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On a Vector Version of a Fundamental Lemma of J. L. Lions*

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Abstract Let Ω be a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary, the set Ω being locally on the same side of $\partial\Omega$. A vector version of a fundamental lemma of J. L. Lions, due to C. Amrouche, the first author, L. Gratie and S. Kesavan, asserts that any vector field $\boldsymbol{v} = (v_i) \in (\mathcal{D}'(\Omega))^N$, such that all the components $\frac{1}{2}(\partial_j v_i + \partial_i v_j)$, $1 \leq i, j \leq N$, of its symmetrized gradient matrix field are in the space $H^{-1}(\Omega)$, is in effect in the space $(L^2(\Omega))^N$. The objective of this paper is to show that this vector version of J. L. Lions lemma is equivalent to a certain number of other properties of interest by themselves. These include in particular a vector version of a well-known inequality due to J. Nečas, weak versions of the classical Donati and Saint-Venant compatibility conditions for a matrix field to be the symmetrized gradient matrix field of a vector field, or a natural vector version of a fundamental surjectivity property of the divergence operator.

Keywords J. L. Lions lemma, Nečas inequality, Donati compatibility conditions, Saint-Venant compatibility conditions
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1 Introduction

All notations and definitions are not explained here, see Section 2.

Given any open subset Ω of $\mathbb{R}^N$, the implication

$$f \in L^2(\Omega) \Rightarrow f \in H^{-1}(\Omega) \text{ and } \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$$

clearly holds. While the converse implication, viz.,

$$f \in H^{-1}(\Omega) \text{ and } \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega),$$

does not necessarily hold in general (see, e.g., the counterexample of Geymonat and Gilardy [11]), it does if Ω is a domain in $\mathbb{R}^N$, i.e., a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary (for details about domains, see Adams [1] or Nečas [22]).

This fundamental observation is due to J. L. Lions (see the footnote 27 in Magenes and Stampacchia [19]). Its first published proofs appeared in Theorem 3.2 in Chapter 3 of Duvaut

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and Lions [10] and in Tartar [23], for domains with sufficiently smooth boundaries. It was later shown by Geymonat and Suquet [14] that this implication holds in fact if Ω is any domain in $\mathbb{R}^N$, i.e., with a boundary that is only Lipschitz-continuous.

This result was further generalized by Borchers and Sohr [6] and Amrouche and Girault [4], who showed that the assumption "$f \in H^{-1}(\Omega)$" can be replaced by the weaker assumption "$f \in \mathcal{D}'(\Omega)$" and that the spaces $\boldsymbol{H}^{-1}(\Omega)$, resp. $L^2(\Omega)$, can be replaced with the more general spaces $\boldsymbol{H}^{-m}(\Omega)$, resp. $H^{-m+1}(\Omega)$, where m is now any integer ≥ 1, according to the following result.

Theorem 1.1 (J. L. Lions Lemma) *Let Ω be a domain in $\mathbb{R}^N$ and let $m \geq 1$ be an integer. Then the following implication holds:*

$$f \in \mathcal{D}'(\Omega) \text{ and } \mathbf{grad}\, f \in \boldsymbol{H}^{-m}(\Omega) \Rightarrow f \in H^{-m+1}(\Omega).$$

Note that it was also shown in [4, 6] that, more generally,

$$f \in \mathcal{D}'(\Omega) \text{ and } \mathbf{grad}\, f \in W^{-m,p}(\Omega) \Rightarrow f \in W^{-m+1,p}(\Omega)$$

for any integer $m \geq 1$ and any $p \in \mathbb{R}$ such that $p > 1$.

J. L. Lions lemma was then generalized to vector-valued distributions, i.e., in $\boldsymbol{\mathcal{D}}'(\Omega)$ (instead of "scalar" distributions in $\mathcal{D}'(\Omega)$), by Amrouche, Ciarlet, Gratie and Kesavan [3] according to the following result.

Theorem 1.2 (Vector Version of J. L. Lions Lemma) *Let Ω be a domain in $\mathbb{R}^N$ and let $m \geq 1$ be an integer. Then the following implication holds:*

$$\boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega) \text{ and } \nabla_s \boldsymbol{v} = \Big(\frac{1}{2}(\partial_j v_i + \partial_i v_j)\Big) \in \mathbb{H}^{-m}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{H}^{-m+1}(\Omega).$$

Proof Let $\boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega)$ be such that $(\nabla_s \boldsymbol{v})_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \in H^{-m}(\Omega)$. Then the well-known identity

$$\partial_j(\partial_k v_i) = \partial_j(\nabla_s \boldsymbol{v})_{ik} + \partial_k(\nabla_s \boldsymbol{v})_{ij} - \partial_i(\nabla_s \boldsymbol{v})_{jk},$$

which holds here in the space $H^{-m-1}(\Omega) \subset \mathcal{D}'(\Omega)$, implies that $\partial_k v_i \in H^{-m}(\Omega)$, which in turn implies that $v_i \in H^{-m+1}(\Omega)$, thanks to two successive applications of J. L. Lions lemma (see Theorem 1.1).

Incidentally, Theorem 1.2 shows that the natural analog in the vector version of J. L. Lions lemma of the gradient operator **grad** of the scalar case is the symmetrized gradient operator ∇_s.

We shall be concerned in this paper with the special case $m = 1$ of Theorem 1.2, viz., with the implication

$$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega) \text{ and } \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

More specifically, our main objective is to show that this implication is in effect equivalent to a certain number of other fundamental properties (cf. Theorems 3.1, 4.1–4.2), by means of natural "vector versions" of similar equivalences that were shown to hold in [2] in the scalar case.

2 Notations and Other Preliminaries

Throughout the article, an integer $N \geq 2$ is given and, unless otherwise specified, Latin indices take their values in the set $\{1, 2, \cdots, N\}$.

All the functions, matrices, etc., considered here are real. The notations $\operatorname{Ker} A$ and $\operatorname{Im} A$ respectively designate the kernel and the image, also known as the range, of a linear operator A. If B is a subset of a vector space X and $\lambda \in \mathbb{R}$, then $\lambda B := \{\lambda x \in X;\ x \in B\}$.

Let X be a normed vector space. Given any $x \in X$ and any $r > 0$, the notation $B(x; r)$ designates the open ball with center x and radius r; given $x \in X$ and a subset B of X, the distance from x to B is defined as $d(x, B) := \inf\{d(x; y); y \in B\}$; a subset B of X is said to be starlike with respect to an open ball $B(x; r)$ if, for every $z \in B$, the convex hull of the set $\{z\} \cup B(x; r)$ is contained in B.

The dual of a topological space X is denoted X' and the duality between X' and X is denoted ${}_{X'}\langle \cdot, \cdot \rangle_X$; given a subspace Z of X, the polar set Z^0 of Z is defined by

$$Z^0 := \{x' \in X';\ {}_{X'}\langle x', x \rangle_X = 0 \text{ for all } x \in Z\}.$$

Given two normed vector spaces X and Y, the notation $\mathcal{L}(X; Y)$ designates the space of all continuous linear operators from X into Y, equipped with the operator norm. Given $A \in \mathcal{L}(X; Y)$, its dual is the operator $A' \in \mathcal{L}(Y'; X')$ defined by ${}_{X'}\langle A'y', x \rangle_X = {}_{Y'}\langle y', Ax \rangle_Y$ for all $x \in X$ and all $y' \in Y'$.

