Chin. Ann. Math. Ser. B
39(1), 2018, 129–144
DOI: 10.1007/s11401-018-1056-6

Chinese Annals of Mathematics, Series B

Quenching Phenomenon for a Parabolic MEMS Equation

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Abstract This paper deals with the electrostatic MEMS-device parabolic equation

$$u_t - \Delta u = \frac{\lambda f(x)}{(1-u)^p}$$

in a bounded domain Ω of $\mathbb{R}^N$, with Dirichlet boundary condition, an initial condition $u_0(x) \in [0,1)$ and a nonnegative profile f, where $\lambda > 0$, $p > 1$. The study is motivated by a simplified micro-electromechanical system (MEMS for short) device model. In this paper, the author first gives an asymptotic behavior of the quenching time T^* for the solution u to the parabolic problem with zero initial data. Secondly, the author investigates when the solution u will quench, with general λ, $u_0(x)$. Finally, a global existence in the MEMS modeling is shown.

Keywords MEMS equation, Quenching time, Global existence
2000 MR Subject Classification 35A01, 35B44, 35K58

1 Introduction

We consider the parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda f(x)}{(1-u)^p}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $\lambda > 0$, Ω is a smooth bounded domain in $\mathbb{R}^N$, f is a nonnegative bounded smooth function, $u_0(x) \in [0,1)$ is a smooth function and $u_0(x) = 0$ on $\partial\Omega$. The associated stationary equation is as follows:

$$\begin{cases} -\Delta w = \dfrac{\lambda f(x)}{(1-w)^p}, & x \in \Omega, \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

The problem (1.1) arises in the study of micro-electromechanical system devices. The parameter $\lambda > 0$ is a constant which is increasing with respect to the applied voltage (see [13, 19] for more details). These systems are microsize integrated devices or tiny systems that combine mechanical and electrical components. They are used in systems ranging across automotive, medical, electronic, chemistry, biology, communication and defence applications. Recently, a

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mathematical modeling proposed by Pelesko and Bernstein [18] in 2002 leads to elliptic and parabolic PDEs of second or fourth order. In these lectures, some interesting problems, results and open questions have been presented on the MEMS modeling. We can refer to [7–8, 13, 16, 18–19] and the references therein for detailed discussions on MEMS devices modeling.

We say that the solution w to (1.2) is regular or classical, if $\|w\|_\infty < 1$; the solution w to (1.2) is singular, if $\|w\|_\infty = 1$. It is well known (see [3, 10–11, 22] and the references therein) that for any given f, there exists a critical value $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, the problem (1.2) has a unique stable classical solution w_λ and the solution to (1.1) is global with $u_0 = 0$. Moreover, w_λ is the minimal solution and $\lambda \to w_\lambda$ is increasing. Here the minimal solution means that $w_\lambda \leq v$ for any solution v to (1.2). For $\lambda = \lambda^*$, the problem (1.2) admits a unique weak solution $w^* := \lim\limits_{\lambda\to\lambda^*} w_\lambda$, called the extremal solution, in the sense that

$$-\int_\Omega w^*\Delta\phi \mathrm{d}x = \lambda\int_\Omega \frac{f\phi}{(1-w^*)^p}\mathrm{d}x$$

for any $\phi \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$, where $w^* \in L^1(\Omega)$ and $\dfrac{f(x)d(x,\partial\Omega)}{(1-w^*)^p} \in L^1(\Omega)$. Moreover, w^* is stable, which means the first eigenvalue μ_{1,λ^*} of the linearized operator

$$L_{w,\lambda^*} := -\Delta - \frac{p\lambda^* f}{(1-w)^{p+1}}$$

is nonnegative. While for $\lambda > \lambda^*$, no solution to (1.2) exists, and with any u_0 satisfying $\|u_0\|_\infty < 1$ the solution u to (1.1) reaches the value 1 at a finite time T^*, called quenching time; i.e., the so called quenching or touchdown phenomenon occurs. More precisely, $\|u(\cdot,t)\|_\infty < 1$ for $t \in [0, T^*)$ and $\lim\limits_{t\to(T^*)^-} \|u(\cdot,t)\|_\infty = 1$. We say that the solution u to (1.1) quenches if it reaches 1 at a finite time. The more precise definition of the quenching time T^* is

$$T^* = \sup\{t > 0 \mid \|u(\cdot,s)\|_\infty < 1,\ \forall s \in [0,t]\}.$$

The corresponding quenching set is defined as

$$\Sigma = \{x \in \overline{\Omega} \mid \exists (x_n, t_n) \in \Omega \times (0, T^*),\ \text{s.t. } x_n \to x,\ t_n \to T^*,\ u(x_n, t_n) \to 1\}. \tag{1.3}$$

From [22], it can be deduced that

$$\frac{p^p}{(p+1)^{p+1}\|\xi\|_\infty} \leq \lambda^* \leq \frac{1}{(p+1)\|\xi\|_\infty},$$

where $\xi \in H_0^1(\Omega)$ is the unique solution to $-\Delta\xi = f$, and if $u_0 = 0$,

$$\frac{1}{\lambda(p+1)\|f\|_\infty} \leq T^* \leq \frac{\|\phi\|_{L^1(\Omega)}}{\lambda(p+1)\|f\phi\|_{L^1(\Omega)} - \|\Delta\phi\|_{L^1(\Omega)}}$$

for large λ, where ϕ is any nonnegative C^2 function such that $f\phi \not\equiv 0$ and $\phi = 0$ on $\partial\Omega$.

In general, we know that w^* could be regular or singular. Usually, w^* is a regular solution in lower dimension and becomes singular in higher dimension (see [7, 10]).

