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Homological Epimorphisms, Compactly Generated t-Structures and Gorenstein-Projective Modules

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Abstract The aim of this paper is two-fold. Given a recollement $(\mathcal{T}', \mathcal{T}, \mathcal{T}'', i^*, i_*, i^!, j_!, j^*, j_*)$, where $\mathcal{T}'$, $\mathcal{T}$, $\mathcal{T}''$ are triangulated categories with small coproducts and $\mathcal{T}$ is compactly generated. First, the authors show that the BBD-induction of compactly generated t-structures is compactly generated when i_* preserves compact objects. As a consequence, given a ladder $(\mathcal{T}', \mathcal{T}, \mathcal{T}'', \mathcal{T}, \mathcal{T}')$ of height 2, then the certain BBD-induction of compactly generated t-structures is compactly generated. The authors apply them to the recollements induced by homological ring epimorphisms. This is the first part of their work. Given a recollement $(D(B\text{-Mod}), D(A\text{-Mod}), D(C\text{-Mod}), i^*, i_*, i^!, j_!, j^*, j_*)$ induced by a homological ring epimorphism, the last aim of this work is to show that if A is Gorenstein, ${}_AB$ has finite projective dimension and $j_!$ restricts to $D^b(C\text{-mod})$, then this recollement induces an unbounded ladder $(B\text{-}\underline{\mathcal{G}\mathrm{proj}}, A\text{-}\underline{\mathcal{G}\mathrm{proj}}, C\text{-}\underline{\mathcal{G}\mathrm{proj}})$ of stable categories of finitely generated Gorenstein-projective modules. Some examples are described.

Keywords Compactly generated t-structure, Recollement, BBD-induction, BPP-induction, Homological ring epimorphism, Gorenstein-projective module

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1 Introduction

The concepts of recollements of triangulated categories and a t-structure on a triangulated category were introduced by Beilinson, Bernstein and Deligne [7]. To build the category of perverse sheaves, they provided a way of gluing t-structures with respect to recollements of triangulated categories. Such a gluing procedure is called the BBD-induction for simplicity. This subject has been developed thoroughly. See for examples [1–3, 9, 11, 18, 21–22, 24–25].

Recently, Broomhead, Pauksztello and Ploog [9] studied how to generate a new t-structure for a given finite set of t-structures on a triangulated category. They showed that given a finite set of compactly generated t-structures in a triangulated category with set-indexed coproducts, the extension closure of aisles is also an aisle. We call this new t-structure the BPP-induction for simplicity. Given a recollement $(\mathcal{Y}, \mathcal{D}, \mathcal{X})$ of triangulated categories and two finite sets S_1 and S_2 of compactly generated t-structures with the same index on $\mathcal{X}$ and $\mathcal{Y}$. Then by taking a compactly generated t-structure in S_1 and a compactly generated t-structure of the same index in S_2, we get a finite set S of their BBD-inductions. Qin and Gao [20] showed that if

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each object in S is compactly generated, then the BPP-induction of objects in S is exactly the BBD-induction of the BPP-induction of objects in S_1 and the BPP-induction of objects in S_2.

Given a recollement $(\mathcal{T}',\mathcal{T},\mathcal{T}'',i^*,i_*,i^!,\ j_!,j^*,j_*)$ of triangulated categories with small coproducts, the first part of our work is to show that the BBD-induction of compactly generated t-structures is compactly generated, if $\mathcal{T}$ is compactly generated and i_* preserves compact objects. Then we generalize this to the case of a ladder of height 2 and apply it to the ladder induced by homological ring epimorphisms. Our first main result is Theorem 2.1.

Gorenstein-projective modules introduced by Enochs and Jenda [14] play a fundamental role in Gorenstein homological algebra. It is well-known that the stable category of finitely generated Gorenstein-projective modules is a triangulated category, see for example [13].

Given a recollement of unbounded derived categories of algebras, a natural question arises: When it can induce a recollement of stable categories of finitely generated Gorenstein-projective modules over corresponding algebras. The last part of our paper is to show that one case does work, that is, it works if this recollement is induced by some homological ring epimorphism. Our second main result is Theorem 3.1.

2 The Compact Generation of BBD-Inductions

The goal of this section is to show that the BBD-induction of compactly generated t-structures is compactly generated for a recollement of derived categories induced by some homological ring epimorphism.

Let us begin by recalling some definitions and key lemmas for the proof of our first main theorem.

Recall from [7] that a t-structure on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}',\mathcal{D}'')$ of full subcategories satisfying

(1) $\mathcal{D}'[1]\subseteq\mathcal{D}'$ and $\mathcal{D}''[-1]\subseteq\mathcal{D}''$;

(2) $\mathrm{Hom}_{\mathcal{D}}(X,Y)=0$ for all $X\in\mathcal{D}'$ and $Y\in\mathcal{D}''[-1]$;

(3) for each Z in $\mathcal{D}$ there is a distinguished triangle $X\to Z\to Y\to X[1]$ with $X\in\mathcal{D}'$ and $Y\in\mathcal{D}''[-1]$.

Let $\mathcal{D}$ be a triangulated category with set-indexed coproducts and S a set of compact objects of $\mathcal{D}$. Denote by $\mathrm{Susp}(S)$ the smallest full subcategory of $\mathcal{D}$ containing S closed under suspension, extensions, set-indexed coproducts and direct summands. Put $\mathcal{D}':=\mathrm{Susp}(S)$ and $\mathcal{D}'':=\{Y\in\mathcal{D}\mid \mathrm{Hom}_{\mathcal{D}}(X,Y[n])=0,\ \text{for all } X\in S \text{ and } n\leq 0\}$.