The notations $(\boldsymbol{a})_i$ and $(\boldsymbol{A})_{ij}$ respectively designate the i-th component of a vector $\boldsymbol{a} \in \mathbb{R}^N$ and the components at the i-th row and j-th column of an $N \times N$ matrix $\boldsymbol{A}$. The notation $\boldsymbol{A} = (a_{ij})$ means that $a_{ij} = (\boldsymbol{A})_{ij}$. The Kronecker's symbol is denoted δ_{ij}. The notation $\boldsymbol{a} \cdot \boldsymbol{b}$ designates the Euclidean inner product of two vectors $\boldsymbol{a} \in \mathbb{R}^N$ and $\boldsymbol{b} \in \mathbb{R}^N$. The set of all $N \times N$ symmetric, resp. antisymmetric, matrices is denoted $\mathbb{S}^N$, resp. $\mathbb{A}^N$.

In what follows, Ω denotes a non-empty open subset of $\mathbb{R}^N$, $x = (x_i)$ designates a generic point in Ω, and $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ respectively designate the partial derivative operators of the first and second order, in the classical sense or in the sense of distributions. The notation $\partial^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index, designates the partial differential operator $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}$, with $|\boldsymbol{\alpha}| = \sum_i \alpha_i$. The support of a function $f : \Omega \to \mathbb{R}$ is denoted $\operatorname{supp} f$, and $f|_A$ designates the restriction of f to a subset A of Ω.

The notation $\mathcal{D}(\Omega)$ designates the space of functions $\varphi \in \mathcal{C}^\infty(\Omega)$ such that $\operatorname{supp} \varphi$ is a compact subset of Ω. The space $\mathcal{D}(\Omega)$ is equipped with its inductive limit topology (cf., e.g., Vo Khac [25]). Then a sequence $(\varphi_n)_{n=1}^\infty$ of functions $\varphi_n \in \mathcal{D}(\Omega)$ converges to a function $\varphi \in \mathcal{D}(\Omega)$ in this topology if there exists a compact subset K of Ω such that

$$\begin{aligned}
&\operatorname{supp} \varphi \subset K \text{ and } \operatorname{supp} \varphi_n \subset K \quad \text{for all } n \geq 1, \\
&\sup_{x \in K} |\partial^{\boldsymbol{\alpha}} \varphi_n(x) - \partial^{\boldsymbol{\alpha}} \varphi(x)| \to 0 \quad \text{as } n \to \infty \text{ for each multi-index } \boldsymbol{\alpha}.
\end{aligned}$$

Such a convergence is denoted

$$\varphi_n \to \varphi \quad \text{in } \mathcal{D}(\Omega) \text{ as } n \to \infty.$$

The notation $\mathcal{D}'(\Omega)$ designates the dual space of $\mathcal{D}(\Omega)$, i.e., the space of distributions on Ω. The notations $H^m(\Omega)$ for each integer $m \geq 0$, with $H^0(\Omega) := L^2(\Omega)$, and the notations $H_0^m(\Omega)$ and $H^{-m}(\Omega)$ for each integer $m \geq 1$, designate the usual Sobolev spaces and their duals.

Spaces of functions or distributions, vector fields, and symmetric matrix fields, i.e., with values in $\mathbb{S}^N$, are respectively denoted by italic capitals, boldface capitals, and special Roman capitals. For instance,

$$\boldsymbol{\mathcal{D}}(\Omega) = \mathcal{D}(\Omega;\mathbb{R}^N), \quad \boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega;\mathbb{R}^N),$$
$$\mathbb{D}'(\Omega) = \mathcal{D}'(\Omega;\mathbb{S}^N), \quad \mathbb{H}^{-1}(\Omega) = H^{-1}(\Omega;\mathbb{S}^N), \text{ etc.}$$

Such spaces are equipped with their natural product norms. For instance,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} := \Big\{ \int_\Omega \Big(\sum_i |v_i|^2 + \sum_{i,j} |\partial_j v_i|^2 \Big) \mathrm{d}x \Big\}^{\frac{1}{2}} \quad \text{for each } \boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega),$$
$$\|e_{ij}\|_{\mathbb{H}^1(\Omega)} := \Big\{ \int_\Omega \Big(\sum_{i,j} |e_{ij}|^2 + \sum_{i,j,k} |\partial_k e_{ij}|^2 \Big) \mathrm{d}x \Big\}^{\frac{1}{2}} \quad \text{for each } \boldsymbol{e} = (e_{ij}) \in \mathbb{H}^1(\Omega), \text{ etc.}$$

The following differential operators will be used in the sequel:

$$\mathbf{grad} : \mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}}'(\Omega), \quad \text{with } (\mathbf{grad}\, f)_i := \partial_i f \text{ for each } f \in \mathcal{D}'(\Omega),$$
$$\nabla_s : \boldsymbol{\mathcal{D}}'(\Omega) \to \mathbb{D}'(\Omega), \quad \text{with } (\nabla_s \boldsymbol{v})_{ij} := \frac{1}{2}(\partial_j v_i + \partial_i v_j) \text{ for each } \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega),$$
$$\mathbf{div} : \mathbb{D}'(\Omega) \to \boldsymbol{D}'(\Omega), \quad \text{with } (\mathbf{div}\, \boldsymbol{e})_i := \sum_j \partial_j e_{ij} \text{ for each } \boldsymbol{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

Note that, when Ω is connected, the kernel of the symmetrized gradient operator ∇_s has a well-known characterization, viz.,

$$\operatorname{Ker} \nabla_s = \{ \boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega); \text{there exist } \boldsymbol{A} \in \mathbb{A}^N \text{ and } \boldsymbol{b} \in \mathbb{R}^N \text{ such that}$$
$$\boldsymbol{v}(x) = \boldsymbol{A}x + \boldsymbol{b} \text{ for each } x \in \Omega \}.$$

3 An Equivalence Theorem

We now show that the vector version of J. L. Lions lemma of Theorem 1.2 with $m = 1$ is equivalent to a certain number of other properties, respectively noted (a), (b), (c), (d), (e), (f), and (g), in the next theorem.

In addition to their interest per se, some of these properties have interesting interpretations.

For instance, property (b) constitutes a "weak" vector version of J. L. Lions lemma, which is nothing but the natural vector version of the "original" lemma of J. L. Lions.

Property (c) constitutes a natural vector version of a well-known inequality due to Nečas in the "scalar" case (see [21–22], or Bramble [7] for a different proof).

Property (e) constitutes a weak version of the classical Donati compatibility conditions; see [3, 12–13], where such conditions were used to define and analyze an intrinsic approach to linearized elasticity (a quick introduction to Donati compatibility conditions is found in Subsection 6.18 of [8]). These conditions are said here to be "weak" to reflect that the given matrix field $\boldsymbol{e}$ is now in the space $\mathbb{H}^{-1}(\Omega)$ instead of the space $\mathbb{L}^2(\Omega)$ and the duality ${}_{\mathbb{H}^{-1}(\Omega)}\langle \boldsymbol{e}, \boldsymbol{s} \rangle_{\mathbb{H}_0^1(\Omega)}$ replaces the inner product $\int_\Omega \boldsymbol{e}.\boldsymbol{s} \mathrm{d}x$.

