Chin. Ann. Math. Ser. B
39(1), 2018, 97–110
DOI: 10.1007/s11401-018-1054-8

Chinese Annals of Mathematics, Series B

New Homogeneous Einstein Metrics on SO(7)/T *

Yu WANG[1] Tianzeng LI[1] Guosong ZHAO[2]

Abstract The authors compute non-zero structure constants of the full flag manifold $M = \mathrm{SO}(7)/T$ with nine isotropy summands, then construct the Einstein equations. With the help of computer they get all the forty-eight positive solutions (up to a scale) for SO(7)/T, up to isometry there are only five G-invariant Einstein metrics, of which one is Kähler Einstein metric and four are non-Kähler Einstein metrics.

Keywords Generalized flag manifold, Einstein metric, Ricci tensor, Isotropy representation, Isometry
2000 MR Subject Classification 53C30, 17B22

1 Introduction

An important class of homogeneous manifolds is the orbits of the adjoint action of a semisimple compact Lie group, called generalized flag manifolds. Such manifolds can be described by a quotient $M = G/C(T)$, where $C(T)$ is the centralizer of a torus T of the Lie group G. If $C(T) = T$, then $M = G/T$ is called a full flag manifold.

Invariant Einstein metrics on full flag manifolds corresponding to classical Lie groups were studied by several authors (see [1–3]). Nevertheless when the isotropy summands of the full flag manifolds increase, it is very difficult to find all the G-invariant Einstein metrics. Since the system of the Einstein equations is very complex, it is a non-trivial problem to get all the positive real solutions of the system of the Einstein equations. In this paper, we give the classification problem of homogeneous Einstein metrics on the full flag manifold SO(7)/T, which admits precisely five Einstein metrics (up to isometry), where one is Kähler Einstein metric and four are non-Kähler Einstein metrics.

This paper is organized as follows. In Section 2 we recall the Lie theoretic description of a generalized flag manifold G/K of a compact and connected semisimple Lie group G. In Section 3 we compute the non-zero structure constants of the full flag manifold SO(7)/T and consider

Manuscript received September 25, 2014. Revised September 21, 2016.
[1]School of Science, Sichuan University of Science and Engineering, Zigong 643000, Sichuan, China.
E-mail: wangyu_813@163.com litianzeng27@163.com
[2]Department of Mathematics, Sichuan University, Chengdu 610064, China. E-mail: gszhao@scu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11501390, 61573010), the Fund of Sichuan University of Science and Engineering (No. 2015RC10), the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No. 2014QZJ03) and the Opening Project of Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things (No. 2016WYJ04).

the isometric problem, then prove that $\mathrm{SO}(7)/T$ admits five (up to isometry) $\mathrm{SO}(7)$-invariant Einstein metrics.

2 Generalized Flag Manifold

In this section we recall the Lie-theoretic description of $M = G/K$.

Let $\mathfrak{k}$ and $\mathfrak{g}$ be the Lie algebras of K and G respectively, and $(\cdot,\cdot)$ be the Cartan Killing form on the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. The complexification $\eta^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, where η is the Cartan subalgebra of $\mathfrak{g}$.

We denote by $(\eta^{\mathbb{C}})^*$ the dual space of $\eta^{\mathbb{C}}$, and let $R \subset (\eta^{\mathbb{C}})^*$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\eta^{\mathbb{C}}$. We consider the root space decomposition $\mathfrak{g}^{\mathbb{C}} = \eta^{\mathbb{C}} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_{\alpha}{}^{\mathbb{C}}$.

Set

$$R_K = R \cap \langle \Pi_K \rangle, \quad R_K^+ = R^+ \cap \langle \Pi_K \rangle, \tag{2.1}$$

where $\langle \Pi_K \rangle$ denotes the set of roots generated by Π_K. Let R_M be a set such that $R = R_K \cup R_M$, which is called the set of complementary roots of M. Then one can get $R_M^+ = R^+ \backslash R_K^+$.

We choose a Weyl basis $\{E_\alpha, H_\alpha : \alpha \in R\}$ of $\mathfrak{g}^{\mathbb{C}}$ with $(E_\alpha, E_{-\alpha}) = 1,\ [E_\alpha, E_{-\alpha}] = H_\alpha$ and

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \text{if } \alpha+\beta \notin R, \\ N_{\alpha,\beta} E_{\alpha+\beta}, & \text{if } \alpha+\beta \in R, \end{cases} \tag{2.2}$$

where the constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $N_{\beta,\alpha} = -N_{\alpha,\beta}$. Then we obtain that

$$g = \eta \oplus \bigoplus_{\alpha\in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \tag{2.3}$$

where $A_\alpha = E_\alpha - E_{-\alpha}, B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Assume that $\mathfrak{p}$ is a parabolic Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{k} = \mathfrak{p} \cap \mathfrak{g} \subset \mathfrak{g}$ which is given by $\mathfrak{k} = \eta \oplus \bigoplus_{\alpha\in R_K^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$. According to (2.3), it follows that the direct decomposition $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{n}$, where $\mathfrak{k}^{\mathbb{C}} = \eta^{\mathbb{C}} \oplus \bigoplus_{\alpha\in R_K} \mathfrak{g}_\alpha^{\mathbb{C}}$ and $\mathfrak{n} = \bigoplus_{\alpha\in R_M^+} \mathfrak{g}_\alpha^{\mathbb{C}}$. Then, we obtain that

$$\mathfrak{m} = \bigoplus_{\alpha\in R_M^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{2.4}$$

For convenience, we fix a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r, \phi_1, \cdots, \phi_k\}$ of R, so that $\Pi_K = \{\phi_1, \cdots, \phi_k\}$ is a basis of the root system R_K and $\Pi_M = \Pi \backslash \Pi_K = \{\alpha_1, \cdots, \alpha_r\}$ $(r+k = l)$. We consider the decomposition $R = R_K \cup R_M$, and define the set

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{k}^{\mathbb{C}}) \cap \mathrm{i}\eta = \{X \in \eta : \phi(X) = 0 \quad \text{for all } \phi \in R_K\}, \tag{2.5}$$

where η is the real ad-diagonal subalgebra $\eta = \eta^{\mathbb{C}} \cap \mathrm{i}\mathfrak{k}$, $\mathfrak{z}$ presents the center of $\mathfrak{k}^{\mathbb{C}}$. Consider the linear restriction map $\kappa : \eta^* \to \mathfrak{t}^*$ defined by $\kappa(\alpha) = \alpha|_{\mathfrak{t}}$, and set $R_{\mathfrak{t}} = \kappa(R) = \kappa(R_M)$. The elements of $R_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots. Note that $\kappa(R_K) = 0$ and $\kappa(0) = 0$.

For an invariant ordering $R_M^+ = R^+ \setminus R_K^+$ in R_M, we set $R_{\mathfrak{t}}^+ = \kappa(R_M^+)$ and $R_{\mathfrak{t}}^- = -R_{\mathfrak{t}}^+ = \{-\xi : \xi \in R_{\mathfrak{t}}^+\}$. It is obvious that $R_{\mathfrak{t}}^- = \kappa(R_M^-)$, thus the splitting $R_{\mathfrak{t}} = R_{\mathfrak{t}}^- \cup R_{\mathfrak{t}}^+$ defines an ordering in $R_{\mathfrak{t}}$. The $\mathfrak{t}$-roots $\xi \in R_{\mathfrak{t}}^+$ (resp. $\xi \in R_{\mathfrak{t}}^-$) will be called positive (resp. negative).