For general $u_0 \not\equiv 0$, it is known in [3, 20] that if $\lambda < \lambda^*$, $\|u_0\|_\infty < 1$, then there exists a unique solution $u(x,t)$ to (1.1) which converges pointwise to its unique minimal steady-state w

as $t \to +\infty$, provided that $u_0 \geq 0$ is a subsolution of (1.2). Furthermore, $u(x,t)$ is monotone nondecreasing for $t > 0$. It is essential to understand the quenching phenomenon, such as the quenching set, the rate or asymptotic behavior of the quenching time T^*. Some interesting results have been obtained in several recent works (see for instance [9, 11–12, 21] and the references therein). In [9, 21], we know that if w^* is regular, there exists an eigenfunction ϕ^* of L_{w^*,λ^*}, satisfying

$$\begin{cases} -\Delta\phi^* = \dfrac{p\lambda^* f\phi^*}{(1-w^*)^{p+1}}, & x \in \Omega, \\ \phi^* > 0, & x \in \Omega, \\ \phi^* = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega (\phi^*)^2 \mathrm{d}x = 1, \end{cases} \tag{1.4}$$

and there is an estimate for the quenching time T^* if $u_0(x) = 0$, that is

$$C_1(\lambda - \lambda^*)^{-\frac{1}{2}} \leq T^* \leq C_2(\lambda - \lambda^*)^{-\frac{1}{2}}, \tag{1.5}$$

as $\lambda > \lambda^*$ and close to λ^*, where C_1 and C_2 are independent of λ. This tells us that

$$\lim_{\lambda \to (\lambda^*)^+} T^* = +\infty.$$

Our purpose is to deal with the problem (1.1), and this paper will be organized as follows. In Section 2, motivated by the bounds (1.5) on the quenching time T^*, we find an asymptotic approximation for T^* in the limiting case when w^* is regular, $u_0(x) = 0$, $\lambda > \lambda^*$ and $\lambda \to \lambda^*$. In Section 3, we consider what conditions λ and initial data $u_0(x)$ satisfy can lead to the quenching of u. In Section 4, we find conditions on the initial data u_0 which are sufficient to lead to the nonexistence of u after a finite time, for $\lambda < \lambda^*$. The case of that w^* is singular is considered in Section 5. In Section 6, we discuss the global existence of the solution to (1.1). In Section 7, we will give some comments of our results.

2 Asymptotic Estimate for T^* for Small $\lambda - \lambda^* > 0$ if w^* is Regular

In this section, assume that $u_0 = 0$, and w^* is a regular solution to (1.2) with $\lambda = \lambda^*$. Motivated by (1.5), we shall consider the limit of $\lambda - \lambda^* \to 0^+$ to estimate the quenching time T^*, as an asymptotic expression for $\lambda - \lambda^* \ll 1$. We shall adapt and improve some of the arguments in [15] to get the following theorem.

Theorem 2.1 *Suppose $u_0 = 0$. Assume the unique extremal solution w^* of* (1.2) *is regular, $\Omega = B_R(0)$, $f(x) = f(|x|)$, $f' \leq 0$. Let ϕ^* be the L^2-normalized eigenfunction satisfying* (1.4). *Let* $\mathrm{I}_1 = \displaystyle\int_{B_R(0)} \frac{f\phi^*}{(1-w^*)^p}\mathrm{d}x$, $\mathrm{I}_2 = \displaystyle\int_{B_R(0)} \frac{(\phi^*)^3 f}{(1-w^*)^{p+2}}\mathrm{d}x$. *Then for $\lambda > \lambda^*$, the finite quenching time T^* of the solution u to* (1.1) *satisfies*

$$\lim_{\lambda \to \lambda^*} \left| T^* - \frac{\pi + 2\arctan\Big(\big(\frac{p(p+1)\lambda^* \mathrm{I}_2}{2\mathrm{I}_1(\lambda - \lambda^*)}\big)^{\frac{1}{2}} \frac{1 - w^*(0)}{\phi^*(0)}\Big)}{\sqrt{2}(p(p+1)\mathrm{I}_1\mathrm{I}_2)^{\frac{1}{2}}(\lambda - \lambda^*)^{\frac{1}{2}}} \right| = 0.$$

Proof First recall from [22] that since $\Omega = B_R(0)$, $f(x) = f(|x|)$, $f' \leq 0$, we have that the quenching set is just $\{0\}$.

Let $u^*(x,t)$ be a solution to (1.1) with $\lambda = \lambda^*$. Since $u_0 = 0$, by the standard parabolic comparison principle, we have $u^*(x,t) \le w^*(x)$. Since $w^*(x)$ is a solution to (1.2) with $\lambda = \lambda^*$, then $u^*(x,t)$ converges to a regular solution to (1.2) with $\lambda = \lambda^*$. By the uniqueness of solution to (1.2), we get that $\lim\limits_{t\to+\infty} u^*(x,t) = w^*(x)$.

By (1.1)–(1.2), we obtain that $u^*(x,t)$ and $w^*(x)$ satisfy

$$\begin{cases} u_t^* - \Delta u^* = \dfrac{\lambda^* f(x)}{(1-u^*)^p}, & (x,t) \in B_R(0)\times(0,T), \\ u^*(x,t) = 0, & (x,t) \in \partial B_R(0)\times(0,T), \\ u^*(x,0) = 0, & x \in B_R(0), \end{cases}$$

$$\begin{cases} -\Delta w^* = \dfrac{\lambda^* f(x)}{(1-w^*)^p}, & x \in B_R(0), \\ w^* = 0, & x \in \partial B_R(0), \end{cases}$$

respectively. Hence similarly to (43) in [15], we can find that

$$u^* \sim w^* - \frac{2\phi^*}{\lambda^* p(p+1)t\displaystyle\int_{B_R(0)} \frac{(\phi^*)^3 f}{(1-w^*)^{p+2}}\mathrm{d}x} \quad \text{as } t\to+\infty.$$