Property (f) constitutes the natural vector version of the surjectivity of the operator div : $\boldsymbol{H}_0^1(\Omega) \to \{ f \in L^2(\Omega); \int_\Omega f \mathrm{d}x = 0 \}$, a fundamental property which goes back to Ladyzhenskaya [17], Ladyzhenskaya and Solonnikov [18], Temam [24], Bagovskii [5].

Note that the implications (b) implies (c), (c) implies (d), and (d) implies (e), in Theorem 3.1 were already established, albeit in a slightly different form, in [3]; in particular, the introduction of the space denoted $\boldsymbol{L}_0^2(\Omega)$, instead of the quotient space $\boldsymbol{L}^2(\Omega)/\operatorname{Ker}\nabla_s$ in [3], somewhat simplifies the arguments. Otherwise, the implications (e) implies (f), (f) implies (g), and (g) implies (a), are new.

The definition of a domain in $\mathbb{R}^N$ has been given in Section 1.

Theorem 3.1 (Equivalence Theorem) *Let Ω be a domain in $\mathbb{R}^N$. The following properties are equivalent:*

(a) *Vector version of J. L. Lions lemma*:

$$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega) \text{ and } \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

(b) *Weak vector version of J. L. Lions lemma*:

$$\boldsymbol{v} \in \boldsymbol{H}^{-1}(\Omega) \text{ and } \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

(c) *Vector version of Nečas inequality*: *There exists a constant $C_0(\Omega)$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C_0(\Omega)(\|\boldsymbol{v}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\nabla_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)}) \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

(d) *The operator ∇_s has a closed range*: *The image of the space*

$$\boldsymbol{L}_0^2(\Omega) := \Big\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \int_\Omega \boldsymbol{v} \cdot \boldsymbol{r} \,\mathrm{d}x = 0 \quad \textit{for all } \boldsymbol{r} \in \operatorname{Ker}\nabla_s\Big\}$$

under the operator ∇_s is a closed subspace of the space $\mathbb{H}^{-1}(\Omega)$.

(e) *Weak Donati compatibility conditions*: *Given a symmetric matrix field $\boldsymbol{e} \in \mathbb{H}^{-1}(\Omega)$, there exists a vector field $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$ such that*

$$\nabla_s \boldsymbol{v} = \boldsymbol{e}$$

if (*and clearly only if*)

$${}_{\mathbb{H}^{-1}(\Omega)}\langle \boldsymbol{e}, \boldsymbol{s}\rangle_{\mathbb{H}_0^1(\Omega)} = 0 \quad \textit{for all } \boldsymbol{s} \in \mathbb{H}_0^1(\Omega) \textit{ that satisfy } \ \mathbf{div}\, \boldsymbol{s} = \mathbf{0} \textit{ in } \Omega.$$

If this is the case, the vector field $\boldsymbol{v}$ is uniquely determined in the space $\boldsymbol{L}_0^2(\Omega)$.

(f) *The operator*

$$\mathbf{div} : \mathbb{H}_0^1(\Omega) \to \boldsymbol{L}_0^2(\Omega)$$

is onto: *Consequently, for each vector field $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$, there exists a unique symmetric matrix field $\boldsymbol{e}(\boldsymbol{v})$ in the orthogonal complement* $(\operatorname{Ker}\mathbf{div})^\perp$ *of* $\operatorname{Ker}\mathbf{div}$ *in the Hilbert space $\mathbb{H}_0^1(\Omega)$ that satisfies*

$$\mathbf{div}(\boldsymbol{e}(\boldsymbol{v})) = \boldsymbol{v} \quad \textit{in } \Omega,$$

and there exists a constant $C_1(\Omega)$ such that

$$\|\boldsymbol{e}(\boldsymbol{v})\|_{\mathbb{H}_0^1(\Omega)} \leq C_1(\Omega)\|\boldsymbol{v}\|_{\boldsymbol{L}_0^2(\Omega)} \quad \textit{for all } \ \boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega).$$

(g) *Approximation property*: *Assume that the domain Ω is starlike with respect to an open ball. Then there exists a constant $C_2(\Omega)$ with the following property: Given any vector field $\boldsymbol{\varphi}$ in the space*

$$\boldsymbol{\mathcal{D}}_0(\Omega) := \Big\{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega); \ \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{r} \,\mathrm{d}x = 0 \textit{ for all } \boldsymbol{r} \in \operatorname{Ker}\nabla_s\Big\},$$

there exist symmetric matrix fields $\boldsymbol{e}_n(\boldsymbol{\varphi}) \in \mathbb{D}(\Omega)$, $n \geq 1$, *such that*

$$\|\boldsymbol{e}_n(\boldsymbol{\varphi})\|_{\mathbb{H}^1(\Omega)} \leq C_2(\Omega)\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \quad \textit{for all } n \geq 1,$$
$$\mathbf{div}(\boldsymbol{e}_n(\boldsymbol{\varphi})) \to \boldsymbol{\varphi} \quad \textit{in } \boldsymbol{\mathcal{D}}(\Omega) \text{ as } n \to \infty.$$

Proof (a) implies (b): Obvious.

(b) implies (c). It is easily verified that the space

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^{-1}(\Omega);\ \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega)\}$$

equipped with the norm $\|\cdot\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\nabla_s \cdot\|_{\mathbb{H}^{-1}(\Omega)}$ is a Banach space. Since the canonical injection $\iota : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}(\Omega)$, which is clearly continuous, is onto by (b), the open mapping theorem shows that the inverse mapping ι^{-1} is continuous, which is precisely what inequality (c) expresses.

(c) implies (d). First, note that $\boldsymbol{L}_0^2(\Omega)$ is the orthogonal complement of $\operatorname{Ker}\nabla_s$ in $\boldsymbol{L}^2(\Omega)$; consequently $\boldsymbol{L}_0^2(\Omega)$ is closed in $\boldsymbol{L}^2(\Omega)$ and the restriction to $\boldsymbol{L}_0^2(\Omega)$ of the operator $\nabla_s : \boldsymbol{L}^2(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is one-to-one.

Second, by a well-known property of linear operators between Banach spaces, property (d) holds if

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C(\Omega)\|\nabla_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega).$$

Assume by contradiction that such a constant $C(\Omega)$ does not exist. Then there exists in this case a sequence $(\boldsymbol{v}_k)_{k=1}^{\infty}$ of vector fields $\boldsymbol{v}_k \in \boldsymbol{L}_0^2(\Omega)$ that satisfy

$$\|\boldsymbol{v}_k \cdot\|_{\boldsymbol{L}^2(\Omega)} = 1 \quad \text{for all } k \geq 1 \quad \text{and} \quad \|\nabla_s \boldsymbol{v}_k \cdot\|_{\mathbb{H}^{-1}(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

By the Rellich-Kondrasov theorem "in the space $L^2(\Omega)$" (cf., e.g., Theorem 6.11–3 in [8]), there thus exists a subsequence $(\boldsymbol{v}_\ell)_{\ell=1}^{\infty}$ of $(\boldsymbol{v}_k)_{k=1}^{\infty}$ that converges in the space $\boldsymbol{H}^{-1}(\Omega)$, on the one hand.

