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Homogenization of Periodically Heterogeneous Thin Beams

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(*Dedicated to Philippe G. Ciarlet on the occasion of his 80th birthday*)

Abstract Consider an elastic thin three-dimensional body made of a periodic distribution of elastic inclusions When both the thickness of the beam and the size of the heterogeneities tend simultaneously to zero the authors obtain three d erent one-dimensional models of beam depending upon the limit of the ratio of these two small parameters

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1 Introduction

The aim of this work was to establish the modeling of a thin elastic beam made of elastic inclusions periodically distributed along its length. More precisely let δ and ε be two small parameters respectively the thickness of the beam and the size of the inclusions, and both are supposed to vanish. Several works have shown how to derive rigorously the reduced one-dimensional models of elastic beams, starting from the three-dimensional equilibrium equations and letting the size δ of the cross-section tend to zero (just to mention some of the pioneer papers for plates and rods (see [2, 13])); on the other hand the limit model of structures made of periodically heterogeneous elastic material has been established by the homogenization approach, when the number of inclusions tends to infinity, i.e., ε tends to zero (just to mention some of the pioneer papers in elasticity (see [1, 11])). In this paper we let both parameters tend to zero simultaneously, which gives rise to three different one-dimensional models of homogeneous beams depending upon the limit of the ratio $\lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon}$.

In the classical Euclidean space the Cartesian coordinate system attached to the beam is denoted $Ox_1x_2x_3$ and we associate an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (this will be defined more precisely later). The three-dimensional body is thin in the direction $\mathbf{e}_1$, $\mathbf{e}_2$; let δ be a small parameter which takes into account the thinness of the beam. The scaled cross-section of the beam occupies the bounded domain (with Lipschitz boundary) $\omega \subset \mathbb{R}^2$. Hence the straight beam occupies the cylinder $\overline{\Omega_\delta} \subset \mathbb{R}^3$, $\Omega_\delta = \omega_\delta \times]0, L[$ of length L and section $\omega_\delta = \delta\omega$. A generic point in Ω_δ is denoted by $x = (x_1, x_2, x_3)$ with $(x_1, x_2) \in \omega_\delta$, $x_3 \in (0, L)$, and therefore $\big(\frac{x_1}{\delta}, \frac{x_2}{\delta}\big) \in \omega$.

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Let[1] $u = (u_i)$ be the three-dimensional displacement field attached to an elastic solid under the action of applied body forces; it can be decomposed as the sum of a displacement field $U_e = (U_{ei})$ (rigid in the cross-sections) and a warping $\overline{u} = (\overline{u}_i)$ (see (2.5)). The elementary displacement field U_e is the sum of a Bernoulli-Navier displacement described by a vector $\mathbb{U} = (\mathbb{U}_\alpha, 0)$ and a scalar Θ and a contribution $\underline{u} = (\underline{u}_i)$ to complete the centerline displacement $\mathbb{U}$ (see (2.11)):

$$\begin{cases} u(x) = U_e(x) + \overline{u}(x), \\ U_e(x) = \mathbb{U}(x_3) + \underline{u}(x_3) + \begin{pmatrix} -\dfrac{\mathrm{d}\mathbb{U}_2}{\mathrm{d}x_3}(x_3) \\ +\dfrac{\mathrm{d}\mathbb{U}_1}{\mathrm{d}x_3}(x_3) \\ \Theta(x_3) \end{pmatrix} \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \end{cases} \quad \text{for a.e. } x \in \Omega_\delta.$$

The structure is clamped on a part $\Gamma_{0,\delta} = \omega_\delta \times \{0\}$ of the boundary $\partial\Omega_\delta$.

In Section 2 we begin to recall Korn's type inequalities (2.13) for thin beams (see [7, 10]):

$$\begin{cases} \|\Theta\|_{L^2(0,L)} + \Big\|\dfrac{\mathrm{d}\mathbb{U}_\alpha}{\mathrm{d}x_3}\Big\|_{L^2(0,L)} \le \dfrac{CL}{\delta^2}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times 3}}, \\ \|\mathbb{U}\|_{[H^2(0,L)]^2} \le \dfrac{CL^2}{\delta^2}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times 3}}, \\ \Big\|\dfrac{\mathrm{d}\underline{u}}{\mathrm{d}x_3}\Big\|_{[L^2(0,L)]^3} \le \dfrac{C}{\delta}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times 3}}, \quad \|\underline{u}\|_{[L^2(0,L)]^3} \le \dfrac{CL}{\delta}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times 3}}, \end{cases}$$

where $e(u)$ represents the linear strain tensor.

Under the action of applied volume force F_δ, the beam, made of an elastic material characterized by its elastic tensor a^δ, undergoes a displacement field u^δ solution to the variational problem (the regularities of a^δ and F_δ are detailed later):

Find $u^\delta \in V(\Omega_\delta) = \{v \in [H^1(\Omega_\delta)]^3 \mid v = 0 \text{ on } \Gamma_{\delta,0}\}$ such that[2]

$$\int_{\Omega_\delta} a^\delta\, e(u^\delta) : e(v)\mathrm{d}x = \int_{\Omega_\delta} F_\delta \cdot v \,\mathrm{d}x, \quad \forall v \in V(\Omega_\delta).$$

We assume that the applied forces have a specific dependence (2.14) with respect to δ, so that the strain tensor $\|e(u^\delta)\|_{[L^2(\Omega_\delta)]^{3\times 3}}$ is of order δ^2 (see (2.18)), thus we are in a position to infer that the sequence u^δ converges in an appropriate space.

In Section 3 we introduce the second small parameter ε which is also supposed to tend to zero. We describe the thin beam as made of an heterogeneous material whose elasticity tensor $a^{\varepsilon,\delta}$ depends also upon ε; the heterogeneities are distributed along the $\mathbf{e}_3$ axis with periodicity ε. In order to study the displacement field, now denoted $u^{\varepsilon,\delta}$, we introduce the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$; for all $p \in [1, +\infty]$ and $\varphi \in L^p(\Omega_\delta)$ it associates a function $\mathcal{T}_{\varepsilon,\delta}(\varphi) \in L^p((0, L) \times Y)$ where $\Omega \doteq (0, L) \times \omega$ is the dilated domain and $Y \doteq \omega \times (0, 1)$ is the unit cell occupied by the heterogeneities. In Section 4 we state the existence of a unique limit displacement field U_e and

[1] Greak indices or exponents, except ε and δ, take their values in the set $\{1, 2\}$, Latin indices (except e and y) take their values in the set $\{1, 2, 3\}$. The Einstein summation convention of repeated indices is applied.

[2] The "dot" notation is for vector product and the "colon" notation is for the tensor product:

$$AE : F = A^{ijkl}E_{ij}F_{kl},$$

where $A = (A^{ijkl})$ is a symmetric fourth-order tensor and $E = (E_{ij})$, $F = (F_{ij})$ are two second-order symmetric tensors.

a displacement corrector $\widehat{U}_e$ and we establish a weak convergence result (4.3),

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(U_e^{\varepsilon,\delta})) \rightharpoonup E(U_e) + E_{y_3}(\widehat{U}_e) \quad \text{in } [L^2((0,L)\times Y)]^{3\times 3},$$

where the symmetric third-order tensors[3] E and E_{y_3} are the strain tensors (3.4), (4.1) (the functional spaces are completely described in Section 3).

In Section 5 we show that the convergence of the second part of the strain tensor $\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta}))$ depends upon the ratio $\dfrac{\delta}{\varepsilon}$ and we study the three possibilities taken by $\theta = \lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon}$. More precisely we establish the existence of limit fields $\overline{u}$ and $\widehat{\overline{u}}$ in appropriate spaces such that the following weak convergences take place:

$$\begin{cases} \theta = +\infty, & \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e_{y_3}^{\infty}(\widehat{\overline{u}}^{\infty}) \quad \text{in } [L^2((0,L)\times Y)]^{3\times 3}, \\ \theta \text{ finite}, & \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e_{y}^{\theta}(\widehat{\overline{u}}^{\theta}) \quad \text{in } [L^2((0,L)\times Y)]^{3\times 3}, \\ \theta = 0, & \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e_{y'}^{0}(\widehat{\overline{u}}^{0}) \quad \text{in } [L^2((0,L)\times Y)]^{3\times 3}, \end{cases}$$

where the limit strain tensors are given in (3.5), (5.5), (5.7) and (5.9). In Section 6 we study the convergence of the sequence $\{u^{\varepsilon,\delta}\}_{(\varepsilon,\delta)\to(0,0)}$ in the three cases: $\theta=+\infty$, θ is finite and $\theta=0$. However, it could be of interest to compute a lower dimensional approximation of the "real displacement field", i.e., with a finite value of the small (but not "equal to zero") parameters ε and δ; this is done in Section 7 where the first terms of an asymptotic expansion of $u^{\varepsilon,\delta}$ (see (7.1)) are given.

Finally we mention the continuity of the function $\theta \to u^{\theta}$ in an appropriate functional space.

2 Displacement Field in a Thin Structure

Let δ be a small parameter which takes into account the thinness of a straight beam. The beam occupies the cylinder $\overline{\Omega_\delta} \subset \mathbb{R}^3, \Omega_\delta = \omega_\delta \times]0,L[$ of length L and section $\omega_\delta = \delta\omega$ (where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary). The Cartesian coordinate system attached to ω has the gravity center of the structure for origin and the direction of its main inertia axes as the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, i.e.,

$$\int_\omega x_\alpha \,\mathrm{d}x_1\mathrm{d}x_2 = \int_\omega x_1x_2 \,\mathrm{d}x_1\mathrm{d}x_2 = 0, \quad I_\alpha = \int_\omega (x_\alpha)^2 \,\mathrm{d}x_1\mathrm{d}x_2, \tag{2.1}$$

and I_1, I_2 are the two principal moments of inertia.

The beam is supposed to be fixed on its extremity $\Gamma_{\delta,0} = \omega_\delta \times \{0\}$.

The structure is made of a material characterized by its elasticity tensor $a^\delta = (a^{ijkl,\delta})$ with the classical properties of symmetry, boundedness and coercivity, i.e., for all symmetric second order tensor e there exist two positive constants c, C such that

$$a^{ijkl,\delta} = a^{jikl,\delta} = a^{klij,\delta}, \quad c\ e_{ij}e_{ij} \le a^\delta e : e = a^{ijkl,\delta}e_{ij}e_{kl} \le C\ e_{ij}e_{ij}. \tag{2.2}$$

Under the action of applied volume forces F_δ the beam undergoes a displacement field $u_\delta \in V(\Omega_\delta)$ solution to the variational problem

$$\int_{\Omega_\delta} a^\delta\ e(u^\delta) : e(v)\,\mathrm{d}x = \int_{\Omega_\delta} F_\delta \cdot v\,\mathrm{d}x, \quad \forall v \in V(\Omega_\delta),$$

[3]The subscript y stands for derivative with respect to (y_1, y_2, y_3), the subscript y' is for derivative with respect to (y_1, y_2), and the subscript y_3 is for derivative with respect to y_3.

posed in the functional space

$$V(\Omega_\delta) \doteq \{v \in [H^1(\Omega_\delta)]^3 \mid v = 0 \text{ on } \Gamma_{\delta,0}\}. \tag{2.3}$$

In this space the classical Korn's inequality reads: For all $v \in V(\Omega_\delta)$, there exists a positive constant $C(\Omega_\delta)$ which depends upon the domain Ω_δ such that

$$\|\nabla v\|_{L^2(\Omega_\delta)} + \|v\|_{L^2(\Omega_\delta)} \leq C(\Omega_\delta)\|e(v)\|_{L^2(\Omega_\delta)}. \tag{2.4}$$

Hence, for $F_\delta \in [L^2(\Omega_\delta)]^3$ and $a^\delta \in [L^\infty(\Omega_\delta)]^{3\times3\times3\times3}$, we can apply Lax-Milgram theorem to obtain the existence and uniqueness of the solution $u^\delta \in V(\Omega_\delta)$.

2.1 Decomposition of the displacement field in a cylinder

In order to study the behavior of the sequence $\{u^\delta\}_\delta$ when δ goes to zero it is of interest to introduce the fixed bi-dimensional domain $\omega \subset \mathbb{R}^2$ (the reference cross-section of the beam), hence $\omega_\delta = \delta\omega$, and the three-dimensional cylinder $\Omega = \omega \times (0, L)$. We also consider the decomposition [4] of any displacement field $u \in [L^1(\Omega_\delta)]^3$ as the sum of an elementary displacement field $U_e \in [L^1(\Omega_\delta)]^3$ and a warping $\overline{u} \in [L^1(\Omega_\delta)]^3$ (see [5, 12]):

$$u(x) = U_e(x) + \overline{u}(x) \quad \text{for a.e. } x \in \Omega_\delta, \tag{2.5}$$

where $\overline{u}$ satisfies

$$\begin{cases} \displaystyle\int_{\omega_\delta} \overline{u}(x)\, \mathrm{d}x_1\mathrm{d}x_2 = 0, \quad \int_{\omega_\delta} x_\alpha \overline{u}_3(x)\, \mathrm{d}x_1\mathrm{d}x_2 = 0, \\ \displaystyle\int_{\omega_\delta} (x_1\overline{u}_2(x) - x_2\overline{u}_1(x))\, \mathrm{d}x_1\mathrm{d}x_2 = 0 \end{cases} \quad \text{for a.e.} x_3 \in (0, L). \tag{2.6}$$

The last equality above means that the warping $\overline{u}$ does not capture the couple of torsion forces (see Section 2.4). The same approach was considered for plates (see [8]). The elementary displacement U_e is given by the displacement of the middle line $U \in [L^1(0, L)]^3$ and the small rotation along the vector $R \in [L^1(0, L)]^3$:

$$U_e(x) = U(x_3) + R(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \quad \text{for a.e. } x \in \Omega_\delta. \tag{2.7}$$

We recall here the Definitions 2–3 of [9–10].

$$\begin{cases} U(x_3) = \dfrac{1}{|\omega_\delta|}\displaystyle\int_{\omega_\delta} u(x)\, \mathrm{d}x_1\mathrm{d}x_2 = \frac{1}{\delta^2|\omega|}\int_{\omega_\delta} u(x)\, \mathrm{d}x_1\mathrm{d}x_2, \\ R_1(x_3) = \dfrac{1}{I_2\delta^4}\displaystyle\int_{\omega_\delta} x_2u_3(x)\, \mathrm{d}x_1\mathrm{d}x_2, \\ R_2(x_3) = -\dfrac{1}{I_1\delta^4}\displaystyle\int_{\omega_\delta} x_1u_3(x)\mathrm{d}x_1\mathrm{d}x_2, \\ R_3(x_3) = \Theta(x_3) = \dfrac{1}{(I_1 + I_2)\delta^4}\displaystyle\int_{\omega_\delta} (x_1u_2(x) - x_2u_1(x))\, \mathrm{d}x_1\mathrm{d}x_2, \end{cases} \tag{2.8}$$

where the two principal moments of inertia I_1, I_2 are given by (2.1).

Remark 2.1 From now on, we assume the displacements to be in $V(\Omega_\delta)$. As a consequence, for every $u \in V(\Omega_\delta)$, the terms U and R of the decomposition belong to $[H^1(0, L)]^3$, while the warping $\overline{u}$ belongs to $V(\Omega_\delta)$. The terms of the decomposition also satisfy the boundary clamping condition

$$U(0) = R(0) = 0, \quad \overline{u}(x_1, x_2, 0) = 0 \quad \text{for a.e. } (x_1, x_2) \in \omega_\delta.$$

[4]The case of a curved beam was consider in [7].

We recall the bounds on the displacement fields given in [10, Theorem 3.1, p. 206].

For all $u \in H^1(\Omega_\delta;\mathbb{R}^3)$ and $\delta \le L$, we have

$$\begin{cases} \|\nabla\overline{u}\|_{L^2(\Omega_\delta)} \le C\|e(u)\|_{L^2(\Omega_\delta)}, \quad \|\overline{u}\|_{L^2(\Omega_\delta)} \le C\delta\|e(u)\|_{L^2(\Omega_\delta)}, \\ \delta\Big\|\dfrac{\mathrm{d}R}{\mathrm{d}x_3}\Big\|_{L^2(0,L)} + \Big\|\dfrac{\mathrm{d}U}{\mathrm{d}x_3} - R\wedge \mathbf{e}_3\Big\|_{L^2(\Omega_\delta)} \le \dfrac{C}{\delta}\|e(u)\|_{L^2(\Omega_\delta)}, \end{cases}$$

where the positive constant C depends neither on δ nor on the length L of the beam.