Now denote $\lambda - \lambda^*$ by η. We first use the formal Taylor expansion on u, which means

$$u(x,t) = u^*(x,t) + \sum_{n=1}^{+\infty} \eta^n u_n(x,t), \tag{2.1}$$

where u^* is a solution to (1.1) with $\lambda = \lambda^*$. According to the equation with u^* and u, we get that

$$\begin{aligned} \sum_{n=1}^{+\infty} \eta^n (u_n)_t = \sum_{n=1}^{+\infty} \eta^n \Delta(u_n) + \sum_{k=1}^{+\infty} \frac{\lambda^* p(p+1)\cdots(p+k-1)f}{k!(1-u^*)^{p+k}} \Big(\sum_{n=1}^{+\infty} \eta^n u_n\Big)^k \\ + \frac{f}{(1-u^*)^p}\eta + \eta\Big(\sum_{k=1}^{+\infty} \frac{p(p+1)\cdots(p+k-1)f}{k!(1-u^*)^{p+k}} \Big(\sum_{n=1}^{+\infty} \eta^n u_n\Big)^k\Big) \end{aligned}$$

and

$$u_n(x,0) = 0.$$

From the (η^1), (η^2), (η^3), (η^4), $\cdots$ terms, we arrive at

$$\begin{aligned} Lu_1 &= \frac{f}{(1-u^*)^p}, \\ Lu_2 &= \frac{\lambda^* p(p+1)f}{2(1-u^*)^{p+2}} u_1^2 + \frac{pf}{(1-u^*)^{p+1}} u_1, \\ Lu_3 &= \frac{\lambda^* p(p+1)(p+2)f}{6(1-u^*)^{p+3}} u_1^3 + \frac{p(p+1)f}{2(1-u^*)^{p+2}} u_1^2 \\ &\quad + \frac{\lambda^* p(p+1)f}{(1-u^*)^{p+2}} u_1 u_2 + \frac{pf}{(1-u^*)^{p+1}} u_2, \\ Lu_4 &= \frac{\lambda^* p(p+1)(p+2)(p+3)f}{4!(1-u^*)^{p+4}} u_1^4 + \frac{p(p+1)(p+2)f}{6(1-u^*)^{p+3}} u_1^3 \end{aligned}$$

$$+\frac{\lambda^* p(p+1)(p+2)f}{2(1-u^*)^{p+3}}u_1^2 u_2 + \frac{p(p+1)f}{(1-u^*)^{p+2}}u_1 u_2$$
$$\frac{\lambda^* p(p+1)f}{2(1-u^*)^{p+2}}u_2^2 + \frac{pf}{(1-u^*)^{p+1}}u_3,$$
$$\cdots$$

where the operator L is defined as $\frac{\partial}{\partial t} - \Delta - \frac{\lambda^* pf}{(1-u^*)^{p+1}}$. By the standard parabolic regularity theory (see [14]), as $\|u^*\|_\infty \le \|w^*\|_\infty$, we obtain that for any $T_0 < T^*$,

$$\|u_1\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B_R(0)\times(0,T_0)})} \le C_1\|f\|_\infty A^p T_0^{\frac{1}{q}} |B_R(0)|^{\frac{1}{q}} =: \mathrm{D}_1,$$

where $A = \frac{1}{(1-\|w^*\|_\infty)}$, C_1 depends only on $N,\ q,\ B_R(0),\ \frac{\lambda^* p\|f\|_\infty}{(1-\|w^*\|_\infty)^{p+1}}$ and $q > N+2$.

Similarly, we are also able to obtain that

$$\|u_2\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B_R(0)\times(0,T_0)})} \le \frac{C_1^3\lambda^* p(p+1)\|f\|_\infty A^{3p+2}(T_0|B_R(0)|^{\frac{3}{q}})}{2} + C_1^2 p\|f\|_\infty^2 A^{2p+1}(T_0|B_R(0)|^{\frac{2}{q}}) =: \mathrm{D}_2$$

and

$$\|u_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B_R(0)\times(0,T_0)})} \le \mathrm{D}_n, \quad n = 3, 4, \cdots.$$

In fact by iteration we can get that the highest exponent of A in D_n is $p + 2(n-1)(p+1)$, the highest exponent of p in D_n is $2n$, the highest exponent of $(T_0|B_R(0)|)$ in D_n is $2n-1$, the highest exponent of C_1 in D_n is $2n-1$, the highest exponent of $\|f\|_\infty$ in D_n is $2n-1$.

So we can obtain that

$$\limsup_{n\to+\infty} \sqrt[n]{\|u_n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B_R(0)\times(0,T_0)})}} < +\infty,$$

which means the convergence domain of series (2.1) has a positive measure. Therefore, series (2.1) is convergent uniformly, provided that η is small enough. For the same reason, we get

$$u_t = u_t^* + \sum_{n=1}^{+\infty} \eta^n (u_n)_t$$

and

$$\Delta u = \Delta u^* + \sum_{n=1}^{+\infty} \eta^n \Delta u_n$$

are also convergent uniformly for small η.

It is known in [9, 21] that there exist constants $C_1,\ C_2 > 0$ such that (1.5) holds, which tells us that $\lim_{\eta\to 0} T^*\eta^{\frac{1}{2}} \in [C_1, C_2]$, and $\lim_{\eta\to 0} T^* = +\infty$. Therefore for any $t \in \left[\frac{T^*}{2}, T^*\right]$, we have $\frac{\eta^{\frac{1}{2}}}{C_2} \le \frac{1}{t} \le \frac{2\eta^{\frac{1}{2}}}{C_1}$, which is near 0, when $\eta \to 0$. With (2.1) and similarly to [15], when $t \in \left[\frac{C_1}{2\eta^{\frac{1}{2}}}, \frac{C_2}{\eta^{\frac{1}{2}}}\right]$, we expand

$$u(x,t) \sim w^*(x) + \eta^{\frac{1}{2}} z(x,t) + \sum_{n=2}^{+\infty} \eta^{\frac{n}{2}} v_n(x,t) \quad \text{as } \eta \to 0,$$

where $z(x,t)$ is to be determined. On making a change in timescale $t = \frac{\tau}{\eta^{\frac{1}{2}}}$, the equation (1.1) gives

$$
\begin{aligned}
&\eta(z)_\tau + \sum_{n=2}^{+\infty} \eta^{\frac{n}{2}+\frac{1}{2}}(v_n)_\tau = u_t = \Delta u + \frac{\lambda f(x)}{(1-u)^p} \\
&= \Delta w^* + \eta^{\frac{1}{2}} \Delta z + \sum_{n=2}^{+\infty} \eta^{\frac{n}{2}} \Delta v_n + (\lambda^* + \eta) \frac{f(x)}{(1-w^*)^p} \\
&\quad + (\lambda^* + \eta) f(x) \sum_{k=1}^{+\infty} \frac{p(p+1)\cdots(p+k-1)}{k!(1-w^*)^{p+k}} \Big(\eta^{\frac{1}{2}} z + \sum_{n=2}^{+\infty} \eta^{\frac{n}{2}} v_n \Big)^k, \qquad (2.2)
\end{aligned}
$$

where we have used Taylor expansion.