Since the sequence $(\nabla_s \boldsymbol{v}_\ell)_{\ell=1}^{\infty}$ converges in the space $\mathbb{H}^{-1}(\Omega)$ (to $\mathbf{0}$) on the other hand, the sequence $(\boldsymbol{v}_\ell)_{\ell=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\nabla_s \cdot\|_{\mathbb{H}^{-1}(\Omega)}$, hence also with respect to the norm $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$ by (c).

Therefore, there exists an element $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$ such that

$$\boldsymbol{v}_\ell \to \boldsymbol{v} \quad \text{in } \boldsymbol{L}_0^2(\Omega) \text{ as } \ell \to \infty,$$

and thus,

$$\nabla_s \boldsymbol{v}_\ell \to \nabla_s \boldsymbol{v} = \mathbf{0} \quad \text{in } \mathbb{H}^{-1}(\Omega) \text{ as } \ell \to \infty,$$

since the operator $\nabla_s : \boldsymbol{L}^2(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is continuous. Consequently, $\boldsymbol{v} = \mathbf{0}$ since $\nabla_s : \boldsymbol{L}_0^2(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is one-to-one. But this contradicts the relation $\|\boldsymbol{v}_\ell\|_{\boldsymbol{L}^2(\Omega)} = 1$ for all $\ell \geq 1$.

(d) implies (e). First, the operator $\mathbf{div} : \mathbb{H}_0^1(\Omega) \to \boldsymbol{L}^2(\Omega)$ maps the space $\mathbb{H}_0^1(\Omega)$ into the subspace $\boldsymbol{L}_0^2(\Omega)$ of $\boldsymbol{L}^2(\Omega)$. To see this, note that, for all $\boldsymbol{e} = (e_{ij}) \in \mathbb{H}_0^1(\Omega)$ and all $\boldsymbol{r} = (r_i) \in \operatorname{Ker}\nabla_s$,

$$\begin{aligned}\int_\Omega \mathbf{div}\,\boldsymbol{e} \cdot \boldsymbol{r}\,\mathrm{d}x &= \sum_{i,j} \int_\Omega (\partial_j e_{ij}) r_i \,\mathrm{d}x = -\sum_{i,j} \int_\Omega e_{ij} \partial_j r_i \,\mathrm{d}x \\ &= -\frac{1}{2} \sum_{i,j} \int_\Omega e_{ij}(\partial_j r_i + \partial_i r_j)\,\mathrm{d}x = 0.\end{aligned}$$

Next, given any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{L}^2_0(\Omega)$ and any matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{H}^1_0(\Omega)$,

$$
\begin{aligned}
{}_{\mathbb{H}^{-1}(\Omega)}\langle \nabla_s \boldsymbol{v}, \boldsymbol{e}\rangle_{\mathbb{H}^1_0(\Omega)} &= \frac{1}{2}\sum_{i,j} {}_{H^{-1}(\Omega)}\langle \partial_j v_i + \partial_i v_j, e_{ij}\rangle_{H^1_0(\Omega)} \\
&= \sum_{i,j} {}_{H^{-1}(\Omega)}\langle \partial_j v_i, e_{ij}\rangle_{H^1_0(\Omega)} = -\sum_{i,j} {}_{L^2_0(\Omega)}\langle v_i, \partial_j e_{ij}\rangle_{L^2_0(\Omega)} \\
&= -{}_{\boldsymbol{L}^2_0(\Omega)}\langle \boldsymbol{v}, \mathbf{div}\, \boldsymbol{e}\rangle_{\boldsymbol{L}^2_0(\Omega)}
\end{aligned}
$$

(the space $\boldsymbol{L}^2_0(\Omega)$ is identified here with its dual space). These relations mean that $\nabla_s : \boldsymbol{L}^2_0(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is the dual operator of $-\mathbf{div} : \mathbb{H}^1_0(\Omega) \to \boldsymbol{L}^2_0(\Omega)$.

Banach closed range theorem then shows that the operator ∇_s has a closed range (property (d)) if and only if

$$\operatorname{Im} \nabla_s = (\operatorname{Ker} \mathbf{div})^0,$$

which is exactly property (e).

Finally, the solution $\boldsymbol{v} \in \boldsymbol{L}^2_0(\Omega)$ to the equation $\nabla_s \boldsymbol{v} = \boldsymbol{e}$ is unique since $\nabla_s : \boldsymbol{L}^2_0(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is one-to-one.

(e) implies (f). The relations above also mean that $-\mathbf{div}: \mathbb{H}^1_0(\Omega) \to \boldsymbol{L}^2_0(\Omega)$ is the dual operator of $\nabla_s : \boldsymbol{L}^2_0(\Omega) \to \mathbb{H}^{-1}(\Omega)$. Banach closed range theorem then shows that

$$\operatorname{Im} \mathbf{div} = (\operatorname{Ker} \nabla_s)^0.$$

But $\nabla_s : \boldsymbol{L}^2_0(\Omega) \to \mathbb{H}^{-1}(\Omega)$ is one-to-one, as already noted. Therefore

$$\operatorname{Im} \mathbf{div} = \boldsymbol{L}^2_0(\Omega),$$

as was to be proved. The continuous operator $\mathbf{div} : (\operatorname{Ker} \mathbf{div})^\perp \to \boldsymbol{L}^2_0(\Omega)$ is thus onto.

Since the same operator is also one-to-one, the open mapping theorem shows that it has a continuous inverse, which is precisely what the other assertions in (f) express.

(f) implies (g). There is no loss of generality in assuming that Ω is starlike with respect to a ball $B(0; r)$, i.e., centered at the origin.

Let $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_0(\Omega)$ be given and kept fixed in the ensuing argument. Since $\boldsymbol{\mathcal{D}}_0(\Omega) \subset \boldsymbol{L}^2_0(\Omega)$, there exists a unique matrix field $\boldsymbol{e} = (e_{ij}) \in (\operatorname{Ker} \mathbf{div})^\perp \subset \mathbb{H}^1_0(\Omega)$ such that

$$\mathbf{div}\, \boldsymbol{e} = \boldsymbol{\varphi} \text{ and } \|\boldsymbol{e}\|_{\mathbb{H}^1(\Omega)} \le C_1(\Omega\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2_0(\Omega)}$$

with a constant $C_1(\Omega)$ independent of $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_0(\Omega)$. Of course, the matrix field $\boldsymbol{e}$, like the matrix fields $\boldsymbol{s}_n$ and $\boldsymbol{e}_n$ introduced below, all depend on $\boldsymbol{\varphi}$.