Proposition 2.1 (see [4–5]) *There is a one-to-one correspondence between* $\mathfrak{t}$*-roots and complex irreducible* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-submodules* $\mathfrak{m}_\xi$ *of* $\mathfrak{m}^{\mathbb{C}}$*. This correspondence is given by*

$$R_{\mathfrak{t}} \ni \xi \leftrightarrow \mathfrak{m}_\xi = \bigoplus_{\alpha \in R_M : \kappa(\alpha) = \xi} \mathbb{C} E_\alpha.$$

Thus $\mathfrak{m}^{\mathbb{C}} = \bigoplus_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi$*. Moreover, these submodules are inequivalent as* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-modules.*

Since the complex conjugation $\tau : g^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $X + \mathrm{i}Y \mapsto X - \mathrm{i}Y$ $(X, Y \in \mathfrak{g})$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to the compact real form $\mathfrak{g}$ interchanges the root spaces, i.e., $\tau(E_\alpha) = -E_{-\alpha}$ and $\tau(E_{-\alpha}) = E_\alpha$, a decomposition of the real $\mathrm{ad}(\mathfrak{k})$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbb{C}})^\tau$ into real irreducible $\mathrm{ad}(\mathfrak{k})$-submodule is given by

$$\mathfrak{m} = \bigoplus_{\xi \in R^+ = \kappa(R_M^+)} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau, \tag{2.6}$$

where $\mathfrak{n}^\tau$ denotes the set of fixed points of the complex conjugation τ in a vector subspace $\mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$. If, for simplicity, we set $R_{\mathfrak{t}}^+ = \{\xi_1, \cdots, \xi_s\}$, then according to (2.6) each real irreducible $\mathrm{ad}(\mathfrak{k})$-submodule $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau$ $(1 \le i \le s)$ corresponding to the positive $\mathfrak{t}$-roots ξ_i is given by

$$\mathfrak{m}_i = \bigoplus_{\alpha \in R_M^+ : \kappa(\alpha) = \xi_i} \mathbb{R} A_\alpha + \mathbb{R} B_\alpha. \tag{2.7}$$

A $\mathfrak{t}$-root is called simple if it is not a sum of two positive $\mathfrak{t}$-root. Let $\Pi_M = \Pi \setminus \Pi_K = \{\alpha_1, \cdots, \alpha_r\}$. Then the set $\{\overline{\alpha}_i = \alpha_i|_{\mathfrak{t}} : \alpha_i \in \Pi_M\}$ is a $\mathfrak{t}$-base of $\mathfrak{t}^*$.

A G-invariant Riemannian metric g on M is identified with an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$, which can be written as $\langle X, Y \rangle = -(\Lambda X, Y)$ $(X, Y \in \mathfrak{m})$, where $\Lambda : \mathfrak{m} \to \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant positive definite symmetric endomorphism on $\mathfrak{m}$. Due to the decomposition (2.6), we can express Λ as $\Lambda = \sum_{\xi \in R_{\mathfrak{t}}^+} x_\xi \cdot \mathrm{Id}|_{(\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau}$, where each element in $\{x_\xi : \xi \in R_{\mathfrak{t}}^+\}$ is an eigenvalue of Λ. Due to decomposition (2.7), Λ is given by

$$\Lambda = \sum_{\xi_i \in R_{\mathfrak{t}}^+} x_{\xi_i} \cdot \mathrm{Id}|_{\mathfrak{m}_i} = \sum_{i=1}^{s} x_{\xi_i} \cdot \mathrm{Id}|_{\mathfrak{m}_i}, \tag{2.8}$$

where $x_i \equiv x_{\xi_i}$ for any $\xi_i \in R_{\mathfrak{t}}^+ = \{\xi_1, \cdots, \xi_s\}$.

It is obvious that the vectors $\{A_\alpha, B_\alpha : \alpha \in R_M^+\}$ are eigenvectors of Λ corresponding to the eigenvalue $x_i \equiv x_{\xi_i}$. We also denote this eigenvalue by $x_\alpha \in \mathbb{R}^+$, where $\alpha \in R_M^+$ is such that $\kappa(\alpha) = \xi_i$ for any $1 \le i \le s$. We extend Λ to $\mathfrak{m}^{\mathbb{C}}$ without any change in notation. Hence the inner product $g = \langle \cdot, \cdot \rangle$ admits a natural extension to an $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant bilinear symmetric form on

$\mathfrak{m}^{\mathbb{C}}$. Then the root vectors $\{E_\alpha : \alpha \in R_M\}$ are eigenvectors of $\Lambda : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ corresponding to the eigenvalues $x_\alpha = x_{-\alpha} > 0$. If we denote by $\{\omega^\alpha\}$ the basis of the dual space $(\mathfrak{m}^{\mathbb{C}})^*$, which is dual to the basis $\{E_\beta,\ \beta \in R_M\}$, i.e., $\omega^\alpha(E_\beta) = \delta^\alpha_\beta$, then we obtain the proposition below.

Proposition 2.2 (see [4, 6]) *Every real* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-invariant inner product* $g = \langle\cdot,\cdot\rangle$ *on* $\mathfrak{m}^{\mathbb{C}}$ *has the form*

$$g = \langle\cdot,\cdot\rangle = \sum_{\alpha\in R_M^+} x_\alpha \omega^\alpha \vee \omega^\beta = \sum_{\xi\in R_{\mathfrak{t}}^+} x_\xi \sum_{\alpha\in\kappa^-(\xi)} \omega^\alpha \vee \omega^\beta, \tag{2.9}$$

where $\omega^\alpha \vee \omega^\beta = \frac{1}{2}(\omega^\alpha \otimes \omega^\beta + \omega^\beta \otimes \omega^\alpha)$ *and the positive real numbers* x_α *satisfy* $x_\alpha = x_\beta$ *if* $\alpha|_{\mathfrak{t}} = \beta|_{\mathfrak{t}}$ *for any* $\alpha, \beta \in R_M^+$.

The space of G-invariant Riemannian metric $g = \langle\cdot,\cdot\rangle = -(\Lambda\cdot,\cdot)$ on M is given by

$$\{x_1 \cdot (-(\cdot,\cdot))|_{\mathfrak{m}_1} + \cdots + x_s \cdot (-(\cdot,\cdot))|_{\mathfrak{m}_s} : x_1, \cdots, x_s > 0\}, \tag{2.10}$$

where $x_1 \equiv x_{\xi_1} > 0, \cdots, x_s \equiv x_{\xi_s} > 0$.

Then the Ricci tensor Ric_g of G/K (as a G-invariant symmetric covariant 2-tensor on G/K) is identified with an $\mathrm{Ad}(K)$-invariant symmetric bilinear form on $\mathfrak{m}$ given by

$$\mathrm{Ric}_g = \gamma_1 x_1(-(\cdot,\cdot))|_{\mathfrak{m}_1} + \cdots + \gamma_s x_s(-(\cdot,\cdot))|_{\mathfrak{m}_s} \tag{2.11}$$

Here $\gamma_1, \cdots, \gamma_s$ are the components of the Ricci tensor on each $\mathfrak{m}_i$.

Proposition 2.3 (see [7]) *Let* $g = \langle\cdot,\cdot\rangle$ *be a* G*-invariant metric given by* (2.10)*, and* J *be a* G*-invariant complex structure induced by an invariant ordering* R_M^+*. Then,* g *is a Kählerian metric with respect to the complex structure* J*, if and only if the positive real numbers* x_ξ *satisfy* $x_{\xi+\zeta} = x_\xi + x_\zeta$ *for any* $\xi, \zeta, \xi+\zeta \in R_{\mathfrak{t}}^+ = \kappa(R_M^+)$*. Equivalently,* g *is Kähler, if and only if* $x_{\alpha+\beta} = x_\alpha + x_\beta$*, where* $\alpha, \beta, \alpha+\beta \in R_M^+$ *satisfy* $\kappa(\alpha) = \xi$ *and* $\kappa(\beta) = \zeta$*.*

Let $\{e_\alpha\}$ be an orthogonal basis with respect to $-(\cdot,\cdot)$ adapted to the decomposition of $\mathfrak{m}$: $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$ with $i < j$ then $\alpha < \beta$. Following [8], $A^\gamma_{\alpha,\beta} := -([e_\alpha, e_\beta], e_\gamma)$, thus $[e_\alpha, e_\beta]_{\mathfrak{m}} = \sum_\gamma A^\gamma_{\alpha,\beta} e_\gamma$. Consider

$$c^k_{ij} := \sum (A^\gamma_{\alpha,\beta})^2, \tag{2.12}$$

where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i,\ e_\beta \in \mathfrak{m}_j,\ e_\gamma \in \mathfrak{m}_k$ and $i, j, k \in \{1, \cdots, s\}$.