Corollary of [10, Theorem 3.1] (Korn's Inequality with Boundary Conditions) *For all $v \in V(\Omega_\delta)$, we have*

$$\|\nabla v\|_{L^2(\Omega_\delta)} + \|v\|_{L^2(\Omega_\delta)} \le \frac{C}{\delta}\|e(v)\|_{L^2(\Omega_\delta)}. \tag{2.9}$$

We note that in this expression of Korn's inequality the bound depends explicitly on the thickness of the thin beam (compare to (2.4)). More precisely, one has

$$\begin{cases} \|u_\alpha\|_{L^2(\Omega_\delta)} \le \dfrac{C}{\delta}\|e(u)\|_{L^2(\Omega_\delta)}, \quad \|u_3\|_{L^2(\Omega_\delta)} \le C\|e(u)\|_{L^2(\Omega_\delta)}, \\ \|\nabla u\|_{L^2(\Omega_\delta)} \le \dfrac{C}{\delta}\|e(u)\|_{L^2(\Omega_\delta)}. \end{cases} \tag{2.10}$$

The constant C is independent of δ.

2.2 Introduction of a new decomposition

We introduce a new decomposition of the elementary displacement part U_e, in order to simplify the expression of the strain tensor $e(U_e) = \frac{1}{2}(\nabla U_e + \nabla^{\mathrm{T}} U_e)$,

$$e(U_e) = \frac{1}{2}\begin{pmatrix} 0 & | & 0 & | & \dfrac{\mathrm{d}U_1}{\mathrm{d}x_3} - R_2 - x_2\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3} \\ & | & & | & \\ 0 & | & 0 & | & \dfrac{\mathrm{d}U_2}{\mathrm{d}x_3} + R_1 + x_1\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3} \\ & | & & | & \\ \dfrac{\mathrm{d}U_1}{\mathrm{d}x_3} - R_2 - x_2\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3} & | & \dfrac{\mathrm{d}U_2}{\mathrm{d}x_3} + R_1 + x_1\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3} & | & 2\Big(\dfrac{\mathrm{d}U_3}{\mathrm{d}x_3} + x_2\dfrac{\mathrm{d}R_1}{\mathrm{d}x_3} - x_1\dfrac{\mathrm{d}R_2}{\mathrm{d}x_3}\Big) \end{pmatrix}.$$

More precisely let $(\mathbb{U}, \underline{u})$ be the new functions defined by

$$\mathbb{U}(x_3) = \int_0^{x_3} R(t)\wedge \mathbf{e}_3 \,\mathrm{d}t, \quad \underline{u}(x_3) = U(x_3) - \mathbb{U}(x_3) \quad \text{for a.e. } x_3 \in (0,L).$$

Then we can eliminate the first two components of the rotation and get

$$\begin{cases} \dfrac{\mathrm{d}R_2}{\mathrm{d}x_3} = \dfrac{\mathrm{d}^2\mathbb{U}_1}{\mathrm{d}x_3^2}, \quad -\dfrac{\mathrm{d}R_1}{\mathrm{d}x_3} = \dfrac{\mathrm{d}^2\mathbb{U}_2}{\mathrm{d}x_3^2}, \\ \dfrac{\mathrm{d}U_1}{\mathrm{d}x_3} - R_2 = \dfrac{\mathrm{d}(U_1 - \mathbb{U}_1)}{\mathrm{d}x_3} = \dfrac{\mathrm{d}\underline{u}_1}{\mathrm{d}x_3}, \quad \dfrac{\mathrm{d}U_2}{\mathrm{d}x_3} + R_1 = \dfrac{\mathrm{d}(U_2 - \mathbb{U}_2)}{\mathrm{d}x_3} = \dfrac{\mathrm{d}\underline{u}_2}{\mathrm{d}x_3}, \end{cases}$$

and $\mathbb{U}_3 \equiv 0$. From now on, we denote $\Theta = R_3$ the third component of the rotation; therefore we have a new decomposition of the field U_e:

$$U_e(x) = \mathbb{U}(x_3) + \underline{u}(x_3) + \begin{pmatrix} -\dfrac{\mathrm{d}\mathbb{U}_2}{\mathrm{d}x_3}(x_3) \\ +\dfrac{\mathrm{d}\mathbb{U}_1}{\mathrm{d}x_3}(x_3) \\ \Theta(x_3) \end{pmatrix} \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \quad \text{for a.e. } x \in \Omega_\delta$$

or component-wise

$$U_e(x) = \begin{pmatrix} \mathbb{U}_1(x_3) + \underline{u}_1(x_3) - x_2\Theta(x_3) \\ \mathbb{U}_2(x_3) + \underline{u}_2(x_3) + x_1\Theta(x_3) \\ \underline{u}_3(x_3) - x_1\dfrac{d\mathbb{U}_1}{dx_3}(x_3) - x_2\dfrac{d\mathbb{U}_2}{dx_3}(x_3) \end{pmatrix}. \tag{2.11}$$

This decomposition yields the new expression of the strain tensor as

$$e(U_e) = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dfrac{d\underline{u}_1}{dx_3} - x_2\dfrac{d\Theta}{dx_3} \\ 0 & 0 & \dfrac{d\underline{u}_2}{dx_3} + x_1\dfrac{d\Theta}{dx_3} \\ \dfrac{d\underline{u}_1}{dx_3} - x_2\dfrac{d\Theta}{dx_3} & \dfrac{d\underline{u}_2}{dx_3} + x_1\dfrac{d\Theta}{dx_3} & 2\Big(\dfrac{d\underline{u}_3}{dx_3} - x_2\dfrac{d^2\mathbb{U}_2}{dx_3^2} - x_1\dfrac{d^2\mathbb{U}_1}{dx_3^2}\Big) \end{pmatrix}.$$

We note that the clamping condition $u = 0$ on $\Gamma_{0,\delta}$ implies boundary conditions on the decomposition:

$$\underline{u}(0) = \mathbb{U}(0) = \frac{d\mathbb{U}}{dx_3}(0) = \Theta(0) = 0, \quad \overline{u}(x) = 0 \quad \text{for a.e. } x \in \Gamma_{0,\delta} = \omega_\delta \times \{0\}. \tag{2.12}$$

We also note that, since $R_\alpha \in H^1(0,L)$, one has $\mathbb{U}_\alpha \in H^2(0,L)$.

2.3 First bounds

In the sequel C represents any positive constant which depends neither on δ nor on the length L of the beam.

When u belongs to $V(\Omega_\delta)$ then, taking into account the boundary conditions (2.12), one obtains the bounds on the new fields $(\mathbb{U}_\alpha, \underline{u}_i, \Theta)$:

$$\begin{cases} \|\Theta\|_{L^2(0,L)} + \Big\|\dfrac{d\mathbb{U}_\alpha}{dx_3}\Big\|_{L^2(0,L)} \le \dfrac{CL}{\delta^2}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times3}}, \\ \|\mathbb{U}\|_{[H^2(0,L)]^2} \le \dfrac{CL^2}{\delta^2}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times3}}, \\ \Big\|\dfrac{d\underline{u}}{dx_3}\Big\|_{[L^2(0,L)]^3} \le \dfrac{C}{\delta}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times3}}, \quad \|\underline{u}\|_{[L^2(0,L)]^3} \le \dfrac{CL}{\delta}\|e(u)\|_{[L^2(\Omega_\delta)]^{3\times3}}. \end{cases} \tag{2.13}$$

2.4 Assumption on the applied volume forces

Let v be a general elastic displacement field belonging to $V(\Omega_\delta)$ decomposed as follows:

$$v = V_e + \overline{v},$$

$$V_e(x) = \mathbb{V}(x_3) + \underline{v}(x_3) + \begin{pmatrix} -\dfrac{d\mathbb{V}_2}{dx_3}(x_3) \\ +\dfrac{d\mathbb{V}_1}{dx_3}(x_3) \\ \Psi(x_3) \end{pmatrix} \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \quad \text{for a.e. } x \in \Omega_\delta.$$

We rely on the bounds (2.13) and (2.10) and we consider, for simplicity, the scaled forces and moments.

(1) Influence of a volume force $f_\delta(x_3) = (\delta^2 f_1(x_3), \delta^2 f_2(x_3), \delta f_3(x_3))$:

$$\int_{\Omega_\delta} f_\delta \cdot V_e \, \mathrm{d}x = |\omega| \int_0^L (\delta^4 f_\alpha(\mathbb{V}_\alpha + \underline{v}_\alpha) + \delta^3 f_3 \underline{v}_3) \, \mathrm{d}x_3.$$

(2) Influence of a moment $f_{M,\delta}(x) = (-x_2 f_T(x_3), x_1 f_T(x_3), x_1 g_1(x_3) + x_2 g_2(x_3))$:

$$\int_{\Omega_\delta} f_{M,\delta} \cdot V_e \, \mathrm{d}x = \delta^4 \int_0^L ((I_1 + I_2) f_T \Psi - I_1 g_1 \mathbb{V}_1' - I_2 g_2 \mathbb{V}_2') \, \mathrm{d}x_3.$$

The quantity $-x_2 f_T(x_3)\mathbf{e}_1 + x_1 f_T(x_3)\mathbf{e}_2$ represents the torsion forces acting on the beam.

(3) Influence of $\underline{f}_\delta(x) = \big(\delta \underline{f}_1(x_3), \delta \underline{f}_2(x_3), \frac{1}{\delta}\underline{f}_3(x)\big)$, $\underline{f}_3(x) = x_1 \underline{g}_1(x_3) + x_2 \underline{g}_2(x_3)$ with $\underline{g}_\alpha$ given by the solution of the ODE, where I_α is given by (2.1):

$$I_\alpha \frac{\mathrm{d}\underline{g}_\alpha}{\mathrm{d}x_3} + |\omega| \underline{f}_\alpha = 0 \quad \text{in } (0,L), \quad \underline{g}_\alpha(L) = 0, \quad \text{no summation on } \alpha = 1,2,$$

$$\begin{aligned}\int_{\Omega_\delta} \underline{f}_\delta \cdot V_e \, \mathrm{d}x &= \int_0^L (\delta^3 |\omega| \underline{f}_\alpha(\mathbb{V}_\alpha + \underline{v}_\alpha) - \delta^3 (I_1 \underline{g}_1 \mathbb{V}_1' + I_2 \underline{g}_2 \mathbb{V}_2')) \, \mathrm{d}x_3 \\ &= |\omega| \delta^3 \int_0^L (\underline{f}_\alpha(\mathbb{V}_\alpha + \underline{v}_\alpha) - \underline{f}_\alpha \mathbb{V}_\alpha) \, \mathrm{d}x_3 = |\omega| \delta^3 \int_0^L \underline{f}_\alpha \underline{v}_\alpha \, \mathrm{d}x_3.\end{aligned}$$

Now let us combine all those elementary forces to define the global one

$$F_\delta = f_\delta + f_{M,\delta} + \underline{f}_\delta$$

or component-wise

$$F_\delta(x) = \begin{pmatrix} \delta^2 f_1(x_3) + \delta \underline{f}_1(x_3) - x_2 f_T(x_3) \\ \delta^2 f_2(x_3) + \delta \underline{f}_2(x_3) + x_1 f_T(x_3) \\ \delta f_3(x_3) + x_1 g_1(x_3) + x_2 g_2(x_3) + \dfrac{1}{\delta}(x_1 \underline{g}_1(x_3) + x_2 \underline{g}_2(x_3)) \end{pmatrix}. \tag{2.14}$$

Note that, due to the definition (2.6) of $\overline{v}$, one has $\int_\Omega F_\delta \cdot \overline{v} \, \mathrm{d}x = 0$. With the decomposition (2.11) of the displacement field and assumption (2.14) on the forces we get the expression of the potential energy

$$\begin{aligned}\int_{\Omega_\delta} F_\delta \cdot v \, \mathrm{d}x &= \delta^4 \int_0^L (|\omega| f_\alpha (\mathbb{V}_\alpha + \underline{v}_\alpha) + (I_1 + I_2) f_T \Psi - (I_1 g_1 \mathbb{V}_1' + I_2 g_2 \mathbb{V}_2')) \, \mathrm{d}x_3 \\ &\quad + \delta^3 \int_0^L |\omega| (\underline{f}_\alpha \underline{v}_\alpha + f_3 \underline{v}_3) \, \mathrm{d}x_3, \end{aligned} \tag{2.15}$$

and the bound follows:

$$\begin{aligned}&\Big| \int_\Omega F_\delta \cdot v \, \mathrm{d}x \Big| \\ &\leq C\delta^2 (\|f\|_{[L^2(0,L)]^3} + \|\underline{f}\|_{[H^1(0,L)]^3} + \|f_T\|_{L^2(0,L)} + \|g\|_{[L^2(0,L)]^2}) \|e(v)\|_{L^2(\Omega_\delta)} \\ &\leq C\delta^2 \|e(v)\|_{L^2(\Omega_\delta)} \quad \text{for all } v \in V(\Omega_\delta). \end{aligned} \tag{2.16}$$

Remark 2.2 The specific choice (2.14) is made so that every component of the force contributes equally to the total elastic energy as we can see in (2.16) and, consequently, will appear through (6.2) in the expression of the limit models.

The variational problem: Find $u^\delta \in V(\Omega_\delta)$.

$$\int_{\Omega_\delta} a^\delta \, e(u^\delta) : e(v)\mathrm{d}x = \int_{\Omega_\delta} F_\delta \cdot v \, \mathrm{d}x, \quad \forall v \in V(\Omega_\delta) \tag{2.17}$$

has a unique solution and the a priori bound on the strain tensor $e(u^\delta)$ can be derived from (2.16):

$$\begin{aligned} & c\|e(u^\delta)\|^2_{L^2(\Omega_\delta)} \\ & \le \int_{\Omega_\delta} a^\delta \, e(u^\delta) : e(u^\delta)\mathrm{d}x = \int_{\Omega} F_\delta \cdot u^\delta \, \mathrm{d}x \\ & \le C\delta^2(\|f\|_{[L^2(0,L)]^3} + \|\underline{f}\|_{[H^1(0,L)]^3} + \|f_T\|_{L^2(0,L)} + \|g\|_{[L^2(0,L)]^2})\|e(u^\delta)\|_{L^2(\Omega_\delta)} \\ & \le C\delta^2\|e(u^\delta)\|_{[L^2(\Omega_\delta)]^{3\times 3}}, \end{aligned}$$

which in turns yields

$$\|e(u^\delta)\|_{[L^2(\Omega_\delta)]^{3\times 3}} \le C\delta^2, \tag{2.18}$$

where the constant $C > 0$ does not depend upon δ.