Given any integer $n_0 \ge 1$ that satisfies $n_0 > \frac{2}{r}$, let

$$
\begin{aligned}
&\lambda_n := 1 - \frac{2}{nr}, \quad \Omega_n := \lambda_n \Omega \subset \Omega \quad \text{for each } n \ge n_0, \\
&\boldsymbol{s}_n(y) := \lambda_n \boldsymbol{e}\Big(\frac{y}{\lambda_n}\Big), \quad \text{if } y \in \Omega_n, \\
&\boldsymbol{s}_n(y) := 0, \quad \text{if } y \in \Omega - \Omega_n \ \text{ for each } n \ge n_0.
\end{aligned}
$$

For all $n \ge n_0$, we thus have

$$\operatorname{dist}(x, \partial\Omega) > \frac{2}{n} \quad \text{for all } x \in \Omega_n$$

(the assumption that Ω is starlike with respect to $B(0;r)$ is used here), and

$$\boldsymbol{s}_n|_{\Omega_n} \in \mathbb{H}_0^1(\Omega_n), \quad \boldsymbol{s}_n|_{\Omega-\Omega_n} = \boldsymbol{0}, \quad \mathbf{div}\, \boldsymbol{s}_n = \boldsymbol{\varphi}\Big(\frac{\cdot}{\lambda_n}\Big) \quad \text{in } \Omega,$$

where $\boldsymbol{\varphi}\big(\frac{\cdot}{\lambda_n}\big)$ designates the extension by $\boldsymbol{0}$ on $\Omega - \Omega_n$ of the vector field defined by $\boldsymbol{\varphi}\big(\frac{y}{\lambda_n}\big)$ if $y \in \Omega_n$ and by $\boldsymbol{0}$ if $y \in \Omega - \Omega_n$.

Let $(\rho_n)_{n=1}^{\infty}$ be a sequence of mollifiers, i.e., that satisfy

$$\rho_n \in \mathcal{C}^{\infty}(\mathbb{R}^N), \quad \operatorname{supp} \rho_n \subset \overline{B\Big(0; \frac{1}{n}\Big)}, \quad \text{and} \quad \int_{\Omega} \rho_n(y)\mathrm{d}y = 1,$$

and let, for each $n \geq n_0$,

$$\begin{aligned}
\widetilde{\Omega}_n &:= \Big\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\Big\}, \\
\boldsymbol{e}_n(x) &:= \int_{B(x;\frac{1}{n})} \boldsymbol{s}_n(y)\rho_n(x-y)\mathrm{d}y, \quad \text{if } x \in \widetilde{\Omega}_n, \\
\boldsymbol{e}_n(x) &:= \boldsymbol{0}, \quad \text{if } x \in \Omega - \widetilde{\Omega}_n.
\end{aligned}$$

In other words,

$$\boldsymbol{e}_n = \boldsymbol{s}_n * \rho_n,$$

where $*$ designates the convolution product.

Then the smoothing property of such convolution products and the inclusions $\operatorname{supp} \boldsymbol{e}_n \subset \{\widetilde{\Omega}_n\}^-$ together imply that

$$\boldsymbol{e}_n \in \boldsymbol{\mathcal{D}}(\Omega) \quad \text{for each } n \geq n_0.$$

Besides, by another well-known property of convolution products,

$$\|\boldsymbol{e}_n\|_{\mathbb{H}^1(\Omega)} \leq \|\boldsymbol{s}_n\|_{\mathbb{H}^1(\Omega)}.$$

Taking as a new variable $y = \frac{x}{\lambda_n} \in \Omega$ for each $x \in \Omega_n$ gives

$$\begin{aligned}
\|\boldsymbol{s}_n\|^2_{\mathbb{H}^1(\Omega)} = \|\boldsymbol{s}_n\|^2_{\mathbb{H}^1(\Omega_n)} &= \sum_{i,j} \int_{\Omega_n} \Big(\Big|\lambda_n e_{ij}\Big(\frac{x}{\lambda_n}\Big)\Big|^2 + \sum_k \Big|\partial_k e_{ij}\Big(\frac{x}{\lambda_n}\Big)\Big|^2\Big)\mathrm{d}x \\
&= \sum_{i,j} \int_{\Omega} \Big(\lambda_n^{N+2}|e_{ij}(y)|^2 + \sum_k \lambda_n^N |\partial_k e_{ij}(y)|^2\Big)\mathrm{d}y \leq \|\boldsymbol{e}\|^2_{\mathbb{H}^1(\Omega)},
\end{aligned}$$

so that, as announced,

$$\|\boldsymbol{e}_n\|_{\mathbb{H}^1(\Omega)} \leq \|\boldsymbol{s}_n\|_{\mathbb{H}^1(\Omega)} \leq \|\boldsymbol{e}\|_{\mathbb{H}^1(\Omega)} \leq C_1(\Omega)\|\boldsymbol{\varphi}\|_{\boldsymbol{L}_0^2(\Omega)} \quad \text{for all } n \geq n_0.$$

It remains to show that

$$\mathbf{div}\, \boldsymbol{e}_n \to \boldsymbol{\varphi} \quad \text{in } \boldsymbol{\mathcal{D}}(\Omega) \text{ as } n \to \infty$$

(the definition of convergence in the space $\mathcal{D}(\Omega)$ is recalled in Section 2).

Since $\operatorname{supp} \boldsymbol{\varphi} \Subset \Omega$ and

$$\mathbf{div}\, \boldsymbol{e}_n = \mathbf{div}(\boldsymbol{s}_n * \rho_n) = (\mathbf{div}\boldsymbol{s}_n) * \rho_n = \boldsymbol{\varphi}\Big(\frac{\cdot}{\lambda_n}\Big) * \rho_n \quad \text{for all } n \geq n_0,$$

there exist a constant $\alpha > 0$ and an integer $n_1 \geq n_0$ such that

$$(\operatorname{supp} \mathbf{div}\, \boldsymbol{e}_n) \cup (\operatorname{supp} \boldsymbol{\varphi}) \subset K := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \geq \alpha\} \quad \text{for all } n \geq n_1,$$

on the one hand.

Using yet another well-known property of convolution products with mollifiers, we infer that, for each multi-index $\boldsymbol{\alpha}$ and each integer $n \geq n_1$,

$$\partial^{\boldsymbol{\alpha}}(\mathbf{div}\, \boldsymbol{e}_n) = \partial^{\boldsymbol{\alpha}}\Big(\boldsymbol{\varphi}\Big(\frac{\cdot}{\lambda_n}\Big) * \rho_n\Big) = \partial^{\boldsymbol{\alpha}}\Big(\boldsymbol{\varphi}\Big(\frac{\cdot}{\lambda_n}\Big)\Big) * \rho_n.$$

Using that $\int_\Omega \rho_n(y)\mathrm{d}y = 1$, we may thus write

$$\begin{aligned}\partial^{\boldsymbol{\alpha}}(\mathbf{div}\boldsymbol{e}_n)(x) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x) &= \int_\Omega \frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}}\Big(\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}\Big(\frac{x-y}{\lambda_n}\Big) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x)\Big)\rho_n(y)\mathrm{d}y \\ &= \int_\Omega \Big\{\Big(\frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}} - 1\Big)\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}\Big(\frac{x-y}{\lambda_n}\Big) + \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}\Big(\frac{x-y}{\lambda_n}\Big) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x)\Big\}\rho_n(y)\mathrm{d}y\end{aligned}$$

at each $x \in \Omega$. Consequently,

$$\begin{aligned}&\sup_{x\in K} |\partial^{\boldsymbol{\alpha}}(\mathbf{div}\boldsymbol{e}_n)(x) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x)| \\ &\leq \Big(\frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}} - 1\Big)\sup_{z\in\Omega}|\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(z)| \\ &\quad + \sup_{x\in K}\Big|\int_{B(0;\frac{1}{n})}\Big\{\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}\Big(x + \Big(\frac{1-\lambda_n}{\lambda_n}\Big)x - \frac{y}{\lambda_n}\Big) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x)\Big\}\rho_n(y)\mathrm{d}y\Big|.\end{aligned}$$