Hence c^k_{ij} is nonnegative, symmetric in all the three entries, and is independent of the orthogonal basis chosen for $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$ (but it depends on the choice of the decomposition of $\mathfrak{m}$).

Now we introduce the notion of symmetric $\mathfrak{t}$-triples of $\mathfrak{t}$-roots.

Definition 2.1 *A symmetric* $\mathfrak{t}$*-triple in* $\mathfrak{t}^*$ *is a triple* $\Omega = (\xi_i, \xi_j, \xi_k)$ *of* $\mathfrak{t}$*-roots* $\xi_i, \xi_j, \xi_k \in R_{\mathfrak{t}}$ *such that* $\xi_i + \xi_j + \xi_k = 0$.

Lemma 2.1 (see [9]) *Let (ξ_i, ξ_j, ξ_k) be symmetric $\mathfrak{t}$-triples. Then there exist roots $\alpha, \beta, \gamma \in R_M$ with $\kappa(\alpha) = \xi_i$, $\kappa(\beta) = \xi_j$, $\kappa(\gamma) = \xi_k$ such that $\alpha + \beta + \gamma = 0$.*

The calculus of the coefficients c_{ij}^k can be laborious. However the next result shows exactly which of them are non-zero.

Lemma 2.2 (see [7, Corollary 1.9]) *Let G/K be a generalized flag manifold of a compact simple Lie group G and $R_\mathfrak{t}$ be the associated $\mathfrak{t}$-root system. Assume that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ is a $-(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible* $\mathrm{ad}(\mathfrak{k})$*-module, and let $\xi_i, \xi_j, \xi_k \in R_\mathfrak{t}$ be the $\mathfrak{t}$-roots associated to the components $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$ respectively. Then, $c_{ij}^k \neq 0$, if and only if (ξ_i, ξ_j, ξ_k) is a symmetric $\mathfrak{t}$-triples, i.e., $\xi_i + \xi_j + \xi_k = 0$.*

3 Invariant Einstein Metrics on SO(7)/*T*

Let $M = G/T$ be a full flag manifold, and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be a $-(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$. Then the set

$$\left\{ X_\alpha = \frac{A_\alpha}{\sqrt{2}} = \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}},\ Y_\alpha = \frac{B_\alpha}{\sqrt{2}} = \sqrt{-1}\frac{E_\alpha + E_{-\alpha}}{\sqrt{2}} : \alpha \in R^+,\ \kappa(\alpha) = \xi_i \in R_\mathfrak{t}^+ \right\} \tag{3.1}$$

is a $-(\cdot,\cdot)$-orthogonal basis of $\mathfrak{m}_i$.

Lemma 3.1 (see [10, Proposition 1.3]) *The non-zero structure constant c_{ij}^k for a full flag manifold G/T is given by*

$$c_{ij}^k = (A_{\alpha,\beta}^{\alpha+\beta})^2 = 2N_{\alpha,\beta}^2,$$

where $\alpha, \beta \in R^+$ with $\kappa(\alpha) = \xi_i,\ \kappa(\beta) = \xi_j,\ \kappa(\alpha+\beta) = \xi_k$.

Lemma 3.2 (see [11]) *Let $M = G/K$ be a reductive homogeneous space of a compact semisimple Lie group G and let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be a decomposition of $\mathfrak{m}$ into mutually inequivalent irreducible* $\mathrm{ad}(\mathfrak{k})$*-submodules. Then the components $\gamma_1, \cdots, \gamma_s$ of the Ricci tensor of a G-invariant metric* (2.10) *on M are given by*

$$\gamma_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{i,j}\frac{x_k}{x_i x_j}c_{ij}^k - \frac{1}{2d_k}\sum_{i,j}\frac{x_j}{x_k x_i}c_{ki}^j, \quad k = 1, \cdots, s. \tag{3.2}$$

Next, we consider invariant Einstein metrics on the full flag manifold of SO(7)/*T* with painted Dynkin diagraph

$$\underset{\alpha_1}{\bullet}\!\!-\!\!\!-\!\!\underset{\alpha_2}{\bullet}\!\Longrightarrow\!\underset{\alpha_3}{\bullet}$$

Here $\Pi_M = \{\alpha_1, \alpha_2, \alpha_3\}$. Letting $\overline{\alpha}_1 = \kappa(\alpha_1)$, $\overline{\alpha}_2 = \kappa(\alpha_2)$ and $\overline{\alpha}_3 = \kappa(\alpha_3)$, it follows that $R_\mathfrak{t}^+ = \kappa(R_M^+) = \{\overline{\alpha}_1,\ \overline{\alpha}_2,\ \overline{\alpha}_3,\ \overline{\alpha}_1 + \overline{\alpha}_2,\ \overline{\alpha}_2 + \overline{\alpha}_3,\ \overline{\alpha}_2 + 2\overline{\alpha}_3,\ \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\alpha}_3,\ \overline{\alpha}_1 + \overline{\alpha}_2 + 2\overline{\alpha}_3,\ \overline{\alpha}_1 + 2\overline{\alpha}_2 + 2\overline{\alpha}_3\}$, thus we conclude the isotropy representation $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6 \oplus \mathfrak{m}_7 \oplus \mathfrak{m}_8 \oplus \mathfrak{m}_9$.

By Lemma 2.2, it follows that non-zero structure constants are c_{12}^4, c_{15}^7, c_{16}^8, c_{23}^5, c_{28}^9, c_{34}^7, c_{35}^6, c_{37}^8, c_{46}^9, c_{57}^9.

Lemma 3.3 *The non-zero structure constants of the full flag manifold* $\mathrm{SO}(7)/T$ *are given by* $c_{12}^4 = c_{15}^7 = c_{16}^8 = c_{23}^5 = c_{28}^9 = c_{34}^7 = c_{35}^6 = c_{37}^8 = c_{46}^9 = c_{57}^9 = \frac{1}{5}$.

Proof From the theory of Lie algebra, we can get $N_{\alpha,\beta}^2 = \frac{q(p+1)}{2}(\alpha,\alpha)$, $(\alpha,\beta) = -\frac{q-p}{2}(\alpha,\alpha)$, where p, q are the largest nonnegative integers such that $\beta + k\alpha \in R$ with $-p \le k \le q$ (see [12]).

By Lemma 3.1, we can calculate the non-zero structure constants of M as follows:

$$\begin{aligned}
c_{12}^4 &= 2N_{\alpha_1,\alpha_2}^2 = (\alpha_1,\alpha_1),\\
c_{15}^7 &= 2N_{\alpha_1,\alpha_2+\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{16}^8 &= 2N_{\alpha_1,\alpha_2+2\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{23}^5 &= 2N_{\alpha_2,\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{28}^9 &= 2N_{\alpha_2,\alpha_1+\alpha_2+2\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{34}^7 &= 2N_{\alpha_3,\alpha_1+\alpha_2}^2 = (\alpha_1,\alpha_1),\\
c_{35}^6 &= 2N_{\alpha_3,\alpha_2+\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{37}^8 &= 2N_{\alpha_3,\alpha_1+\alpha_2+\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{46}^9 &= 2N_{\alpha_1+\alpha_2,\alpha_2+2\alpha_3}^2 = (\alpha_1,\alpha_1),\\
c_{57}^9 &= 2N_{\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3}^2 = (\alpha_1,\alpha_1).
\end{aligned}$$

As $(\alpha_1,\alpha_1) = \frac{1}{5}$ (see [13]), we obtain $c_{12}^4 = c_{15}^7 = c_{16}^8 = c_{23}^5 = c_{28}^9 = c_{34}^7 = c_{35}^6 = c_{37}^8 = c_{46}^9 = c_{57}^9 = \frac{1}{5}$.