Lemma 2.1 (see [1, Theorem A.1]) *The pair $(\mathcal{D}',\mathcal{D}'')$ is a t-structure on $\mathcal{D}$, which is called a compactly generated t-structure.*

Lemma 2.2 (see [9, Theorem A]) *Let $\mathcal{D}$ be a triangulated category with set-indexed coproducts and $\{(\mathcal{D}'_i,\mathcal{D}''_i)\mid i\in I\}$ be a finite set of compactly generated t-structures on $\mathcal{D}$. Then $\big(\langle\mathcal{D}'_i\mid i\in I\rangle,\bigcap\limits_{i\in I}\mathcal{D}''_i\big)$ is a t-structure, where $\langle\mathcal{D}'_i\mid i\in I\rangle$ is the smallest full subcategory of $\mathcal{D}$ containing all $\mathcal{D}'_i$ closed under extensions and direct summands and $\bigcap\limits_{i\in I}\mathcal{D}''_i$ is the intersection of all $\mathcal{D}''_i$.*

For convenience, we call $\big(\langle\mathcal{D}'_i\mid i\in I\rangle,\bigcap\limits_{i\in I}\mathcal{D}''_i\big)$ the BPP-induction of t-structures in the set

$\{(\mathcal{D}'_i, \mathcal{D}''_i) \mid i \in I\}$.

Let $\mathcal{Y}$, $\mathcal{D}$ and $\mathcal{X}$ be triangulated categories. Recall from [7] that $\mathcal{D}$ is a recollement with respect to $\mathcal{Y}$ and $\mathcal{X}$, if there is a diagram of six triangle functors

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{X} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(2) i_*, j_* and $j_!$ are fully faithful;

(3) $i^! \circ j_* = 0$ (and hence $j^* \circ i_* = 0$ and $i^* \circ j_! = 0$);

(4) for each $Z \in \mathcal{D}$ there are distinguished triangles

$$j_!j^*Z \overset{\epsilon_Z}{\to} Z \overset{\eta_Z}{\to} i_*i^*Z \to (j_!j^*Z)[1],$$
$$i_*i^!Z \overset{\omega_Z}{\to} Z \overset{\zeta_Z}{\to} j_*j^*Z \to (i_*i^!Z)[1],$$

where ϵ_Z is the counit of $(j_!, j^*)$, η_Z is the unit of (i^*, i_*), ω_Z is the counit of $(i_*, i^!)$ and ζ_Z is the unit of (j^*, j_*).

For short, we denote this recollement by $(\mathcal{Y}, \mathcal{D}, \mathcal{X}, i^*, i_*, i^!, j_!, j^*, j_*)$, or by $(\mathcal{Y}, \mathcal{D}, \mathcal{X})$. Recall from [8], [27] or [22] that $\mathcal{D}$ is an upper recollement with respect to $\mathcal{Y}$ and $\mathcal{X}$, if there is a diagram of four triangle functors

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "i_*"'] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[r, "j^*"'] & \mathcal{X} \arrow[l, bend right, "j_!"'] \end{tikzcd}$$

such that

(1) (i^*, i_*) and $(j_!, j^*)$ are adjoint pairs;

(2) i_* and $j_!$ are fully faithful;

(3) $j^* \circ i_* = 0$;

(4) for each $Z \in \mathcal{D}$ there is a distinguished triangle

$$j_!j^*Z \xrightarrow{\epsilon_Z} Z \xrightarrow{\eta_Z} i_*i^*Z \to (j_!j^*Z)[1],$$

where ϵ_Z is the counit of $(j_!, j^*)$ and η_Z is the unit of (i^*, i_*).

Lemma 2.3 (see [7, Theorem 1.4.10]) *Let $(\mathcal{Y}, \mathcal{D}, \mathcal{X})$ be a recollement of triangulated categories. Suppose that $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ are t-structures on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then $(\mathcal{D}', \mathcal{D}'')$ given by*

$$\mathcal{D}' := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}',\ i^*Z \in \mathcal{Y}'\},$$
$$\mathcal{D}'' := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}'',\ i^!Z \in \mathcal{Y}''\}$$

is a t-structure on $\mathcal{D}$, which is called the BBD-induction of $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ in [25].

Proposition 2.1 *Suppose that there exists the following recollement of triangulated categories:*

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_*"] & \mathcal{T} \arrow[l, shift right=3, "i^*"'] \arrow[l, shift left=3, "i^!"] \arrow[r, "j^*"] & \mathcal{T}'', \arrow[l, shift right=3, "j_!"'] \arrow[l, shift left=3, "j_*"] \end{tikzcd}$$

where $\mathcal{T}'$, $\mathcal{T}$, $\mathcal{T}''$ admit small coproducts and $\mathcal{T}$ is compactly generated. Let $\{(\mathcal{X}'_i, \mathcal{X}''_i) \mid i \in I\}$ and $\{(\mathcal{Y}'_i, \mathcal{Y}''_i) \mid i \in I\}$ be two finite sets of compactly generated t-structures on $\mathcal{T}''$ and $\mathcal{T}'$, and $(\mathcal{D}'_i, \mathcal{D}''_i)$ be the BBD-induction of $(\mathcal{X}'_i, \mathcal{X}''_i)$ and $(\mathcal{Y}'_i, \mathcal{Y}''_i)$ for each i. Suppose that i_* preserves compact objects. Then

(1) *$(\mathcal{D}'_i, \mathcal{D}''_i)$ is compactly generated for each $i \in I$;*

(2) *the BBD-induction of $\big(\langle \mathcal{X}'_i \mid i \in I\rangle, \bigcap_{i\in I} \mathcal{X}''_i\big)$ and $\big(\langle \mathcal{Y}'_i \mid i \in I\rangle, \bigcap_{i\in I} \mathcal{Y}''_i\big)$ is exactly $\big(\langle \mathcal{D}'_i \mid i \in I\rangle, \bigcap_{i\in I} \mathcal{D}''_i\big)$.*

Proof Since $\mathcal{T}'$, $\mathcal{T}$, $\mathcal{T}''$ admit small coproducts and $\mathcal{T}$ is compactly generated, it follows from [6] that i^* and $j_!$ preserve compact objects. Since i_* preserves compact objects, we obtain from [6] again that j^* preserves compact objects. Since $(j_!, j^*)$ is an adjoint pair, we get from [26, Theorems 4.1 and 5.1] that $j_!$ and j^* preserve small coproducts. Since (i^*, i_*) is an adjoint pair and i_* preserves compact objects, we get from [26, Theorems 4.1 and 5.1] that i_* and i^* preserve small coproducts.