From the $(\eta^{\frac{1}{2}})$ terms in (2.2), we have

$$
0 = \Delta z + \frac{p\lambda^* f(x)}{(1-w^*)^{p+1}} z.
$$

Since the problem above has the form of problem (1.4), for some continuous function $a_1(\tau)$, we can write

$$
z(x,\tau) = a_1(\tau)\phi^*(x), \qquad (2.3)
$$

where ϕ^* satisfies the condition of this theorem.

From the (η^1) terms in (2.2), by (2.3) we arrive at

$$
\begin{aligned}
(a_1)_\tau \phi^* &= \frac{\mathrm{d}}{\mathrm{d}\tau} z \\
&= \Delta v_2 + \frac{f(x)}{(1-w^*)^p} + \frac{\lambda^* p f}{(1-w^*)^{p+1}} v_2 + \frac{\lambda^* p(p+1) f}{2(1-w^*)^{p+2}} (a_1)^2 (\phi^*)^2.
\end{aligned}
$$

By multiplying ϕ^* on both side and integrating over $B_R(0)$, we get

$$
\begin{aligned}
(a_1)_\tau &= \int_{B_R(0)} \frac{f\phi^*}{(1-w^*)^p} \mathrm{d}x + \frac{1}{2} p(p+1)\lambda^* a_1^2 \int_{B_R(0)} \frac{(\phi^*)^3 f}{(1-w^*)^{p+2}} \mathrm{d}x \\
&:= \mathrm{I}_1 + \frac{1}{2} p(p+1)\lambda^* a_1^2 \mathrm{I}_2
\end{aligned}
$$

by the condition $\displaystyle\int_{B_R(0)} (\phi^*)^2 \mathrm{d}x = 1$. Therefore

$$
a_1(t) = \Big(\frac{2\mathrm{I}_1}{p(p+1)\lambda^* \mathrm{I}_2} \Big)^{\frac{1}{2}} \tan \Big(t \Big(\frac{1}{2} p(p+1) \mathrm{I}_1 \mathrm{I}_2 \eta \Big)^{\frac{1}{2}} - \frac{\pi}{2} \Big)
$$

and the quenching set being just $\{0\}$ indicates that

$$
\lim_{\lambda \to \lambda^*} \Big| T^* - \frac{\pi + 2\arctan\big(\big(\frac{p(p+1)\lambda^* \mathrm{I}_2}{2\mathrm{I}_1(\lambda - \lambda^*)} \big)^{\frac{1}{2}} \frac{1-w^*(0)}{\phi^*(0)} \big)}{\sqrt{2}(p(p+1)\mathrm{I}_1\mathrm{I}_2)^{\frac{1}{2}} (\lambda - \lambda^*)^{\frac{1}{2}}} \Big| = 0.
$$

This completes the proof.

3 Quenching Phenomenon of u for General λ and Initial Data $u_0(x)$

In this section, we do not need to assume $u_0(x) \equiv 0$ any more. We are going to consider for which λ and $u_0(x)$ the solution to (1.1) will hit 1. We have the following theorem.

Theorem 3.1 *If λ and initial data $u_0(x)$ satisfy*

$$\frac{\lambda}{\Big(1-\displaystyle\int_\Omega u_0\phi \mathrm{d}x\Big)^p} > \lambda_1\Big(\int_\Omega u_0\phi \mathrm{d}x\Big)\Big(\int_\Omega \frac{\phi}{f}\mathrm{d}x\Big) \tag{3.1}$$

and

$$\frac{\lambda p}{\Big(1-\displaystyle\int_\Omega u_0\phi \mathrm{d}x\Big)^{p+1}} > \lambda_1\Big(\int_\Omega \frac{\phi}{f}\mathrm{d}x\Big), \tag{3.2}$$

then the unique solution u to (1.1) will reach 1 at a finite time. Here λ_1 is the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, and ϕ is the corresponding eigenfunction with $\displaystyle\int_\Omega \phi \mathrm{d}x = 1$.

Proof Set

$$G(t) = \int_\Omega u(x,t)\phi(x)\mathrm{d}x < 1,$$

so

$$G(0) = \int_\Omega u_0\phi \mathrm{d}x \le \int_\Omega \phi \mathrm{d}x = 1.$$

It is clear that $G(t)$ is well defined on the existence interval of the solution u.

Differentiating $G(t)$ yields that

$$\begin{aligned} G'(t) &= \int_\Omega u_t\phi \mathrm{d}x \\ &= \int_\Omega \phi\Big(\Delta u + \frac{\lambda f}{(1-u)^p}\Big)\mathrm{d}x \\ &= -\lambda_1\int_\Omega u\phi \mathrm{d}x + \lambda\int_\Omega \frac{f\phi}{(1-u)^p}\mathrm{d}x \\ &\ge -\lambda_1\int_\Omega u\phi \mathrm{d}x + \frac{\lambda\Big(\displaystyle\int_\Omega \frac{\phi}{(1-u)^{\frac{p}{2}}}\mathrm{d}x\Big)^2}{\displaystyle\int_\Omega \frac{\phi}{f}\mathrm{d}x}, \end{aligned} \tag{3.3}$$

where Hölder inequality is used in the last inequality. By Jensen's inequality, if $p > 1$,

$$\Big(\int_\Omega \frac{\phi}{(1-u)^{\frac{p}{2}}}\mathrm{d}x\Big)^2 \ge \frac{1}{\big(1-\int_\Omega u\phi \mathrm{d}x\big)^p}.$$

Substituting it into (3.3), we obtain

$$G'(t) \ge -\lambda_1\int_\Omega u\phi \mathrm{d}x + \frac{\lambda}{\Big(1-\displaystyle\int_\Omega u\phi \mathrm{d}x\Big)^p\Big(\displaystyle\int_\Omega \frac{\phi}{f}\mathrm{d}x\Big)}$$

$$= -\lambda_1 G(t) + \frac{\lambda}{(1-G(t))^p \Big(\displaystyle\int_\Omega \frac{\phi}{f}\mathrm{d}x\Big)}. \tag{3.4}$$

If u remains smaller than 1 for all t, then $G(t)$ is defined and smaller than 1 for all t. However, from the ODE theory, $G(t)$ will blow up (reach 1) at a finite time, provided that λ and the initial data u_0 satisfy (3.1) and (3.2), because $-\lambda_1 G(t) + \frac{\lambda}{(1-G(t))^p\big(\int_\Omega \frac{\phi}{f}\mathrm{d}x\big)}$ is increasing with respect to $G(t)$ and $G'(0) \geq -\lambda_1 G(0) + \frac{\lambda}{(1-G(0))^p\big(\int_\Omega \frac{\phi}{f}\mathrm{d}x\big)}$ under the conditions (3.1) and (3.2). Then the proof of this theorem is complete.