2.5 A priori estimates

Any displacement field u^δ satisfying (2.18) can be decomposed as presented in the previous subsections. Therefore, the appropriate choice of the applied forces and moments (2.14) leads to the following bounds on the different fields $(u^\delta, \mathbb{U}^\delta, \underline{u}^\delta, \Theta^\delta, \overline{u}^\delta)$ associated to (2.13):

(1) Bound on the total displacement u^δ:

$$\begin{aligned} \|u^\delta_\alpha\|_{L^2(\Omega_\delta)} &\le \frac{C}{\delta}\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C\delta, \\ \|u^\delta_3\|_{L^2(\Omega_\delta)} &\le C\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C\delta^2. \end{aligned}$$

(2) Bound on the principal flexion $\mathbb{U}^\delta$:

$$\|\mathbb{U}^\delta\|_{[H^2(0,L)]^2} \le \frac{CL^2}{\delta^2}\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C.$$

(3) Bound on the stretching or complementary flexion $\underline{u}^\delta$:

$$\|\underline{u}^\delta\|_{[H^1(0,L)]^3} \le \frac{CL}{\delta}\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C\delta.$$

(4) Bound on the angle of torsion of the sections (around the middle straight line) Θ^δ:

$$\|\Theta^\delta\|_{H^1(0,L)} \le \frac{CL}{\delta^2}\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C.$$

(5) Bound on the warping $\overline{u}^\delta$:

$$\begin{aligned} \|\overline{u}^\delta\|_{[L^2(\Omega_\delta)]^3} &\le C\delta\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C\delta^3, \\ \|\nabla\overline{u}^\delta\|_{[L^2(\Omega_\delta)]^9} &\le C\|e(u^\delta)\|_{L^2(\Omega_\delta)} \le C\delta^2. \end{aligned}$$

To sum up, we get

$$\begin{aligned} & \|\Theta^\delta\|_{H^1(0,L)} + \|\mathbb{U}^\delta\|_{[H^2(0,L)]^2} \le C, \quad \|\underline{u}^\delta\|_{[H^1(0,L)]^3} \le C\delta, \\ & \|\overline{u}^\delta\|_{[L^2(\Omega_\delta)]^3} \le C\delta^3, \quad \|\nabla\overline{u}^\delta\|_{L^2(\Omega_\delta)} \le C\delta^2. \end{aligned} \tag{2.19}$$

In the sequel, we consider periodically heterogeneous thin beams: More precisely, the material is described by an elasticity tensor $a^{\varepsilon,\delta} \in [L^\infty(\Omega_\delta)]^{3\times3\times3\times3}$ (with the same kind of coercivity and symmetry properties as in (2.2)) whose components are periodic along the $\mathbf{e_3}$ direction and depend, now, upon two small parameters δ and ε (the applied forces are independent of ε). The variational problem: Find $u^{\varepsilon,\delta} \in V(\Omega_\delta)$ such that

$$\int_{\Omega_\delta} a^{\varepsilon,\delta}\ e(u^{\varepsilon,\delta}) : e(v)\mathrm{d}x = \int_{\Omega_\delta} F_\delta \cdot v\ \mathrm{d}x, \quad \forall v \in V(\Omega_\delta)$$

has a unique solution $u^{\varepsilon,\delta}$. This paper aims at studying the behaviour of the sequence of solutions $\{u^{\varepsilon,\delta}\}_{\varepsilon,\delta}$ when both parameters ε and δ go to zero independently.

3 Introduction of Another Small Parameter: The Size ε of the Inclusions

3.1 The unfolding operator $\mathcal{T}_\varepsilon$, definition and first properties

The beam is made of periodic cells distributed along the direction $\mathbf{e_3}$, in such a manner that each of these identical cells occupies a domain of thin section ω_δ and of small length ε. In order to simplify the presentation we assume that the macroscopic domain $(0, L)$ is covered by an integer number of elementary cells[5]: $\varepsilon = L/N$, $N \in \mathbb{N}^*$.

We define the unique decomposition of almost every real number $z \in \mathbb{R}$ as the sum of its integer part $[z]$ (also called the "slow" evolving part) and the remainder $\{z\}$ (also called the "fast" evolving part) which belongs to the microscopic domain $(0, 1)$):

$$z = [z] + \{z\}, \quad [z] \in \mathbb{Z}, \quad \{z\} \in (0, 1).$$

The unfolding operator $\mathcal{T}_\varepsilon$ maps $L^p(0, L)$ into $L^p((0, L) \times (0, 1))$ for all $p \in [1, +\infty]$:

$$\forall\varphi \in L^p(0, L), \quad \mathcal{T}_\varepsilon(\varphi)(x, y) = \varphi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) \quad \text{for a.e. } (x, y) \in (0, L) \times (0, 1).$$

An immediate property of this linear operator is that for all functions $\varphi, \phi \in L^1(0, L)$ we have

$$\int_0^L \varphi(x)\mathrm{d}x = \int_0^L \mathrm{d}x \int_0^1 \mathcal{T}_\varepsilon(\varphi)(x, y)\mathrm{d}y, \quad \mathcal{T}_\varepsilon(\varphi\phi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\phi).$$

Let us introduce the spaces

$$H^1_{\mathrm{per}}(0, 1) = \Big\{\psi \in H^1(0, 1) \mid \psi(\cdot, 0) = \psi(\cdot, 1), \int_0^1 \psi(y)\mathrm{d}y = 0\Big\},$$
$$H^2_{\mathrm{per}}(0, 1) = \{\psi \in H^2(0, 1) \cap H^1_{\mathrm{per}}(0, 1) \mid \psi'(\cdot, 0) = \psi'(\cdot, 1)\}.$$

In the sequel we will make use of important results of convergence gathered in the following three theorems (see [3, p. 1599, p. 1603]), in which the variable x lies in $(0, L)$ and y in $(0, 1)$.

[5] The domain Ω_δ does not depend upon ε.

Theorem 3.1 *Let $\{v_\varepsilon\}_\varepsilon$ be a sequence in $L^2(0,L)$ satisfying $\|v_\varepsilon\|_{L^2(0,L)} \le C$. There exists a subsequence, still denoted $\{\varepsilon\}$, and limits $v \in L^2(0,L)$ and $\widehat{v} \in L^2((0,L)\times(0,1))$ such that*

$$\begin{cases} v_\varepsilon \rightharpoonup v & \text{weakly in } L^2(0,L), \\ \mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup v + \widehat{v} & \text{weakly in } L^2((0,L)\times(0,1)), \quad \displaystyle\int_0^1 \widehat{v}(\cdot,y)\mathrm{d}y = 0. \end{cases}$$

Moreover, if $\big\|\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}x}\big\|_{L^2(0,L)} \le \frac{C}{\varepsilon}$, then $\widehat{v} \in L^2(0,L;H^1_{\mathrm{per}}(0,1))$ and

$$\frac{\partial \mathcal{T}_\varepsilon(v_\varepsilon)}{\partial y} \equiv \varepsilon \mathcal{T}_\varepsilon\Big(\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}x}\Big) \rightharpoonup \frac{\partial \widehat{v}}{\partial y} \quad \text{weakly in } L^2((0,L)\times(0,1)).$$

Theorem 3.2 *Let $\{v_\varepsilon\}_\varepsilon$ be a sequence in $H^1(0,L)$ satisfying $\|v_\varepsilon\|_{H^1(0,L)} \le C$. There exists a subsequence, still denoted $\{\varepsilon\}$, $v \in H^1(0,L)$ and $\widehat{v} \in L^2(0,L;H^1_{\mathrm{per}}(0,1))$ such that*

$$\begin{cases} v_\varepsilon \rightharpoonup v & \text{weakly in } H^1(0,L), \\ \mathcal{T}_\varepsilon(v_\varepsilon) \to v & \text{strongly in } L^2(0,L;H^1(0,1)), \\ \mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}v_\varepsilon}{\mathrm{d}x}\Big) \rightharpoonup \dfrac{\mathrm{d}v}{\mathrm{d}x} + \dfrac{\partial \widehat{v}}{\partial y} & \text{weakly in } L^2((0,L)\times(0,1)). \end{cases}$$

Theorem 3.3 *Let $\{v_\varepsilon\}_\varepsilon$ be a sequence in $H^2(0,L)$ satisfying $\|v_\varepsilon\|_{H^2(0,L)} \le C$. There exists a subsequence, still denoted $\{\varepsilon\}$, $v \in H^2(0,L)$ and $\widehat{v} \in L^2(0,L;H^2_{\mathrm{per}}(0,1))$ such that*

$$\begin{cases} v_\varepsilon \rightharpoonup v & \text{weakly in } H^2(0,L), \\ \mathcal{T}_\varepsilon(v_\varepsilon) \to v & \text{strongly in } L^2(0,L;H^2(0,1)), \\ \mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}v_\varepsilon}{\mathrm{d}x}\Big) \to \dfrac{\mathrm{d}v}{\mathrm{d}x} & \text{strongly in } L^2(0,L;H^1(0,1)), \\ \mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}^2v_\varepsilon}{\mathrm{d}x^2}\Big) \rightharpoonup \dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + \dfrac{\partial^2 \widehat{v}}{\partial y^2} & \text{weakly in } L^2((0,L)\times(0,1)). \end{cases}$$

We are then in a position to extend these results to the study of sequences $\{w^{\varepsilon,\delta}\}_{\varepsilon,\delta}$ according to the limit of the ratio $\frac{\delta}{\varepsilon}$ when both the two small parameters converge to zero.

In the lemma below, we consider a sequence $\{(\varepsilon,\delta)\}$ converging to $(0,0)$.

Lemma 3.1 *Let $\{w^{\varepsilon,\delta}\}_{\varepsilon,\delta}$ be a sequence converging weakly to w in $L^2(0,L)$ and satisfying*

$$\delta\Big\|\frac{\mathrm{d}w^{\varepsilon,\delta}}{\mathrm{d}x}\Big\|_{L^2(0,L)} \le C.$$

If $\frac{\delta}{\varepsilon} \to 0$ then there exists $\widehat{w}$ in $L^2((0,L)\times(0,1))$ satisfying $\int_0^1 \widehat{w}(\cdot,y)\mathrm{d}y = 0$ such that the following convergences hold:

$$\begin{cases} \text{(i) } \mathcal{T}_\varepsilon(w^{\varepsilon,\delta}) \rightharpoonup w + \widehat{w} & \text{weakly in } L^2(0,L;H^1(0,1)), \\ \text{(ii) } \delta\mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}w^{\varepsilon,\delta}}{\mathrm{d}x}\Big) \rightharpoonup 0 & \text{weakly in } L^2((0,L)\times(0,1)). \end{cases}$$

If $\frac{\delta}{\varepsilon} \to +\infty$ then there exists $\widehat{w}$ in $L^2(0,L;H^1_{\mathrm{per}}(0,1))$ such that the following convergences hold:

$$\begin{cases} \text{(iii) } \mathcal{T}_\varepsilon(w^{\varepsilon,\delta}) \rightharpoonup w & \text{weakly in } L^2(0,L;H^1(0,1)), \\ \text{(iv) } \delta\mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}w^{\varepsilon,\delta}}{\mathrm{d}x}\Big) \rightharpoonup \dfrac{\partial \widehat{w}}{\partial y} & \text{weakly in } L^2((0,L)\times(0,1)). \end{cases}$$

Proof We begin with the obvious estimates

$$\|\mathcal{T}_\varepsilon(w^{\varepsilon,\delta})\|_{L^2((0,L)\times(0,1))} \leq C, \quad \Big\|\frac{\partial \mathcal{T}_\varepsilon(w^{\varepsilon,\delta})}{\partial y}\Big\|_{L^2((0,L)\times(0,1))} = \varepsilon\Big\|\frac{\mathrm{d}w^{\varepsilon,\delta}}{\mathrm{d}x}\Big\|_{L^2(0,L)} \leq C\frac{\varepsilon}{\delta}.$$

Step 1 We prove (i) and (ii). Convergence (i) is an immediate consequence of the first part of Theorem 3.1. Since the sequence $\frac{\delta}{\varepsilon}$ is bounded from above we get

$$\Big\|\frac{\delta}{\varepsilon}\mathcal{T}_\varepsilon(w^{\varepsilon,\delta})\Big\|_{L^2((0,L)\times(0,1))} \leq C\frac{\delta}{\varepsilon}, \quad \Big\|\frac{\delta}{\varepsilon}\mathcal{T}_\varepsilon(w^{\varepsilon,\delta})\Big\|_{L^2(0,L;H^1(0,1))} \leq C.$$

From the above estimates and the fact that $\frac{\delta}{\varepsilon} \to 0$ we deduce

$$\frac{\delta}{\varepsilon}\mathcal{T}_\varepsilon(w^{\varepsilon,\delta}) \rightharpoonup 0 \quad \text{weakly in } L^2(0,L;H^1(0,1)).$$

And

$$\frac{\delta}{\varepsilon}\frac{\partial \mathcal{T}_\varepsilon(w^{\varepsilon,\delta})}{\partial y} = \delta\mathcal{T}_\varepsilon\Big(\frac{\mathrm{d}w^{\varepsilon,\delta}}{\mathrm{d}x}\Big)$$

gives convergence (ii).

Step 2 We prove (iii) and (iv). In this second case, since $\frac{\varepsilon}{\delta}$ is bounded from above, from Theorem 3.1, up to a subsequence, there exists $W \in L^2(0,L;H^1_{\text{per}}(0,1))$ such that

$$\mathcal{T}_\varepsilon(w^{\varepsilon,\delta}) \rightharpoonup w + W \quad \text{weakly in } L^2(0,L;H^1(0,1)).$$

Taking into account the fact that $\frac{\varepsilon}{\delta} \to 0$, we get $\frac{\partial W}{\partial y} = 0$, then W is independent of y. And $\int_0^1 W(x,y)\mathrm{d}y = 0$ for a.e. $x \in (0,L)$ gives $W = 0$ and convergence (iii).

Applying Theorem 3.2 to the sequence $\{\delta w^{\varepsilon,\delta}\}_{\varepsilon,\delta}$, which is uniformly bounded in $H^1(0,L)$, and weakly convergent to 0 in $H^1(0,L)$, we obtain a function $\widehat{w}$ in $L^2(0,L;H^1_{\text{per}}(0,1))$ such that (up to a subsequence)

$$\mathcal{T}_\varepsilon\Big(\frac{\mathrm{d}(\delta w^{\varepsilon,\delta})}{\mathrm{d}x}\Big) \rightharpoonup \frac{\partial \widehat{w}}{\partial y} \quad \text{weakly in } L^2((0,L)\times(0,1)),$$

whence convergence (iv).

3.2 The dilation operator Π_δ and the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$

Associated to the scaled domain Ω_δ we introduce another unfolding operator $\Pi_\delta : L^2(\Omega_\delta) \to L^2(\Omega)$ defined for all $\phi \in L^2(\Omega_\delta)$ by

$$\Pi_\delta(\phi)(x_3,y_1,y_2) = \phi(\delta y_1, \delta y_2, x_3), \quad (x_3,y_1,y_2) \in \Omega,$$

and we note that

$$\|\Pi_\delta(\phi)\|_{L^2(\Omega)} = \frac{1}{\delta}\|\phi\|_{L^2(\Omega_\delta)}. \tag{3.1}$$

Moreover, for every $\varphi \in H^1(\Omega_\delta)$ one has

$$\Pi_\delta\Big(\frac{\partial\varphi}{\partial x_\alpha}\Big) = \frac{1}{\delta}\,\frac{\partial}{\partial y_\alpha}(\Pi_\delta(\varphi)), \quad \Pi_\delta\Big(\frac{\partial\varphi}{\partial x_3}\Big) = \frac{\partial}{\partial x_3}(\Pi_\delta(\varphi)). \tag{3.2}$$

Next, we combine the two previous scalings due to the thinness of the geometry and the periodicity of the elasticity tensor and we introduce the third unfolded operator

$$\mathcal{T}_{\varepsilon,\delta} = \mathcal{T}_\varepsilon \circ \Pi_\delta = \Pi_\delta \circ \mathcal{T}_\varepsilon.$$

From now on, the reference microscopic domain is denoted $Y = \omega \times (0,1)$ and we get the definition

$$\begin{aligned}
&\mathcal{T}_{\varepsilon,\delta} : \varphi \in L^1(\Omega_\delta) \to \mathcal{T}_{\varepsilon,\delta}(\varphi) \in L^1((0,L) \times Y),\\
&\mathcal{T}_{\varepsilon,\delta}(\varphi)(x_3, y) = \varphi\Big(\varepsilon\Big[\frac{x_3}{\varepsilon}\Big]\mathbf{e}_3 + \delta y_1 \mathbf{e}_1 + \delta y_2 \mathbf{e}_2 + \varepsilon y_3 \mathbf{e}_3\Big)
\end{aligned}$$

and the main properties

$$\begin{cases}
\mathcal{T}_{\varepsilon,\delta}(\varphi\phi) = \mathcal{T}_{\varepsilon,\delta}(\varphi)\mathcal{T}_{\varepsilon,\delta}(\phi) & \text{for all } (\varphi,\phi) \in L^1(\Omega_\delta) \times L^1(\Omega_\delta),\\
\displaystyle\int_{\Omega_\delta} \varphi \,\mathrm{d}x = \delta^2 \int_{(0,L)\times Y} \mathcal{T}_{\varepsilon,\delta}(\varphi)\,\mathrm{d}y\,\mathrm{d}x_3 & \text{for all } \varphi \in L^1(\Omega_\delta),\\
\displaystyle\int_{\Omega_\delta} |\varphi|^2 \mathrm{d}x = \delta^2 \int_{(0,L)\times Y} |\mathcal{T}_{\varepsilon,\delta}(\varphi)|^2\,\mathrm{d}y\,\mathrm{d}x_3 & \text{for all } \varphi \in L^2(\Omega_\delta).
\end{cases}$$

Remark 3.1 Let us observe that when dealing with the operator $\mathcal{T}_\varepsilon$, only the variable x_3 is unfolded (ω is just a set of parameters), when dealing with the operator Π_δ, only the variables x_1, x_2 are unfolded, while with the operator $\mathcal{T}_{\varepsilon,\delta}$ the three variables (x_1, x_2, x_3) are unfolded.