(1) Let $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ be two compactly generated t-structures on $\mathcal{T}''$ and $\mathcal{T}'$ respectively, and $(\mathcal{D}', \mathcal{D}'')$ be the BBD-induction. Then by the definition there is a set S_1 of compact objects of $\mathcal{T}''$ such that $\mathcal{X}' = \mathrm{Susp}(S_1)$ and a set S_2 of compact objects of $\mathcal{T}'$ such that $\mathcal{Y}' = \mathrm{Susp}(S_2)$. Since i_* and $j_!$ preserve compact objects, we get that $j_!S_1 \cup i_*S_2$ is a set of compact objects of $\mathcal{T}$.

We claim that $\mathcal{D}' = \mathrm{Susp}(j_!S_1 \cup i_*S_2)$. Indeed, let Z be an object in $\mathcal{D}'$. Then by the definition $j^*Z \in \mathcal{X}' = \mathrm{Susp}(S_1)$ and $i^*Z \in \mathcal{Y}' = \mathrm{Susp}(S_2)$. Consider the distinguished triangle $j_!j^*Z \to Z \to i_*i^*Z \to (j_!j^*Z)\,[1]$. Since the triangle functors $j_!$ and i_* preserve coproducts and direct summands, it follows that $j_!j^*Z \in \mathrm{Susp}(j_!S_1)$ and $i_*i^*Z \in \mathrm{Susp}(i_*S_2)$. So $Z \in \mathrm{Susp}(j_!S_1 \cup i_*S_2)$, this means that $\mathcal{D}' \subseteq \mathrm{Susp}(j_!S_1 \cup i_*S_2)$. On the other hand, let X be an object in S_1. Then $j^*j_!X \cong X \in \mathrm{Susp}(S_1) = \mathcal{X}'$ and $i^*j_!X = 0 \in \mathcal{Y}'$. So $j_!X \in \mathcal{D}'$. Let Y be an object in S_2. Then $j^*i_*Y = 0 \in \mathcal{X}'$ and $i^*i_*Y \cong Y \in \mathrm{Susp}(S_2) = \mathcal{Y}'$. So $i_*Y \in \mathcal{D}'$. It follows that $j_!S_1 \cup i_*S_2 \subseteq \mathcal{D}'$. Since $(i_*, i^!)$ is an adjoint pair and i_* preserves small coproducts, we get from [26, Theorem 5.1] that $i^!$ preserves coproducts. Since j^* and i^* preserve coproducts, we have $\mathrm{Susp}(j_!S_1 \cup i_*S_2) \subseteq \mathcal{D}'$. Thus $\mathcal{D}' = \mathrm{Susp}(j_!S_1 \cup i_*S_2)$ and $(\mathcal{D}', \mathcal{D}'')$ is a compactly generated t-structure.

(2) Since each $(\mathcal{D}'_i, \mathcal{D}''_i)$ is compactly generated for each $i \in I$ by (1), it suffices to show that $(\langle \mathcal{D}'_1, \mathcal{D}'_2\rangle, \mathcal{D}''_1 \cap \mathcal{D}''_2)$ is the BBD-induction of $(\langle \mathcal{X}'_1, X'_2\rangle, \mathcal{X}''_1 \cap \mathcal{X}''_2)$ and $(\langle \mathcal{Y}'_1, \mathcal{Y}'_2\rangle, \mathcal{Y}''_1 \cap \mathcal{Y}''_2)$. Suppose that $(\mathcal{D}', \mathcal{D}'')$ is the BBD-induction of $(\langle \mathcal{X}'_1, \mathcal{X}'_2\rangle, \mathcal{X}''_1 \cap \mathcal{X}''_2)$ and $(\langle \mathcal{Y}'_1, \mathcal{Y}'_2\rangle, \mathcal{Y}''_1 \cap \mathcal{Y}''_2)$. Then

$$\mathcal{D}' = \{Z \in \mathcal{T} : j^*Z \in \langle \mathcal{X}'_1, \mathcal{X}'_2\rangle,\ i^*Z \in \langle \mathcal{Y}'_1, \mathcal{Y}'_2\rangle\}$$

and

$$\mathcal{D}'' = \{Z \in \mathcal{T} : j^*Z \in \mathcal{X}_1'' \cap \mathcal{X}_2'',\ i^!Z \in \mathcal{Y}_1'' \cap \mathcal{Y}_2''\}.$$

By the assumption on $(\mathcal{D}_i', \mathcal{D}_i'')$ for $i = 1, 2$,

$$\mathcal{D}_i' = \{Z \in \mathcal{T} : j^*Z \in \mathcal{X}_i',\ i^*Z \in \mathcal{Y}_i'\}$$

and

$$\mathcal{D}_i'' = \{Z \in \mathcal{T} : j^*Z \in \mathcal{X}_i'',\ i^!Z \in \mathcal{Y}_i''\}.$$

We claim that $\mathcal{D}_1'' \cap \mathcal{D}_2'' = \mathcal{D}''$. In fact, let Z be an object in $\mathcal{D}''$. Then $j^*Z \in \mathcal{X}_1'' \cap \mathcal{X}_2''$ and $i^!Z \in \mathcal{Y}_1'' \cap \mathcal{Y}_2''$. So $j^*Z \in \mathcal{X}_1''$ and $i^!Z \in \mathcal{Y}_1''$, this means that $Z \in \mathcal{D}_1''$. Also, $j^*Z \in \mathcal{X}_2''$ and $i^!Z \in \mathcal{Y}_2''$, this means that $Z \in \mathcal{D}_2''$. It follows that $\mathcal{D}'' \subseteq \mathcal{D}_1'' \cap \mathcal{D}_2''$. Conversely, let Z be an object in $\mathcal{D}_1'' \cap \mathcal{D}_2''$. Then $Z \in \mathcal{D}_1''$ and $Z \in \mathcal{D}_2''$ at the same time. So by the definition $j^*Z \in \mathcal{X}_1'' \cap \mathcal{X}_2''$ and $i^!Z \in \mathcal{Y}_1'' \cap \mathcal{Y}_2''$, this means that $\mathcal{D}_1'' \cap \mathcal{D}_2'' \subseteq \mathcal{D}''$. Thus $\mathcal{D}'' = \langle \mathcal{D}_1'', \mathcal{D}_2'' \rangle$, and so $(\langle \mathcal{D}_1', \mathcal{D}_2' \rangle, \mathcal{D}_1'' \cap \mathcal{D}_2'')$ is the BBD-induction of $(\langle \mathcal{X}_1', X_2' \rangle, \mathcal{X}_1'' \cap \mathcal{X}_2'')$ and $(\langle \mathcal{Y}_1', \mathcal{Y}_2' \rangle, \mathcal{Y}_1'' \cap \mathcal{Y}_2'')$. We finish the proof by induction on the number of elements of I.