Lemma 3.4 *The components* γ_i $(i = 1, \cdots, 9)$ *of Ricci tensor associated to the* $\mathrm{SO}(7)$*-invariant Riemmanian metric* g *on* $\mathrm{SO}(7)/T$ *are the following:*

$$\left\{\begin{aligned}
\gamma_1 &= \frac{1}{2x_1} + \frac{x_1^2 - x_2^2 - x_4^2}{20x_1x_2x_4} + \frac{x_1^2 - x_5^2 - x_7^2}{20x_1x_5x_7} + \frac{x_1^2 - x_6^2 - x_8^2}{20x_1x_6x_8},\\
\gamma_2 &= \frac{1}{2x_2} + \frac{x_2^2 - x_1^2 - x_4^2}{20x_1x_2x_4} + \frac{x_2^2 - x_3^2 - x_5^2}{20x_2x_3x_5} + \frac{x_2^2 - x_8^2 - x_9^2}{20x_2x_8x_9},\\
\gamma_3 &= \frac{1}{2x_3} + \frac{x_3^2 - x_2^2 - x_5^2}{20x_2x_3x_5} + \frac{x_3^2 - x_4^2 - x_7^2}{20x_3x_4x_7} + \frac{x_3^2 - x_5^2 - x_6^2}{20x_3x_5x_6} + \frac{x_3^2 - x_7^2 - x_8^2}{20x_3x_7x_8},\\
\gamma_4 &= \frac{1}{2x_4} + \frac{x_4^2 - x_3^2 - x_7^2}{20x_3x_4x_7} + \frac{x_4^2 - x_1^2 - x_2^2}{20x_1x_2x_4} + \frac{x_4^2 - x_6^2 - x_9^2}{20x_4x_6x_9},\\
\gamma_5 &= \frac{1}{2x_5} + \frac{x_5^2 - x_2^2 - x_3^2}{20x_2x_3x_5} + \frac{x_5^2 - x_1^2 - x_7^2}{20x_1x_5x_7} + \frac{x_5^2 - x_3^2 - x_6^2}{20x_3x_5x_6} + \frac{x_5^2 - x_7^2 - x_9^2}{20x_5x_7x_9},\\
\gamma_6 &= \frac{1}{2x_6} + \frac{x_6^2 - x_1^2 - x_8^2}{20x_1x_6x_8} + \frac{x_6^2 - x_3^2 - x_5^2}{20x_3x_5x_6} + \frac{x_6^2 - x_4^2 - x_9^2}{20x_4x_6x_9},\\
\gamma_7 &= \frac{1}{2x_7} + \frac{x_7^2 - x_1^2 - x_5^2}{20x_1x_5x_7} + \frac{x_7^2 - x_3^2 - x_4^2}{20x_3x_4x_7} + \frac{x_7^2 - x_3^2 - x_8^2}{20x_3x_7x_8} + \frac{x_7^2 - x_5^2 - x_9^2}{20x_5x_7x_9},\\
\gamma_8 &= \frac{1}{2x_8} + \frac{x_8^2 - x_1^2 - x_6^2}{20x_1x_6x_8} + \frac{x_8^2 - x_2^2 - x_9^2}{20x_2x_8x_9} + \frac{x_8^2 - x_3^2 - x_7^2}{20x_3x_7x_8},\\
\gamma_9 &= \frac{1}{2x_9} + \frac{x_9^2 - x_2^2 - x_8^2}{20x_2x_8x_9} + \frac{x_9^2 - x_4^2 - x_6^2}{20x_4x_6x_9} + \frac{x_9^2 - x_5^2 - x_7^2}{20x_5x_7x_9}.
\end{aligned}\right.$$

Proof By substituting the dimension $d_i = \dim(\mathfrak{m}_i) = 2$ and the non-zero structure constants c_{ij}^k into (3.2), we can get the results.

A G-invariant Riemmanian metric g on $M = \mathrm{SO}(7)/T$ is Einstein, if and only if, there is a positive constant e such that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = e$, or equivalently,

$$\gamma_1 - \gamma_2 = \gamma_2 - \gamma_3 = \gamma_3 - \gamma_4 = \gamma_4 - \gamma_5 = \gamma_5 - \gamma_6 = \gamma_6 - \gamma_7 = \gamma_7 - \gamma_8 = \gamma_8 - \gamma_9 = 0. \tag{3.3}$$

By Lemma 3.4 and system (3.3), one can obtain the following polynomial system (we apply the normalization $x_1 = 1$):