3.3 The periodically heterogeneous beam

In order to take into account the periodicity of the elasticity tensor $a^{\varepsilon,\delta}$ we assume that there exists a tensor $A \in L^\infty(Y)^{3\times3\times3\times3}$ (with the classical properties of symmetry, boundedness and coercivity) such that

$$a^{ijkl,\varepsilon,\delta}(x) = A^{ijkl}\Big(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \Big\{\frac{x_3}{\varepsilon}\Big\}\Big) \quad \text{a.e. in } \Omega_\delta.$$

This yields

$$\mathcal{T}_{\varepsilon,\delta}(a^{\varepsilon,\delta})(x_3, y) = A(y) \quad \text{for a.e. } (x_3, y) \in (0,L) \times Y. \tag{3.3}$$

The problem we have to solve now is to find the limit of the sequence $\{u^{\varepsilon,\delta}\}_{\varepsilon,\delta}$ whose elements are solution to the variational problem: Find $u^{\varepsilon,\delta} \in V(\Omega_\delta)$,

$$\int_{\Omega_\delta} a^{\varepsilon,\delta}\, e(u^{\varepsilon,\delta}) : e(v)\mathrm{d}x = \int_{\Omega_\delta} F_\delta \cdot v\,\mathrm{d}x, \quad \forall v \in V(\Omega_\delta).$$

As in (2.5), (2.11) we have the decomposition

$$\begin{cases}
u^{\varepsilon,\delta}(x) = U_e^{\varepsilon,\delta}(x) + \overline{u}^{\varepsilon,\delta}(x),\\
U_e^{\varepsilon,\delta}(x) = \begin{pmatrix}
\mathbb{U}_1^{\varepsilon,\delta}(x_3) + \underline{u}_1^{\varepsilon,\delta}(x_3) - x_2\Theta^{\varepsilon,\delta}(x_3)\\
\mathbb{U}_2^{\varepsilon,\delta}(x_3) + \underline{u}_2^{\varepsilon,\delta}(x_3) + x_1\Theta^{\varepsilon,\delta}(x_3)\\
\underline{u}_3^{\varepsilon,\delta}(x_3) - x_1\dfrac{\mathrm{d}\mathbb{U}_1^{\varepsilon,\delta}}{\mathrm{d}x_3}(x_3) - x_2\dfrac{\mathrm{d}\mathbb{U}_2^{\varepsilon,\delta}}{\mathrm{d}x_3}(x_3)
\end{pmatrix}.
\end{cases}$$

3.4 First convergences

Let us introduce the following vector spaces:

$$\mathbf{V_M} \doteq \Big\{(\mathbb{V}, \Psi, \underline{v}) \in [H^2(0,L)]^2 \times H^1(0,L) \times [H^1(0,L)]^3 \mid \mathbb{V}(0) = \frac{\mathrm{d}\mathbb{V}}{\mathrm{d}x_3}(0) = \Psi(0) = \underline{v}(0) = 0\Big\},$$

$$\overline{\mathbf{V}} \doteq \Big\{\overline{v} \in [L^2(0,L;H^1(\omega))]^3 \,\Big|\, \int_\omega \overline{v}_i(x_3,y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 = \int_\omega y_\alpha \overline{v}_3(x_3,y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 = 0,$$
$$\int_\omega \big(y_1\overline{v}_2(x_3,y_1,y_2) - y_2\overline{v}_1(x_3,y_1,y_2)\big)\mathrm{d}y_1\mathrm{d}y_2 = 0 \quad \text{for a.e. } x_3 \in (0,L)\Big\}.$$

For every $(\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M}$, we define the symmetric tensor E by

$$E(\mathbb{V},\Psi,\underline{v}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\Big(\frac{\mathrm{d}\underline{v}_1}{\mathrm{d}x_3} - y_2\frac{\mathrm{d}\Psi}{\mathrm{d}x_3}\Big) \\ 0 & 0 & \frac{1}{2}\Big(\frac{\mathrm{d}\underline{v}_2}{\mathrm{d}x_3} + y_1\frac{\mathrm{d}\Psi}{\mathrm{d}x_3}\Big) \\ \frac{1}{2}\Big(\frac{\mathrm{d}\underline{v}_1}{\mathrm{d}x_3} - y_2\frac{\mathrm{d}\Psi}{\mathrm{d}x_3}\Big) & \frac{1}{2}\Big(\frac{\mathrm{d}\underline{v}_2}{\mathrm{d}x_3} + y_1\frac{\mathrm{d}\Psi}{\mathrm{d}x_3}\Big) & \frac{\mathrm{d}\underline{v}_3}{\mathrm{d}x_3} - y_1\frac{\mathrm{d}^2\mathbb{V}_1}{\mathrm{d}x_3^2} - y_2\frac{\mathrm{d}^2\mathbb{V}_2}{\mathrm{d}x_3^2} \end{pmatrix}, \tag{3.4}$$

and for every $\overline{v} \in \overline{\mathbf{V}}$, we define the symmetric tensor $e_{y'}$ by

$$e_{y'}(\overline{v}) = \begin{pmatrix} e_{11,y'}(\overline{v}) & e_{12,y'}(\overline{v}) & \frac{1}{2}\frac{\partial\overline{v}_3}{\partial y_1} \\ e_{12,y'}(\overline{v}) & e_{22,y'}(\overline{v}) & \frac{1}{2}\frac{\partial\overline{v}_3}{\partial y_2} \\ \frac{1}{2}\frac{\partial\overline{v}_3}{\partial y_1} & \frac{1}{2}\frac{\partial\overline{v}_3}{\partial y_2} & 0 \end{pmatrix}, \tag{3.5}$$

where the subscript y' emphasizes the derivation with respect to variables (y_1, y_2) only.

Then we can state the first convergence result.

Lemma 3.2 *There exists a subsequence of (ε,δ), still denoted (ε,δ), and limit displacement fields*

$$(\mathbb{U}, \Theta, \underline{u}) \in \mathbf{V_M}, \quad \overline{u} \in \overline{\mathbf{V}} \tag{3.6}$$

such that

$$\begin{cases} \mathbb{U}_\alpha^{\varepsilon,\delta} \rightharpoonup \mathbb{U}_\alpha & \textit{weakly in } [H^2(0,L)]^2, \\ \frac{1}{\delta}\underline{u}^{\varepsilon,\delta} \rightharpoonup \underline{u} & \textit{weakly in } [H^1(0,L)]^3, \\ \Theta^{\varepsilon,\delta} \rightharpoonup \Theta & \textit{weakly in } H^1(0,L), \\ \frac{1}{\delta^2}\Pi_\delta(\overline{u}^{\varepsilon,\delta}) \rightharpoonup \overline{u} & \textit{weakly in } \overline{\mathbf{V}}, \\ \frac{1}{\delta}\frac{\partial\Pi_\delta(\overline{u}^{\varepsilon,\delta})}{\partial x_3} \rightharpoonup 0 & \textit{weakly in } [L^2((0,L)\times\omega)]^3. \end{cases} \tag{3.7}$$

Moreover, one has

$$\begin{cases} \Pi_\delta(u_\alpha^{\varepsilon,\delta}) \rightharpoonup \mathbb{U}_\alpha & \textit{weakly in } L^2(0,L;H^1(\omega)),\\ \dfrac{1}{\delta}\Pi_\delta(u_3^{\varepsilon,\delta}) \rightharpoonup \underline{u}_3 - y_1\dfrac{\mathrm{d}\mathbb{U}_1}{\mathrm{d}x_3} - y_2\dfrac{\mathrm{d}\mathbb{U}_2}{\mathrm{d}x_3} & \textit{weakly in } L^2(0,L;H^1(\omega)),\\ \dfrac{1}{\delta}\Pi_\delta(e(U_e^{\varepsilon,\delta})) \rightharpoonup E(\mathbb{U},\Theta,\underline{u}) & \textit{weakly in } [L^2(\Omega)]^{3\times 3},\\ \dfrac{1}{\delta}\Pi_\delta(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) & \textit{weakly in } [L^2(\Omega)]^{3\times 3}, \end{cases}$$

4 Expression of the Unfolded Strain Tensor $\mathcal{T}_{\varepsilon,\delta}(e(U_e^{\varepsilon,\delta}))$ and Convergence Result

Let

$$\widehat{\mathbf{V}}_{\mathbf{per}} \doteq \Big\{(\widehat{\mathbb{V}},\widehat{\Psi},\widehat{\underline{v}}) \in L^2(0,L;[H^2_{\rm per}(0,1)]^2\times H^1_{\rm per}(0,1)\times[H^1_{\rm per}(0,1)]^3)\Big| \int_0^1 \widehat{\mathbb{V}}(\cdot,y_3){\rm d}y_3=0,\quad \int_0^1 \widehat{\Psi}(\cdot,y_3){\rm d}y_3=0,\quad \int_0^1 \widehat{\underline{v}}(\cdot,y_3){\rm d}y_3=0\Big\},$$

and for every $(\widehat{\mathbb{V}},\widehat{\Psi},\widehat{\underline{v}})\in\widehat{\mathbf{V}}_{\mathbf{per}}$, we define the symmetric tensor $E_{y_3}(\widehat{\mathbb{V}},\widehat{\Psi},\widehat{\underline{v}})$ by

$$\begin{pmatrix} 0 & 0 & \dfrac12\Big(\dfrac{\partial\widehat{\underline{v}}_1}{\partial y_3}-y_2\dfrac{\partial\widehat{\Psi}}{\partial y_3}\Big)\\ 0 & 0 & \dfrac12\Big(\dfrac{\partial\widehat{\underline{v}}_2}{\partial y_3}+y_1\dfrac{\partial\widehat{\Psi}}{\partial y_3}\Big)\\ \dfrac12\Big(\dfrac{\partial\widehat{\underline{v}}_1}{\partial y_3}-y_2\dfrac{\partial\widehat{\Psi}}{\partial y_3}\Big) & \dfrac12\Big(\dfrac{\partial\widehat{\underline{v}}_2}{\partial y_3}+y_1\dfrac{\partial\widehat{\Psi}}{\partial y_3}\Big) & \dfrac{\partial\widehat{\underline{v}}_3}{\partial y_3}-y_1\dfrac{\partial^2\widehat{\mathbb{V}}_1}{\partial y_3^2}-y_2\dfrac{\partial^2\widehat{\mathbb{V}}_2}{\partial y_3^2}\end{pmatrix}. \tag{4.1}$$

In an obvious way, to the displacement field $U_e^{\varepsilon,\delta}$ we associate the new unknowns $(\mathbb{U}^{\varepsilon,\delta},\Theta^{\varepsilon,\delta},\underline{u}^{\varepsilon,\delta})$ and a straightforward computation yields the unfolded scaled symmetric tensor

$$\frac1\delta\mathcal{T}_{\varepsilon,\delta}(e(U_e^{\varepsilon,\delta}))=\frac12\left(\begin{array}{c|c|c} 0 & 0 & -y_2\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}\Theta^{\varepsilon,\delta}}{{\rm d}x_3}\Big)+\dfrac1\delta\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}\underline{u}_1^{\varepsilon,\delta}}{{\rm d}x_3}\Big)\\ * & 0 & +y_1\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}\Theta^{\varepsilon,\delta}}{{\rm d}x_3}\Big)+\dfrac1\delta\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}\underline{u}_2^{\varepsilon,\delta}}{{\rm d}x_3}\Big)\\ * & * & 2\Big[-y_1\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}^2\mathbb{U}_1^{\varepsilon,\delta}}{{\rm d}x_3^2}\Big)-y_2\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}^2\mathbb{U}_2^{\varepsilon,\delta}}{{\rm d}x_3^2}\Big)+\dfrac1\delta\mathcal{T}_\varepsilon\Big(\dfrac{{\rm d}\underline{u}_3^{\varepsilon,\delta}}{{\rm d}x_3}\Big)\Big]\end{array}\right)$$

and we get the following convergence result.

Lemma 4.1 *There exists a subsequence of (ε,δ), still denoted (ε,δ), the limit of Bernoulli-Navier displacement field $(\mathbb{U},\Theta,\underline{u})$ and the correctors $(\widehat{\mathbb{U}},\widehat{\Theta},\widehat{\underline{u}})$*

$$(\mathbb{U},\Theta,\underline{u})\in\mathbf{V_M},\quad (\widehat{\mathbb{U}},\widehat{\Theta},\widehat{\underline{u}})\in\widehat{\mathbf{V}}_{\mathbf{per}} \tag{4.2}$$

such that

$$\begin{cases}\mathcal{T}_{\varepsilon,\delta}(u_\alpha^{\varepsilon,\delta})\rightharpoonup\mathbb{U}_\alpha & \textit{weakly in } L^2(0,L;H^1(Y)),\\ \dfrac1\delta\mathcal{T}_{\varepsilon,\delta}(u_1^{\varepsilon,\delta}-\mathbb{U}_1^{\varepsilon,\delta})\rightharpoonup\underline{u}_1-y_2\Theta & \textit{weakly in } L^2(0,L;H^1(Y)),\\ \dfrac1\delta\mathcal{T}_{\varepsilon,\delta}(u_2^{\varepsilon,\delta}-\mathbb{U}_2^{\varepsilon,\delta})\rightharpoonup\underline{u}_2+y_1\Theta & \textit{weakly in } L^2(0,L;H^1(Y)),\\ \dfrac1\delta\mathcal{T}_{\varepsilon,\delta}(u_3^{\varepsilon,\delta})\rightharpoonup\underline{u}_3-y_1\dfrac{{\rm d}\mathbb{U}_1}{{\rm d}x_3}-y_2\dfrac{{\rm d}\mathbb{U}_2}{{\rm d}x_3} & \textit{weakly in } L^2(0,L;H^1(Y)),\\ \dfrac1\delta\mathcal{T}_{\varepsilon,\delta}(e(U_e^{\varepsilon,\delta}))\rightharpoonup E(\mathbb{U},\Theta,\underline{u})+E_{y_3}(\widehat{\mathbb{U}},\widehat{\Theta},\widehat{\underline{u}}) & \textit{weakly in } [L^2((0,L)\times Y)]^9,\end{cases} \tag{4.3}$$

where the symmetric strain tensors $E(\mathbb{U},\Theta,\underline{u})$, $E_{y_3}(\widehat{\mathbb{U}},\widehat{\Theta},\widehat{\underline{u}})$ are given by (3.4) *and* (4.1).

