathrm{P}(H) \cap \mathrm{Sp}(H)$ *and* $U \in \mathrm{U}(H) \cap \mathrm{Sp}(H)$*. So we have* $\mathrm{Sp}(H) \cong (\mathrm{P}(H) \cap \mathrm{Sp}(H)) \times (\mathrm{U}(H) \cap \mathrm{Sp}(H))$*.*

(b) *Each* $A \in \mathrm{P}(H) \cap \mathrm{Sp}(H)$ *is uniquely represented by* $A = \exp(S)$*, where* $S \in \mathrm{S}(H) \cap \mathrm{sp}(H)$ *is of the form* $S = \begin{pmatrix} 0 & S_{12} \\ S_{12}^* & 0 \end{pmatrix}$*,* $S_{12} \in \mathcal{B}(H^-, H^+)$*. So we have* $\mathrm{P}(H) \cap \mathrm{Sp}(H) \cong \mathcal{B}(H^-, H^+)$*.*

(c) *Each* $U \in \mathrm{U}(H) \cap \mathrm{Sp}(H)$ *is of the form* $U = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix}$*, where* $U_{11} \in \mathrm{U}(H^+)$*,* $U_{22} \in \mathrm{U}(H^-)$*. So we have* $\mathrm{U}(H) \cap \mathrm{Sp}(H) \cong \mathrm{U}(H^+) \times \mathrm{U}(H^-)$*.*

(d) $\mathrm{Sp}(H) \cong \mathcal{B}(H^-, H^+) \times \mathrm{U}(H^+) \times \mathrm{U}(H^-)$ *is path connected.*

(e) $\mathrm{Sp}(H)$ *is path-connected, and the fundamental group of* $\mathrm{Sp}(H)$ *is given by*

$$\pi_1(\mathrm{Sp}(H)) = \begin{cases} 0, & \text{if } \dim H^+ = \dim H^- = 0 \text{ or } +\infty, \\ \mathbb{Z}, & \text{if } 0 < \dim H^\pm < +\infty,\ \dim H^\mp = 0 \text{ or } +\infty, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } 0 < \dim H^+, \dim H^- < +\infty. \end{cases} \tag{3.1}$$

(f) $\mathrm{Sp}(H)$ *is a real analytic Hilbert manifold modelled on the real Hilbert space* $\mathrm{sp}(H)$*. Its real dimension is* m^2 *if* $H = \mathbb{C}^m$*.*

Proof (a) By the polar decomposition, each $M \in \mathrm{Sp}(H)$ is uniquely represented by $M = AU$, where $A \in \mathrm{P}(H)$ and $U \in \mathrm{U}(H)$. Then we obtain

$$M = J^{-1}(M^*)^{-1}J = J^{-1}A^{-1}JJ^{-1}(U^*)^{-1}J.$$

By the uniqueness of the decomposition, we obtain $A = J^{-1}AJ$ and $U = J^{-1}(U^*)^{-1}J$. Thus both A and U are symplectic. The converse is obvious.

(b) Each $A \in \mathrm{P}(H) \cap \mathrm{Sp}(H)$ is uniquely represented by $A = \exp(S)$, where $S \in \mathrm{S}(H)$. Since A is symplectic,

$$\exp(S) = J^{-1}(\exp(S))^{-1}J = \exp(-J^{-1}SJ).$$

By the uniqueness we obtain $S = -J^{-1}SJ$ and $S \in \mathrm{sp}(H)$. The converse is obvious.

(c) Note that $U \in \mathrm{U}(H) \cap \mathrm{Sp}(H)$ if and only if $UJ = JU$ and $U \in \mathrm{U}(H)$. Our results then follow from direct calculation.

(d) and (e) (d) follows from (a), (b) and (c), and (e) follows from (d).

(f) The first part of (f) follows from (d). If $H = \mathbb{C}^m$, we denote by $m^\pm := \dim H^\pm$. Then the real dimension of $\mathrm{Sp}(H)$ is

$$2m^+m^- + (m^+)^2 + (m^-)^2 = (m^+ + m^-)^2 = m^2.$$

4 The Iteration Formulae for Symplectic Paths

In this section we derive some iteration formulae for symplectic paths from Theorem 2.2.