$$\left\{\begin{aligned}
&x_2x_3x_4x_5x_7x_9 + x_2x_3x_4x_6x_8x_9 + 2x_3x_5x_6x_7x_8x_9 - x_2^2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8^2\\
&+x_3x_4x_5x_6x_7x_9^2 - x_2^2x_4x_6x_7x_8x_9 + x_3^2x_4x_6x_7x_8x_9 + x_4x_5^2x_6x_7x_8x_9 - x_2x_3x_4x_5x_6^2x_7x_9\\
&-x_2x_3x_4x_5^2x_6x_8x_9 - x_2x_3x_4x_5x_7x_8^2x_9 - x_2x_3x_4x_6x_7^2x_8x_9 - 2x_2^2x_3x_5x_6x_7x_8x_9\\
&-10x_3x_4x_5x_6x_7x_8x_9 + 10x_2x_3x_4x_5x_6x_7x_8x_9 = 0;\\
&x_2^2x_3x_4x_5x_6x_7 + x_2^2x_3x_5x_6x_7x_8x_9 + 2x_2^2x_4x_6x_7x_8x_9 - x_2x_3^2x_4x_5x_6x_9 - x_2x_3^2x_4x_7x_8x_9\\
&-x_2x_3^2x_5x_6x_8x_9 + x_2x_4^2x_5x_6x_8x_9 + x_2x_4x_5^2x_7x_8x_9 + x_2x_4x_5x_6x_7^2x_9 - 10x_2x_4x_5x_6x_7x_8x_9\\
&+x_2x_4x_5x_6x_8^2x_9 + x_2x_4x_6^2x_7x_8x_9 + x_2x_5x_6x_7^2x_8x_9 - 2x_3^2x_4x_6x_7x_8x_9 - x_3x_4x_5x_6x_7x_8^2\\
&+10x_3x_4x_5x_6x_7x_8x_9 - x_3x_4x_5x_6x_7x_9^2 - x_3x_5x_6x_7x_8x_9 = 0;\\
&x_2^2x_3x_5x_6x_7x_8x_9 - x_2^2x_4x_6x_7x_8x_9 + x_2x_3^2x_4x_5x_6x_9 + x_2x_3^2x_4x_7x_8x_9 + 2x_2x_3^2x_5x_6x_8x_9\\
&-x_2x_3x_4^2x_5x_7x_8 + x_2x_3x_5x_6^2x_7x_8 - 10x_2x_3x_5x_6x_7x_8x_9 + x_2x_3x_5x_7x_8x_9^2 - 2x_2x_4^2x_5x_6x_8x_9\\
&-x_2x_4x_5^2x_7x_8x_9 - x_2x_4x_5x_6x_7^2x_9 + 10x_2x_4x_5x_6x_7x_8x_9 - x_2x_4x_5x_6x_8^2x_9 - x_2x_4x_6^2x_7x_8x_9\\
&+x_3^2x_4x_6x_7x_8x_9 - x_3x_4^2x_5x_6x_7x_8x_9 + x_3x_5x_6x_7x_8x_9 - x_4x_5^2x_6x_7x_8x_9 = 0;\\
&-x_2^2x_3x_5x_6x_7x_9 - x_2^2x_4x_6x_7x_9 + x_2x_3^2x_4x_7x_9 - x_2x_3^2x_5x_6x_9 + x_2x_3x_4^2x_5x_7 - x_2x_3x_4x_5^2x_6x_9\\
&-x_2x_3x_4x_5^2x_6 + x_2x_3x_4x_6x_7^2x_9 + x_2x_3x_4x_6x_7^2 - 10x_2x_3x_4x_6x_7x_9 + x_2x_3x_4x_6x_9^2 + x_2x_3x_4x_6x_9\\
&-x_2x_3x_5x_6^2x_7 + 10x_2x_3x_5x_6x_7x_9 - x_2x_3x_5x_7x_9^2 + x_2x_4^2x_5x_6x_9 - x_2x_4x_5^2x_7x_9 - x_2x_5x_6x_7^2x_9\\
&+x_3^2x_4x_6x_7x_9 + x_3x_4^2x_5x_6x_7x_9 - x_3x_5x_6x_7x_9 - x_4x_5^2x_6x_7x_9 = 0;\\
&-x_2^2x_4x_6x_7x_8x_9 + x_2x_3x_4^2x_5x_7x_8 + x_2x_3x_4x_5^2x_6x_8x_9 + x_2x_3x_4x_5^2x_6x_8 - x_2x_3x_4x_5x_6^2x_7x_9\\
&+x_2x_3x_4x_5x_7x_8^2x_9 - 10x_2x_3x_4x_5x_7x_8x_9 + x_2x_3x_4x_5x_7x_9 - x_2x_3x_4x_6x_7^2x_8x_9 - x_2x_3x_4x_6x_7^2x_8\\
&+10x_2x_3x_4x_6x_7x_8x_9 - x_2x_3x_4x_6x_8x_9^2 - x_2x_3x_4x_6x_8x_9 - x_2x_3x_5x_6^2x_7x_8 + x_2x_3x_5x_7x_8x_9^2\\
&+2x_2x_4x_5^2x_7x_8x_9 - 2x_2x_4x_6^2x_7x_8x_9 - x_3^2x_4x_6x_7x_8x_9 + x_4x_5^2x_6x_7x_8x_9 = 0;\\
&x_3^2x_4x_5x_6x_9 - x_3^2x_4x_7x_8x_9 + x_3^2x_5x_6x_8x_9 - x_3x_4^2x_5x_7x_8 + x_3x_4x_5^2x_6x_8x_9 + x_3x_4x_5^2x_6x_8\\
&+x_3x_4x_5x_6^2x_7x_9 - 10x_3x_4x_5x_6x_8x_9 - x_3x_4x_5x_7x_8^2x_9 + 10x_3x_4x_5x_7x_8x_9 - x_3x_4x_5x_7x_9\\
&-x_3x_4x_6x_7^2x_8x_9 - x_3x_4x_6x_7^2x_8 + x_3x_4x_6x_8x_9^2 + x_3x_4x_6x_8x_9 + x_3x_5x_6^2x_7x_8 - x_3x_5x_7x_8x_9^2\\
&+x_4^2x_5x_6x_8x_9 - x_4x_5^2x_7x_8x_9 - x_4x_5x_6x_7^2x_9 + x_4x_5x_6x_8^2x_9 + x_4x_6^2x_7x_8x_9 - x_5x_6x_7^2x_8x_9 = 0;\\
&x_2^2x_3x_4x_5x_6x_7 - x_2x_3^2x_5x_6x_8x_9 - x_2x_3x_4x_5^2x_6x_8x_9 - x_2x_3x_4x_5^2x_6x_8 + x_2x_3x_4x_5x_6^2x_7x_9\\
&+10x_2x_3x_4x_5x_6x_8x_9 - 10x_2x_3x_4x_5x_6x_7x_9 - x_2x_3x_4x_5x_7x_8^2x_9 + x_2x_3x_4x_5x_7x_9\\
&+x_2x_3x_4x_6x_7^2x_8x_9 + x_2x_3x_4x_6x_7^2x_8 - x_2x_3x_4x_6x_8x_9^2 - x_2x_3x_4x_6x_8x_9 - x_2x_4^2x_5x_6x_8x_9\\
&+2x_2x_4x_5x_6x_7^2x_9 - 2x_2x_4x_5x_6x_8^2x_9 + x_2x_5x_6x_7^2x_8x_9 - x_3x_4x_5x_6x_7x_8^2 + x_3x_4x_5x_6x_7x_9^2 = 0;\\
&-x_2x_3^2x_4x_5x_6x_9 + x_2x_3x_4^2x_5x_7x_8 + x_2x_3x_4x_5^2x_6x_8 - x_2x_3x_4x_5x_6^2x_7x_9 + 2x_3x_4x_5x_6x_7x_8^2\\
&-10x_2x_3x_4x_5x_6x_7x_8 + 10x_2x_3x_4x_5x_6x_7x_9 - 2x_3x_4x_5x_6x_7x_9^2 + x_2x_3x_4x_5x_7x_8^2x_9\\
&-x_2x_3x_4x_5x_7x_9 + x_2x_3x_4x_6x_7^2x_8 - x_2x_3x_4x_6x_8x_9^2 + x_2x_3x_5x_6^2x_7x_8 - x_2x_3x_5x_7x_8x_9^2\\
&-x_2x_4x_5x_6x_7^2x_9 + x_2x_4x_5x_6x_8^2x_9 = 0;
\end{aligned}\right.$$

Every positive real solution $x_2 > 0,\ x_3 > 0,\ x_4 > 0,\ x_5 > 0,\ x_6 > 0,\ x_7 > 0,\ x_8 > 0,\ x_9 > 0$ of the system above determines a SO(7)-invariant Einstein metric $(1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathbb{R}_+^9$ on $M = \mathrm{SO}(7)/T$. With the help of computer we get all the forty-eight positive solutions (up to a scale) for above system, i.e., there are forty-eight G-invariant Einstein metrics (up to a scale) on the full flag manifold $\mathrm{SO}(7)/T$.

Next we talk about the isometric problem about the metrics, in general, this is a non-trivial problem.

Let G/K be a generalized flag manifold with isotropy decomposition $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_s$, and $d = \sum_{i=1}^{s} d_i = \dim M$. For any G-invariant Einstein metric $g = (x_1, \cdots, x_s)$ on M, we determine a scale invariant given by $H_g = V^{\frac{1}{d}} S_g$, where S_g is the scalar curvature of g, $V = V_g/V_B$ is the quotient of the volumes $V_g = \prod_{i=1}^{s} x_i^{d_i}$ of the given metric g, and V_B is the volume of the normal metric induced by the negative of the Killing form of G. We normalize $V_B = 1$, so

$$H_g = V_g^{\frac{1}{d}} S_g.$$

The scalar curvature S_g of a G-invariant metric g on M is given by the following well-known formula (see [8]):

$$S_g = \sum_{i=1}^{s} d_i \gamma_i = \frac{1}{2} \sum_{i=1}^{s} \frac{d_i}{x_i} - \frac{1}{4} \sum_{1 \le i,j,k \le s} c_{ij}^k \frac{x_k}{x_i x_j}, \tag{3.4}$$

where the components γ_i of the Ricci tensor are given by (3.2). The scalar curvature is a homogeneous polynomial of degree -1 on the variables x_i $(i = 1, \cdots, s)$. The volume V_g is a monomial of degree d, so $H_g = V_g^{\frac{1}{d}} S_g$ is a homogeneous polynomial of degree 0. Therefore, H_g is invariant under a common scaling of the variables x_i. As

$$H_g = V_g^{\frac{1}{d}} S_g,$$

according to (3.4) we obtain

$$H_g = \Big(\prod_{i=1}^{9} x_i^{\frac{1}{9}}\Big) \Big(\sum_{i=1}^{9} 2\gamma_i\Big). \tag{3.5}$$

If two metrics are isometric, then they have the same scale invariant, so if the scale invariant H_g and $H_{g'}$ are different, then the metrics g and g' can not be isometric. However, if $H_g = H_{g'}$, we can not immediately conclude if the metrics g and g' are isometric or not. For such a case we have to look at the group of automorphisms of G and check if there is an automorphism which permutes the isotopy summands and takes one metric to another.