4 Quenching Phenomenon of u for General $0 < \lambda < \lambda^*$

In this section, we are going to check when the solution u to (1.1) will quench even for $\lambda < \lambda^*$. We will compare u_0 with some suitable function.

4.1 (1.2) has a second nonminimal solution $w > 0$

If w is a nonminimal solution to (1.2), from [10], we deduce that the principal eigenvalue $\mu_{1,\lambda}(w)$ associated with the eigenfunction ϕ_1, for the problem

$$\begin{cases} -\Delta\phi_1 - \dfrac{p\lambda f(x)}{(1-w)^{p+1}}\phi_1 = \mu_{1,\lambda}(w)\phi_1, & x \in \Omega, \\ \phi_1 > 0, & x \in \Omega, \\ \phi_1 = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega \phi_1 \mathrm{d}x = 1 \end{cases} \tag{4.1}$$

is negative. Then we have the following theorem.

Theorem 4.1 *Assume that* (1.2) *has a nonminimal solution w for $\lambda < \lambda^*$, and assume $\|w\|_\infty < 1$. Then for $\lambda < \lambda^*$, the solution u to* (1.1) *will quench at a finite time T^*, provided that $u_0(x) \geq w(x)$ and $u_0(x) \not\equiv w(x)$.*

Proof Setting $v(x,t) = u(x,t) - w(x)$, we have

$$\begin{aligned} v_t = u_t &= \Delta u + \frac{\lambda f(x)}{(1-u)^p} \\ &= \Delta v + \Delta w + \frac{\lambda f(x)}{(1-u)^p} \\ &= \Delta v + \lambda f\Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}}v\Big) \\ &\quad + \frac{\lambda f p}{(1-w)^{p+1}}v + \mu_{1,\lambda}v - \mu_{1,\lambda}v. \end{aligned}$$

Choose now $a(t) = \displaystyle\int_\Omega v\phi_1 \mathrm{d}x \leq \sup_{x\in\Omega} u \int_\Omega \phi_1 \mathrm{d}x \leq 1$, where ϕ_1 is defined in (4.1). Differentiating $a(t)$ leads to

$$a_t = \int_\Omega v_t \phi_1 \mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega \phi_1 \Big(\Delta v + \lambda f(x)\Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}} v\Big)\Big) \mathrm{d}x \\
&\quad + \int_\Omega \phi_1 \Big(\frac{\lambda f p}{(1-w)^{p+1}} v + \mu_{1,\lambda} v - \mu_{1,\lambda} v\Big) \mathrm{d}x \\
&= \int_\Omega \Delta\phi_1 v \mathrm{d}x + \int_\Omega \Big(\frac{\lambda f p}{(1-w)^{p+1}} \phi_1 + \mu_{1,\lambda}\phi_1\Big) v \mathrm{d}x \\
&\quad + \lambda \int_\Omega f\phi_1 \Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}} v\Big) \mathrm{d}x - \mu_{1,\lambda} \int_\Omega v\phi_1 \mathrm{d}x \\
&= -\mu_{1,\lambda} a + \lambda \int_\Omega f\phi_1 \Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}} v\Big) \mathrm{d}x.
\end{aligned}
$$

The inequalities

$$
\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}} v \geq \begin{cases} \dfrac{p(p+1)v^2}{2(1-w)^{p+2}}, & v \geq 0, \\ \dfrac{p(p+1)v^2}{2(1-u)^{p+2}}, & v < 0, \end{cases} \tag{4.2}
$$

and Hölder inequality yield that

$$
\begin{aligned}
a_t &\geq -\mu_{1,\lambda} a + \frac{p(p+1)}{2} \lambda \int_\Omega v^2 f \phi_1 \mathrm{d}x \\
&\geq -\mu_{1,\lambda} a + \frac{p(p+1)\lambda}{2\displaystyle\int_\Omega \frac{\phi_1}{f} \mathrm{d}x} a^2.
\end{aligned}
$$

Since $a(0) = \int_\Omega (u_0(x) - w(x))\phi_1 \mathrm{d}x$, we have that if $u_0(x) \geq w(x)$ and $u_0(x) \not\equiv w(x)$, the quenching time T^* verifies

$$
T^* \leq \int_{a(0)}^1 \frac{1}{-\mu_{1,\lambda} y + \frac{p(p+1)\lambda}{2\int_\Omega \frac{\phi_1}{f}\mathrm{d}x} y^2} \mathrm{d}y < +\infty,
$$

and the proof is complete.

Remark 4.1 It can be deduced from [6] that when $\lambda \to (\lambda^*)^-$, there exists a nonminimal solution w to (1.2); and if $\mu_{k,\lambda}(w) \geq 0$, then $\|w\|_\infty < 1$, where $\mu_{k,\lambda}(w)$ is the k-th eigenvalue of problem (1.4), $k \geq 2$.

This remark says that the assumption in Theorem 4.1 is reasonable, provided that $\lambda \to (\lambda^*)^-$ and there exists a $k \geq 2$ such that $\mu_{k,\lambda}(w) \geq 0$.

4.2 (1.2) may admit a unique solution w for $\lambda \in (0, \lambda^*)$ small enough

In [5, 10], we know that when λ is small enough, (1.2) has only a unique minimal solution. Therefore, we need to choose another suitable function to compare with the initial data $u_0(x)$. In this part, let $\Omega = B_1(0)$, $f(x) \equiv 1$. And we need assume $N = 3$, $p = 2$ for simplicity. For general N, p, the idea is similar. Now we establish the theorem below.