4.1 Some general facts on the Maslov-type index in the finite dimensional case

The following lemmas are useful for deriving the iteration formulae from Theorem 2.2. For each $\tau > 0$, we define

$$\mathcal{P}_\tau(H) := \{\gamma \in C([0, \tau], \mathrm{Sp}(H)),\ \gamma(0) = \mathrm{I}_H\}. \tag{4.1}$$

Lemma 4.1 *Let (H,ω) and $(X,\widetilde{\omega})$ be symplectic Hilbert spaces defined in Section 2. Assume that $\dim H<+\infty$. Let $\phi:\mathcal{P}_{\tau_1}(H)\to\mathcal{P}_{\tau_2}(H)$ be a continuous map such that $\phi(c_0)=c_0$, where τ_1, τ_2 are two positive numbers and c_0 denotes the constant path I_H. Let V be a Lagrangian subspace of X. Denote the curve $\gamma(s\cdot):[0,1]\to\mathrm{Sp}(H)$ by $\gamma(s\cdot)(t):=\gamma(st)$. Then for each $\gamma\in\mathcal{P}_{\tau_1}(H)$, we have*

$$i_V(\phi(\gamma))=i_V(\{\phi(\gamma(s\cdot))(\tau_2),\ s\in[0,1]\}). \tag{4.2}$$

Proof Define the homotopy $h(s,t):[0,1]\times[0,\tau_2]\to\mathrm{Sp}(H)$ by $h(s,t):=\phi(\gamma(s\cdot))(t)$. Then we have $h(0,t)=h(s,0)=\mathrm{I}_H$. By the homotopy invariance of the Maslov index, the equation (4.2) follows.

Lemma 4.2 *Let (H,ω) and $(X,\widetilde{\omega})$ be symplectic Hilbert spaces defined in Section 2. Let $f_1,f_2:\mathrm{Sp}(H)\to\mathrm{Sp}(H)$ be a two continuous maps such that $f_1(\mathrm{I}_H)=f_2(\mathrm{I}_H)=\mathrm{I}_H$. Let V_1, V_2 be two Lagrangian subspaces of X. Assume that $\dim H<+\infty$. Then the following hold:*

(a) *Assume that $(f_1)_*=(f_2)_*:\pi_1(\mathrm{Sp}(H))\to\mathbb{Z}$. Then there is an integer valued function $\delta:\mathrm{Sp}(H)\to\mathbb{Z}$ defined by*

$$\delta(M):=i_{V_2}(f_2(\gamma))-i_{V_1}(f_1(\gamma)), \tag{4.3}$$

where $\gamma\in\mathcal{P}_\tau(H)$, $\tau>0$ with $\gamma(\tau)=M$.

(b) *Assume that $(f_1)_*=(f_2)_*:\pi_1(\mathrm{Sp}(H))\to\mathbb{Z}$. Let $\delta:\mathrm{Sp}(H)\to\mathbb{Z}$ be the function defined in* (a). *Then for each $\gamma\in C([a,b],\mathrm{Sp}(H))$, we have*

$$\delta(\gamma(b))-\delta(\gamma(a))=i_{V_2}(f_2(\gamma))-i_{V_1}(f_1(\gamma)). \tag{4.4}$$

(c) *We have $(f_1f_2)_*=(f_1)_*+(f_2)_*:\pi_1(\mathrm{Sp}(H))\to\mathbb{Z}$.*

Proof (a) Let γ_1 and γ_2 be in $\mathcal{P}_\tau(H)$ with $\gamma_1(\tau)=\gamma_2(\tau)=M$. Let $\gamma_3\in\mathcal{P}_{2\tau}(H)$ be the loop in $\mathrm{Sp}(H)$ defined by γ_1 followed by the reverse of γ_2. Then we have $[f_1(\gamma_3)]=[f_2(\gamma_3)]\in\pi_1(\mathrm{Sp}(H))$. Since $\mathcal{L}(X)$ is path connected, there is a path $\lambda:[0,1]\to\mathcal{L}(X)$ of Lagrangian subspaces such that $\lambda(0)=V_1$, $\lambda(1)=V_2$. Then we have a homotopy $h:[0,1]\times[0,2\tau]\to(\mathcal{L}(X))^2$ defined by $h(s,t)=(\lambda(s),\mathrm{Gr}(\gamma_3(t)))$. By the homotopy invariance of the Maslov index, we have $i_{V_1}(f_1(\gamma_3))=i_{V_2}(f_1(\gamma_3))=i_{V_2}(f_2(\gamma_3))$. Since $i_{V_j}(f_j(\gamma_3))=i_{V_j}(f_j(\gamma_1))-i_{V_j}(f_j(\gamma_2))$ for $j=1,2$, the equation (4.3) follows.

(b) Let γ be in $C([a,b],\mathrm{Sp}(H))$. By Proposition 3.1(d), there is a path $\alpha\in\mathcal{P}_\tau(H)$ such that $\alpha(\tau)=\gamma(a)$. Denote by $\widetilde{\gamma}\in\mathcal{P}_{2\tau}(H)$ the path defined by α followed by γ. Since $i_{V_j}(f_j(\widetilde{\gamma}))=i_{V_j}(f_j(\alpha))+i_{V_j}(f_j(\gamma))$ for $j=1,2$, the equation (4.4) follows.