Proof Again, from the bounds (2.19), and now with the help of Theorem 3.2, we infer the existence of a subsequence of (ε,δ) and of limit corrector fields $(\widehat{\mathbb{U}},\widehat{\Theta},\widehat{\underline{u}})\in\widehat{\mathbf{V}}_{\mathbf{per}}$ such that the following weak convergences hold:

$$\begin{cases}\mathcal{T}_\varepsilon(\mathbb{U}^{\varepsilon,\delta})\rightharpoonup\mathbb{U} & \text{weakly in } [L^2 0,L;H^2(0,1))]^2,\\ \mathcal{T}_\varepsilon(\Theta^{\varepsilon,\delta})\rightharpoonup\Theta & \text{weakly in } L^2(0,L;H^1(0,1)),\\ \dfrac{1}{\delta}\mathcal{T}_\varepsilon(\underline{u}^{\varepsilon,\delta})\rightharpoonup\underline{u} & \text{weakly in } [L^2(0,L;H^1(0,1))]^3\end{cases}$$

and

$$\begin{cases}\mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}\mathbb{U}^{\varepsilon,\delta}}{\mathrm{d}x_3}\Big)\rightharpoonup\dfrac{\mathrm{d}\mathbb{U}}{\mathrm{d}x_3} & \text{weakly in } [L^2(0,L;H^1(0,1))]^2,\\ \mathcal{T}_\varepsilon\Big(\dfrac{d^2\mathbb{U}^{\varepsilon,\delta}}{\mathrm{d}x_3^2}\Big)\rightharpoonup\dfrac{\mathrm{d}^2\mathbb{U}}{\mathrm{d}x_3^2}+\dfrac{\partial^2\widehat{\mathbb{U}}}{\partial y_3^2} & \text{weakly in } [L^2((0,L)\times(0,1))]^2,\\ \mathcal{T}_\varepsilon\Big(\dfrac{d\Theta^{\varepsilon,\delta}}{\mathrm{d}x_3}\Big)\rightharpoonup\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3}+\dfrac{\partial\widehat{\Theta}}{\partial y_3} & \text{weakly in } L^2((0,L)\times(0,1)),\\ \dfrac{1}{\delta}\mathcal{T}_\varepsilon\Big(\dfrac{\mathrm{d}\underline{u}^{\varepsilon,\delta}}{\mathrm{d}x_3}\Big)\rightharpoonup\dfrac{\mathrm{d}\underline{u}}{\mathrm{d}x_3}+\dfrac{\partial\widehat{\underline{u}}}{\partial y_3} & \text{weakly in } [L^2((0,L)\times(0,1))]^3.\end{cases}$$

Hence, the convergences in (4.3) are obtained.

Remark 4.1 The limit $(\mathbb{U},\Theta,\underline{u})$ is the same as the one obtained in Lemma 3.2.

5 Expression of the Unfolded Strain Tensor $\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta}))$ and Convergence Results

This section is devoted to the contribution of the warping part of the displacement field $\overline{u}^{\varepsilon,\delta}$, for which we use the bounds given in (2.19),

$$\|\overline{u}^{\varepsilon,\delta}\|_{L^2(\Omega_\delta)}\le C\delta^3,\quad \|\nabla\overline{u}^{\varepsilon,\delta}\|_{L^2(\Omega_\delta)}\le C\delta^2. \tag{5.1}$$

Let us recall the chain rule which gives the transformation of the gradient for any vector-field $\varphi\in H^1(\Omega_\delta)$,

$$\mathcal{T}_{\varepsilon,\delta}\Big(\frac{\partial\varphi}{\partial x_\alpha}\Big)=\frac{1}{\delta}\,\frac{\partial}{\partial y_\alpha}(\mathcal{T}_{\varepsilon,\delta}(\varphi)),\quad \mathcal{T}_{\varepsilon,\delta}\Big(\frac{\partial\varphi}{\partial x_3}\Big)=\frac{1}{\varepsilon}\,\frac{\partial}{\partial y_3}(\mathcal{T}_{\varepsilon,\delta}(\varphi)). \tag{5.2}$$

Then, a straightforward computation yields

$$\begin{aligned}\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta}))&=\frac{1}{\delta^2}\begin{pmatrix} e_{11,y}(\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})) & e_{12,y}(\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})) & \dfrac{1}{2}\dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_3^{\varepsilon,\delta})}{\partial y_1}\\ e_{12,y}(\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})) & e_{22,y}(\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})) & \dfrac{1}{2}\dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_3^{\varepsilon,\delta})}{\partial y_2}\\ \dfrac{1}{2}\dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_3^{\varepsilon,\delta})}{\partial y_1} & \dfrac{1}{2}\dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_3^{\varepsilon,\delta})}{\partial y_2} & 0\end{pmatrix}\\ &\quad+\frac{1}{2\varepsilon\delta}\begin{pmatrix} 0 & 0 & \dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_1^{\varepsilon,\delta})}{\partial y_3}\\ 0 & 0 & \dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_2^{\varepsilon,\delta})}{\partial y_3}\\ \dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_1^{\varepsilon,\delta})}{\partial y_3} & \dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_2^{\varepsilon,\delta})}{\partial y_3} & 2\dfrac{\partial\mathcal{T}_{\varepsilon,\delta}(\overline{u}_3^{\varepsilon,\delta})}{\partial y_3}\end{pmatrix}.\end{aligned} \tag{5.3}$$

Because of the chain rule (5.2) and estimates (3.1) and (5.1), we deduce that there exists a constant $C > 0$ such that we have the bounds

$$\begin{cases} \Big\| \frac{1}{\delta^2} \mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta}) \Big\|_{[L^2((0,L)\times Y)]^3} = \Big\| \frac{1}{\delta^3} \overline{u}^{\varepsilon,\delta} \Big\|_{[L^2(\Omega_\delta)]^3} \leq C, \\ \Big\| \frac{1}{\delta^2} \frac{\partial \mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})}{\partial y_\alpha} \Big\|_{[L^2((0,L)\times Y)]^3} = \Big\| \frac{1}{\delta^2} \frac{\partial \overline{u}^{\varepsilon,\delta}}{\partial x_\alpha} \Big\|_{[L^2(\Omega_\delta)]^3} \leq C, \\ \Big\| \frac{1}{\delta^2} \frac{\partial \mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta})}{\partial y_3} \Big\|_{[L^2((0,L)\times Y)]^3} = \frac{\varepsilon}{\delta} \Big\| \frac{1}{\delta^2} \frac{\partial \overline{u}^{\varepsilon,\delta}}{\partial x_3} \Big\|_{[L^2(\Omega_\delta)]^3} \leq \frac{\varepsilon}{\delta} C. \end{cases} \tag{5.4}$$

Therefore, the limit of the unfolded strain tensor $\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta}))$ and, consequently that of the unfolded tensor $\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(u^{\varepsilon,\delta}))$ will depend upon the relationship between ε and δ. Let us denote $\theta = \lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon}$; there exist three possible cases which are

- $\theta = +\infty$,
- θ is finite,
- $\theta = 0$.

The remain of this section is devoted to the study of these three possibilities.

5.1 The case $\lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = +\infty$

In order to study the convergence of the unfolded strain tensor $\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta}))$, in the case $\theta = +\infty$, we introduce the vector space:

$$\begin{aligned} \widehat{\widehat{\mathbf{V}}}^{\infty} \doteq \Big\{ & v \in [L^2((0,L)\times\omega; H^1_{\text{per}}(0,1))]^3 \;\Big|\; \int_\omega v_i(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = \int_\omega y_\alpha v_3(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = 0, \\ & \int_\omega (y_1 v_2(x_3,y) - y_2 v_1(x_3,y))\mathrm{d}y_1\mathrm{d}y_2 = 0 \quad \text{for a.e. } (x_3,y_3) \in (0,L)\times(0,1), \\ & \int_0^1 v_i(x_3,y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3,y_1,y_2) \in \Omega \Big\}. \end{aligned}$$

For any $v \in [L^2(\omega; H^1(0,1))]^3$, let $e^\infty_{y_3}$ be the strain tensor defined by

$$e^\infty_{y_3}(v) = \begin{pmatrix} 0 & | & 0 & | & \frac{1}{2}\frac{\partial v_1}{\partial y_3} \\ & | & & | & \\ 0 & | & 0 & | & \frac{1}{2}\frac{\partial v_2}{\partial y_3} \\ & | & & | & \\ \frac{1}{2}\frac{\partial v_1}{\partial y_3} & | & \frac{1}{2}\frac{\partial v_2}{\partial y_3} & | & \frac{\partial v_3}{\partial y_3} \end{pmatrix}. \tag{5.5}$$

Lemma 5.1 *There exists $\widehat{\overline{u}}^{\infty} \in \widehat{\widehat{\mathbf{V}}}^{\infty}$ such that (up to a subsequence)*

$$\begin{cases} \frac{1}{\delta^2}\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta}) \rightharpoonup \overline{u} & \textit{weakly in } [L^2((0,L)\times\omega; H^1(0,1))]^3, \\ \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e^\infty_{y_3}(\widehat{\overline{u}}^{\infty}) & \textit{weakly in } [L^2((0,L)\times Y)]^{3\times 3}, \end{cases} \tag{5.6}$$

where $\overline{u}$ was introduced in Lemma 3.2, with $e_{y'}$ defined by (3.5) and where $e^\infty_{y_3}$ is given by (5.5).

Proof We still have the bounds as given in (2.19),

$$\|\overline{u}^{\varepsilon,\delta}\|_{[L^2(\Omega_\delta)]^3} \le C\delta^3, \quad \|\nabla\overline{u}^{\varepsilon,\delta}\|_{[L^2(\Omega_\delta)]^{3\times3}} \le C\delta^2$$

and the conditions given by (2.6). Therefore, on the one hand, recall $(3.7)_{4-5}$,

$$\begin{cases} \dfrac{1}{\delta^2}\Pi_\delta(\overline{u}^{\varepsilon,\delta}) \rightharpoonup \overline{u} & \text{weakly in } [L^2(0,L;H^1(\omega))]^3, \\ \dfrac{1}{\delta}\dfrac{\partial\Pi_\delta(\overline{u}^{\varepsilon,\delta})}{\partial x_3} \rightharpoonup 0 & \text{weakly in } [L^2(\Omega)]^3. \end{cases}$$

On the other hand, the sequence $\big\{w^\varepsilon = \frac{1}{\delta^2}\Pi_\delta(\overline{u}^{\varepsilon,\delta})\big\}_\varepsilon$ satisfies the assumption of Lemma 3.1 (second part):

$$\|w^\varepsilon\|_{L^2(\Omega)} = \frac{1}{\delta}\Big\|\frac{1}{\delta^2}\overline{u}^{\varepsilon,\delta}\Big\|_{L^2(\Omega_\delta)} \le C, \quad \delta\Big\|\frac{\partial w^\varepsilon}{\partial x_3}\Big\|_{L^2(\Omega)} = \Big\|\frac{1}{\delta^2}\frac{\partial\overline{u}^{\varepsilon,\delta}}{\partial x_3}\Big\|_{L^2(\Omega_\delta)} \le C.$$

Hence, there exists $\widehat{\overline{u}}^\infty \in \widehat{\overline{\mathbf{V}}}^\infty$ such that the weak convergences (5.6) hold.

5.2 The case $\lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = \theta \in]0,\infty[$

Denote

$$\begin{aligned} \widehat{\overline{\mathbf{V}}}^\theta \doteq \Big\{ & v \in [L^2(0,L;H^1_{\rm per}(Y))]^3 \;\Big|\; \int_\omega v_i(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = \int_\omega y_\alpha v_3(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = 0, \\ & \int_\omega \big(y_1v_2(x_3,y) - y_2v_1(x_3,y)\big)\mathrm{d}y_1\mathrm{d}y_2 = 0 \quad \text{for a.e. } (x_3,y_3)\in(0,L)\times(0,1), \\ & \int_0^1 v_i(x_3,y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3,y_1,y_2)\in\Omega\Big\}. \end{aligned}$$

For any $v \in [H^1(\omega\times Y)]^3$, let e^θ_y be the strain tensor defined by

$$e^\theta_y(v) = \begin{pmatrix} \dfrac{\partial v_1}{\partial y_1} & \Big| & \dfrac{1}{2}\Big(\dfrac{\partial v_1}{\partial y_2} + \dfrac{\partial v_2}{\partial y_1}\Big) & \Big| & \dfrac{1}{2}\Big(\dfrac{\partial v_3}{\partial y_1} + \theta\dfrac{\partial v_1}{\partial y_3}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial v_1}{\partial y_2} + \dfrac{\partial v_2}{\partial y_1}\Big) & \Big| & \dfrac{\partial v_2}{\partial y_2} & \Big| & \dfrac{1}{2}\Big(\dfrac{\partial v_3}{\partial y_2} + \theta\dfrac{\partial v_2}{\partial y_3}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial v_3}{\partial y_1} + \theta\dfrac{\partial v_1}{\partial y_3}\Big) & \Big| & \dfrac{1}{2}\Big(\dfrac{\partial v_3}{\partial y_2} + \theta\dfrac{\partial v_2}{\partial y_3}\Big) & \Big| & \theta\,\dfrac{\partial v_3}{\partial y_3} \end{pmatrix}. \tag{5.7}$$

Lemma 5.2 *There exists $\widehat{\overline{u}}^\theta \in \widehat{\overline{\mathbf{V}}}^\theta$ such that (up to a subsequence)*

$$\begin{cases} \dfrac{1}{\delta^2}\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta}) \rightharpoonup \overline{u} + \widehat{\overline{u}}^\theta & \textit{weakly in } [L^2(0,L;H^1(Y))]^3, \\ \dfrac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e^\theta_y(\widehat{\overline{u}}^\theta) & \textit{weakly in } [L^2((0,L)\times Y)]^{3\times3}, \end{cases} \tag{5.8}$$

where $\overline{u}$ is introduced in Lemma 3.2 *with $e_{y'}$ defined by* (3.5) *and where e^θ_y is given by* (5.7).

Proof Proceeding as in Lemma 5.1, we introduce the sequence $\{w^\varepsilon = \frac{1}{\delta^2}\Pi_\delta(\overline{u}^{\varepsilon,\delta})\}_\varepsilon$. From (5.4), the sequence $\mathcal{T}_\varepsilon(w_\varepsilon)_\varepsilon$ is uniformly bounded in $[L^2(0,L;H^1(Y))]^3$, satisfing the assumption of Theorem 3.1. Hence, there exists $\widehat{\overline{u}}^\theta \in \widehat{\overline{\mathbf{V}}}^\theta$ such that the weak convergences (5.8) hold.

5.3 The case $\lim_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = 0$

Denote

$$\widehat{\widehat{\mathbf{V}}}^0 \doteq \Big\{ v \in [L^2((0,L)\times(0,1);H^1(\omega))]^3 \;\Big|\; \int_\omega v_i(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = \int_\omega y_\alpha v_3(x_3,y)\mathrm{d}y_1\mathrm{d}y_2 = 0,$$
$$\int_\omega \big(y_1 v_2(x_3,y) - y_2 v_1(x_3,y)\big)\mathrm{d}y_1\mathrm{d}y_2 = 0 \quad \text{for a.e. } (x_3,y_3)\in(0,L)\times(0,1),$$
$$\int_0^1 v_i(x_3,y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3,y_1,y_2)\in\Omega \Big\}.$$

For every $v \in [L^2(0,1;H^1(\omega))]^3$ we define $e^0_{y'}$ by

$$e^0_{y'}(v) = \begin{pmatrix} e_{11,y'}(v) & | & e_{12,y'}(v) & | & \frac{1}{2}\frac{\partial v_3}{\partial y_1} \\ & | & & | & \\ e_{12,y'}(v) & | & e_{22,y'}(v) & | & \frac{1}{2}\frac{\partial v_3}{\partial y_2} \\ & | & & | & \\ \frac{1}{2}\frac{\partial v_3}{\partial y_1} & | & \frac{1}{2}\frac{\partial v_3}{\partial y_2} & | & 0 \end{pmatrix}. \tag{5.9}$$

Lemma 5.3 *There exists* $\widehat{\widehat{u}}^0 \in \widehat{\widehat{\mathbf{V}}}^0$ *such that (up to a subsequence)*

$$\begin{cases} \frac{1}{\delta^2}\mathcal{T}_{\varepsilon,\delta}(\overline{u}^{\varepsilon,\delta}) \rightharpoonup \overline{u} + \widehat{\widehat{u}}^0 & \text{weakly in } [L^2((0,L)\times(0,1);H^1(\omega))]^3, \\ \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\overline{u}^{\varepsilon,\delta})) \rightharpoonup e_{y'}(\overline{u}) + e^0_{y'}(\widehat{\widehat{u}}^0) & \text{weakly in } [L^2((0,L)\times Y)]^{3\times 3}, \end{cases}$$

where $\overline{u}$ *is introduced in Lemma* 3.2 *with* $e_{y'}$ *defined by* (3.5) *and where* $e^0_{y'}$ *is given by* (5.9).

Proof We proceed as in the two previous Lemmas 5.1–5.2, but here using part 1 of Lemma 3.1.

The following section is devoted to the study of the whole limit field u according to the different values of θ.

6 The Limit Problems

Let us make more precise the framework in which the convergences and, consequently, the limit problems are obtained.

