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Asymptotic Derivation of a Linear Plate Model for Soft Ferromagnetic Materials

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(*Dedicated to Philippe G. Ciarlet on the occasion of his 80th birthday*)

Abstract The authors use the asymptotic expansion method by P. G. Ciarlet to obtain a Kirchhoff-Love-type plate model for a linear soft ferromagnetic material. They also give a mathematical justification of the obtained model by means of a strong convergence result.

Keywords Asymptotic methods, Plates, Magnetoelasticity
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1 Introduction

The use of the framework of continuum mechanics for the study of the influence of electromagnetic effects on solids has been largely stimulated by Truesdell and Toupin [14]. Their research on the coupling between the mechanical and magnetic responses of magnetoelastic solids was the first of a long list (without any attempt to be exhaustive, see, e.g. [1, 4–7, 10, 12–13] etc.). An important stimulus for the development of these researches was the study of the magnetoelastic buckling problem. Indeed a plate, made of a magnetoelastic material, subject to a transverse magnetic field, buckles when the magnetic field attains a critical value; see also [3, 5, 9], for a general analysis of the buckling of some magnetoelastic structures. Following a pioneering experimental and theoretical research of Moon and Pao, the first rigorous attempt to analyze this problem is due to Pao and Yeh [12]. Maugin and Goudjo [8] considered a plate model with particular attention on the regularity of the boundary. More recently, in the case linear soft magnetoelastic materials, Zhou and Zheng [15–17] have revisited the subject by adapting the usual Kirchhoff-Love and von Kármán models only modifying the equivalent transverse force with the addition of the magnetic effects.

The paper is organized as follows. In Section 2, we briefly recall the governing equilibrium equations of magnetoelasticity and, then, in Section 3, we state the problem on a variable domain assuming that the magnetic forces are given. In order to apply the Ciarlet's method (see, e.g., [2]), we must at first prove that the magnetic forces give rise to a linear and continuous form. This can be achieved under suitable assumptions on the magnetic forces (see lemma 3.1).

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In Section 4, we introduce the usual scaling on the mechanical quantities and we scale the magnetic quantities in such a way that the Gauss' Law (2.1) is conserved. Using the classical change of variables, we deduce the scaled equations. Then we can use the asymptotic methods, following [2], to obtain the limit problem and the strong convergence result. It is interesting to remark that in the simplest situation of a transversal magnetic field, we recover the Kirchhoff-Love model of Zhou and Zheng, which is, hence, completely justified.

2 Governing Equations of Magnetoelasticity for Linear Soft Ferromagnetic Materials

In the sequel, Greek indices range in the set $\{1,2\}$, Latin indices range in the set $\{1,2,3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. Let us consider a three-dimensional Euclidean space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. We introduce the following notations for the vector product: $\mathbf{a} \wedge \mathbf{b} = a_i \mathbf{e}_i \wedge b_j \mathbf{e}_j = a_i b_j \epsilon_{ijk} \mathbf{e}_k$, for all vectors $\mathbf{a} = (a_i) \in \mathbb{R}^3$ and $\mathbf{b} = (b_i) \in \mathbb{R}^3$, where ϵ_{ijk} denotes the alternator Ricci's symbol.

When a magnetizable, deformable elastic solid Ω is placed in a magnetic field, magnetic moments are induced inside the body. The action of the external magnetic induction $\mathbf{B}_0$ manifests itself in magnetization $\mathbf{M}$ (magnetic moment per unit volume). Within the body, the magnetic induction $\mathbf{B}$ is not necessarily equal to $\mathbf{B}_0$. The induced magnetization $\mathbf{M} = (M_i)$ is related to $\mathbf{B} = (B_i)$ by $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where $\mathbf{H} = (H_i)$ is called magnetic intensity and μ_0 is the magnetic permeability of vacuum. Generally, we have $\mathbf{H} = \mathbf{H}(\mathbf{M})$, but, in the sequel, we restrict our attention to a class of linear isotropic magnetoelastic materials called soft ferromagnetic materials, which are characterized by the fact that their local average magnetization becomes zero when the external field is set to zero. In this particular case, the hysteresis loops are narrow and the influence of induced currents is small in comparison with the effect of magnetization. Therefore, it is possible to use the quasi-static approximation, i.e., the equations of magneto-statics:

$$\begin{aligned} &\partial_i B_i = 0 && \text{in } \Omega \text{ (Gauss' Law)},\\ &\epsilon_{kij} \partial_j H_i = 0 && \text{in } \Omega \text{ (Ampère's Law)}. \end{aligned} \tag{2.1}$$

The magnetic constitutive law takes the following linear form

$$\mathbf{M} = \chi \mathbf{H} \quad \text{or} \quad \mathbf{B} = \mu_0 \mu_r \mathbf{H}, \tag{2.2}$$

where χ represents the magnetic susceptibility and $\mu_r := \chi + 1$ is the relative magnetic permeability. For linear soft ferromagnetic materials, such as steels, iron, cobalt and various alloys, the relative permeability is very large, μ_r or $\chi = 10^2 \sim 10^5$.