(c) Let $\gamma\in C([0,1],\mathrm{Sp}(H))$ be such that $\gamma(0)=\gamma(1)=\mathrm{I}$. Then we have a homotopy $h:[0,1]^2\to\mathrm{Sp}(H)$ such that $h(s,t)=f_1(\gamma)f_2(\gamma)$. So we have $[f_1(\gamma)f_2(\gamma)]=[f_1(\gamma)]+[f_2(\gamma)]$ in $\pi_1(\mathrm{Sp}(H))$ and (c) follows.

4.2 The iteration formulae for the A-iteration of the symplectic path

Firstly, we define the iteration of a symplectic path $\gamma\in\mathcal{P}_\tau(H)$.

Definition 4.1 (see [12, (4.3)–(4.4)]) *Let (H,ω) be a symplectic Hilbert space defined in Section 2. Given an $A\in\mathrm{Sp}(2n)$, a $\tau>0$, a positive integer k and a path $\gamma\in\mathcal{P}_\tau(H)$, we define*

$$\widetilde{\gamma}(t)=A^j\gamma(t-j\tau)(A^{-1}\gamma(\tau))^j,\quad t\in[j\tau,(j+1)\tau],\ j=0,\cdots,k-1. \tag{4.5}$$

(a) *We call* $\widetilde{\gamma}$ *the k-th A-iteration of* γ. *The map* $P_A(\gamma) := A^{-1}\gamma(\tau)$ *is called the Poincaré map of* γ *at* τ.

(b) *If* $\dim H$ *is finite, we define*

$$i_z(\gamma, k, A) := i_{zA^k}(\widetilde{\gamma}), \quad \nu_z(\gamma, k, A) := \nu_{zA^k}(\widetilde{\gamma}(k\tau)) \tag{4.6}$$

for each $z \in \mathbb{C}$ *with* $|z| = 1$.

We need the following lemmas.

Lemma 4.3 *Let* X *be a vector space and* $A, B \in \mathrm{End}(X)$ *be two linear maps. Assume that* $AB = BA$. *Then* AB *is Freholm if and only if* A *and* B *are Fredholm. In this case we have*

$$\mathrm{index}(AB) = \mathrm{index}\, A + \mathrm{index}\, B. \tag{4.7}$$

Proof If A and B are Fredholm, AB is Fredholm and (4.7) holds. Since $AB = BA$, we have

$$\ker(AB) \supset \ker A + \ker B, \quad \mathrm{im}(AB) \subset \mathrm{im}\, A \cap \mathrm{im}\, B.$$

So A and B are Fredholm if AB is Fredholm.

The following fact is clear.

Lemma 4.4 *Let* X *and* Y *be two Abelian groups and* $A, B \in \mathrm{Hom}(X, Y)$ *be two group homomorphisms. Then we have* $\ker(A - B) \cong \mathrm{Gr}(A) \cap \mathrm{Gr}(B)$ *and* $Y/\mathrm{im}(A - B) \cong (X \times Y)/(\mathrm{Gr}(A) + \mathrm{Gr}(B))$.

We have the following form of iteration formula.

Theorem 4.1 (The Iteration Formula for the Power of the Symplectic Path) *Let* (H, ω) *be a symplectic Hilbert space defined in Section* 2. *Let* $\gamma \in C([a, b], \mathrm{Sp}(H))$ *be a symplectic path such that* $(\gamma(t))^k - \mathrm{I}_H$ *is Fredholm of index* 0 *for each* $t \in [a, b]$, *where* k *is a positive integer. Then we have*

$$i_1(\gamma^k) = \sum_{z^k=1} i_z(\gamma), \tag{4.8}$$

$$\nu_1((\gamma(t))^k) = \sum_{z^k=1} \nu_z(\gamma(t)), \quad \forall t \in [a, b]. \tag{4.9}$$

Proof Set $V_j := \mathrm{Gr}\big(\exp\big(\frac{2\pi j\sqrt{-1}}{}$

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Corollary 4.1 (The Iteration Formula for the A-Iteration of the Symplectic Path) *Let* (H, ω) *be a finite dimensional symplectic Hilbert space. Given an* $A \in \mathrm{Sp}(H)$*, a* $\tau >$

$(NS)^2 = \mathrm{I}_X$. Set $U^\pm := \ker(N \mp \mathrm{I}_X)$, $V^\pm := \ker(NS \mp \mathrm{I}_X)$, and $K := NM^{-1}NM$. Let M be in the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : V^+ \oplus V^- \to U^+ \oplus U^-. \tag{4.17}$$

Then we have

$$\ker(K - S) = \ker C \oplus \ker B, \quad \operatorname{im}(MN(K-S)) = \operatorname{im} B \oplus \operatorname{im} C, \tag{4.18}$$
$$\ker C = V^+ \cap (M^{-1}U^+) = \ker(K-S) \cap V^+, \tag{4.19}$$
$$\ker B = V^- \cap (M^{-1}U^-) = \ker(K-S) \cap V^-. \tag{4.20}$$