(1) First, based on the decomposition

$$E(\mathbb{U},\Theta,\underline{u}) = \frac{\mathrm{d}^2\mathbb{U}_1}{\mathrm{d}x_3^2}\begin{pmatrix} 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & -y_1 \end{pmatrix} + \frac{\mathrm{d}^2\mathbb{U}_2}{\mathrm{d}x_3^2}\begin{pmatrix} 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & -y_2 \end{pmatrix}$$
$$+ \frac{1}{2}\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\begin{pmatrix} 0 & | & 0 & | & -y_2 \\ 0 & | & 0 & | & y_1 \\ -y_2 & | & y_1 & | & 0 \end{pmatrix} + \frac{1}{2}\frac{\mathrm{d}\underline{u}_1}{\mathrm{d}x_3}\begin{pmatrix} 0 & | & 0 & | & 1 \\ 0 & | & 0 & | & 0 \\ 1 & | & 0 & | & 0 \end{pmatrix}$$
$$+ \frac{1}{2}\frac{\mathrm{d}\underline{u}_2}{\mathrm{d}x_3}\begin{pmatrix} 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & 1 \\ 0 & | & 1 & | & 0 \end{pmatrix} + \frac{\mathrm{d}\underline{u}_3}{\mathrm{d}x_3}\begin{pmatrix} 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & 0 \\ 0 & | & 0 & | & 1 \end{pmatrix},$$

we can represent this strain tensor in a more compact form (recall that $Y = \omega \times (0,1)$)

$$E(\mathbb{U}, \Theta, \underline{u})(x_3, y) = g_i(x_3)\mathcal{A}_i(y) \quad \text{for a.e. } (x_3, y) \in (0, L) \times Y \tag{6.1}$$

with an obvious definition of the matrices $\mathcal{A}_i$ and unknowns $g_i, i = 1, \cdots, 6$, formed with the unknowns $(\mathbb{U}'', \Theta', \underline{u}')$.

(2) Under the assumptions on the forces given in Subsection 2.4, $(f_i, f_T, g_\alpha, \underline{f}_i) \in [L^2(0, L)]^9$ and for all displacement field $(\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M}$ we have introduced the potential energy

$$\int_0^L F(\mathbb{V}, \Psi, \underline{v})\mathrm{d}x_3$$
$$= |\omega| \int_0^L \Big(f_\alpha \mathbb{V}_\alpha + \Big(\frac{I_1 + I_2}{|\omega|}\Big) f_T \Psi - \Big(\frac{I_1}{|\omega|} g_1 \mathbb{V}_1' + \frac{I_2}{|\omega|} g_2 \mathbb{V}_2'\Big) + \underline{f}_\alpha \underline{v}_\alpha + f_3 \underline{v}_3 \Big)\mathrm{d}x_3. \tag{6.2}$$

In the sequel we will also make use of another set of unknowns, namely, for the displacement field $\boldsymbol{\mathcal{V}} = (\mathbb{V}', \Psi, \underline{v})$, with $(\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M}$ we will associate the equivalent formulation of the potential energy

$$\int_0^L |\omega| \boldsymbol{\mathcal{F}} \cdot \boldsymbol{\mathcal{V}} \, \mathrm{d}x_3 \tag{6.3}$$

with

$$\boldsymbol{\mathcal{F}}^{\mathrm{T}} = \Big(\Big(\int_{x_3}^L f_1(t)\mathrm{d}t - \frac{I_1}{|\omega|} g_1 \Big),\ \Big(\int_{x_3}^L f_2(t)\mathrm{d}t - \frac{I_2}{|\omega|} g_2 \Big),\ \frac{I_1 + I_2}{|\omega|} f_T,\ \underline{f}_1,\ \underline{f}_2,\ \underline{f}_3 \Big). \tag{6.4}$$

Even though the treatment of the three different values of θ is quite similar, we decided to present them in three self-contained sub-sections.

6.1 The case $\lim_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = +\infty$

To any $((\widehat{\mathbb{V}}, \widehat{\Psi}, \widehat{\underline{v}}), \widehat{\overline{v}}) \in \widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\overline{\mathbf{V}}}^{\infty}$ we associate the corrector field

$$\widehat{V} = \begin{pmatrix} \widehat{\mathbb{V}}_1 + \widehat{\underline{v}}_1 - y_2 \widehat{\Psi} \\ \widehat{\mathbb{V}}_2 + \widehat{\underline{v}}_2 + y_1 \widehat{\Psi} \\ \widehat{\underline{v}}_3 - y_1 \dfrac{\partial \widehat{\mathbb{V}}_1}{\partial y_3} - y_2 \dfrac{\partial \widehat{\mathbb{V}}_2}{\partial y_3} \end{pmatrix} + \widehat{\overline{v}} \tag{6.5}$$

belonging to

$$\widehat{\mathbf{V}} \doteq \Big\{ \widehat{V} \in [L^2((0, L); H^1_{\mathrm{per}}(Y))]^3 \ \Big| \ \int_0^1 \widehat{V}(x_3, y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3, y_1, y_2) \in \Omega \Big\}.$$

The spaces $\widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\overline{\mathbf{V}}}^{\infty}$ (endowed with their usual norms) and $\widehat{\mathbf{V}}$ are isometric. To any $\widehat{V} \in \widehat{\mathbf{V}}$ we associate the symmetric tensor (see (4.1) and (5.5))

$$E^{\infty}_{y_3}(\widehat{V}) = \begin{pmatrix} 0 & 0 & \dfrac{1}{2}\dfrac{\partial \widehat{V}_1}{\partial y_3} \\ 0 & 0 & \dfrac{1}{2}\dfrac{\partial \widehat{V}_2}{\partial y_3} \\ \dfrac{1}{2}\dfrac{\partial \widehat{V}_1}{\partial y_3} & \dfrac{1}{2}\dfrac{\partial \widehat{V}_2}{\partial y_3} & \dfrac{\partial \widehat{V}_3}{\partial y_3} \end{pmatrix}. \tag{6.6}$$

By collecting the convergence results given by Lemmas 4.1 and 5.1 we can state the following result.

Corollary 6.1 *There exists a subsequence of* (ε,δ)*, still denoted* (ε,δ)*, limit displacement fields and correctors* $(\mathbb{U},\Theta,\underline{u})\in\mathbf{V_M}$*,* $\overline{u}\in\overline{\mathbf{V}}$ *and* $\widehat{U}\in\widehat{\mathbf{V}}$ *such that*

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(u^{\varepsilon,\delta}))\rightharpoonup E(\mathbb{U},\Theta,\underline{u})+e_{y'}(\overline{u})+E^{\infty}_{y_3}(\widehat{U})\quad \text{weakly in } [L^2((0,L)\times Y)]^{3\times3},\tag{6.7}$$

where the strain tensors are given by (3.4)–(3.5) *and* (6.6)*.*

We endow $\mathbf{V_M}\times\overline{\mathbf{V}}\times\widehat{\mathbf{V}}$ with the following norm:

$$\begin{aligned}&\|((\mathbb{V},\Psi,\underline{v}),\overline{v},\widehat{V})\|\\&=\sqrt{\int_{(0,L)\times Y}(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v})+E^{\infty}_{y_3}(\widehat{V})):(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v})+E^{\infty}_{y_3}(\widehat{V}))\mathrm{d}x_3\mathrm{d}y}.\end{aligned}$$

Lemma 6.1 *On the space* $\mathbf{V_M}\times\overline{\mathbf{V}}\times\widehat{\mathbf{V}}$*, the norm* $\|\cdot\|$ *is equivalent to the usual norm of this product spaces.*

Proof Due to the fact that $\int_0^1\widehat{V}(x_3,y)\mathrm{d}y_3=0$ for a.e. $(x_3,y_1,y_2)\in\Omega$, we first have

$$\begin{aligned}\|((\mathbb{V},\Psi,\underline{v}),\overline{v},\widehat{V})\|^2=&\int_\Omega(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v})):(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\\&+\int_{(0,L)\times Y}E^{\infty}_{y_3}(\widehat{V}):E^{\infty}_{y_3}(\widehat{V})\big)\mathrm{d}x_3\mathrm{d}y.\end{aligned}$$

The 3D Korn's inequality, the periodicity of $\widehat{V}$ and its mean value property imply the existence of a constant $C>0$ such that

$$\|\widehat{V}\|^2_{[L^2(0,L;H^1(Y))]^3}\le C\int_{(0,L)\times Y}E^{\infty}_{y_3}(\widehat{V}):E^{\infty}_{y_3}(\widehat{V})\mathrm{d}x_3\mathrm{d}y.$$

Then we easily prove (see [7])

$$\begin{aligned}&\|\mathbb{V}\|^2_{[H^2(0,L)]^2}+\|\underline{v}\|^2_{[H^1(0,L)]^3}+\|\Psi\|_{H^1(0,L)}+\|\overline{v}\|^2_{[L^2((0,L)\times\omega)]^3}\\&\le\int_\Omega(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v})):(E(\mathbb{V},\Theta,\underline{v})+e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

The existence and uniqueness of the limit fields and correctors $(\mathbb{U},\Theta,\underline{u},\overline{u})$ and $\widehat{U}$ are given in the following theorem.

Theorem 6.1 *The limit fields* $(\mathbb{U},\Theta,\underline{u})\in\mathbf{V_M}$*,* $\overline{u}\in\overline{\mathbf{V}}$*,* $\widehat{U}\in\widehat{\mathbf{V}}$ *solve the coupled variational problems:*

$$\begin{aligned}&\int_{(0,L)\times Y}A(y)(E(\mathbb{U},\Theta,\underline{u})+e_{y'}(\overline{u})+E^{\infty}_{y_3}(\widehat{U})):(E(\mathbb{V},\Psi,\underline{v})+e_{y'}(\overline{v})+E^{\infty}_{y_3}(\widehat{V}))\mathrm{d}x_3\mathrm{d}y\\&=\int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3,\quad\forall(\mathbb{V},\Psi,\underline{v})\in\mathbf{V_M},\ \overline{v}\in\overline{\mathbf{V}},\ \widehat{V}\in\widehat{\mathbf{V}},\end{aligned}\tag{6.8}$$

where the potential energy is given by (6.2)*.*

Proof Based on the decomposition (2.11) we choose $(\mathbb{V},\Psi,\underline{v})\in\mathbf{V_M}$, $\overline{v}\in[\mathbb{C}^1(\overline{\Omega})]^3\cap\overline{\mathbf{V}}$ vanishing on $\Gamma_{0,\delta}$ and we select the test-function

$$v^{\varepsilon\delta}(x)=\begin{pmatrix}\mathbb{V}_1(x_3)+\underline{v}_1(x_3)-x_2\Psi(x_3)\\ \mathbb{V}_2(x_3)+\underline{v}_2(x_3)+x_1\Psi(x_3)\\ \underline{v}_3(x_3)-x_1\dfrac{\mathrm{d}\mathbb{V}_1}{\mathrm{d}x_3}-x_2\dfrac{\mathrm{d}\mathbb{V}_2}{\mathrm{d}x_3}\end{pmatrix}+\delta^2\overline{v}\Big(x_3,\frac{x_1}{\delta},\frac{x_2}{\delta}\Big).$$

A straightforward calculation leads to the convergences

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(v^{\varepsilon,\delta}))\to E(\mathbb{V},\Psi,\underline{v})+e_{y'}(\overline{v})\quad\text{strongly in }[L^2((0,L)\times Y)]^{3\times 3},$$
$$\frac{1}{\delta^2}\int_{\Omega_\delta}F_\delta\cdot v^{\varepsilon,\delta}\,\mathrm{d}x\to\int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3.$$

Now, dividing (2.17) by δ^2 then transforming with $\mathcal{T}_{\varepsilon,\delta}$ (the left hand-side) leads to

$$\int_{(0,L)\times Y}\mathcal{T}_{\varepsilon,\delta}(a^{\varepsilon,\delta})\ \frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(u^{\varepsilon,\delta})):\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(v^{\varepsilon,\delta}))\mathrm{d}x_3\mathrm{d}y=\frac{1}{\delta^2}\int_{\Omega_\delta}F_\delta\cdot v^{\varepsilon,\delta}\mathrm{d}x,$$

and by passing to the limit, we get

$$\int_{(0,L)\times Y}A(y)(E(\mathbb{U},\Theta,\underline{u})+e_{y'}(\overline{u})+E^\infty_{y_3}(\widehat{U})):(E(\mathbb{V},\Psi,\underline{v})+e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y$$
$$=\int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3.$$

Let us introduce the correctors $\widehat{V}\in[\mathcal{C}^\infty_c(0,L;H^1_{\rm per}(Y))]^6\cap\widehat{\mathbf{V}}$ and consider the second test-functions $\widehat{v}^{\varepsilon,\delta}$ as

$$\widehat{v}^{\varepsilon,\delta}(x)=\delta^2\widehat{V}\Big(x_3,\frac{x_1}{\delta},\frac{x_2}{\delta},\Big\{\frac{x_3}{\varepsilon}\Big\}\Big)\quad\text{for a.e. }x\in\Omega_\delta.$$

The simple computation $\frac{1}{\delta}\frac{\partial\widehat{v}^{\varepsilon,\delta}}{\partial x_\alpha}=\frac{\partial\widehat{V}}{\partial y_\alpha}$ and $\frac{1}{\delta}\frac{\partial\widehat{v}^{\varepsilon,\delta}}{\partial x_3}=\delta\frac{\partial\widehat{V}}{\partial x_3}+\frac{\delta}{\varepsilon}\frac{\partial\widehat{V}}{\partial y_3}$ yields the strong convergence

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(\widehat{v}^{\varepsilon,\delta}))\to E^\infty_{y_3}(\widehat{V})\quad\text{strongly in }[L^2((0,L);H^1(Y))]^{3\times 3}$$

which, since $\int_{\Omega_\delta}F_\delta\cdot\widehat{v}^{\varepsilon,\delta}\mathrm{d}x\to 0$, achieves establishing (6.8).

Set $\mathcal{C}^\infty$ the 6×6 symmetric matrix defined by its elements

$$\mathcal{C}^\infty_{mp}=\int_Y A(y)\mathcal{B}^\infty_m(y):\mathcal{A}_p(y)\mathrm{d}y=\int_Y A(y)\mathcal{B}^\infty_m(y):\mathcal{B}^\infty_p(y)\mathrm{d}y,\tag{6.9}$$

where the second order tensors $\mathcal{B}^\infty_m$ are given in (6.13).

Theorem 6.2 *Under the assumptions on the forces given in Section* 2.4, $(f_i,f_T,g_\alpha,\underline{f}_i)\in[L^2(0,L)]^9$, *the limit displacement field* $(\mathbb{U},\Theta,\underline{u})\in\mathbf{V_M}$ *is the solution of the variational problem*

$$\int_0^L\mathcal{C}^\infty\boldsymbol{\mathcal{U}}'\cdot\boldsymbol{\mathcal{V}}'\mathrm{d}x_3=\int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3,\quad\forall(\mathbb{V},\Psi,\underline{v})\in\mathbf{V_M},\tag{6.10}$$

where the symmetric matrix $\mathcal{C}^\infty$ *is given in* (6.9) *and where*

$$\boldsymbol{\mathcal{U}}=(\mathbb{U}_1'\mid\mathbb{U}_2'\mid\Theta\mid\underline{u}_1\mid\underline{u}_2\mid\underline{u}_3)^{\rm T},\quad\boldsymbol{\mathcal{V}}=(\mathbb{V}_1'\mid\mathbb{V}_2'\mid\Psi\mid\underline{v}_1\mid\underline{v}_2\mid\underline{v}_3)^{\rm T}.$$

Proof For all $\overline{v} \in \overline{\mathbf{V}}$, $\widehat{V} \in \widehat{\mathbf{V}}$, set $\widetilde{v} = \overline{v} + \widehat{V}$ and denote

$$G_y^\infty(\widetilde{v}) = e_{y'}(\overline{v}) + E_y^\infty(\widehat{V}).$$

We rewrite (6.8) with $(\mathbb{V}, \Psi, \underline{v}) = 0$ that as:

Find $\widetilde{v} \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$ such that for all $\widetilde{v} \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$,

$$\int_Y AG_y^\infty(\widetilde{u}) : G_y^\infty(\widetilde{v})\mathrm{d}y = -\int_Y AE(\mathbb{U}, \Theta, \underline{u}) : G_y^\infty(\widetilde{v})\mathrm{d}y. \tag{6.11}$$

We consider the decomposition of the strain tensor E as given by (6.1), and we introduce the 6 auxiliary fields $\widetilde{\xi_i^\infty} \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$, $i = 1, \cdots, 6$, as the unique solution to the following six independent variational problems:

$$\int_Y AG_y^\infty(\widetilde{\xi_i^\infty}) : G_y^\infty(\widetilde{v})\mathrm{d}y = -\int_Y A\,\mathcal{A}_i : G_y^\infty(\widetilde{v})\mathrm{d}y. \tag{6.12}$$

The solution of problem (6.11) is under the form $\widetilde{u}(x_3, y) = g_i(x_3)\ \widetilde{\xi_i^\infty}(y)$.