In this work we use the model proposed by Brown [1] where the action of the magnetic field is given by a magnetic body force (per unit volume) $\mathbf{f}^m = (f_i^m)$ and a magnetic body couple (per unit volume) $\mathbf{I}^m = (I_i^m)$:

$$\mathbf{f}^m = \mu_0 (\boldsymbol{\nabla} \mathbf{H}) \mathbf{M} \quad \text{and} \quad \mathbf{I}^m = \mu_0 \mathbf{M} \wedge \mathbf{H},$$

i.e., component-wise, $f_i^m = \mu_0 M_j \partial_j H_i$ and $I_i^m = \mu_0 \epsilon_{kji} M_k H_j$. This choice of the action of the magnetic field is sometimes called the dipole model of microcurrents and has been used in particular by Pao-Yeh for soft ferromagnetic elastic solids (see [12]). Using the Gauss' law and the Ampère's law, the magnetic body force $\mathbf{f}^m$ can also be written as the divergence of a second order tensor $\mathbf{T}^m = (T^m_{ij})$, the so-called Maxwell's stress tensor

$$\mathbf{f}^m = \mathbf{div}\mathbf{T}^m, \tag{2.3}$$

or, component-wise, $f^m_j = \partial_i T^m_{ij}$ with $T^m_{ij} := B_i H_j - \frac{1}{2}\mu_0 H^2 \delta_{ij}$ and $H^2 := H_k H_k$. As pointed out by [1], other different choices of the Maxwell's stress tensor are possible and, indeed, they depend on the choice of the Helmomtz free energy; see in particular [5], for a clear explanation of the influence of the choice of the arguments in the free energy on the Maxwell stress, magnetic body forces and traction boundary conditions.

Considering the expressions above, in the absence of electric field, charge distribution and conduction current, the mechanical governing equations defined in a magnetized body Ω can be expressed by

$$\begin{cases} \partial_i t_{ij} + f^m_j = 0 & \text{in } \Omega, \\ t_{ij} - t_{ji} + \mu_0(M_i H_j - M_j H_i) = 0 & \text{in } \Omega, \\ t_{ij} n_i = \dfrac{1}{2}\mu_0 (M_n)^2 n_j & \text{in } \Xi, \end{cases} \tag{2.4}$$

where t_{ij} is the non-symmetric total stress tensor, (n_i) represents the unit normal vector to the boundary $\Xi \subset \partial\Omega$ and $M_n := M_i n_i$ is the normal surface boundary magnetization. The non-symmetry of the total stress tensor is due to the presence of a magnetic body couple. In order to simplify the model, we neglect magnetostriction and piezostrictive terms in the constitutive laws and we consider an isotropic linear elastic material. Thus, we obtain that

$$t_{ij} = \sigma_{ij} + \mu_0 M_i H_j \quad \text{with} \quad \sigma_{ij} = \lambda e_{pp}\delta_{ij} + 2\mu e_{ij}, \tag{2.5}$$

where σ_{ij} denotes the symmetric Cauchy stress tensor, associated with the linearized strain tensor $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, being (u_i) the displacement field, through the classical Lamé's constitutive equations. The presence of the term $\mu_0 M_i H_j$ in the decomposition of the total stress always follows from the form of the Helmoltz free energy (see [1, 5]).

In the Pao-Yeh's case of soft ferromagnetic materials, thanks to (2.2), the magnetic body couple $\mathbf{l}^m = \mu_0\chi\mathbf{H}\wedge\mathbf{H} = \mathbf{0}$, and, hence, the stress tensor t_{ij} becomes symmetric, i.e., $t_{ij} = t_{ji}$. Moreover, substituting the expression of the Maxwell's stress tensor into the divergence relation (2.3) and using (2.1), we can find an alternative form of the magnetic body force:

$$\mathbf{f}^m = \mu_0(\boldsymbol{\nabla}\mathbf{H})\mathbf{M} = \frac{\mu_0\chi}{2}\nabla(H^2).$$

In the sequel, we will focus our attention on the reduced mechanical model arising from the use of the asymptotic methods, assuming that the magnetization $\mathbf{M}$ and the magnetic intensity $\mathbf{H}$ are a given external magnetic source.