Proof Since $(NS)^2 = \mathrm{I}_X$, we have

$$\begin{aligned}(NM - MNS)(\mathrm{I}_X \pm NS) &= NM(\mathrm{I}_X \pm NS) - M(NS \pm \mathrm{I}_X) \\ &= (NM \mp M)(\mathrm{I}_X \pm NS) \\ &= (N \mp \mathrm{I}_X)M(\mathrm{I}_X \pm NS).\end{aligned}$$

Since $N^2 = (NS)^2 = \mathrm{I}_X = \frac{\mathrm{I}_X + N}{2} + \frac{\mathrm{I}_X - N}{2} = \frac{\mathrm{I}_X + NS}{2} + \frac{\mathrm{I}_X - NS}{2}$, we have

$$MN(K-S) = NM - MNS = \begin{pmatrix} 0 & 2B \\ -2C & 0 \end{pmatrix}. \tag{4.21}$$

Our results then follows from (4.21) and direct computations.

Lemma 4.7 *Let (H, ω) be a symplectic Hilbert space defined in Section 2. Let $N \in \mathcal{B}(H)$ be a bounded opertor such that $N^*JN = -J$ and $N^2 = \mathrm{I}_H$. Let $\gamma \in C([a,b], \mathrm{Sp}(H))$ be a positive symplectic path. Then we have $N^*J = -JN$, and the path $\alpha := N\gamma^{-1}N$ is a positive symplectic path.*

Proof Since $N^*JN = -J$ and $N^2 = \mathrm{I}_H$, we have $N^*J = -JN^{-1} = -JN$. Since γ is a symplectic path, α is also a symplectic path. It follows that

$$\begin{aligned}-J\dot{\alpha}\alpha^{-1} &= -J(N(-\gamma^{-1}\dot{\gamma}\gamma^{-1})N)(N^{-1}\gamma N^{-1}) \\ &= N^*\gamma^*(-J\dot{\gamma}\gamma^{-1})\gamma N > 0.\end{aligned}$$

The proof is complete.

We have the following form of two times iteration formula for the generalized brake symmetry.

Theorem 4.2 *Let (H, ω) be a symplectic Hilbert space defined in Section 2. Let $N \in \mathcal{B}(H)$ be a bounded opertor such that $N^*JN = -J$ and $N^2 = \mathrm{I}_H$. Let $S \in \mathrm{Sp}(H)$ be such that $(NS)^2 = \mathrm{I}_H$. Let $\gamma \in C([a,b], \mathrm{Sp}(2n, H))$ be a symplectic path such that $N\gamma(t)^{-1}N\gamma(t) - S$ is Fredholm of index 0 for each $t \in [a,b]$. Then we have*

$$i_S(N\gamma^{-1}N\gamma) = i_{V^+ \times U^+}(\gamma) + i_{V^- \times U^-}(\gamma). \tag{4.22}$$

Proof Set $V_0 := \mathrm{Gr}(S)$, $V_1 := V^+ \times U^+$, $V_2 := V^- \times U^-$. By Lemmas 4.5–4.6, we have

$$0 = \operatorname{index}(N(\gamma(t))^{-1}N\gamma(t) - S) = \operatorname{index}(\mathrm{Gr}(\gamma(t)), V_1) + \operatorname{index}(\mathrm{Gr}(\gamma(t)), V_2)$$

for each $t \in [a,b]$. By [2, Proposition 1], we have $\operatorname{index}(\operatorname{Gr}(\gamma(t)), V_j) \le 0$ for $j = 1, 2$. So both of the two index are 0. By Lemmas 4.5–4.7, we can apply Theorem 2.2 and our result follows.

Our next result in the real case was obtained by Y. Long, D. Zhang and the second author [15, Proposition C] when $S = \mathrm{I}_H$, and by X. Hu and S. Sun [9, Theorem 1.6] when $S, N \in \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{U}(2n)$.

Corollary 4.2 (Two Times Iteration Formula for the Generalized Brake Symmetry) *Let (H, ω) be a complex symplectic Hilbert space of dimension $2n$. Let $N \in \mathcal{B}(H)$ be a bounded operator such that $N^* J N = -J$ and $N^2 = \mathrm{I}_H$. Let $S \in \mathrm{Sp}(H)$ be such that $(NS)^2 = \mathrm{I}_H$. Given a path $\gamma \in \mathcal{P}_\tau(H)$, we have*

$$i_S(\gamma^{(2)}) = i_{V^+ \times U^+}(\gamma) + i_{V^- \times U^-}(\gamma). \tag{4.23}$$

Proof By Lemma 4.1 and Theorem 4.2, the result follows.