Theorem 3.1 *The full flag manifold $M = \mathrm{SO}(7)/T$ admits (up to a scale) forty-eight $\mathrm{SO}(7)$-invariant Einstein metrics, which approximately are given as follows:*

$$\begin{array}{ll}
(1)\ \left(1, \frac{1}{4}, \frac{1}{8}, \frac{3}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{1}{2}, \frac{1}{4}\right), & (2)\ \left(1, \frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{3}, \frac{2}{3}\right), \\
(3)\ \left(1, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{5}{4}, 2, \frac{1}{4}, 1, \frac{3}{2}\right), & (4)\ \left(1, \frac{3}{4}, \frac{1}{8}, \frac{1}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{1}{2}, \frac{1}{4}\right), \\
(5)\ \left(1, 1, \frac{5}{2}, 2, \frac{3}{2}, 4, \frac{1}{2}, 3, 2\right), & (6)\ \left(1, 1, \frac{1}{2}, 2, \frac{3}{2}, 2, \frac{5}{2}, 3, 4\right), \\
(7)\ \left(1, 2, \frac{1}{2}, 1, \frac{5}{2}, 3, \frac{3}{2}, 2, 4\right), & (8)\ \left(1, 3, \frac{1}{2}, 2, \frac{5}{2}, 2, \frac{3}{2}, 1, 4\right), \\
(9)\ \left(1, 4, \frac{5}{2}, 3, \frac{3}{2}, 1, \frac{1}{2}, 2, 2\right), & (10)\ \left(1, 3, \frac{5}{2}, 4, \frac{1}{2}, 2, \frac{3}{2}, 1, 2\right), \\
(11)\ \left(1, \frac{4}{3}, \frac{1}{2}, \frac{1}{3}, \frac{5}{6}, \frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{2}{3}\right), & (12)\ \left(1, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 1, \frac{5}{4}, 2, \frac{3}{2}\right),
\end{array}$$

(13) $\left(1, 2, \frac{5}{2}, 1, \frac{1}{2}, 3, \frac{3}{2}, 4, 2\right)$, (14) $\left(1, 2, \frac{1}{2}, 3, \frac{3}{2}, 1, \frac{5}{2}, 2, 4\right)$,

(15) $\left(1, \frac{1}{2}, \frac{5}{4}, \frac{3}{2}, \frac{3}{4}, 2, \frac{1}{4}, 1, \frac{1}{2}\right)$, (16) $\left(1, \frac{3}{2}, \frac{5}{4}, \frac{1}{2}, \frac{1}{4}, 1, \frac{3}{4}, 2, \frac{1}{2}\right)$,

(17) $\left(1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{4}{3}, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}\right)$, (18) $\left(1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \frac{2}{3}\right)$,

(19) $\left(1, \frac{1}{2}, \frac{1}{8}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{8}, \frac{3}{4}, \frac{1}{4}\right)$, (20) $\left(1, 2, \frac{3}{4}, 1, \frac{5}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{2}\right)$,

(21) $\left(1, \frac{1}{2}, \frac{1}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}\right)$, (22) $\left(1, 1, \frac{3}{4}, 2, \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{1}{2}, \frac{3}{2}\right)$,

(23) $\left(1, 2, \frac{5}{4}, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{2}, \frac{1}{2}\right)$, (24) $\left(1, 1, \frac{5}{4}, 2, \frac{1}{4}, \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}\right)$,

(25) $(1, 0.4215, 0.1854, 0.6820, 0.4421, 0.4215, 0.6955, 0.6820, 0.3251)$,

(26) $(1, 0.4766, 0.6482, 0.6180, 1.0197, 1.4662, 0.2719, 0.6180, 1)$,

(27) $(1, 0.7712, 1.6501, 1.6182, 1.0489, 2.3726, 0.4400, 1.6182, 1)$,

(28) $(1, 1.6182, 1.6501, 0.7712, 0.4400, 1.6182, 1.0489, 2.3726, 1)$,

(29) $(1, 2.0982, 0.5705, 1.2966, 2.1395, 2.0982, 1.3600, 1.2966, 3.0764)$,

(30) $(1, 2.3726, 1.6501, 1.6182, 1.0489, 0.7712, 0.4400, 1.6182, 1)$,

(31) $(1, 0.6820, 0.1854, 0.4215, 0.6955, 0.6820, 0.4421, 0.4215, 0.3251)$,

(32) $(1, 0.6180, 0.6482, 0.4766, 0.2719, 0.6180, 1.0197, 1.4662, 1)$,

(33) $(1, 1.2966, 0.5705, 2.0982, 1.3600, 1.2966, 2.1395, 2.0982, 3.0764)$,

(34) $(1, 1.6182, 1.6501, 2.3726, 0.4400, 1.6182, 1.0489, 0.7712, 1)$,

(35) $(1, 1.4662, 0.6482, 0.6180, 1.0197, 0.4766, 0.2719, 0.6180, 1)$,

(36) $(1, 0.6180, 0.6482, 1.4662, 0.2719, 0.6180, 1.0197, 0.4766, 1)$,

(37) $(1, 0.6083, 1.0188, 1, 1.0188, 0.6083, 0.2736, 1, 1)$,

(38) $(1, 1.6440, 0.4498, 1.6440, 1.6750, 1.6440, 1.6750, 1.6440, 1)$,

(39) $(1, 1, 1.0188, 0.6083, 0.2736, 1, 1.0188, 0.6083, 1)$,

(40) $(1, 0.6638, 0.2309, 0.6638, 0.6826, 0.6638, 0.6826, 0.6638, 0.5983)$,

(41) $(1, 1.1094, 0.3860, 1.1094, 1.1409, 1.1094, 1.1409, 1.1094, 1.6713)$,

(42) $(1, 1.5066, 1.0284, 1, 1.0284, 0.9014, 0.3479, 1, 1)$,

(43) $(1, 0.9014, 1.0284, 1, 1.0284, 1.5066, 0.3479, 1, 1)$,

(44) $(1, 1, 1.0284, 1.5066, 0.3479, 1, 1.0284, 0.9014, 1)$,

(45) $(1, 1, 1.0284, 0.9014, 0.3479, 1, 1.0284, 1.5066, 1)$,

(46) $(1, 0.7739, 0.2818, 0.7739, 0.7978, 0.7739, 0.7978, 0.7739, 1)$,

(47) $(1, 1.2921, 1.0308, 1, 1.0308, 1.2921, 0.3641, 1, 1)$,

(48) $(1, 1, 1.0308, 1.2921, 0.3641, 1, 1.0308, 1.2921, 1)$,

where $(1),\cdots,(24)$ *are Kähler Einstein metrics. Then, the approximate* H_g *corresponding to every metric* g *from* (1) *to* (48) *are as follows:*

$$\begin{aligned}
&H_{g_{(1)}}=\cdots=H_{\mathfrak{g}_{(24)}}\approx 5.935271057,\\
&H_{\mathfrak{g}_{(25)}}=\cdots=H_{\mathfrak{g}_{(36)}}\approx 5.925652920,\\
&H_{\mathfrak{g}_{(37)}}=H_{\mathfrak{g}_{(38)}}=H_{\mathfrak{g}_{(39)}}\approx 5.870792469,\\
&H_{\mathfrak{g}_{(40)}}=\cdots=H_{\mathfrak{g}_{(45)}}\approx 5.9098080970,\\
&H_{\mathfrak{g}_{(46)}}=H_{\mathfrak{g}_{(47)}}=H_{\mathfrak{g}_{(48)}}\approx 5.9121091966.
\end{aligned}$$

As it is known that there is a one-to-one correspondence between non-equivalent G-invariant complex structures and the non-isometric Kähler Einstein metrics on a flag manifold. Since there is only one G-invariant complex structure on any full flag manifold, so there is only one non-isometric Kähler Einstein metric on full flag manifold $\mathrm{SO}(7)/T$, thus the Kähler Einstein metrics from (1) to (24) in Theorem 3.1 are all isometric.