It is clear that $\sup\limits_{x\in K}\ \sup\limits_{y\in B(0;\frac{1}{n})} \big|\big(\frac{1-\lambda_n}{\lambda_n}\big)x - \frac{y}{\lambda_n}\big| \to 0$ as $n \to \infty$ since $\lambda_n = 1 - \frac{2}{nr} \to 1$ as $n \to \infty$. Together, this property and the uniform continuity and boundedness of each partial derivative $\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}$ on Ω therefore imply that

$$\sup_{x\in K}\big|\partial^{\boldsymbol{\alpha}}(\mathbf{div}\boldsymbol{e}_n)(x) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(x)\big| \to 0, \quad \text{as } n \to \infty,$$

on the other hand.

We have thus shown that $\mathbf{div}\, \boldsymbol{e}_n \to \boldsymbol{\varphi}$ in $\boldsymbol{\mathcal{D}}(\Omega)$ as $n \to \infty$.

(g) implies (a). Let a vector field $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega)$. In order to show that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, it suffices to show that there exists a constant $C(\Omega, \boldsymbol{v})$ such that

$$|{}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)}| \leq C(\Omega, \boldsymbol{v})\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),$$

thanks to the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$ and to F. Riesz representation theorem.

To this end, we proceed in three stages, numbered (i)–(iii).

(i) There exists a constant $c_0(\Omega)$ independent of $\boldsymbol{v}$ with the following property: Given any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, there exists a vector field $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{D}}(\Omega)$, which depends on $\boldsymbol{\varphi}$, such that

$$\begin{aligned}&(\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \in \boldsymbol{\mathcal{D}}_0(\Omega) := \Big\{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \int_\Omega \boldsymbol{\psi}\cdot\boldsymbol{r}\mathrm{d}x = 0 \text{ for all } \boldsymbol{r} \in \operatorname{Ker}\nabla_s\Big\}, \\ &\|\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}\|_{\boldsymbol{L}^2(\Omega)} \leq c_0(\Omega)\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Assume without loss of generality that $0 \in \Omega$. It is easily seen, by defining appropriate products of N even or odd functions in $\mathcal{D}(\mathbb{R})$ with small enough support, that there exist functions $\theta_i \in \mathcal{D}(\Omega)$, $1 \le i \le N$, that satisfy

$$\int_\Omega \theta_1 \mathrm{d}x = 1 \text{ and } \int_\Omega \theta_1 x_j \mathrm{d}x = 0, \quad 1 \le j \le N,$$
$$\int_\Omega \theta_i \mathrm{d}x = 0 \text{ and } \int_\Omega \theta_i x_j \mathrm{d}x = 2\delta_{ij}, \quad 1 \le j \le N,\ 2 \le i \le N.$$

Let $\boldsymbol{e}_k$ denote the basis vectors of $\mathbb{R}^N$. Then the vector fields defined by

$$\boldsymbol{\eta}_k := \theta_1 \boldsymbol{e}_k \in \boldsymbol{\mathcal{D}}(\Omega), \quad 1 \le k \le N, \quad \text{and} \quad \boldsymbol{\eta}_{k\ell} := \theta_\ell \boldsymbol{e}_k \in \boldsymbol{\mathcal{D}}(\Omega), \quad 1 \le k < \ell \le N,$$

satisfy respectively

$$\int_\Omega (\boldsymbol{\eta}_k)_i \mathrm{d}x = \delta_{ik} \ \text{ and } \int_\Omega (\boldsymbol{\eta}_k)_i x_j \mathrm{d}x = 0, \quad 1 \le i, j, k \le N,$$
$$\int_\Omega (\boldsymbol{\eta}_{k\ell})_i \mathrm{d}x = 0 \ \text{ and } \int_\Omega (\boldsymbol{\eta}_{k\ell})_i x_j \mathrm{d}x = 2\delta_{ik}\delta_{j\ell}, \quad 1 \le i, j, \le N,\ 1 \le k < \ell \le N.$$

Given any vector field $\boldsymbol{\varphi} = (\varphi_i) \in \boldsymbol{\mathcal{D}}(\Omega)$, let the vector field $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{D}}(\Omega)$ be defined by

$$\widetilde{\boldsymbol{\varphi}} := \sum_k \Big(\int_\Omega \varphi_k \mathrm{d}x \Big) \boldsymbol{\eta}_k + \frac{1}{2} \sum_{k<\ell} \Big(\int_\Omega (\varphi_k x_\ell - \varphi_\ell x_k) \mathrm{d}x \Big) \boldsymbol{\eta}_{k\ell}.$$

We then claim that the vector field $(\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \in \boldsymbol{\mathcal{D}}(\Omega)$ satisfies the announced properties: First,

$$\int_\Omega \widetilde{\varphi}_i \mathrm{d}x = \sum_k \Big(\int_\Omega \varphi_k \mathrm{d}x \Big) \Big(\int_\Omega (\boldsymbol{\eta}_k)_i \mathrm{d}x \Big) = \int_\Omega \varphi_i \mathrm{d}x, \quad 1 \le i \le N.$$

Second, given any numbers a_{ij} satisfying $a_{ij} = -a_{ji}$, $1 \le i,\ j \le N$,

$$\sum_{i,j} \int_\Omega (\varphi_i - \widetilde{\varphi}_i) a_{ij} x_j \mathrm{d}x = \sum_{i<j} \int_\Omega \{(\varphi_i - \widetilde{\varphi}_i) x_j - (\varphi_j - \widetilde{\varphi}_j) x_i\} a_{ij} \mathrm{d}x.$$

Noting that

$$\int_\Omega \widetilde{\varphi}_i x_j \mathrm{d}x = \frac{1}{2} \sum_{k<\ell} \Big(\int_\Omega (\varphi_k x_\ell - \varphi_\ell x_k) \mathrm{d}x \Big) \int_\Omega (\boldsymbol{\eta}_{k\ell})_i x_j \mathrm{d}x$$
$$= \begin{cases} \displaystyle\int_\Omega (\varphi_i x_j - \varphi_j x_i) \mathrm{d}x, & \text{if } i < j, \\ 0, & \text{if } i \ge j, \end{cases}$$

and that, likewise,

$$\int_\Omega \widetilde{\varphi}_j x_i \mathrm{d}x = \begin{cases} \displaystyle\int_\Omega (\varphi_j x_i - \varphi_i x_j) \mathrm{d}x, & \text{if } j < i, \\ 0, & \text{if } i \le j, \end{cases}$$

we infer that

$$\int_\Omega \{(\varphi_i - \widetilde{\varphi}_i) x_j - (\varphi_j - \widetilde{\varphi}_j) x_i\} \mathrm{d}x = 0 \quad \text{for all } i < j.$$

We have thus shown that

$$\int_\Omega (\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \cdot (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}) \mathrm{d}x = 0 \quad \text{for all } \boldsymbol{A} \in \mathbb{A}^N \text{ and all } \boldsymbol{b} \in \mathbb{R}^N,$$

i.e., that $(\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \in \boldsymbol{\mathcal{D}}_0(\Omega)$, by the characterization of the space $\operatorname{Ker} \nabla_s$ (cf. Section 2). The existence of a constant $c_0(\Omega)$ independent of $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\|\boldsymbol{\varphi}-\widetilde{\boldsymbol{\varphi}}\|_{\boldsymbol{L}^2(\Omega)} \leq c_0(\Omega)\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)}$ follows from the triangle inequality and the Cauchy-Schwarz inequality.