Theorem 4.2 *Let* $N = 3$, $p = 2$, $f(x) \equiv 1$, $\Omega = B_1(0)$. *Then the solution* u *to* (1.1) *will* *quench at a finite time* T^**, if* $u_0(x) \geq z(x)$ *and* $u_0(x) \not\equiv z(x)$*, where*

$$\begin{cases} z(x) = C + \dfrac{\lambda}{6(1-C)^2}(R_1^2 - r^2), & 0 < r = |x| < R_1, \\ z(x) = \dfrac{R_1^3}{3}\Big(\dfrac{\lambda}{(1-C)^2} - a\Big)\Big(\dfrac{1}{r} - 1\Big) + \dfrac{a}{6}(1-r^2), & R_1 < r = |x| < 1, \end{cases} \tag{4.3}$$

$$C = \Big(1 - \frac{2\lambda}{\pi^2}\Big)^{\frac{1}{3}}, \quad R_1 = \Big(\frac{\pi^2 - 3C(1-C)\pi^2 - C^3\pi^2}{2\lambda}\Big)^{\frac{1}{2}} \tag{4.4}$$

and

$$a = \frac{2\lambda R_1^2(1-R_1) - 6(1-C)^2 C}{(1-C)^2(1-R_1)(1+R_1+2R_1^2)}. \tag{4.5}$$

Proof Let $z(x)$ satisfy (4.3)–(4.5). Then we have

$$0 < (1-C)^3 \leq \frac{2\lambda R_1^2}{\pi^2} \tag{4.6}$$

and $z \in C^{1,\alpha}(\overline{B_1(0)})$.

Consider now the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi_1 - \dfrac{p\lambda}{(1-z)^3}\varphi_1 = \mu_1\varphi_1, & x \in B_1(0), \\ \varphi_1 = 0, & x \in \partial B_1(0), \\ \displaystyle\int_{B_1(0)} \varphi_1 \mathrm{d}x = 1. \end{cases} \tag{4.7}$$

From (4.6) we reach that $\mu_1 < 0$. Moreover $\|z\|_{L^\infty(B_1(0))} = z(0) < 1$, $a \leq \lambda$.

Next, we set $v(x,t) = u(x,t) - z(x)$, then $v(x,0) = u_0(x) - z(x)$ in $B_1(0)$. Let $a_2(t) = \int_{B_1(0)} v(x,t)\varphi_1(x)\mathrm{d}x$, then we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}a_2(t) &= \int_{B_1(0)} \Big(\Delta u + \frac{\lambda}{(1-u)^2}\Big)\varphi_1 \mathrm{d}x \\ &= -\int_{B_1(0)} \nabla u \nabla\varphi_1 \mathrm{d}x + \int_{B_1(0)} \frac{\lambda\varphi_1}{(1-u)^2}\mathrm{d}x \\ &= -\int_{B_1(0)} \nabla v \nabla\varphi_1 \mathrm{d}x - \int_{B_1(0)} \nabla z \nabla\varphi_1 \mathrm{d}x + \int_{B_1(0)} \frac{\lambda\varphi_1}{(1-u)^2}\mathrm{d}x \\ &\geq \int_{B_1(0)} v\Delta\varphi_1 \mathrm{d}x - \int_{B_1(0)} \frac{\lambda\varphi_1}{(1-z)^2}\mathrm{d}x + \int_{B_1(0)} \frac{\lambda\varphi_1}{(1-u)^2}\mathrm{d}x \\ &= (-\mu_1)\int_{B_1(0)} v\varphi_1 \mathrm{d}x + \int_{B_1(0)} v\Delta\varphi_1 \mathrm{d}x \\ &\quad + \mu_1 \int_{B_1(0)} v\varphi_1 \mathrm{d}x + \int_{B_1(0)} \frac{2\lambda}{(1-z)^3}\varphi_1 v \mathrm{d}x \\ &\quad + \lambda\int_{B_1(0)} \varphi_1\Big(\frac{1}{(1-(z+v))^2} - \frac{1}{(1-z)^2} - \frac{2}{(1-z)^3}v\Big)\mathrm{d}x \\ &\geq -\mu_1 a_2 + 3\lambda\int_{B_1(0)} \varphi_1 v^2 \mathrm{d}x \\ &\geq -\mu_1 a_2 + 3\lambda a_2^2. \end{aligned}$$

The last inequality is due to Hölder inequality. As

$$a_2(0) = \int_{B_1(0)} (u_0(x) - z(x))\varphi_1(x)\mathrm{d}x \le 1 + \max\{-z(x), 0\}, \quad \mu_1 < 0,$$

we conclude that if $u_0(x) \ge z(x)$ and $u_0(x) \not\equiv z(x)$, the quenching time T^* for the solution u verifies

$$T^* \le \int_{a_2(0)}^{1+\max\{-z(x),0\}} \frac{1}{-\mu_1 y + 3\lambda y^2}\mathrm{d}y < +\infty,$$

where $z(x)$ is as stated in (4.3). This finishes the proof of the theorem.

5 Quenching Phenomenon of u for $\lambda > \lambda^*$, if w^* is Singular

Now we discuss the case where w^* is singular. For any minimal stable solution w_Λ to (1.2) with λ being replaced by Λ, $0 \le \Lambda < \lambda^*$, we consider the problem

$$\begin{cases} -\Delta\psi = \mu_1(\Lambda)\dfrac{pf(x)}{(1-w_\Lambda)^{p+1}}\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \\ \psi > 0, & x \in \Omega, \\ \displaystyle\int_\Omega \psi \mathrm{d}x = 1. \end{cases} \tag{5.1}$$

From [1] we know that $\mu_1(\Lambda)$ exists and is decreasing with respect to Λ. Therefore $\lim\limits_{\Lambda\to\lambda^*} \mu_1(\Lambda)$ exists and moreover $\mu_1(\Lambda)$ depends continuously on $\frac{pf}{(1-w_\Lambda)^{p+1}}$ in $L^{\frac{N}{2}}(\Omega)$ topology.