Hence $e_{y'}(\overline{u}) + E_y^\infty(\widehat{U}) = G_y^\infty(\widetilde{u}) = g_i G_y^\infty(\widetilde{\xi_i^\infty})$. Let us return to the limit homogeneous problem (6.8) with $\widehat{\overline{v}} = 0$, $\widehat{V} = 0$:

$$\int_{(0,L)\times Y} A(g_m\mathcal{A}_m + g_i G_y^\infty(\widetilde{\xi_i^\infty})) : h_j\mathcal{A}_j \mathrm{d}x_3\mathrm{d}y = \int_0^L F(\mathbb{V}, \Psi, \underline{v})\mathrm{d}x_3, \quad \forall(\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M},$$

where h is formed with the unknowns $(\mathbb{V}'', \Psi', \underline{v}')$. Letting

$$\mathcal{B}_i^\infty = \mathcal{A}_i + G_y^\infty(\widetilde{\xi_i^\infty}), \quad i = 1, \cdots, 6, \tag{6.13}$$

we get

$$\int_0^L g_i(x_3)h_j(x_3)\Big(\int_Y A(y)\mathcal{B}_i^\infty(y) : \mathcal{A}_j(y)\mathrm{d}y\Big)\mathrm{d}x_3 = \int_0^L F(\mathbb{V}, \Psi, \underline{v})\mathrm{d}x_3.$$

6.2 The case $\lim\limits_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = \theta \in]0, \infty[$

To any $((\widehat{\mathbb{V}}, \widehat{\Psi}, \widehat{\underline{v}}), \widehat{\overline{v}}) \in \widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\overline{\mathbf{V}}}^\theta$ we associate the corrector field

$$\widehat{V} = \frac{1}{\theta}\begin{pmatrix} \frac{1}{\theta}\widehat{\mathbb{V}}_1 + \widehat{\underline{v}}_1 - y_2\widehat{\Psi} \\ \frac{1}{\theta}\widehat{\mathbb{V}}_2 + \widehat{\underline{v}}_2 + y_1\widehat{\Psi} \\ \widehat{\underline{v}}_3 - y_1\dfrac{\partial\widehat{\mathbb{V}}_1}{\partial y_3} - y_2\dfrac{\partial\widehat{\mathbb{V}}_2}{\partial y_3} \end{pmatrix} + \widehat{\overline{v}} \tag{6.14}$$

belonging to

$$\widehat{\mathbf{V}} \doteq \Big\{\widehat{V} \in [L^2((0,L); H^1_{\mathrm{per}}(Y))]^3 \ \Big|\ \int_0^1 \widehat{V}(x_3, y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3, y_1, y_2) \in \Omega\Big\}.$$

The spaces $\widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\widehat{\mathbf{V}}}^{\theta}$ and $\widehat{\mathbf{V}}$ (endowed with their usual norms) are isometric. To any $\widehat{V} \in \widehat{\mathbf{V}}$ we associate the symmetric tensor (see (4.1) and (5.7)),

$$E^{\theta}_y(\widehat{V}) = \begin{pmatrix} e_{11,y'}(\widehat{V}) & e_{12,y'}(\widehat{V}) & \frac{1}{2}\Big(\theta\frac{\partial \widehat{V}_1}{\partial y_3} + \frac{\partial \widehat{V}_3}{\partial y_1}\Big) \\ e_{12,y'}(\widehat{V}) & e_{22,y'}(\widehat{V}) & \frac{1}{2}\Big(\theta\frac{\partial \widehat{V}_2}{\partial y_3} + \frac{\partial \widehat{V}_3}{\partial y_2}\Big) \\ \frac{1}{2}\Big(\theta\frac{\partial \widehat{V}_1}{\partial y_3} + \frac{\partial \widehat{V}_3}{\partial y_1}\Big) & \frac{1}{2}\Big(\theta\frac{\partial \widehat{V}_2}{\partial y_3} + \frac{\partial \widehat{V}_3}{\partial y_2}\Big) & \theta\frac{\partial \widehat{V}_3}{\partial y_3} \end{pmatrix}. \tag{6.15}$$

By collecting the convergence results given by Lemmas 4.1 and 5.2 we can state the following result.

Corollary 6.2 *There exists a subsequence of* (ε, δ), *still denoted* (ε, δ), *limit displacement fields and correctors* $(\mathbb{U}, \Theta, \underline{u}) \in \mathbf{V_M}$, $\overline{u} \in \overline{\mathbf{V}}$ *and* $\widehat{U} \in \widehat{\mathbf{V}}$ *such that*

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(u^{\varepsilon,\delta})) \rightharpoonup E(\mathbb{U}, \Theta, \underline{u}) + e_{y'}(\overline{u}) + E^{\theta}_y(\widehat{U}) \quad \text{weakly in } [L^2((0, L) \times Y)]^{3\times 3}, \tag{6.16}$$

where the limit of the symmetric strain tensor is obtained using (3.4)–(3.5) *and* (6.15).

We endow $\mathbf{V_M} \times \overline{\mathbf{V}} \times \widehat{\mathbf{V}}$ with the following norm:

$$\begin{aligned} &\|((\mathbb{V}, \Psi, \underline{v}), \overline{v}, \widehat{V})\| \\ &= \sqrt{\int_{(0,L)\times Y} (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}) + E^{\theta}_y(\widehat{V})) : (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}) + E^{\theta}_y(\widehat{V}))\mathrm{d}x_3\mathrm{d}y}\,. \end{aligned}$$

Lemma 6.2 *On the space* $\mathbf{V_M} \times \overline{\mathbf{V}} \times \widehat{\mathbf{V}}$, *the norm* $\|\cdot\|$ *is equivalent to the usual norm of this product spaces.*

Proof Due to the fact that $\int_0^1 \widehat{V}(x_3, y)\mathrm{d}y_3 = 0$ for a.e. $(x_3, y_1, y_2) \in \Omega$, we first have

$$\begin{aligned} \|((\mathbb{V}, \Psi, \underline{v}), \overline{v}, \widehat{V})\|^2 = &\int_{\Omega} (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v})) : (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{(0,L)\times Y} E^{\theta}_y(\widehat{V}) : E^{\theta}_y(\widehat{V})\big)\mathrm{d}x_3\mathrm{d}y. \end{aligned}$$

Now, consider the change of variable $y = (y_1, y_2, y_3) \in Y \Leftrightarrow z = (y_1, y_2, \theta y_3) \in Y^{\theta} = \omega \times (0, \theta)$ which transforms the scaled beam of section ω and length 1 to a beam of section ω and length θ. We define the new function $\widehat{V}^{\theta} \in [L^2((0, L); \boldsymbol{H}^1_{\mathrm{per}}(Y^{\theta}))]^3$ associated to $\widehat{V}$ given by (6.14),

$$\widehat{V}^{\theta}(z) = \widehat{V}\Big(z_1, z_2, \frac{z_3}{\theta}\Big), \quad \forall z \in Y^{\theta}. \tag{6.17}$$

One has

$$E_z(\widehat{V}^{\theta})(z) = (e_{ij,z}(\widehat{V}^{\theta})(z)) = E^{\theta}_y(\widehat{V})\Big(z_1, z_2, \frac{z_3}{\theta}\Big).$$

The 3D Korn's inequality, the periodicity of $\widehat{V}$ and the mean value property of $\widehat{V}$ imply the existence of a constant C (which depends on θ) such that

$$\|\widehat{V}^\theta\|^2_{[L^2(0,L;H^1(Y^\theta))]^3} \leq C \int_{(0,L)\times Y^\theta} E_z(\widehat{V}^\theta) : E_z(\widehat{V}^\theta)\big)\mathrm{d}x_3\mathrm{d}z.$$

Hence, there exists a constant C (which depends on θ) such that

$$\|\widehat{V}\|^2_{[L^2(0,L;H^1(Y))]^3} \leq C \int_{(0,L)\times Y} E^\theta_y(\widehat{V}) : E^\theta_y(\widehat{V})\mathrm{d}x_3\mathrm{d}y. \tag{6.18}$$

We easily prove (see [7]),

$$\begin{aligned}&\|\mathbb{V}\|^2_{[H^2(0,L)]^2} + \|\underline{v}\|^2_{[H^1(0,L)]^3} + \|\Psi\|_{H^1(0,L)} + \|\overline{v}\|^2_{[L^2((0,L)\times\omega)]^3}\\ \leq &\int_\Omega (E(\mathbb{V},\Theta,\underline{v}) + e_{y'}(\overline{v})) : (E(\mathbb{V},\Theta,\underline{v}) + e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

The existence and uniqueness of the limit displacement fields and correctors $(\mathbb{U},\Theta,\underline{u},\overline{u})$ and $\widehat{U}$ are given in the following theorem whose proof is obtained as in the previous section.

Theorem 6.3 *The limit fields* $(\mathbb{U},\Theta,\underline{u}) \in \mathbf{V_M}$, $\overline{u} \in \overline{\mathbf{V}}$, $\widehat{U} \in \widehat{\mathbf{V}}$ *solve the coupled variational problems:*

$$\begin{aligned}&\int_{(0,L)\times Y} A(E(\mathbb{U},\quad,\underline{u}) + e_{y'}(\overline{u}) + E^\theta_y(\widehat{U}))\quad(E(\mathbb{V},\quad,\underline{v}) + e_{y'}(\overline{v}) + E^\theta_y(\widehat{V}))\mathrm{d}x_3\mathrm{d}y\\ &\int_0^L F(\mathbb{V},\quad,\underline{v})\mathrm{d}x_3,\quad \forall(\mathbb{V},\quad,\underline{v}) \in \mathbf{V_M},\ \overline{v} \in \overline{\mathbf{V}},\ \widehat{V} \in \widehat{\mathbf{V}},\end{aligned} \qquad (\quad)$$

where the potential energy is given in (6.2).

Then we are in the position to state the main result in the case $\theta \in]0,\infty[$.

Theorem 6.4 *Under the assumptions on the forces given in Section* 2.4, $(f_i, f_T, g_\alpha, \underline{f}_i) \in [L^2(0,L)]^9$, *the limit displacement field* $(\mathbb{U},\Theta,\underline{u}) \in \mathbf{V_M}$ *is the solution of the variational problem*

$$\int_0^L \mathcal{C}^\theta \boldsymbol{\mathcal{U}}' \cdot \boldsymbol{\mathcal{V}}'\mathrm{d}x_3 = \int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3,\quad \forall(\mathbb{V},\Psi,\underline{v}) \in \mathbf{V_M} \tag{6.20}$$

and the 6×6 *symmetric matrix* $\mathcal{C}^\theta$ *is given in* (6.22) *and where*

$$\boldsymbol{\mathcal{U}} = (\mathbb{U}_1' \mid \mathbb{U}_2' \mid \Theta \mid \underline{u}_1 \mid \underline{u}_2 \mid \underline{u}_3)^{\mathrm{T}},\quad \boldsymbol{\mathcal{V}} = (\mathbb{V}_1' \mid \mathbb{V}_2' \mid \Psi \mid \underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3)^{\mathrm{T}}.$$

Proof The proof follows the same scheme as before. For all $\overline{v} \in \overline{\mathbf{V}}$, $\widehat{V} \in \widehat{\mathbf{V}}$, we set $\widetilde{v} = \overline{v} + \widehat{V}$, denote $G^\theta_y(\widetilde{v}) = e_{y'}(\overline{v}) + E^\theta_y(\widehat{V})$ and introduce the 6 auxiliary fields $\widetilde{\xi}^\theta_i \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$, $i = 1,\cdots,6$, as the unique solution to the following six independent variational problems:

$$\int_Y AG^\theta_y(\widetilde{\xi}^\theta_i) : G^\theta_y(\widetilde{v})\mathrm{d}y = -\int_Y A\,\mathcal{A}_i : G^\theta_y(\widetilde{v})\mathrm{d}y. \tag{6.21}$$

The limit problem (6.20) is obtained by defining the 6×6 matrix $\mathcal{C}^\theta$ by its elements, as

$$\mathcal{C}^\theta_{mp} = \int_Y A(y)\mathcal{B}^\theta_m(y) : \mathcal{A}_p(y)\mathrm{d}y = \int_Y A(y)\mathcal{B}^\theta_m(y) : \mathcal{B}^\theta_p(y)\mathrm{d}y \tag{6.22}$$

with

$$\mathcal{B}^\theta_i = \mathcal{A}_i + G^\theta_y(\widetilde{\xi}^\theta_i),\quad i = 1,\cdots,6.$$

6.3 The case $\lim_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = 0$

To any $((\widehat{\mathbb{V}}, \widehat{\Psi}, \underline{\widehat{v}}), \widehat{\overline{v}}) \in \widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\overline{\mathbf{V}}}^0$ we associate the corrector field

$$\widehat{V} = \begin{pmatrix} \widehat{\mathbb{V}}_1 + \underline{\widehat{v}}_1 - y_2\widehat{\Psi} \\ \widehat{\mathbb{V}}_2 + \underline{\widehat{v}}_2 + y_1\widehat{\Psi} \\ \underline{\widehat{v}}_3 - y_1\dfrac{\partial\widehat{\mathbb{V}}_1}{\partial y_3} - y_2\dfrac{\partial\widehat{\mathbb{V}}_2}{\partial y_3} \end{pmatrix} + \widehat{\overline{v}} \tag{6.23}$$

belonging to

$$\widehat{\mathbf{V}} \doteq \Big\{ \widehat{V} \in [L^2((0,L); H^1_{\mathrm{per}}(Y))]^3 \;\Big|\; \int_{(0,1)} \widehat{V}(x_3, y)\mathrm{d}y_3 = 0 \quad \text{for a.e. } (x_3, y_1, y_2) \in \Omega \Big\}.$$

The spaces (endowed with their usual norms) $\widehat{\mathbf{V}}_{\mathbf{per}} \times \widehat{\overline{\mathbf{V}}}^0$ and $\widehat{\mathbf{V}}$ are isometric. To any $\widehat{V} \in \widehat{\mathbf{V}}$ we associate the symmetric tensor (see (4.1) and (5.9))

$$E^0_{y'}(\widehat{V}) = \begin{pmatrix} e_{11,y'}(\widehat{V}) & | & e_{12,y'}(\widehat{V}) & | & \dfrac{1}{2}\dfrac{\partial\widehat{V}_3}{\partial y_1} \\ & | & & | & \\ e_{12,y'}(\widehat{V}) & | & e_{22,y'}(\widehat{V}) & | & \dfrac{1}{2}\dfrac{\partial\widehat{V}_3}{\partial y_2} \\ & | & & | & \\ \dfrac{1}{2}\dfrac{\partial\widehat{V}_3}{\partial y_1} & | & \dfrac{1}{2}\dfrac{\partial\widehat{V}_3}{\partial y_2} & | & 0 \end{pmatrix}. \tag{6.24}$$

By collecting the convergence results given by Lemmas 4.1 and 5.3 we can state the following result.

Corollary 6.3 *There exists a subsequence of* (ε, δ), *still denoted* (ε, δ), *limit displacement fields and correctors* $(\mathbb{U}, \Theta, \underline{u}) \in \mathbf{V_M}$, $\overline{u} \in \overline{\mathbf{V}}$ *and* $\widehat{U} \in \widehat{\mathbf{V}}$ *such that*

$$\frac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(u^{\varepsilon,\delta})) \rightharpoonup E(\mathbb{U}, \Theta, \underline{u}) + e_{y'}(\overline{u}) + E^0_{y'}(\widehat{U}) \quad \textit{weakly in } [L^2((0,L)\times Y)]^{3\times 3}, \tag{6.25}$$

where the strain tensors are given by (3.4)–(3.5) *and* (6.24).