(ii) Assume that Ω is starlike with respect to a ball. Since $(\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \in \boldsymbol{\mathcal{D}}_0(\Omega)$, there exist by (g) matrix fields $\boldsymbol{e}_n \in \mathbb{D}(\Omega)$, $n \geq 1$, that depend on $(\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}})$, hence on $\boldsymbol{\varphi}$, such that

$$\begin{aligned}
&\|\boldsymbol{e}_n\|_{\mathbb{H}^1(\Omega)} \leq C_2(\Omega)\|\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}\|_{\boldsymbol{L}^2(\Omega)} \leq C_2(\Omega)c_0(\Omega)\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } n \geq 1,\\
&\mathbf{div}\, \boldsymbol{e}_n \to (\boldsymbol{\varphi} - \widetilde{\boldsymbol{\varphi}}) \text{ in } \boldsymbol{\mathcal{D}}(\Omega) \quad \text{as } n \to \infty.
\end{aligned}$$

Given any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, the scalar $\langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle$ can therefore be written as

$$\langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{v}, \widetilde{\boldsymbol{\varphi}} \rangle + \lim_{n\to\infty} \langle \boldsymbol{v}, \mathbf{div} \boldsymbol{e}_n \rangle,$$

where, for notational brevity, we let

$$\langle \cdot, \cdot \rangle := {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{D}}(\Omega)},$$

like in the rest of this part of the proof. Let us examine the two terms appearing in the above expression of $\langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle$.

First, by definition of $\widetilde{\boldsymbol{\varphi}}$ (see (i)),

$$\langle \boldsymbol{v}, \widetilde{\boldsymbol{\varphi}} \rangle = \sum_k \Big(\int_\Omega \varphi_k \mathrm{d}x \Big) \langle \boldsymbol{v}, \boldsymbol{\eta}_k \rangle + \frac{1}{2} \sum_{k,\ell} \Big(\int_\Omega (\varphi_k x_\ell - \varphi_\ell x_k) \mathrm{d}x \Big) \langle \boldsymbol{v}, \boldsymbol{\eta}_{k\ell} \rangle.$$

Hence there exists a constant $c_1(\Omega, \boldsymbol{v})$ such that

$$|\langle \boldsymbol{v}, \widetilde{\boldsymbol{\varphi}} \rangle| \leq c_1(\Omega, \boldsymbol{v})\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega).$$

Second, the assumption that $\nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega)$ implies that

$$\langle \boldsymbol{v}, \mathbf{div}\, \boldsymbol{e}_n \rangle = -\, {}_{\mathbb{H}^{-1}(\Omega)}\langle \nabla_s \boldsymbol{v}, \boldsymbol{e}_n \rangle_{\mathbb{H}^1_0(\Omega)} \quad \text{for all } n \geq 1,$$

hence that

$$|\langle \boldsymbol{v}, \mathbf{div}\, \boldsymbol{e}_n \rangle| \leq \|\nabla_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)} \|\boldsymbol{e}_n\|_{\mathbb{H}^1_0(\Omega)} \leq c_2(\Omega, \boldsymbol{v})\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)},$$

with $c_2(\Omega, \boldsymbol{v}) := \|\nabla_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)} C_2(\Omega) c_0(\Omega)$.

Consequently,

$$|\langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle| \leq C(\Omega, \boldsymbol{v})\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$$

with $C(\Omega, \boldsymbol{v}) := c_1(\Omega, \boldsymbol{v}) + c_2(\Omega, \boldsymbol{v})$. We have thus shown that the vector version of J. L. Lions lemma holds on a domain that is starlike with respect to a ball.

(iii) Finally, assume that Ω is a general domain. In this case, arguments similar to those used in Maz'ya [20] or Costabel and McIntosh [9] show that Ω can be written as a finite union

$$\Omega = \bigcup_{j=1}^m \Omega_j$$

of domains $\Omega_j \subset \Omega$, $1 \leq j \leq m$, each of which is starlike with respect to a ball.

Given any open subset U of Ω and any vector field $\boldsymbol{\theta} \in \boldsymbol{\mathcal{D}}(U)$, the notation $\boldsymbol{\theta}^\sharp$ designates in what follows the extension of $\boldsymbol{\theta}$ by $\mathbf{0}$ on $\Omega - U$, so that $\boldsymbol{\theta}^\sharp \in \boldsymbol{\mathcal{D}}(\Omega)$.

For each $1 \leq j \leq m$, the linear form

$$\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega_j) \to {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \boldsymbol{\varphi}^\sharp \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \in \mathbb{R}$$

defines a distribution, denoted $\boldsymbol{v}_j$, on Ω_j, which therefore satisfies

$${}_{\boldsymbol{\mathcal{D}}'(\Omega_j)}\langle \boldsymbol{v}_j, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega_j)} = {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \boldsymbol{\varphi}^\sharp \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega_j).$$

Consequently, given any matrix field $\boldsymbol{e} \in \mathbb{D}(\Omega_j)$,

$$\begin{aligned}{}_{\mathbb{D}'(\Omega_j)}\langle \nabla_s \boldsymbol{v}_j, \boldsymbol{e} \rangle_{\mathbb{D}(\Omega_j)} &= -{}_{\boldsymbol{\mathcal{D}}'(\Omega_j)}\langle \boldsymbol{v}_j, \mathbf{div}\, \boldsymbol{e} \rangle_{\boldsymbol{\mathcal{D}}(\Omega_j)} \\ &= -{}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, (\mathbf{div} \boldsymbol{e})^\sharp \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = {}_{\mathbb{H}^{-1}(\Omega)}\langle \nabla_s \boldsymbol{v}, \boldsymbol{e}^\sharp \rangle_{\mathbb{H}^1_0(\Omega)},\end{aligned}$$

and thus

$$|{}_{\mathbb{D}'(\Omega_j)}\langle \nabla_s \boldsymbol{v}_j, \boldsymbol{e} \rangle_{\mathbb{D}(\Omega_j)}| \leq \|\nabla_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)} \|\boldsymbol{e}\|_{\mathbb{H}^1_0(\Omega_j)} \quad \text{for all } \boldsymbol{e} \in \mathbb{D}(\Omega_j).$$