4.4 The iteration formula for the brake symmetry

Let (H, ω) be a symplectic Hilbert space defined in Section 2. Let $N \in \mathcal{B}(H)$ be such that $N^2 = \mathrm{I}_H$ and $N^* J N = -J$. Set $U^\pm := \ker(N \mp \mathrm{I}_H)$. Then we have $U^\pm \in \mathcal{L}(H)$. By replacing the inner product of H with $\langle \cdot, \cdot \rangle_{U^-} \oplus \langle \cdot, \cdot \rangle_{U^+}$, we can assume that $N^* N = \mathrm{I}_H$. Here the form ω is unchanged. We fix the orthogonal decomposition $H = U^- \oplus U^+$. Then we have $N = \begin{pmatrix} -\mathrm{I}_{U^-} & 0 \\ 0 & \mathrm{I}_{U^+} \end{pmatrix}$, $J = \begin{pmatrix} 0 & -K^* \\ K & 0 \end{pmatrix}$, where $K \in \mathcal{B}(U^-, U^+)$ and K, K^* are injective maps.

Lemma 4.8 *Let X be a vector space with two linear maps $A, B \in \operatorname{End}(X)$. Assume that one of the following three conditions holds:*

(i) $\ker(AB) \supset \ker A$, *and* $B : \ker A \to \ker A$ *is surjective.*

(ii) $\ker(AB) \supset \ker A$, $\ker A \cap \ker B = \{0\}$, *and* $\ker A$ *is finite dimensional.*

(iii) $\ker A$ *and* $\ker B$ *are finite dimensional, and* $\dim \ker(AB) \ge \dim \ker A + \dim \ker B$.

Then there exist short exact sequences

$$0 \to \ker B \to \ker(AB) \to \ker A \to 0, \tag{4.24}$$

$$0 \to X/\operatorname{im} B \to X/\operatorname{im}(AB) \to X/\operatorname{im} A \to 0. \tag{4.25}$$

Proof By [18, Exercises B.11], it is enough to show that $B : \ker(AB) \to \ker A$ is surjective.

(i) Since $\ker(AB) \supset \ker A$, we have

$$B \ker A \subset B \ker(AB) \subset \ker A.$$

Since $B : \ker A \to \ker A$ is surjective, we have $B \ker(AB) \supset B \ker A = \ker A$. So we obtain $B \ker(AB) = \ker A$. Thus $B : \ker(AB) \to \ker A$ is surjective and our results follow.

(ii) Since $\ker A \cap \ker B = \{0\}$, the map $B : \ker A \to \ker A$ is an injection. Since $\dim \ker A < +\infty$, it is an isomorphism. By (i), our results hold.

(iii) By [18, Exercises B.11], we have $\dim \ker(AB) \le \dim \ker A + \dim \ker B$. So we have $\dim \ker(AB) = \dim \ker A + \dim \ker B < +\infty$. By [18, Exercises B.11], $B : \ker(AB) \to \ker A$ is an isomorphism. So (4.24) and (4.25) hold.

Lemma 4.9 *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(H)$ *be such that* $(NM)^2 = \mathrm{I}_H$ *and* k *be a positive integer. Then the following hold:*

(a) (see [6, Proposition 2.1]) *We have*

$$KA = D^*K, \quad KB = B^*K^*, \quad K^*C = C^*K, \tag{4.26}$$

$$AB = BD, \quad CA = DC, \quad A^2 - BC = \mathrm{I}_{U^-}, \quad D^2 - CB = \mathrm{I}_{U^+}. \tag{4.27}$$

(b) *Let* α *be a real number such that* $\frac{\alpha}{\pi} \notin \mathbb{Z}$. *Then we have*

$$\dim\ker(D - (\cos\alpha)\mathrm{I}_{U^+}) = \dim\ker(M - \mathrm{e}^{\sqrt{-1}\alpha}\mathrm{I}_H), \tag{4.28}$$

$$\dim(U^-/\operatorname{im}(D - (\cos\alpha)\mathrm{I}_{U^+})) = \dim(H/\operatorname{im}(M - \mathrm{e}^{\sqrt{-1}\alpha}\mathrm{I}_H)). \tag{4.29}$$

(c) (see [6, Lemma 3.1]) *Denote by* T_k *and* U_k *the Chebyshev polynomials of the first kind and the second kind respectively. Then we have*

$$M^k = \begin{pmatrix} T_k(A) & U_{k-1}(A)B \\ CU_{k-1}(A) & T_k(D) \end{pmatrix}. \tag{4.30}$$

(d) *Assume that there is* $P \in \mathrm{Sp}(H)$ *such that* $M = NP^{-1}NP$. *Then the pair* $(\mathrm{Gr}(M), U^+ \times U^+)$ *is Fredholm (of index 0) if* $(\mathrm{Gr}(P), U^+ \times U^+)$ *and* $(\mathrm{Gr}(P), U^+ \times U^-)$ *are Fredholm (of index 0). In this case, we have*

$$\nu_{U^+\times U^+}(M) = \nu_{U^+\times U^+}(P) + \nu_{U^+\times U^-}(P), \tag{4.31}$$

$$\widetilde{\nu}_{U^+\times U^+}(M) = \widetilde{\nu}_{U^+\times U^+}(P) + \widetilde{\nu}_{U^+\times U^-}(P). \tag{4.32}$$