Let R and S be rings. Recall from [16] that a ring epimorphism $\lambda : R \to S$ is homological, if $\mathrm{Tor}_j^R(S, S) = 0$ for all $j > 0$. This is equivalent to that the restriction functor $D(\lambda_*) : D(S\text{-Mod}) \to D\ (R\text{-Mod})$ is fully faithful, where $D(S\text{-Mod})$ (resp. $D(R\text{-Mod})$) denotes the unbounded derived category of modules category over S (resp. R). We denote by $\mathrm{pd}_R S$ the projective dimension of S as a left R-module. Denote by $D(R\text{-Mod})^c$ the triangulated subcategory of compact objects in $D(R\text{-Mod})$.

Corollary 2.1 *Let A and B be projective k-algebras over a commutative ring k. Let $\lambda : A \to B$ be a homological ring epimorphism which induces a recollement of unbounded derived categories*

$$\begin{tikzcd} D(B\text{-Mod}) \arrow[r, "i_*"] & D(A\text{-Mod}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D(C\text{-Mod}), \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where C is a projective k-algebra and i_* is the restriction functor induced by λ. Let $\{(\mathcal{X}_i', \mathcal{X}_i'') \mid i \in I\}$ and $\{(\mathcal{Y}_i', \mathcal{Y}_i'') \mid i \in I\}$ be two finite sets of compactly generated t-structures on $D(C\text{-Mod})$ and $D(B\text{-Mod})$, and $(\mathcal{D}_i', \mathcal{D}_i'')$ be the BBD-induction of $(\mathcal{X}_i', \mathcal{X}_i'')$ and $(\mathcal{Y}_i', \mathcal{Y}_i'')$ for each i. Suppose $\mathrm{pd}_A B < \infty$. Then

(1) *$(\mathcal{D}_i', \mathcal{D}_i'')$ is compactly generated for each $i \in I$;*

(2) *the BBD-induction of $\big(\langle \mathcal{X}_i' \mid i \in I \rangle, \bigcap_{i \in I} \mathcal{X}_i''\big)$ and $\big(\langle \mathcal{Y}_i' \mid i \in I \rangle, \bigcap_{i \in I} \mathcal{Y}_i''\big)$ is exactly $\big(\langle \mathcal{D}_i' \mid i \in I \rangle, \bigcap_{i \in I} \mathcal{D}_i''\big)$.*

Proof Note that $D(A\text{-Mod})$, $D(B\text{-Mod})$ and $D(C\text{-Mod})$ admit small coproducts and are compactly generated. We denote by $D(A\text{-Mod})^c$ the full subcategory of compact objects in $D(A\text{-Mod})$, similarly for $D(B\text{-Mod})^c$. Since $D(A\text{-Mod})^c = K^b(A\text{-proj})$, $D(B\text{-Mod})^c = K^b(B\text{-proj})$ and $\mathrm{pd}_A B < \infty$, it follows that ${}_AB$ is a compact object in $D(A\text{-mod})$. This implies that i_* preserves compact objects. Thus by Proposition 2.1 we complete the proof.

Definition 2.1 (see [4, Section 3; 8, Section 1.5]) *A ladder $\mathcal{L}$ is a finite or infinite diagram of triangulated categories and triangle functors*

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ & \xrightarrow{\;i_2\;} & & \xrightarrow{\;j_2\;} & \\ & \xleftarrow{\;j_1\;} & & \xleftarrow{\;i_1\;} & \\ \mathcal{T}' & \xrightarrow{\;i_0\;} & \mathcal{T} & \xrightarrow{\;j_0\;} & \mathcal{T}'' \\ & \xleftarrow{\;j_{\text{-}1}\;} & & \xleftarrow{\;i_{\text{-}1}\;} & \\ & \xrightarrow{\;i_{\text{-}2}\;} & & \xrightarrow{\;j_{\text{-}2}\;} & \\ & \vdots & & \vdots & \end{array}$$

such that any three consecutive rows form a recollement. The rows are labelled by a subset of $\mathbb{Z}$ and multiple occurence of the same recollment is allowed. The height of a ladder is the number of recollements contained in it (counted with multiplicities).

Theorem 2.1 *Suppose that there is a ladder of triangulated categories of height 2*

$$\begin{array}{ccccc} & \xrightarrow{\;i^*\;} & & \xrightarrow{\;j_!\;} & \\ & \xleftarrow{\;i_*\;} & & \xleftarrow{\;j^*\;} & \\ \mathcal{T}' & \xrightarrow{\;i^!\;} & \mathcal{T} & \xrightarrow{\;j_*\;} & \mathcal{T}'', \\ & \xleftarrow{\;i_?\;} & & \xleftarrow{\;j^?\;} & \end{array}$$

where $\mathcal{T}', \mathcal{T}, \mathcal{T}''$ admit small coproducts and $\mathcal{T}$ is compactly generated. Let $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ be two finite sets of compactly generated t-structure on $\mathcal{T}''$ and $\mathcal{T}'$ respectively. Let $(\mathcal{D}_1', \mathcal{D}_1'')$ be their BBD-induction corresponding to the recollement $(\mathcal{T}', \mathcal{T}, \mathcal{T}'')$. Then $(\mathcal{D}_1', \mathcal{D}_1'')$ is a compactly generated t-structure.