By Theorem 3.1 we obtain that there are four non-equal values of H_g corresponding to non-Kähler-Einstein metrics from (25) to (48). Thus there are at least four non-isometric non-Kähler Einstein metrics. Next, we prove there are exactly four non-isometric non-Kähler Einstein metrics.

Let

$$\Pi=\{\alpha_1,\alpha_2,\alpha_3\}$$

be the simple root system of the Lie algebra of $\mathrm{SO}(7)$. then the Weyl group $\mathcal{W}$ of $\mathrm{SO}(7)$ is generated by $r_{\alpha_1}, r_{\alpha_2}, r_{\alpha_3}$, where

$$r_\alpha(\beta)=\beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha,\quad (\alpha,\beta)=-\frac{q-p}{2}(\alpha,\alpha)$$

and p,q are the largest nonnegative integers such that $\beta+k\alpha\in R$ with

$$-p\le k\le q.$$

Let

$$\begin{aligned}
R^+=\{&\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+2\alpha_3,\\
&\alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\}
\end{aligned}$$

be the positive roots of the Lie algebra of $\mathrm{SO}(7)$. Then we obtain

$$\begin{aligned}
&r_{\alpha_1}(\alpha_1)=-\alpha_1,\\
&r_{\alpha_1}(\alpha_2)=\alpha_1+\alpha_2,\\
&r_{\alpha_1}(\alpha_3)=\alpha_3,\\
&r_{\alpha_1}(\alpha_1+\alpha_2)=\alpha_2,\\
&r_{\alpha_1}(\alpha_2+\alpha_3)=\alpha_1+\alpha_2+\alpha_3,
\end{aligned}$$

$$
\begin{aligned}
&r_{\alpha_1}(\alpha_2+2\alpha_3)=\alpha_1+\alpha_2+2\alpha_3,\\
&r_{\alpha_1}(\alpha_1+\alpha_2+\alpha_3)=\alpha_2+\alpha_3,\\
&r_{\alpha_1}(\alpha_1+\alpha_2+2\alpha_3)=\alpha_2+2\alpha_3,\\
&r_{\alpha_1}(\alpha_1+2\alpha_2+2\alpha_3)=\alpha_1+2\alpha_2+2\alpha_3;\\
&r_{\alpha_2}(\alpha_1)=\alpha_1+\alpha_2,\\
&r_{\alpha_2}(\alpha_2)=-\alpha_2,\\
&r_{\alpha_2}(\alpha_3)=\alpha_3+\alpha_2,\\
&r_{\alpha_2}(\alpha_1+\alpha_2)=\alpha_1,\\
&r_{\alpha_2}(\alpha_2+\alpha_3)=\alpha_3,\\
&r_{\alpha_2}(\alpha_2+2\alpha_3)=\alpha_2+2\alpha_3,\\
&r_{\alpha_2}(\alpha_1+\alpha_2+\alpha_3)=\alpha_1+\alpha_2+\alpha_3,\\
&r_{\alpha_2}(\alpha_1+\alpha_2+2\alpha_3)=\alpha_1+2\alpha_2+2\alpha_3,\\
&r_{\alpha_2}(\alpha_1+2\alpha_2+2\alpha_3)=\alpha_1+\alpha_2+2\alpha_3;\\
&r_{\alpha_3}(\alpha_1)=\alpha_1,\\
&r_{\alpha_3}(\alpha_2)=\alpha_2+2\alpha_3,\\
&r_{\alpha_3}(\alpha_3)=-\alpha_3,\\
&r_{\alpha_3}(\alpha_1+\alpha_2)=\alpha_1+\alpha_2+2\alpha_3,\\
&r_{\alpha_3}(\alpha_2+\alpha_3)=\alpha_2+\alpha_3,\\
&r_{\alpha_3}(\alpha_2+2\alpha_3)=\alpha_2,\\
&r_{\alpha_3}(\alpha_1+\alpha_2+\alpha_3)=\alpha_1+\alpha_2+\alpha_3,\\
&r_{\alpha_3}(\alpha_1+\alpha_2+2\alpha_3)=\alpha_1+\alpha_2,\\
&r_{\alpha_3}(\alpha_1+2\alpha_2+2\alpha_3)=\alpha_1+2\alpha_2+2\alpha_3.
\end{aligned}
$$

Since the action of the Weyl group of SO(7) on the root system of the Lie algebra SO(7) induces an action on the components of the SO(7)-invaraint metric

$$g=x_1\cdot(-(\cdot,\cdot))|_{\mathfrak{m}_1}+\cdots+x_s\cdot(-(\cdot,\cdot))|_{\mathfrak{m}_s}$$

In particular, if

$$g=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)=(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9) \tag{3.6}$$

is a positive solution of the system of the Einstein equations of SO(7)/T, then

$$
\begin{aligned}
&r_{\alpha_1}(g)=(a_1,a_4,a_3,a_2,a_7,a_8,a_5,a_6,a_9),\\
&r_{\alpha_2}(g)=(a_4,a_2,a_5,a_1,a_3,a_6,a_7,a_9,a_8),\\
&r_{\alpha_3}(g)=(a_1,a_6,a_3,a_8,a_5,a_2,a_7,a_4,a_9),\\
&r_{\alpha_1}\circ r_{\alpha_2}(g)=(a_2,a_4,a_7,a_1,a_3,a_8,a_5,a_9,a_6),\\
&r_{\alpha_1}\circ r_{\alpha_3}(g)=(a_1,a_8,a_3,a_6,a_7,a_4,a_5,a_2,a_9),
\end{aligned}
$$

$$
\begin{aligned}
&r_{\alpha_2}\circ r_{\alpha_3}(g)=(a_4,a_6,a_5,a_9,a_3,a_2,a_7,a_1,a_8),\\
&r_{\alpha_2}\circ r_{\alpha_1}(g)=(a_4,a_1,a_5,a_2,a_7,a_9,a_3,a_6,a_8),\\
&r_{\alpha_3}\circ r_{\alpha_2}(g)=(a_8,a_6,a_5,a_1,a_3,a_2,a_7,a_9,a_4),\\
&r_{\alpha_1}\circ r_{\alpha_2}\circ r_{\alpha_1}(g)=(a_2,a_1,a_7,a_4,a_5,a_9,a_3,a_8,a_6),\\
&r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_3}(g)=(a_8,a_2,a_5,a_9,a_3,a_6,a_7,a_1,a_4),\\
&r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}(g)=(a_9,a_6,a_3,a_4,a_5,a_2,a_7,a_8,a_1),\\
&r_{\alpha_1}\circ r_{\alpha_2}\circ r_{\alpha_3}(g)=(a_2,a_8,a_7,a_9,a_3,a_4,a_5,a_1,a_6),\\
&r_{\alpha_1}\circ r_{\alpha_3}\circ r_{\alpha_2}(g)=(a_6,a_8,a_7,a_1,a_3,a_4,a_5,a_9,a_2),\\
&r_{\alpha_2}\circ r_{\alpha_1}\circ r_{\alpha_3}(g)=(a_4,a_9,a_5,a_6,a_7,a_1,a_3,a_2,a_8),\\
&r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_1}(g)=(a_8,a_1,a_5,a_6,a_7,a_9,a_3,a_2,a_4),\\
&r_{\alpha_1}\circ r_{\alpha_2}\circ r_{\alpha_1}\circ r_{\alpha_3}(g)=(a_2,a_9,a_7,a_8,a_5,a_1,a_3,a_4,a_6),\\
&r_{\alpha_1}\circ r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}(g)=(a_9,a_8,a_3,a_2,a_7,a_4,a_5,a_6,a_1),\\
&r_{\alpha_1}\circ r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_1}(g)=(a_6,a_1,a_7,a_8,a_5,a_9,a_3,a_4,a_2),\\
&r_{\alpha_1}\circ r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_3}(g)=(a_6,a_4,a_7,a_9,a_3,a_8,a_5,a_1,a_2),\\
&r_{\alpha_2}\circ r_{\alpha_1}\circ r_{\alpha_3}\circ r_{\alpha_2}(g)=(a_6,a_9,a_7,a_4,a_5,a_1,a_3,a_8,a_2),\\
&r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_3}(g)=(a_9,a_2,a_3,a_8,a_5,a_6,a_7,a_4,a_1),\\
&r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_1}(g)=(a_9,a_4,a_3,a_6,a_7,a_8,a_5,a_2,a_1),\\
&r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_1}\circ r_{\alpha_3}(g)=(a_8,a_9,a_5,a_2,a_7,a_1,a_3,a_6,a_4)
\end{aligned}
\tag{3.7}
$$

are also positive solutions of the system of the Einstein equations of the full flag manifold $\mathrm{SO}(7)/T$. These metrics in system (3.7) are all isometric, and they are all isometric to the metric $g=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$.