(e) *The pair* $(\mathrm{Gr}(M^k), U^+ \times U^+)$ *is Fredholm (of index 0) if and only if* $(\mathrm{Gr}(M), U^+ \times U^+)$ *and* $M - \mathrm{e}^{\frac{\sqrt{-1}j\pi}{k}}\mathrm{I}_H$ *are Fredholm (of index 0) for* $j = 1, \cdots, k-1$. *In this case, we have*

$$\nu_{U^+\times U^+}(M^k) = \nu_{U^+\times U^+}(M) + \sum_{j=1}^{k-1} \nu_{\mathrm{e}^{\frac{\sqrt{-1}j\pi}{k}}}(M), \tag{4.33}$$

$$\widetilde{\nu}_{U^+\times U^+}(M^k) = \widetilde{\nu}_{U^+\times U^+}(M) + \sum_{j=1}^{k-1} \widetilde{\nu}_{\mathrm{e}^{\frac{\sqrt{-1}j\pi}{k}}}(M). \tag{4.34}$$

(f) *Assume that there is* $P \in \mathrm{Sp}(H)$ *such that* $M = NP^{-1}NP$. *Then the pair* $(\mathrm{Gr}(PM^k), U^+ \times U^+)$ *is Fredholm (of index 0) if and only if* $(\mathrm{Gr}(P), U^+ \times U^+)$ *and* $M - \mathrm{e}^{\frac{2\sqrt{-1}j\pi}{2k+1}}\mathrm{I}_H$ *are Fredholm (of index 0) for* $j = 1, \cdots, k$. *In this case, we have*

$$\nu_{U^+\times U^+}(PM^k) = \nu_{U^+\times U^+}(P) + \sum_{j=1}^{k} \nu_{\mathrm{e}^{\frac{2\sqrt{-1}j\pi}{2k+1}}}(M), \tag{4.35}$$

$$\widetilde{\nu}_{U^+\times U^+}(PM^k) = \widetilde{\nu}_{U^+\times U^+}(P) + \sum_{j=1}^{k} \widetilde{\nu}_{\mathrm{e}^{\frac{2\sqrt{-1}j\pi}{2k+1}}}(M). \tag{4.36}$$

Proof (a) Since M is symplectic, we have

$$D^*KB - B^*K^*D = 0, \quad C^*KA - A^*KC = 0, \quad D^*KA - B^*K^*C = K.$$

Since $M^*JM = J$ and $(NM)^2 = \mathrm{I}_H$, we have $M^*J = JM^{-1} = JNMN$. Then (4.26) holds. Then we have

$$\begin{aligned} K(AB - BD) &= D^*KB - B^*K^*D = 0, \\ K^*(CA - DC) &= C^*KA - A^*KC = 0, \\ K(A^2 - BC) &= D^*KA - B^*K^*C = K, \\ K^*(D^2 - CB) &= A^*K^*D - C^*KB = K^*. \end{aligned}$$

By the injectivity of K and K^*, we obtain (4.27).

(b) Set $\lambda := \mathrm{e}^{\sqrt{-1}\alpha}$. Since $(NM)^2 = \mathrm{I}_H$, we have $(MN)^2 = \mathrm{I}_H$ and $\lambda\mathrm{I}_H - MN$ is invertible. Then we have

$$\begin{aligned} (\mathrm{I}_H - \lambda^{-1}MN)(M - \lambda\mathrm{I}_H) &= M(\mathrm{I}_H + N) - (\lambda\mathrm{I}_H + \lambda^{-1}N) \\ &= \begin{pmatrix} -2\sqrt{-1}(\sin\alpha)\mathrm{I}_{U^-} & 0 \\ 0 & 2(D - (\cos\alpha)\mathrm{I}_{U^+}) \end{pmatrix}. \end{aligned} \tag{4.37}$$

Since $\frac{\alpha}{\pi} \notin \mathbb{Z}$, (4.28) and (4.29) hold.

(c) In a similar way to the proof of [6, Proposition 2.1], where we replace A^T by D, the result follows.