Proof By assumption $(\mathcal{T}'', \mathcal{T}, \mathcal{T}', j^*, j_*, j^?, i_*, i^!, i_?)$ is a recollement. Since $\mathcal{T}''$, $\mathcal{T}$, $\mathcal{T}'$ admit small coproducts and $\mathcal{T}$ is compactly generated, it follows from [6] that i_* preserves compact objects. By Proposition 2.1, we immediately get that $(\mathcal{D}_1', \mathcal{D}_1'')$ is a compactly generated t-structure.

For short, we denote this ladder by $(\mathcal{T}', \mathcal{T}, \mathcal{T}'', i^*, i_*, i^!, i_?, j_!, j^*, j_*, j^?)$, or by $(\mathcal{T}', \mathcal{T}, \mathcal{T}'', \mathcal{T}, \mathcal{T}')$.

Corollary 2.2 *Let A, B and C be projective k-algebras over a commutative ring k. Let $\lambda_1 : A \to B$ and $\lambda_2 : A \to C$ be two homological ring epimorphisms which induce a ladder of height 2*

$$\begin{array}{ccccc} & \xrightarrow{\;i^*\;} & & \xrightarrow{\;j_!\;} & \\ & \xleftarrow{\;i_*\;} & & \xleftarrow{\;j^*\;} & \\ D(B\text{-Mod}) & \xrightarrow{\;i^!\;} & D(A\text{-Mod}) & \xrightarrow{\;j_*\;} & D(C\text{-Mod}), \\ & \xleftarrow{\;i_?\;} & & \xleftarrow{\;j^?\;} & \end{array}$$

where $i^!$ and j^* are the restriction functors induced by λ_1 and λ_2, respectively. Let $(\mathcal{X}', \mathcal{X}'')$ be a compactly generated t-structure on $D(C\text{-Mod})$ and $(\mathcal{Y}', \mathcal{Y}'')$ a compactly generated t-structure on $D(B\text{-Mod})$. Let $(\mathcal{D}_1', \mathcal{D}_1'')$ be their BBD-induction corresponding to the recollement $(D(B\text{-Mod}), D(A\text{-Mod}), D(C\text{-Mod}))$. Then $(\mathcal{D}_1', \mathcal{D}_1'')$ is a compactly generated t-structure.

Proof Note that $D(A\text{-Mod})$, $D(B\text{-Mod})$ and $D(C\text{-Mod})$ admit small coproducts and $D(A\text{-Mod})$ is compactly generated. Thus by Theorem 2.1 we obtain the desired result.

Example 2.1 Let $A = \left(\begin{smallmatrix} C & M \\ 0 & B \end{smallmatrix}\right)$ be an upper triangular matrix algebra, where B, C are finite dimensional k-algebras over a field k and M is a finite dimensional C-B-bimodule. Let $(\mathcal{X}', \mathcal{X}'')$ be a compactly generated t-structure on $D(C\text{-Mod})$ and $(\mathcal{Y}', \mathcal{Y}'')$ a compactly generated t-structure on $D(B\text{-Mod})$. Then

(1) there is a ladder of height 2

$$\begin{tikzcd}[column sep=huge]
D(B\text{-Mod}) \arrow[r, shift left=4.5] \arrow[r, leftarrow, shift left=1.5, "fA \otimes_A^{\mathbb{L}} -"] \arrow[r, shift right=1.5, "\mathbb{R}\mathrm{Hom}_{fAf}(fA,-)"] \arrow[r, leftarrow, shift right=4.5] & D(A\text{-Mod}) \arrow[r, shift left=4.5] \arrow[r, leftarrow, shift left=1.5, "Ae \otimes_{eAe}^{\mathbb{L}} -"] \arrow[r, shift right=1.5, "eA \otimes_A^{\mathbb{L}} -"] \arrow[r, leftarrow, shift right=4.5] & D(C\text{-Mod}),
\end{tikzcd}$$

(2) the BBD-induction of $(\mathcal{X}', \mathcal{X}'')$ and $(\mathcal{Y}', \mathcal{Y}'')$ corresponding to the recollement $(D(B\text{-Mod}), D(A\text{-Mod}), D(C\text{-Mod}))$ is a compactly generated t-structure.

Proof Let $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $A/AeA = B = fA = fAf$, $A/AfA = C = Ae = eAe$, $eAf = M$ and $fAe = 0$. It follows that AeA is a projective right A-module and AfA is a projective left A-module, and so by [23, Remark 3.2] AeA and AfA are stratifying ideals of A. Furthermore, $\lambda_1 : A \to B$ and $\lambda_2 : A \to C$ are homological ring epimorphisms and they induce a ladder of height 2

$$\begin{tikzcd}[column sep=huge]
D(B\text{-Mod}) \arrow[r, shift left=4.5] \arrow[r, leftarrow, shift left=1.5, "fA \otimes_A^{\mathbb{L}} -"] \arrow[r, shift right=1.5, "\mathbb{R}\mathrm{Hom}_{fAf}(fA,-)"] \arrow[r, leftarrow, shift right=4.5] & D(A\text{-Mod}) \arrow[r, shift left=4.5] \arrow[r, leftarrow, shift left=1.5, "Ae \otimes_{eAe}^{\mathbb{L}} -"] \arrow[r, shift right=1.5, "eA \otimes_A^{\mathbb{L}} -"] \arrow[r, leftarrow, shift right=4.5] & D(C\text{-Mod}),
\end{tikzcd}$$

where $\mathbb{R}\mathrm{Hom}_{fAf}(fA, -)$ and $Ae \otimes_{eAe}^{\mathbb{L}} -$ are the restriction functors induced by λ_1 and λ_2. By Corollary 2.2 we complete the proof.

Remark 2.1 We mention that (1) in Example 2.1 is implicit in [4, Example 3.4].

3 Recollements of Gorenstein Projective Modules

The goal of this section is to show that a recollement of derived categories of algebras induced by some homological ring epimorphism produces a ladder of the stable categories of Gorenstein-projective modules over corresponding algebras.

Before we state our last main theorem, we fix some notation and recall some definitions and facts.