Now we claim that $\lim\limits_{\Lambda\to\lambda^*} \mu_1(\Lambda) := \mu_0 \ge \lambda^*$. If not, then $\mu_0 < \lambda^*$. By the continuity, there exists a $\delta > 0$, such that for any $\Lambda \in (\lambda^* - \delta, \lambda^*)$, $\mu_1(\Lambda) < \Lambda < \lambda^*$. So there exists a function ψ satisfying

$$\begin{cases} -\Delta\psi - \dfrac{\Lambda pf}{(1-w_\Lambda)^{p+1}}\psi = -(\Lambda - \mu_1(\Lambda))\dfrac{pf}{(1-w_\Lambda)^{p+1}}\psi, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \\ \psi > 0, & x \in \Omega, \\ \displaystyle\int_\Omega \psi \mathrm{d}x = 1, \end{cases}$$

which is impossible, because w_Λ is stable. Hence our claim is concluded.

Take $\lambda_1 \in (\mu_0, \mu_1(0))$, then there exists a unique $\lambda_2 \in (0, \lambda^*)$, such that $\mu_1(\lambda_2) = \lambda_1$, and $\lambda_2 < \lambda_* \le \mu_0 < \lambda_1$. Now we give our theorem.

Theorem 5.1 *Assume $\Omega = B_R(0)$, $f(x) \equiv 1$. Suppose that the extremal solution w^* to (1.2) is singular. Let ψ be the eigenfunction satisfying (5.1). Then for any $\lambda > \lambda^*$, if $N \ge N_p := 2 + \frac{4p}{p+1} + 4\sqrt{\dfrac{p}{p+1}}$, the quenching time T^* of the solution u to (1.1) is finite. Moreover, T^* satisfies*

$$T^* \le \frac{\sqrt{2}\pi}{\sqrt{\lambda^* p(p+1)}}(\lambda - \lambda^*)^{-\frac{1}{2}}$$

for any $\lambda > \lambda^*$, and

$$T^* \leq \frac{2\sqrt{2}\pi}{\lambda\sqrt{p(p+1)}}$$

for λ larger than and close to λ^*.

Proof Let w be the minimal solution to

$$\begin{cases} -\Delta w = \dfrac{\lambda_2}{(1-w)^p}, & x \in B_R(0), \\ w = 0, & x \in \partial B_R(0), \end{cases}$$

where $\mu_1(0) > \mu_0 \geq \lambda^* > \lambda_2$. Let $v(x,t) = u(x,t) - w(x)$ and $\overline{\lambda} \in [\mu_0, \mu_1(0)]$, then $\lambda_2 < \overline{\lambda}$ and for any $\lambda > \overline{\lambda}$, we have

$$\begin{aligned} v_t = u_t &= \Delta u + \frac{\lambda}{(1-u)^p} \\ &= \Delta v + \Delta w + \frac{\lambda}{(1-u)^p} \\ &= \Delta v - \frac{\lambda_2}{(1-w)^p} + \frac{\lambda}{(1-u)^p} \\ &\geq \Delta v - \frac{\overline{\lambda}}{(1-w)^p} + \frac{\lambda}{(1-(v+w))^p} \\ &= \frac{(\lambda - \overline{\lambda})}{(1-u)^p} + \Delta v + \frac{\overline{\lambda}p}{(1-w)^{p+1}} \\ &\quad + \overline{\lambda}\Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}}v\Big). \end{aligned} \tag{5.2}$$

Multiplying (5.2) by ψ, where ψ satisfies (5.1) with $\lambda_2 := \mu_1^{-1}(\overline{\lambda})$, and integrating over $B_R(0)$ yield that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R(0)} v\psi \mathrm{d}x \geq (\lambda - \overline{\lambda}) + \overline{\lambda}\int_{B_R(0)}\Big(\frac{1}{(1-(v+w))^p} - \frac{1}{(1-w)^p} - \frac{p}{(1-w)^{p+1}}v\Big)\mathrm{d}x.$$

Hölder inequality and (4.2) enable us that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{B_R(0)} v\psi \mathrm{d}x &\geq (\lambda - \overline{\lambda}) + \frac{\overline{\lambda}p(p+1)}{2}\int_{B_R(0)} \psi v^2 \mathrm{d}x \\ &\geq (\lambda - \overline{\lambda}) + \frac{\overline{\lambda}p(p+1)\Big(\int_{B_R(0)} v\psi \mathrm{d}x\Big)^2}{2}. \end{aligned}$$

Since

$$\int_{B_R(0)} v(x,0)\psi(x)\mathrm{d}x = \int_{B_R(0)} (u_0(x) - w(x))\psi(x)\mathrm{d}x \geq -\|w\|_\infty \geq -1,$$

then we get

$$T^* \leq \int_{-1}^{1} \frac{1}{(\lambda - \overline{\lambda}) + \frac{\overline{\lambda}p(p+1)}{2}y^2}\mathrm{d}y$$

$$< \frac{\pi\sqrt{2}}{\sqrt{\overline{\lambda}p(p+1)}}(\lambda-\overline{\lambda})^{-\frac{1}{2}},$$

where $\lambda^* \le \mu_0 \le \overline{\lambda} < \lambda$.

Next we claim that $\mu_0 = \lambda^*$. Hence $\overline{\lambda}$ can be equal to λ^*, so T^* satisfies

$$T^* \le \frac{\sqrt{2}\pi}{\sqrt{\lambda^* p(p+1)}}(\lambda-\lambda^*)^{-\frac{1}{2}}$$

for any $\lambda > \lambda^*$, and

$$\begin{aligned} T^* &\le \inf_{\overline{\lambda}<\lambda} \frac{\pi\sqrt{2}}{\sqrt{\overline{\lambda}p(p+1)}}(\lambda-\overline{\lambda})^{-\frac{1}{2}} \\ &\le \frac{2\sqrt{2}\pi}{\lambda\sqrt{p(p+1)}} \end{aligned}$$

for λ larger than and close to λ^*.