As for any G-invariant Einstein metric g, we have

$$
\begin{aligned}
&r_{\alpha_1}\circ r_{\alpha_3}(g)=r_{\alpha_3}\circ r_{\alpha_1}(g),\\
&r_{\alpha_1}\circ r_{\alpha_2}\circ r_{\alpha_1}(g)=r_{\alpha_2}\circ r_{\alpha_1}\circ r_{\alpha_2}(g),\\
&r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_3}(g)=r_{\alpha_3}\circ r_{\alpha_2}\circ r_{\alpha_3}\circ r_{\alpha_2}(g).
\end{aligned}
$$

Thus, if $g=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$ is a G-invariant Einstein metric, $w(g)$ is one of in system (3.7) for any $w\in\mathcal{W}$ (except the identity element Id).

As $H_{g_{(25)}}=H_{g_{(26)}}=\cdots=H_{g_{(36)}}=5.925652920$, we prove that the metrics from (25) to (36) are isometric.

Because

$$g_{(25)}=(1,0.4215,0.1854,0.6820,0.4421,0.4215,0.6955,0.6820,0.3251)$$

is a G-invariant Einstein metric, by the action of the elements of the Weyl group of SO(7) on $g_{(25)}$ (except the identity element Id), we can obtain 23 positive solutions in system (19). But some of them are equal, except the equal solutions we can get the metrics from (26) to (36), and they are all isometric to $g_{(25)}=(1,0.4215,0.1854,0.6820,0.4421,0.4215,0.6955,0.6820,0.3251)$.

As $H_{g_{(37)}}=H_{g_{(38)}}=H_{g_{(39)}}=5.870792469$, we prove that the metrics from (37) to (39) are isometric.

Because $g_{(37)}$=(1, 0.6083, 1.0188, 1, 1.0188, 0.6083, 0.2736, 1, 1) is a G-invariant Einstein metric, by the action of the elements of the Weyl group of SO(7) on $g_{(37)}$ (except the identity map Id), we can obtain 23 positive solutions in system (3.7), but only two of them are not equal to each other. Thus we obtain the metrics (38) and (39), and they are all isometric to $g_{(37)} = (1, 0.6083, 1.0188, 1, 1.0188, 0.6083, 0.2736, 1, 1)$.

As $H_{g_{(40)}} = \cdots = H_{g_{(45)}} = 5.9098080970$, we prove that the metrics from (40) to (45) are isometric.

Because

$$g_{(40)} = (1, 0.6638, 0.2309, 0.6638, 0.6826, 0.6638, 0.6826, 0.6638, 0.5983)$$

is a G-invariant Einstein metric, by the action of the elements of the Weyl group of SO(7) on $g_{(40)}$ (except the identity map Id), we can obtain 23 positive solutions in system (3.7), but only five of them are not equal to each other. Thus we obtain the metrics from (41) to (45), and they are all isometric to $g_{(40)}$=(1, 0.6638, 0.2309, 0.6638, 0.6826, 0.6638, 0.6826, 0.6638, 0.5983).

As $H_{g_{(46)}} = H_{g_{(47)}} = H_{g_{(48)}} = 5.9121091966$, we prove that the metrics from (46) to (48) are isometric.

Because $g_{(46)} = (1, 0.7739, 0.2818, 0.7739, 0.7978, 0.7739, 0.7978, 0.7739, 1)$ is a G-invariant Einstein metric, by the action of the elements of the Weyl group of SO(7) on $g_{(46)}$ (except the identity map Id), we can obtain 23 positive solutions in system (3.7), but only two of them are not equal to each other. Thus we obtain the metrics (47) and (48), and they are all isometric to $g_{(46)} = (1, 0.7739, 0.2818, 0.7739, 0.7978, 0.7739, 0.7978, 0.7739, 1)$.

By the analysis above it follows that there are exactly five non-isometric Einstein metrics on the full flag manifold SO(7)/T, of which one is Kähler Einstein metric and four are non-Kähler Einstein metrics.

Theorem 3.2 *The full flag manifold* SO(7)/T *admits exactly five* SO(7)*-invariant Einstein metrics* (*up to isometry*)*. There is a unique Kähler Einstein metric* (*up to a scale*) *given by* $g = \left(1, 1, \frac{1}{2}, 2, \frac{3}{2}, 2, \frac{5}{2}, 3, 4\right)$ *and other four are non-Kähler Einstein metrics* (*up to a scale*) *given by as follows:*

(a) $(1, 0.4215, 0.1854, 0.6820, 0.4421, 0.4215, 0.6955, 0.6820, 0.3251)$,

(b) $(1, 0.6083, 1.0188, 1, 1.0188, 0.6083, 0.2736, 1, 1)$,

(c) $(1, 0.6638, 0.2309, 0.6638, 0.6826, 0.6638, 0.6826, 0.6638, 0.5983)$,

(d) $(1, 0.7739, 0.2818, 0.7739, 0.7978, 0.7739, 0.7978, 0.7739, 1)$.

References

[1] Arvanitoyeorgos, A., New invariant Einstein metrics on generalizd flag manifolds, *Trans. Amer. Math. Soc.,* **337**(2), 1993, 981–995.

[2] Sakane, Y., Homogeneous Einstein metrics on flag manifold, *Lobachevskii J. Math.*, **4**, 1999, 71–87.

[3] Dos Santos, E. C. F. and Negreiros, C. J. C., Einstein metrics on flag manifolds, *Revista Della, Unión Mathemática Aregetina,* **47**(2), 2006, 77–84.

[4] Alekseevsky, D. V. and Perelomov, A. M., Invariant Kähler-Einstein metrics on compact homogeneous spaces, *Funct. Anal. Appl.*, **20**(3), 1986, 171–182.

[5] Siebenthal, J., Sur certains modules dans une algébre de Lie semisimple, *Comment. Math. Helv.*, **44**(1), 1964, 1–44.

[6] Alekseevsky, D. V., Homogeneous Einstein metrics, *Differential Geometry and Its Applications* (*Brno Conference Proceedings*), 1987, 1–21.

[7] Chrysikos, I., Flag manifolds, symmetric t-triples and Einstein metrics, *Differential Geom. Appl.*, **30**(6), 2012, 642–659.

[8] Wang, Y. and Ziller, W., Existence and non-excistence of homogeneous Einstein metrics, *Invent. Math.*, **84**, 1986, 177–194.

[9] Graev, M. M., On the number of invariant Eistein metrics on a compact homogeneous space, Newton polytopes and contraction of Lie algebras, *Int. J. Geom. Methods Mod. Phys.*, **3**(5–6), 2006, 1047–1075.

[10] Arvanitoyeorgos, A., Chrysikos, I. and Sakane, Y., Homogeneous Einstein metrics on G_2/T, *Proc. Amer. Math. Soc.*, **14**(7), 2013, 2485–2499.

[11] Park, J. S. and Sakane, Y., Invariant Einstein metrics on certain homogeneous space, *Tokyo J. Math.*, **20**(1), 1997, 51–61.

[12] Varadarajan, V. S., Lie Group, Lie Algebra and Their Representations, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1984.

[13] Onishchik, A. L. and Vinberg, E. B., Lie Groups and Lie Algebras III, Structure of Lie Groups and Lie Algebras, Springer-Verlag, Berlin, Heideberg, 1994.