Throughout this section, let A be a finite dimensional k-algebra over a field k and A-Mod the category of left A-modules. Denote by A-mod the full subcategory of finitely generated left A-modules, by A-proj the full subcategory of projective A-modules in A-mod, and by $\mathrm{pd}_A X$ the projective dimension of X in A-mod. Recall that an A-module M in A-mod is said to be Gorenstein-projective, if there is an exact sequence $P^\bullet = \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$ in A-proj with $\mathrm{Hom}_A(P^\bullet, Q)$ exact for any A-module Q in A-proj, such that $M = \ker d^0$ (see [14, Definition 10.2.1]). Denote by A-Gproj the full subcategory of Gorenstein-projective modules in A-mod.

Denote by $K^b(A\text{-proj})$ the bounded homotopy category of A-proj, by $D(A\text{-Mod})$ the unbounded derived category of A-Mod, and by $D^b(A\text{-mod})$ the bounded derived category of A-mod. Denote by $D_{sg}(A)$ the singularity category of A, which is defined as the Verdier quotient category of $D^b(A\text{-mod})$ with respect to the thick subcategory $K^b(A\text{-proj})$.

Theorem 3.1 *Let A be a Gorenstein algebra. Let $\lambda : A \to B$ be a homological ring epimorphism which induces a recollement of derived categories of algebras B, A and C*

$$\begin{tikzcd}[column sep=large] D(B\text{-Mod}) \arrow[r, "i_*" description] & D(A\text{-Mod}) \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, "j^*" description] & D(C\text{-Mod}) \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"] \end{tikzcd} \tag{3.1}$$

such that $j_!$ restricts to $D^b(C\text{-mod})$. If $\mathrm{pd}_A B < \infty$, then there is an unbounded ladder

$$\begin{tikzcd}[column sep=large] B\text{-}\underline{\mathrm{Gproj}} \arrow[r, phantom, shift left=7ex, "\vdots"] \arrow[r, shift left=4ex, "i_?"] \arrow[r, "i_*" description] \arrow[r, shift right=4ex, "i_@"'] \arrow[r, phantom, shift right=7ex, "\vdots"] & A\text{-}\underline{\mathrm{Gproj}} \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, phantom, shift left=7ex, "\vdots"] \arrow[r, shift left=4ex, "j^?"] \arrow[r, "j^*" description] \arrow[r, shift right=4ex, "j^@"'] \arrow[r, phantom, shift right=7ex, "\vdots"] & C\text{-}\underline{\mathrm{Gproj}}. \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"] \end{tikzcd}$$

Proof Since $\lambda : A \to B$ is a homological ring epimorphism, it follows from [2, 1.6 and 1.7] that $i_* = {}_AB \otimes^{\mathbb{L}}_B -$. Since $\mathrm{pd}_A B < \infty$, it follows that $i_*(B) \in K^b(A\text{-proj})$. Since $D(A\text{-Mod})$, $D(B\text{-Mod})$ and $D(C\text{-Mod})$ admit small coproducts and are compactly generated, and also $D(A\text{-Mod})^c = K^b(A\text{-proj})$, $D(B\text{-Mod})^c = K^b(B\text{-proj})$ and $D(C\text{-Mod})^c = K^b(C\text{-proj})$, we obtain from [6] that j^* and i^* restrict to $K^b(A\text{-proj})$, and $j_!$ restricts to $K^b(C\text{-proj})$. This implies from [26, Theorems 4.1 and 5.1] that j_* has a right adjoint. Since $j_!$ restricts to $D^b(C\text{-mod})$ and $i_*(B) \in K^b(A\text{-proj})$, we get from [4, Theorem 4.6] that (3.1) restricts to a recollement

$$\begin{tikzcd}[column sep=large] D^b(B\text{-mod}) \arrow[r, "i_*" description] & D^b(A\text{-mod}) \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, "j^*" description] & D^b(C\text{-mod}). \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"] \end{tikzcd} \tag{3.2}$$

Since A is Gorenstein, it follows from [27, Theorem 3.1] and (3.2) that B and C are Gorenstein. Since $j_!$ restricts to $D^b(C\text{-mod})$, it follows from [4, Proposition 3.2] that $j_!$ admits a left adjoint. Thus there exists a ladder of height 3

$$\begin{tikzcd}[column sep=large] D(B\text{-Mod}) \arrow[r, shift left=4ex, "i_?"] \arrow[r, "i_*" description] \arrow[r, shift right=4ex, "i_@"'] & D(A\text{-Mod}) \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, shift left=4ex, "j^?"] \arrow[r, "j^*" description] \arrow[r, shift right=4ex, "j^@"'] & D(C\text{-Mod}). \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"] \end{tikzcd}$$

Moreover, $i_?$ restricts to $K^b(B\text{-proj})$ and $j^?$ restricts to $K^b(A\text{-proj})$.

Since i^* restricts to $K^b(A\text{-proj})$, it follows from [19, Lemma 1] that $i_?$ restricts to $D^b(B\text{-mod})$. This means from [4, Proposition 3.2] that $i_?$ admits a left adjoint. Since i_* restricts to $D^b(B\text{-mod})$, we get from [19, Lemma 1] again that $i^!$ restricts to $K^b(A\text{-proj})$. So we know from [26, Theorems 4.1 and 5.1] that $i_@$ has a right adjoint. We proceed the same procedure. Then there exists the following unbounded ladder

$$\begin{tikzcd}[column sep=large]
 & \vdots & & \vdots & \\
D(B\text{-Mod}) \arrow[rr, shift left=8, "i_?"] \arrow[rr, leftarrow, shift left=4, "i^*"] \arrow[rr, "i_*"] \arrow[rr, leftarrow, shift right=4, "i^!"] \arrow[rr, shift right=8, "i_@"] & & D(A\text{-Mod}) \arrow[rr, shift left=8, "j^?"] \arrow[rr, leftarrow, shift left=4, "j_!"] \arrow[rr, "j^*"] \arrow[rr, leftarrow, shift right=4, "j_*"] \arrow[rr, shift right=8, "j^@"] & & D(C\text{-Mod}). \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Furthermore, we also obtain from above arguments that there are the following two induced unbounded ladders:

$$\begin{tikzcd}[column sep=large]
 & \vdots & & \vdots & \\
D^b(B\text{-mod}) \arrow[rr, shift left=8, "i_?"] \arrow[rr, leftarrow, shift left=4, "i^*"] \arrow[rr, "i_*"] \arrow[rr, leftarrow, shift right=4, "i^!"] \arrow[rr, shift right=8, "i_@"] & & D^b(A\text{-mod}) \arrow[rr, shift left=8, "j^?"] \arrow[rr, leftarrow, shift left=4, "j_!"] \arrow[rr, "j^*"] \arrow[rr, leftarrow, shift right=4, "j_*"] \arrow[rr, shift right=8, "j^@"] & & D^b(C\text{-mod}) \\
 & \vdots & & \vdots &
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=large]
 & \vdots & & \vdots & \\
K^b(B\text{-proj}) \arrow[rr, shift left=8, "i_?"] \arrow[rr, leftarrow, shift left=4, "i^*"] \arrow[rr, "i_*"] \arrow[rr, leftarrow, shift right=4, "i^!"] \arrow[rr, shift right=8, "i_@"] & & K^b(A\text{-proj}) \arrow[rr, shift left=8, "j^?"] \arrow[rr, leftarrow, shift left=4, "j_!"] \arrow[rr, "j^*"] \arrow[rr, leftarrow, shift right=4, "j_*"] \arrow[rr, shift right=8, "j^@"] & & K^b(C\text{-proj}). \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Thus we get an unbounded ladder of singularity categories

$$\begin{tikzcd}[column sep=large]
 & \vdots & & \vdots & \\
D_{sg}(B) \arrow[rr, leftarrow, shift left=4, "\widetilde{i^*}"] \arrow[rr, "\widetilde{i_*}"] \arrow[rr, leftarrow, shift right=4, "\widetilde{i^!}"] & & D_{sg}(A) \arrow[rr, leftarrow, shift left=4, "\widetilde{j_!}"] \arrow[rr, "\widetilde{j^*}"] \arrow[rr, leftarrow, shift right=4, "\widetilde{j_*}"] & & D_{sg}(C). \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Since A, B and C are Gorenstein, it follows from [10] that there are triangle-equivalences $D_{sg}(B) \cong B\text{-}\underline{\mathrm{Gproj}}$, $D_{sg}(A) \cong A\text{-}\underline{\mathrm{Gproj}}$ and $D_{sg}(C) \cong C\text{-}\underline{\mathrm{Gproj}}$. This implies the desired ladder.

Before stating the following corollary, we recall the notion of stratifying ideals. Let A be an algebra and e an idempotent of A. The two-sided ideal AeA generated by e is called a stratifying ideal if the multiplication map $Ae \otimes_{eAe} eA \to AeA$ is bijective and $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n > 0$, or equivalently, the canonical epimorphism $\lambda : A \to A/AeA$ is homological (see [12]).

Corollary 3.1 *Let A be a Gorenstein algebra. Let e be an idempotent of A such that AeA is a stratifying ideal with $\mathrm{pd}_A AeA < \infty$ and eAe has finite global dimension. Then there is a triangle-equivalence $A/AeA\text{-}\underline{\mathrm{Gproj}} \cong A\text{-}\underline{\mathrm{Gproj}}$.*

Proof Since AeA is a stratifying ideal, there is from [12, Section 2] the following recollement

$$\begin{tikzcd}[column sep=huge]
D(A/AeA\text{-Mod}) \arrow[r] & D(A\text{-Mod}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D(eAe\text{-Mod}). \arrow[l, bend right, "Ae\otimes_{eAe}-"'] \arrow[l, bend left]
\end{tikzcd}$$

Since A is a finite dimensional algebra and the global dimension of eAe is finite, we get that $Ae\otimes^{\mathbb{L}}_{eAe}-$ restricts to $D^b(eAe\text{-mod})$ and $eAe\text{-Gproj} = eAe\text{-proj}$. Since $\mathrm{pd}_A AeA < \infty$, it follows that $\mathrm{pd}_A A/AeA < \infty$. Thus by Theorem 3.1 there exists an unbounded ladder

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \longrightarrow & & \longrightarrow & \\
 & \longleftarrow & & \longleftarrow & \\
A/AeA\text{-}\underline{\mathrm{Gproj}} & \longrightarrow & A\text{-}\underline{\mathrm{Gproj}} & \longrightarrow & eAe\text{-}\underline{\mathrm{Gproj}}. \\
 & \longleftarrow & & \longleftarrow & \\
 & \longrightarrow & & \longrightarrow & \\
 & \vdots & & \vdots &
\end{array}$$

This implies that there is a triangle-equivalence $A/AeA\text{-}\underline{\mathrm{Gproj}} \cong A\text{-}\underline{\mathrm{Gproj}}$.

Let A and B be two rings, ${}_AN_B$ an A-B-bimodule, and ${}_BM_A$ a B-A-bimodule. Assume that $\varphi : M\otimes_A N \to B$ is a B-B-bimodule homomorphism and $\psi : N\otimes_B M \to A$ is an A-A-bimodule homomorphism, such that $\varphi(m\otimes n)m' = m\psi(n\otimes m')$ and $n\varphi(m\otimes n') = \psi(n\otimes m)n'$ for all $m, m' \in M$ and $n, n' \in N$. Following [5], the Morita ring is defined as follows:

$$\Lambda_{(\varphi,\psi)} = \begin{pmatrix} A & N \\ M & B \end{pmatrix},$$

where the addition of elements of $\Lambda_{(\varphi,\psi)}$ is componentwise and the multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \psi(n\otimes m') & an' + nb' \\ ma' + bm' & bb' + \varphi(m\otimes n') \end{pmatrix}.$$

Note that $\Lambda_{(\varphi,\psi)}$ is an associative ring. Then we have the following example.