Now we begin to prove the claim. Firstly, we know that when $N \ge N_p$, $w^*(x) = 1-\left(\frac{|x|}{R}\right)^{\frac{2}{p+1}}$ is the extremal solution to (1.2) with $f \equiv 1$ and $\lambda^* = \frac{2}{(p+1)R^2}\left(N-2+\frac{2}{p+1}\right)$. So

$$\begin{aligned} \mu_0 &= \inf_{\psi\in H_0^1(B_R(0))} \frac{\displaystyle\int_{B_R(0)} |\nabla\psi|^2 \mathrm{d}x}{p\displaystyle\int_{B_R(0)} \frac{\psi^2}{(1-w^*)^{p+1}}\mathrm{d}x} \\ &= \inf_{\psi\in H_0^1(B_R(0))} \frac{\displaystyle\int_{B_R(0)} |\nabla\psi|^2 \mathrm{d}x}{p\displaystyle\int_{B_R(0)} \frac{\psi^2R^2}{|x|^2}\mathrm{d}x}. \end{aligned}$$

If $\psi(x) = 1-\left(\frac{|x|}{R}\right)^a$, then we get that

$$\frac{\displaystyle\int_{B_R(0)} |\nabla\psi|^2 \mathrm{d}x}{p\displaystyle\int_{B_R(0)} \frac{\psi^2R^2}{|x|^2}\mathrm{d}x} = \frac{(N-2)(N-2+a)}{2pR^2}.$$

Moreover when $N > N_p$, we can choose a suitable a, such that

$$\mu_0 \le \frac{(N-2)(N-2+a)}{2pR^2} \le \lambda^*.$$

Therefore $\mu_0 = \lambda^*$. Hence our claim is correct and the theorem is proved.

6 Global Existence for the Solution to (1.1)

In this section, we will check the global existence of the solution. In fact, under some circumstances the solution of (1.1) should exist globally. That means that we have the following theorem.

Theorem 6.1 *Suppose that $\Omega = B_R(0)$ and $f(x) \equiv f$ is a constant. Assume that the initial datum u_0 is a subsolution to* (1.2)*, and let*

$$h(M) := \frac{1}{MR^2}\Big(1 - \Big(\frac{f}{2MN}\Big)^{\frac{1}{p}}\Big).$$

Then for any $\lambda \in \big[0, \frac{2Np^p}{(p+1)^{p+1}R^2 f}\big]$, if u_0 satisfies $u_0(x) \le \lambda M_0(R^2 - |x|^2)$, then the solution u to (1.1) *exists globally, where M_0 verifies $h(M_0) = \lambda$, $h'(M_0) \le 0$.*

Proof Consider the equation of u_t, by the standard maximum principle, we get $u_t > 0$. Thus we just need to construct a suitable supersolution $z(x) < 1$ to (1.1), such that $u_0(x) \le z(x)$.

Let $z(x) = \lambda M(R^2 - |x|^2) < 1$. So we should have $\lambda MR^2 < 1$. In order to ensure $z(x)$ is a supersolution to (1.1), we need $-\Delta z = 2\lambda MN \ge \frac{\lambda f}{(1-z)^p}$. That is, $z(x) \le 1 - \big(\frac{f}{2MN}\big)^{\frac{1}{p}}$. So we only need to let

$$\lambda \le \frac{1}{MR^2}\Big(1 - \Big(\frac{f}{2MN}\Big)^{\frac{1}{p}}\Big) =: h(M). \tag{6.1}$$

By direct calculation, we arrive at

$$h(M_1) = \max_{M > \frac{f}{2N}} h(M) = \frac{2Np^p}{(p+1)^{p+1}R^2 f},$$

where $M_1 = \frac{(p+1)^p f}{2Np^p}$. From [22], we get that

$$\frac{1}{M_1R^2}\Big(1 - \Big(\frac{f}{2M_1N}\Big)^{\frac{1}{p}}\Big) \le \lambda^*.$$

Therefore for any $\lambda \in \big[0, \frac{2Np^p}{(p+1)^{p+1}R^2 f}\big]$, we choose $M_0 > \frac{f}{2N}$ such that $h(M_0) = \lambda$, $h'(M_0) \le 0$, which means that $z(x)$ can be chosen as

$$z(x) = \lambda M_0(R^2 - |x|^2).$$

Then if $u_0(x) \le z(x)$, applying the maximum principle, we reach that $u(x) \le z(x) < 1$, which means that $u(x)$ exists globally. This completes the proof of this theorem.

7 Discussion of Results

We have found in Sections 2–5 the circumstances under which the solution to problem (1.1) exhibits the phenomenon of quenching.

In order to apply these results, we first need to decide the value of λ^*. Although there is not exact expression of λ^*, some people have given some estimates of λ^*. For example, from [22] we can easily deduce that

$$\lambda^* \in \Big[\frac{p^{p-1}}{(p+1)^p\|\phi\|_\infty}, \frac{1}{(p+1)\|\phi\|_\infty}\Big],$$

where $\phi \in H_0^1(\Omega)$ is the unique solution to $-\Delta\phi = f$ in Ω. The second thing we need to consider is the regularity of the extremal solution w^*. In fact, there is a well known result about the regularity of the extremal solution in the negative exponent situation (see [2, 17]). That is, for

$$N < N_p = 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}},$$

the extremal solution w^* to (1.2) is regular.

Hence, if w^* is regular, we apply the results of Section 2 to give

$$T^* \sim O(\lambda - \lambda^*)^{-\frac{1}{2}}$$

as $\lambda \to (\lambda^*)^+$. For general λ and initial data $u_0(x)$, Section 3 tells us the condition under which the solution u to (1.1) will quench. For $\lambda < \lambda^*$, we consider two situations: (1.2) has a nonminimal solution; (1.2) may only admit a unique minimal solution. In both situations, we need $u_0(x)$ satisfy suitable condition, then in Section 4 we obtain $T^* < +\infty$. The rationality of these two situations is as follows: From [6], it follows that when $N < N_p = 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}}$, there exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, problem (1.2) has a nonminimal second solution $w \geq w_\lambda$. Meanwhile, it is well known that if $\Omega \subset \mathbb{R}^N (N \geq 3)$ is star-shaped, or Ω is strictly convex in $\mathbb{R}^2$, then (1.2) admits a unique solution for $\lambda > 0$ small enough (see [5]).

If w^* is singular, in Section 5, we show that the quenching time $T^* < +\infty$, for $\lambda > \lambda^*$.

In Section 6, we make a short discussion about the situation of the global existence for the solution to (1.1). We can get that if the initial data is suitable small, then for $\lambda < \lambda^*$, the quenching phenomenon will not occur.

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