We endow $\mathbf{V_M} \times \overline{\mathbf{V}} \times \widehat{\mathbf{V}}$ with the following norm:

$$\|((\mathbb{V}, \Psi, \underline{v}), \overline{v}, \widehat{V})\| = \sqrt{\int_{(0,L)\times Y} \big(E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}) + E^0_{y'}(\widehat{V})\big) : \big(E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}) + E^0_{y'}(\widehat{V})\big)\mathrm{d}x_3\mathrm{d}y}.$$

Lemma 6.3 *On the space* $\mathbf{V_M} \times \overline{\mathbf{V}} \times \widehat{\mathbf{V}}$, *the norm* $\|\cdot\|$ *is equivalent to the usual norm of this product spaces.*

Proof Due to the fact that $\int_0^1 \widehat{V}(x_3, y)\mathrm{d}y_3 = 0$ for a.e. $(x_3, y_1, y_2) \in \Omega$, we first have

$$\begin{aligned} \|((\mathbb{V}, \Psi, \underline{v}), \overline{v}, \widehat{V})\|^2 &= \int_\Omega (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v})) : (E(\mathbb{V}, \Theta, \underline{v}) + e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad + \int_{(0,L)\times Y} E^0_{y'}(\widehat{V}) : E^0_{y'}(\widehat{V})\big)\mathrm{d}x_3\mathrm{d}y. \end{aligned}$$

The 3D Korn's inequality, the periodicity of $\widehat{V}$ and the mean value property of $\widehat{V}$ imply the existence of a constant C. Hence, there exists a constant C such that

$$\|\widehat{V}\|^2_{[L^2(0,L;H^1(Y))]^3} \le C \int_{(0,L)\times Y} E^0_y(\widehat{V}) : E^0_y(\widehat{V})\big)\mathrm{d}x_3\mathrm{d}y.$$

We easily prove that (see [7])

$$\begin{aligned}&\|\mathbb{V}\|^2_{[H^2(0,L)]^2} + \|\underline{v}\|^2_{[H^1(0,L)]^3} + \|\Psi\|_{H^1(0,L)} + \|\overline{v}\|^2_{[L^2((0,L)\times\omega)]^3}\\ \le &\int_\Omega (E(\mathbb{V},\Theta,\underline{v}) + e_{y'}(\overline{v})) : (E(\mathbb{V},\Theta,\underline{v}) + e_{y'}(\overline{v}))\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

The existence and uniqueness of the limit fields and correctors $(\mathbb{U},\Theta,\underline{u},\overline{u})$ and $\widehat{U}$ are given in the following lemma.

Theorem 6.5 *The limit fields* $(\mathbb{U},\Theta,\underline{u}) \in \mathbf{V_M}$, $\overline{u} \in \overline{\mathbf{V}}$, $\widehat{U} \in \widehat{\mathbf{V}}$ *solve the coupled variational problems:*

$$\begin{aligned}&\int_{(0,L)\times Y} A(E(\mathbb{U}, \quad ,\underline{u}) + e_{y'}(\overline{u}) + E^0_{y'}(\widehat{U})) \quad (E(\mathbb{V}, \quad ,\underline{v}) + e_{y'}(\overline{v}) + E^0_{y'}(\widehat{V}))\mathrm{d}x_3\mathrm{d}y\\ &\int_0^L F(\mathbb{V}, \quad ,\underline{v})\mathrm{d}x_3, \quad \forall(\mathbb{V}, \quad ,\underline{v}) \in \mathbf{V_M},\ \overline{v} \in \overline{\mathbf{V}},\ \widehat{V} \in \widehat{\mathbf{V}},\end{aligned} \qquad (\text{[illegible]})$$

where the potential energy is given by (6.2).

Then we are in the position to state the main result in the case $\theta = 0$.

Theorem 6.6 *Under the assumptions on the forces given in Section* 2.4, $(f_i, f_T, g_\alpha, \underline{f}_i) \in [L^2(0,L)]^9$, *the limit displacement field* $(\mathbb{U},\Theta,\underline{u}) \in \mathbf{V_M}$ *is the solution of the variational problem*

$$\int_0^L \mathcal{C}^0 \boldsymbol{\mathcal{U}}' \cdot \boldsymbol{\mathcal{V}}'\mathrm{d}x_3 = \int_0^L F(\mathbb{V},\Psi,\underline{v})\mathrm{d}x_3, \quad \forall(\mathbb{V},\Psi,\underline{v}) \in \mathbf{V_M}, \qquad (6.27)$$

where the 6×6 *symmetric matrix* $\mathcal{C}^0$ *is given in* (6.30) *and where*

$$\boldsymbol{\mathcal{U}} = (\mathbb{U}'_1 \mid \mathbb{U}'_2 \mid \Theta \mid \underline{u}_1 \mid \underline{u}_2 \mid \underline{u}_3)^{\mathrm{T}}, \quad \boldsymbol{\mathcal{V}} = (\mathbb{V}'_1 \mid \mathbb{V}'_2 \mid \Psi \mid \underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3)^{\mathrm{T}}.$$

Proof The proof follows the same scheme as before. For all $\overline{v} \in \overline{\mathbf{V}}, \widehat{V} \in \widehat{\mathbf{V}}$, set $\widetilde{v} = \overline{v} + \widehat{V}$ and denote

$$G^0_y(\widetilde{v}) = e_{y'}(\overline{v}) + E^0_y(\widehat{V}).$$

We rewrite (6.26) with $(\mathbb{V},\Psi,\underline{v}) = 0$ as: Find $\widetilde{v} \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$ such that for all $\widetilde{v} \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$,

$$\int_Y AG^0_y(\widetilde{u}) : G^0_y(\widetilde{v})\mathrm{d}y = -\int_Y AE(\mathbb{U},\Theta,\underline{u}) : G^0_y(\widetilde{v})\mathrm{d}y. \qquad (6.28)$$

We consider the decomposition of the strain tensor E as given by (6.1) and introduce the 6 auxiliary fields $\widetilde{\xi}^0_i \in \overline{\mathbf{V}} \oplus \widehat{\mathbf{V}}$, $i = 1,\cdots,6$, as the unique solution to the following six independent variational problems:

$$\int_Y AG^\theta_y(\widetilde{\xi}^0_i) : G^0_y(\widetilde{v})\mathrm{d}y = -\int_Y A\,\mathcal{A}_i : G^0_y(\widetilde{v})\mathrm{d}y. \qquad (6.29)$$

The solution of problem (6.28) is under the form

$$\widetilde{u}(x_3, y) = g_i(x_3)\ \widetilde{\xi}_i^0(y).$$

Hence $e_{y'}(\overline{u}) + E^0_{y'}(\widehat{U}) = G^0_y(\widetilde{u}) = g_i G^0_y(\widetilde{\xi}_i^0)$. Let us return to the limit homogeneous problem (6.26) with $\widehat{\overline{v}} = 0,\ \widehat{V} = 0$,

$$\int_{(0,L)\times Y} A(g_m \mathcal{A}_m + g_i G^0_y(\widetilde{\xi}_i^0)) : h_j \mathcal{A}_j \mathrm{d}x_3 \mathrm{d}y = \int_0^L F(\mathbb{V}, \Psi, \underline{v}) \mathrm{d}x_3, \quad \forall (\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M},$$

where h is formed with the unknowns $(\mathbb{V}'', \Psi', \underline{v}')$. Letting

$$\mathcal{B}_i^0 = \mathcal{A}_i + G^0_y(\widetilde{\xi}_i^0), \quad i = 1, \cdots, 6,$$

we get

$$\int_0^L g_i(x_3) h_j(x_3) \Big(\int_Y A(y) \mathcal{B}_i^0(y) : \mathcal{A}_j(y) \mathrm{d}y \Big) \mathrm{d}x_3 = \int_0^L F(\mathbb{V}, \Psi, \underline{v}) \mathrm{d}x_3.$$

The limit problem (6.27) is obtained by defining the 6×6 matrix $\mathcal{C}^0$ by its elements

$$\mathcal{C}^0_{mp} = \int_Y A(y) \mathcal{B}^0_m(y) : \mathcal{A}_p(y) \mathrm{d}y = \int_Y A(y) \mathcal{B}^0_m(y) : \mathcal{B}^0_p(y) \mathrm{d}y. \tag{6.30}$$

6.4 Strong formulation

Let us remark that the six order unknown $\boldsymbol{\mathcal{U}} = (\boldsymbol{\mathcal{U}}_i) = (\mathbb{U}', \Theta, \underline{u}) \in [H^1(0, L)]^6$ obtained for $\lim_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon} = +\infty, \theta, 0$ solve respectively the limit problems (6.10), (6.20), (6.27) which can be written in the form

$$\int_0^L \mathcal{C} \boldsymbol{\mathcal{U}}' \cdot \boldsymbol{\mathcal{V}}' \, \mathrm{d}x_3 = \int_0^L \boldsymbol{\mathcal{F}} \cdot \boldsymbol{\mathcal{V}} \, \mathrm{d}x_3,$$

where

$$\boldsymbol{\mathcal{V}} = (\mathbb{V}_1' \mid \mathbb{V}_2' \mid \Psi \mid \underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3)^{\mathrm{T}}, \quad (\mathbb{V}, \Psi, \underline{v}) \in \mathbf{V_M}$$

and where $\boldsymbol{\mathcal{F}}$ is given in (6.4), and the fourth order symmetric positive elasticity tensor takes respectively the value $\mathcal{C}^\infty, \mathcal{C}^\theta, \mathcal{C}^0$ associated to the fields $\boldsymbol{\mathcal{U}}^\infty, \boldsymbol{\mathcal{U}}^\theta, \boldsymbol{\mathcal{U}}^0$.

Since, whatever the value of $\lim_{(\varepsilon,\delta)\to(0,0)} \frac{\delta}{\varepsilon}$, we have the boundary condition $\boldsymbol{\mathcal{U}}(0) = 0$. One can also write these problems as

$$-\mathcal{C} \frac{\mathrm{d}^2 \boldsymbol{\mathcal{U}}}{\mathrm{d}x_3^2} = \boldsymbol{\mathcal{F}}, \quad \boldsymbol{\mathcal{U}}(0) = 0, \quad \frac{\mathrm{d}\boldsymbol{\mathcal{U}}}{\mathrm{d}x_3}(0) = 0.$$

Remark 6.1 Even though the initial elasticity tensor $a^{\varepsilon,\delta}$ is diagonal and depending only upon the variable x_3, generally, the homogenized six-order elasticity tensor $\mathcal{C}^\infty$, $\mathcal{C}^\theta$ and $\mathcal{C}^0$, given respectively by (6.9), (6.22) and (6.30), will be coupled so that the limit fields $\boldsymbol{\mathcal{U}}^\infty, \boldsymbol{\mathcal{U}}^\theta, \boldsymbol{\mathcal{U}}^0$ will be the solution of coupled systems.

7 Asymptotic Expansion

We recall the decomposition (2.11),

$$u^{\varepsilon,\delta}(x)=\mathbb{U}^{\varepsilon,\delta}(x_3)+\underline{u}^{\varepsilon,\delta}(x_3)+\begin{pmatrix}-\dfrac{\mathrm{d}\mathbb{U}_2^{\varepsilon,\delta}}{\mathrm{d}x_3}(x_3)\\ +\dfrac{\mathrm{d}\mathbb{U}_1^{\varepsilon,\delta}}{\mathrm{d}x_3}(x_3)\\ \Theta^{\varepsilon,\delta}(x_3)\end{pmatrix}\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2)+\overline{u}^{\varepsilon,\delta}(x)\quad \text{a.e. } x\in\Omega_\delta.$$

With appropriate definition of the vectors $\widehat{\overline{u}},\widehat{U}$ and of the functional spaces $\widehat{\overline{\mathbf{V}}},\widehat{\mathbf{V}}$ (according to the value of the limit θ), we have established in (4.3), (6.7), (6.16) and (6.25) the existence of limits

$$(\mathbb{U},\Theta,\underline{u})\in\mathbf{V_M},\quad \overline{u}\in\overline{\mathbf{V}},\quad \widehat{\overline{u}}\in\widehat{\overline{\mathbf{V}}},\quad \widehat{U}\in\widehat{\mathbf{V}}$$

such that

$$\begin{cases}\mathcal{T}_{\varepsilon,\delta}(u^{\varepsilon,\delta})\rightharpoonup\mathbb{U} & \text{weakly in } [L^2(0,L;H^1(Y))]^3,\\ \dfrac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(u^{\varepsilon,\delta}-\mathbb{U}^{\varepsilon,\delta})\rightharpoonup\underline{u}+\mathcal{R} & \text{weakly in } [L^2(0,L;H^1(Y))]^3,\\ \dfrac{1}{\delta^2}\mathcal{T}_{\varepsilon,\delta}(u^{\varepsilon,\delta}-U_e^{\varepsilon,\delta})\rightharpoonup\overline{u}+\widehat{\overline{u}} & \text{weakly in } [L^2(0,L;H^1(Y))]^3,\\ \dfrac{1}{\delta}\mathcal{T}_{\varepsilon,\delta}(e(U_e^{\varepsilon,\delta}))\rightharpoonup E(\mathbb{U},\Theta,\underline{u})+e_{y'}(\overline{u})+E_y(\widehat{U}) & \text{weakly in } [L^2((0,L)\times Y)]^9\end{cases}$$

with

$$\mathcal{R}(x)=\begin{pmatrix}-\dfrac{\mathrm{d}\mathbb{U}_2}{\mathrm{d}x_3}(x_3)\\ +\dfrac{\mathrm{d}\mathbb{U}_1}{\mathrm{d}x_3}(x_3)\\ \Theta(x_3)\end{pmatrix}\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\quad\text{for a.e. } (x\in\Omega_\delta).$$

This suggest an asymptotic expansion up to the second order in ε,δ,

$$\begin{aligned}u^{\varepsilon,\delta}(x)&=\mathbb{U}(x_3)+\mathcal{R}(x)+\varepsilon\widehat{\mathcal{R}}_\varepsilon(x)+\delta\underline{u}(x_3)\\ &\quad+\varepsilon\delta\widehat{\underline{u}}\Big(x_3,\Big\{\frac{x_3}{\varepsilon}\Big\}\Big)+\delta^2\overline{u}\Big(x_3,\frac{x_1}{\delta},\frac{x_2}{\delta}\Big)+\delta^2\widehat{\overline{u}}\Big(x_3,\frac{x_1}{\delta},\frac{x_2}{\delta},\Big\{\frac{x_3}{\varepsilon}\Big\}\Big)+\cdots\end{aligned}$$

with

$$\widehat{\mathcal{R}}_\varepsilon(x)=\begin{pmatrix}-\dfrac{\mathrm{d}\widehat{\mathbb{U}}_2}{\mathrm{d}y_3}\Big(x_3,\Big\{\dfrac{x_3}{\varepsilon}\Big\}\Big)\\ +\dfrac{\mathrm{d}\widehat{\mathbb{U}}_1}{\mathrm{d}y_3}\Big(x_3,\Big\{\dfrac{x_3}{\varepsilon}\Big\}\Big)\\ \widehat{\Theta}\Big(x_3,\Big\{\dfrac{x_3}{\varepsilon}\Big\}\Big)\end{pmatrix}\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2).\tag{7.1}$$

It is of interest to note that the corrector $\widehat{\overline{u}}$ appears in the expansion at the same order δ^2 as $\overline{u}$.

Remark 7.1 Following the lines of [4, Chapter 10, Section 9] we obtain these two results.

(1) The function $\theta\mapsto\mathbf{C}^\theta$ from $[0,+\infty]$ into $\mathbb{R}^{6\times 6}$ is continuous and uniformly elliptic. And as an immediate consequence we can establish that the function $\theta\mapsto\mathbf{U}^\theta=(\mathbb{U}^\theta,\Theta^\theta,\underline{u}^\theta)$ from $[0,+\infty]$ into $\mathbf{V_M}$ is continuous.

(2) We can also prove that the limit problem corresponding to $\theta = 0$ is the one obtained when first ε goes to 0 and then δ while the limit problem for $\theta = +\infty$ is the one obtained when first δ goes to 0 and then ε (see also [6]).

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