(d) Let P be of the form

$$P = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

Then we have $B = 2A_2B_1 = -2B_2D_1$, and $\ker B_1 \cap \ker D_1 = \{0\}$. So we have

$$\begin{aligned} \ker B = MU^+ \cap U^+ &= \{x \in U^+,\ Px \in NPN(U^+)\} \\ &= \{x \in U^+,\ B_1x = -B_1y, D_1x = D_1y \text{ for some } y \in U^+\} \\ &= \ker B_1 \oplus \ker D_1. \end{aligned} \tag{4.38}$$

By a direct calculation, we have

$$\begin{aligned} \mathrm{Gr}(M) + U^+ \times U^+ &= \begin{pmatrix} \mathrm{I}_H & 0 \\ 0 & NP^{-1}N \end{pmatrix} (\mathrm{Gr}(P) + U^+ \times NPN(U^+)) \\ &= \begin{pmatrix} \mathrm{I}_H & 0 \\ 0 & NP^{-1}N \end{pmatrix} \Bigg(U^+ \times \{0\} \\ &\quad + \begin{pmatrix} \mathrm{I}_{U^-} & 0 & 0 \\ A_1 & \mathrm{I}_{U^-} & 0 \\ C_1 & 0 & \mathrm{I}_{U^+} \end{pmatrix} (U^- \times (\operatorname{im} B_1 \oplus \operatorname{im} D_1)) \Bigg). \end{aligned}$$

By Lemma 4.5, we have

$$U^-/\operatorname{im} B \cong H^2/(\mathrm{Gr}(M) + U^+ \times U^+) \cong U^-/\operatorname{im} B_1 \oplus U^+/\operatorname{im} D_1. \tag{4.39}$$

By (4.38)–(4.39), B is Fredholm if and only if B_1 and D_1 are Fredholm. In this case, we have

$$\dim\ker B = \dim\ker B_1 + \dim\ker D_1, \tag{4.40}$$

$$\dim U^-/\operatorname{im} B = \dim U^-/\operatorname{im} B_1 + \dim U^+/\operatorname{im} D_1, \tag{4.41}$$

$$\operatorname{index} B = \operatorname{index} B_1 + \operatorname{index} D_1. \tag{4.42}$$

By Lemma 4.5 and (2.12), we have $\operatorname{index} B_1 = \operatorname{index}(\operatorname{Gr}(P), U^+ \times U^+) \le 0$, $\operatorname{index} D_1 = \operatorname{index}(\operatorname{Gr}(P), U^+ \times U^-) \le 0$ and $\operatorname{index} B = \operatorname{index}(\operatorname{Gr}(M), U^+ \times U^+) \le 0$. So our result follows from (4.40)–(4.42).

(e) We have $\ker(U_{k-1}(A)B) \supset \ker B$. Since $AB = BD$ and U_{k-1} is a polynomial, we have $U_{k-1}(A)B = BU_{k-1}(D)$. It follows that $\ker(U_{k-1}(A)B) \supset \ker U_{k-1}(D)$.

We recall the identities of Chebyshev polynomials (see [6, Lemma 4.1]) for $a \in \mathbb{C}$,

$$\begin{cases} T_k(\cos\alpha) = \cos k\alpha, \\ U_k(\cos\alpha) = \dfrac{\sin((k+1)\alpha)}{\sin\alpha}. \end{cases} \tag{4.43}$$

It follows that the set of zeros of U_{k-1} is $\{\cos(\frac{j\pi}{k}) \mid 0 < j < k,\ j \in \mathbb{Z}\}$. Then we have

$$\ker U_{k-1}(D) = \sum_{j=1}^{k-1} \ker\Big(D - \cos\Big(\frac{j\pi}{k}\Big)\mathrm{I}_{U^+}\Big). \tag{4.44}$$

By (4.27), we have $D^2 - \mathrm{I}_n = CB$. It follows that

$$\ker B \subset \ker(CB) \subset \ker(D - \mathrm{I}_{U^+}) \oplus \ker(D + \mathrm{I}_{U^+}). \tag{4.45}$$

Then we can conclude that

$$\ker B \cap \ker(U_{k-1}(D)) = \{0\}.$$

By Lemma 4.8 and (b), $(\operatorname{Gr}(M^k), U^+ \times U^+)$ is Fredholm if and only if $(\operatorname{Gr}(M), U^+ \times U^+)$ and $M - e^{\frac{\sqrt{-1}j\pi}{k}}\mathrm{I}_H$ are Fredholm for $k = 1, \cdots, k-1$. In this case (4.33)–(4.34) hold. By (2.12), $(\operatorname{Gr}(M^k), U^+ \times U^+)$ is Fredholm of index 0 if and only if $(\operatorname{Gr}(M), U^+ \times U^+)$ and $M - e^{\frac{\sqrt{-1}j\pi}{k}}\mathrm{I}_H$ are Fredholm of index 0 for $k = 1, \cdots, k-1$.