The last relation shows that $\nabla_s \boldsymbol{v}_j \in \mathbb{H}^{-1}(\Omega_j)$. The vector version of J. L. Lions lemma on the domain Ω_j (see (ii)) thus implies that each vector field $\boldsymbol{v}_j \in \mathbb{D}'(\Omega_j)$ can be identified with a vector field in $\mathbb{L}^2(\Omega_j)$, in the sense that

$${}_{\boldsymbol{\mathcal{D}}'(\Omega_j)}\langle \boldsymbol{v}_j, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega_j)} = \int_{\Omega_j} \boldsymbol{v}_j \cdot \boldsymbol{\varphi} \mathrm{d}x \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega_j).$$

Besides, given any $1 \leq j,\, k \leq N$, the relation

$$\begin{aligned}\int_{\Omega_j \cap \Omega_k} \boldsymbol{v}_j \cdot \boldsymbol{\varphi} \mathrm{d}x &= \int_{\Omega_j} \boldsymbol{v}_j \cdot \boldsymbol{\varphi}^\sharp|_{\Omega_j} \mathrm{d}x = {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \boldsymbol{\varphi}^\sharp \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\ &= \int_{\Omega_k} \boldsymbol{v}_k \cdot \boldsymbol{\varphi}^\sharp|_{\Omega_k} \mathrm{d}x = \int_{\Omega_j \cap \Omega_k} \boldsymbol{v}_k \cdot \boldsymbol{\varphi} \mathrm{d}x \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega_j \cap \Omega_k)\end{aligned}$$

shows that $\boldsymbol{v}_j = \boldsymbol{v}_k$ in $\Omega_j \cap \Omega_k$. The relations $\boldsymbol{w}|_{\Omega_j} := \boldsymbol{v}_j$, $1 \leq j \leq m$, therefore unambiguously define a vector field $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$.

It remains to show that $\boldsymbol{v} = \boldsymbol{w}$. Given any vector field $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, let $(\alpha_j)_{j=1}^m$ denote a partition of unity associated with the open covering $\operatorname{supp} \boldsymbol{\varphi} \subset \bigcup_{j=1}^m \Omega_j$, i.e., made up with functions $\alpha_j \in \mathcal{D}(\Omega)$, $1 \leq j \leq m$, that satisfy

$$\operatorname{supp} \alpha_j \subset \Omega_j, \quad 1 \leq j \leq m, \quad \text{and} \quad \sum_{j=1}^m \alpha_j(x) = 1 \quad \text{for all } \ x \in \operatorname{supp} \boldsymbol{\varphi}.$$

We then have

$$\begin{aligned}{}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} &= \sum_{j=1}^m {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{v}, \alpha_j \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\ &= \sum_{j=1}^m {}_{\boldsymbol{\mathcal{D}}'(\Omega_j)}\langle \boldsymbol{v}_j, (\alpha_j \boldsymbol{\varphi})|_{\Omega_j} \rangle_{\boldsymbol{\mathcal{D}}(\Omega_j)} = \sum_{j=1}^m \int_{\Omega_j} \boldsymbol{v}_j \cdot (\alpha_j \boldsymbol{\varphi})|_{\Omega_j} \mathrm{d}x \\ &= \sum_{j=1}^m \int_{\Omega_j} \boldsymbol{w}|_{\Omega_j} \cdot (\alpha_j \boldsymbol{\varphi})|_{\Omega_j} \mathrm{d}x = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{\varphi} \mathrm{d}x \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega).\end{aligned}$$

Consequently, $\boldsymbol{v} = \boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ as announced. This completes the proof.

4 Two Further Equivalences

To conclude, we also state two further properties that are also equivalent to the vector version of J. L. Lions lemma.

The first equivalence asserts that the "weak" version of the Donati compatibility conditions (e) of Theorem 3.1 can be replaced by an even "weaker" version, in the sense that the "trial fields" $\boldsymbol{s}$ need be only in the space $\mathbb{D}(\Omega)$ instead of in the space $\mathbb{H}_0^1(\Omega)$. More specifically, we have the following theorem.

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^N$. The vector version of J. L. Lions lemma, viz.,*

$$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega) \text{ and } \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$$

is equivalent to the following version of the Donati compatibility conditions: Given a matrix field $\boldsymbol{e} \in \mathbb{H}^{-1}(\Omega)$, there exists a vector field $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$ such that

$$\nabla_s \boldsymbol{v} = \boldsymbol{e}$$

if (and clearly only if)

$$_{\mathbb{H}^{-1}(\Omega)}\langle \boldsymbol{e}, \boldsymbol{s}\rangle_{\mathbb{H}_0^1(\Omega)} = 0 \text{ for all } \boldsymbol{s} \in \mathbb{D}(\Omega) \text{ that satisfy } \mathbf{div}\, \boldsymbol{s} = \mathbf{0} \text{ in } \Omega.$$

If this is the case, the vector field $\boldsymbol{v}$ is uniquely determined in the space $\boldsymbol{L}_0^2(\Omega)$.

Proof The principle of the proof is identical to that of the proof of Theorem 4.1 in [2], or similar to that of the proof of Theorem 2.3 in Chapter 1 of Girault and Raviart [15]; for this reason, the proof is omitted. Suffices it to mention that establishing the sufficiency of the above Donati conditions relies not only on the vector version of J. L. Lions lemma itself, but also on the one of its consequences, viz., the sufficiency of the weak Donati compatibility conditions (e) established in Theorem 3.1.

The second equivalence asserts that the vector version of J. L. Lions lemma is equivalent to a weak version of the well-known Saint-Venant compatibility conditions, "weak" in the sense that the matrix field $\boldsymbol{e}$ is now given in the space $\mathbb{H}^{-1}(\Omega)$, instead of in the space $\mathbb{L}^2(\Omega)$ as in [3], where such conditions played a crucial role for defining an intrinsic approach to linearized elasticity (a quick introduction to the Saint-Venant compatibility condition is found in Subsection 6.19 of [8]).

Theorem 4.2 *Let Ω be a simply-connected domain in $\mathbb{R}^N$. Then the vector version of J. L. Lions lemma, viz.,*

$$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega) \text{ and } \nabla_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$$

is equivalent to the following weak Saint-Venant compatibility conditions: Given a matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{H}^{-1}(\Omega)$, there exists a vector field $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$ such that $\nabla_s \boldsymbol{v} = \boldsymbol{e}$ if (and clearly only if)

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } H^{-3}(\Omega).$$

If this is the case, the vector field $\boldsymbol{v}$ is uniquely determined in the space $\boldsymbol{L}_0^2(\Omega)$.

Proof This equivalence was already established in Theorem 7.3 of [3] in the special case where $N = 3$, by means of arguments, essentially based on a clever idea due to Kesavan [16], that can be easily extended to any integer $N \geq 2$ and which, for this reason, will not be reproduced here. We simply emphasize that the assumption of simple-connectedness of the domain Ω is crucially needed here, as the proof relies on the sufficiency of the "classical" Saint-Venant compatibility condition, which itself relies on the "classical" Poincaré lemma (for details, see again Subsection 6.19 of [8]).

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