Example 3.1 Let A be a Gorenstein algebra, and let e and f be two idempotent elements of A such that $fAe = 0$. Let $N := Ae\otimes_k fA$ and $\Lambda_{(0,0)} := \left(\begin{smallmatrix} A & N \\ N & A \end{smallmatrix}\right)$. Then there exists an unbounded ladder

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \longrightarrow & & \longrightarrow & \\
 & \longleftarrow & & \longleftarrow & \\
A\text{-}\underline{\mathrm{Gproj}} & \longrightarrow & \Lambda_{(0,0)}\text{-}\underline{\mathrm{Gproj}} & \longrightarrow & A\text{-}\underline{\mathrm{Gproj}}. \\
 & \longleftarrow & & \longleftarrow & \\
 & \longrightarrow & & \longrightarrow & \\
 & \vdots & & \vdots &
\end{array}$$

Proof Since $fAe = 0$, it follows that $N\otimes_A N = 0$. This implies that $\Lambda_{(0,0)}$ is a Morita ring and furthermore there is the following recollement of module categories by [15, Proposition 2.4] or [17]

$$\begin{tikzcd}[column sep=huge]
A\text{-mod} \arrow[r, "\mathrm{Z}_A"] & \Lambda_{(0,0)}\text{-mod} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathrm{U}_A"] & A\text{-mod}. \arrow[l, bend right, "\mathrm{T}_A"'] \arrow[l, bend left, "\mathrm{H}_A"]
\end{tikzcd}$$

Let $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then for the left side algebra A in the above recollement, we have $A \cong \Lambda_{(0,0)}/\Lambda_{(0,0)}e\Lambda_{(0,0)}$. Also, $\mathrm{T}_A = \Lambda_{(0,0)}e \otimes_{e\Lambda_{(0,0)}e} -$ and $\mathrm{H}_A = \mathrm{Hom}_{e\Lambda_{(0,0)}e}(e\Lambda_{(0,0)}, -)$. Since N is a both left and right projective (A, A)-bimodule, we get that $\Lambda_{(0,0)}e$ is a projective right $e\Lambda_{(0,0)}e$-module and $e\Lambda_{(0,0)}$ is a projective left $e\Lambda_{(0,0)}e$-module. Note from [17] that the indecomposable projective left $\Lambda_{(0,0)}$-module is either of the form $(P, N \otimes_A P, \mathrm{Id}_{N\otimes_A P}, 0)$ or of the form $(N \otimes_A P, P, 0, \mathrm{Id}_{N\otimes_A P})$. Let $\eta : \mathrm{T}_A \circ \mathrm{U}_A \to \mathrm{Id}_{\Lambda_{(0,0)}}$ be the counit of the adjoint pair $(\mathrm{T}_A, \mathrm{U}_A)$. Since $\mathrm{T}_A \circ \mathrm{U}_A(P, N \otimes_A P, \mathrm{Id}_{N\otimes_A P}, 0) = (P, N \otimes_A P, \mathrm{Id}_{N\otimes_A P}, 0)$ and $\mathrm{T}_A \circ \mathrm{U}_A(N \otimes_A P, P, 0, \mathrm{Id}_{N\otimes_A P}) = (N \otimes_A P, 0, 0, 0)$, it follows that $\eta_{(P,N\otimes_A P,\mathrm{Id}_{N\otimes_A P},0)} = (\mathrm{Id}_P, \mathrm{Id}_{N\otimes_A P})$ and $\eta_{(N\otimes_A P,P,0,\mathrm{Id}_{N\otimes_A P})} = (\mathrm{Id}_{N\otimes_A P}, 0)$ are monic. This means that $\Lambda_{(0,0)}e \otimes_{e\Lambda_{(0,0)}e} e\Lambda_{(0,0)} \to \Lambda_{(0,0)}e\Lambda_{(0,0)}$ is bijective. It follows that the canonical epimorphism $\Lambda_{(0,0)} \to A$ is homological. Thus there is an induced recollement of unbounded derived categories

$$
\begin{tikzcd}
D(A\text{-Mod}) \arrow[r, "D(\mathrm{Z}_A)"] & D(\Lambda_{(0,0)}\text{-Mod}) \arrow[l, shift right=3ex] \arrow[l, shift left=3ex] \arrow[r] & D(A\text{-Mod}) \arrow[l, shift right=3ex, "D(\mathrm{T}_A)"'] \arrow[l, shift left=3ex]
\end{tikzcd}
$$

such that $D(\mathrm{T}_A)$ restricts to $D^b(A\text{-mod})$. Since A is Gorenstein, we obtain from [15, Corollary 4.15] that $\Lambda_{(0,0)}$ is Gorenstein. By Theorem 3.1 we complete the proof.

The following example is implicit in [29, Theorem 2.1]. Now we explain it from another point of view for our purpose.

Example 3.2 Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be a Gorenstein algebra with $\mathrm{pd}_A M < \infty$. Then there is an unbounded ladder

$$
\begin{tikzcd}
 & \vdots & & \vdots & \\
A\text{-}\underline{\mathrm{Gproj}} \arrow[r] & \Lambda\text{-}\underline{\mathrm{Gproj}} \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & B\text{-}\underline{\mathrm{Gproj}}. \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \\
 & \vdots & & \vdots &
\end{tikzcd}
$$

Proof Let $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $e + f = 1_\Lambda$. Then $A = \Lambda/\Lambda e\Lambda$ and $B = e\Lambda e$. Since $\Lambda e\Lambda = \Lambda e$ is a projective left Λ-module, we get from [23, Remark 3.2] that $\Lambda e\Lambda$ is a stratifying ideal of Λ. So there is an induced homological ring epimorphism $\lambda : A \to \Lambda$.

Since $A = \Lambda/\Lambda e\Lambda = f\Lambda f = \Lambda f$, we know that A is a projective left Λ-module. Since Λ is Gorenstein and $\mathrm{pd}_A M < \infty$, it follows from [28, Theorem 2.2] that $\mathrm{pd} M_B < \infty$. By $j_! = \Lambda e \otimes^{\mathbb{L}}_{e\Lambda e} - = (M \oplus B) \otimes^{\mathbb{L}}_B -$, $j_!$ restricts to $D^b(B\text{-mod})$. Thus by Theorem 3.1 we get the desired ladder.

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