(f) Let P and PM^k be of the form

$$P = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad PM^k = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}.$$

Since $M = NP^{-1}NP$, we have $PNM = NP$. So $A_1 B = B_1(D + \mathrm{I}_{U^+})$ holds. Note that $AB = BD$. By (c), we have

$$B_3 = A_1 U_{k-1}(A)B + B_1 T_k(D) = A_1 B U_{k-1}(D) + B_1 T_k(D) = B_1 R_k(D), \tag{4.46}$$

where $R_k(x) := (x+1)U_{k-1}(x) + T_k(x)$ is a k-th order polynomial. By (4.43), we have

$$R_k(\cos\alpha) = \frac{\sin((k+\frac{1}{2})\alpha)}{\sin\frac{\alpha}{2}}. \tag{4.47}$$

So the set of zeros of R_k is $\{\cos(\frac{2j\pi}{2k+1}) \mid 1 \le j \le k,\ j \in \mathbb{Z}\}$. Since $M = NP^{-1}NP$ and $NU^+ = U^+$, we have $PU^+ \cap U^+ \subset MU^+ \cap PM^k U^+ \cap U^+$. So we have $\ker B_1 \subset \ker B \cap \ker B_3$. By a proof similar to that of (e), (f) is obtained.

We have the following form of iteration formula for the brake symmetry.

Theorem 4.3 *Let (H,ω) be a symplectic Hilbert space defined in Section 2. Let $N\in\mathcal{B}(H)$ be a bounded opertor such that $N^*JN=-J$ and $N^2=\mathrm{I}_H$. Let $\gamma\in C([a,b],\ \mathrm{Sp}(H))$ be a symplectic path such that $(N\gamma(t))^2=\mathrm{I}_H$. Set $U^{\pm}:=\ker(N\mp\mathrm{I}_H)$. Let k be a positive integer. Then the following hold:*

(a) *Assume that there is a symplectic path $\gamma_1\in C([a,b],\mathrm{Sp}(H))$ such that $\gamma=N\gamma_1^{-1}N\gamma_1$ and the pair $(\gamma(t)U^+,U^+)$ is Fredholm of index 0 for each $t\in[a,b]$. Then we have*

$$i_{U^+\times U^+}(\gamma)=i_{U^+\times U^+}(\gamma_1)+i_{U^+\times U^-}(\gamma_1). \tag{4.48}$$

(b) *Assume that the pair $(\gamma(t)^kU^+,U^+)$ is Fredholm of index 0 for each $t\in[a,b]$. Then we have*

$$i_{U^+\times U^+}(\gamma^k)=i_{U^+\times U^+}(\gamma)+\sum_{j=1}^{k-1}i_{\mathrm{e}^{\frac{\sqrt{-1}j\pi}{k}}}(\gamma). \tag{4.49}$$

(c) *Assume that there is a symplectic path $\gamma_1\in C([a,b],\mathrm{Sp}(H))$ such that $\gamma=N\gamma_1^{-1}N\gamma_1$ and the pair $(\gamma_1(t)\gamma(t)^kU^+,U^+)$ is Fredholm of index 0 for each $t\in[a,b]$. Then we have*

$$i_{U^+\times U^+}(\gamma_1\gamma^k)=i_{U^+\times U^+}(\gamma_1)+\sum_{j=1}^{k}i_{\mathrm{e}^{\frac{2\sqrt{-1}j\pi}{2k+1}}}(\gamma). \tag{4.50}$$

Proof Note that we can assume $N^*N=\mathrm{I}_H$. By Lemma 4.9, we can apply Theorem 2.2 and our result follows.

Our next result in the real case was obtained by [12, Theorem 1.3].

Corollary 4.3 (The Iteration Formula for the Brake Symmetry) *Let (H,ω) be a complex symplectic Hilbert space of dimension $2n$. Let $N\in\mathcal{B}(H)$ be a bounded opertor such that $N^*JN=-J$ and $N^2=\mathrm{I}_H$. Let k be a positive integer. Given a path $\gamma\in\mathcal{P}_\tau(H)$, the following hold:*

(a) *We have*

$$i_{U^+\times U^+}(\gamma^{(2)})=i_{U^+\times U^+}(\gamma)+i_{U^+\times U^-}(\gamma). \tag{4.51}$$

(b) *Assume that $(N\gamma(\tau))^2=\mathrm{I}_H$. Denote by $\widetilde{\gamma}$ the k-th I_H-iteration of γ. We have*

$$i_{U^+\times U^+}(\widetilde{\gamma})=i_{U^+\times U^+}(\gamma)+\sum_{j=1}^{k-1}i_{\mathrm{e}^{\frac{\sqrt{-1}j\pi}{k}}}(\gamma). \tag{4.52}$$

(c) *We have*

$$i_{U^+\times U^+}(\gamma^{(2k+1)})=i_{U^+\times U^+}(\gamma)+\sum_{j=1}^{k}i_{\mathrm{e}^{\frac{2\sqrt{-1}j\pi}{2k+1}}}(\gamma^{(2)}). \tag{4.53}$$

Proof By Lemma 4.1 and Theorem 4.3, the result follows.

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