3 Position of the Problem

Let $\omega \subset \mathbb{R}^2$ denote a smooth domain in the plane spanned by vectors $\mathbf{e}_\alpha$, with boundary γ; $\gamma_0 \subset \gamma$ is a measurable subset of γ with strictly positive length measure; $\gamma_1 := \gamma \setminus \gamma_0$ is the complement of γ_0 with respect to γ; finally, $0 < \varepsilon < 1$ is a dimensionless small real parameter which shall tend to zero. For each ε, we define

$$\Omega^\varepsilon := \omega \times (-h^\varepsilon, h^\varepsilon), \quad \Gamma^\varepsilon := \gamma \times (-h^\varepsilon, h^\varepsilon),$$

$$\Gamma_0^\varepsilon := \gamma_0 \times (-h^\varepsilon, h^\varepsilon), \quad \Gamma_\pm^\varepsilon := \omega \times \{\pm h^\varepsilon\}$$

with $h^\varepsilon > 0$. Hence the boundary $\partial\Omega^\varepsilon$ of Ω^ε is partitioned into the lateral surface Γ^ε and the upper and lower faces Γ_+^ε and Γ_-^ε; the lateral surface is itself partitioned as $\Gamma^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$, with $\Gamma_1^\varepsilon := \gamma_1 \times (-h^\varepsilon, h^\varepsilon)$. Moreover, we let $\widehat{\Gamma}^\varepsilon := \Gamma_\pm^\varepsilon \cup \Gamma_1^\varepsilon = \partial\Omega^\varepsilon \setminus \Gamma_0^\varepsilon$, the complement of Γ_0^ε with respect to $\partial\Omega^\varepsilon$.

We assume that Ω^ε is constituted by a homogeneous isotropic linear soft ferromagnetic material, whose constitutive law is given in (2.5). We suppose that the Lamé's coefficients satisfy the classical positivity properties. The plate is clamped on Γ_0^ε, so that $\mathbf{u}^\varepsilon = \mathbf{0}$, and, for simplicity, we consider that no mechanical charges are applied to the body. The only source terms are given by M_i^ε and H_i^ε.

Let $V(\Omega^\varepsilon) := \{\mathbf{v}^\varepsilon \in H^1(\Omega^\varepsilon; \mathbb{R}^3);\ \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\}$ be the functional space of admissible displacements. The variational formulation of problem (2.4), defined on the variable domain Ω^ε, takes the following form:

$$\begin{aligned} &\text{Find } \mathbf{u}^\varepsilon = (u_i^\varepsilon) \in V(\Omega^\varepsilon) && \text{such that} \\ &A^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = L^\varepsilon(\mathbf{v}^\varepsilon) && \text{for all } \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in V(\Omega^\varepsilon), \end{aligned} \tag{3.1}$$

where the bilinear form $A^\varepsilon(\cdot,\cdot)$ and the linear form $L^\varepsilon(\cdot)$ are, respectively, defined by

$$A^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) := \int_{\Omega^\varepsilon} t_{ij}^\varepsilon e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\mathrm{d}x^\varepsilon, \quad L^\varepsilon(\mathbf{v}^\varepsilon) := \int_{\Omega^\varepsilon} \mu_0 M_j^\varepsilon \partial_j^\varepsilon H_i^\varepsilon v_i^\varepsilon \mathrm{d}x^\varepsilon + \int_{\widehat{\Gamma}^\varepsilon} \frac{1}{2}\mu_0 (M_n^\varepsilon)^2 n_i^\varepsilon v_i^\varepsilon \mathrm{d}\Gamma^\varepsilon$$

with $t_{ij}^\varepsilon := \lambda e_{pp}^\varepsilon(\mathbf{u}^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(\mathbf{u}^\varepsilon) + \mu_0 M_i^\varepsilon H_j^\varepsilon$.

In order to prove the wellposedness of the problem, by virtue of the Lax-Milgram's lemma, we rewrite (3.1) in an alternative form:

$$\begin{aligned} &\text{Find } \mathbf{u}^\varepsilon = (u_i^\varepsilon) \in V(\Omega^\varepsilon) && \text{such that} \\ &\overline{A}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \overline{L}^\varepsilon(\mathbf{v}^\varepsilon) && \text{for all } \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in V(\Omega^\varepsilon), \end{aligned} \tag{3.2}$$

where

$$\overline{A}^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) := \int_{\Omega^\varepsilon} \{\lambda e_{pp}^\varepsilon(\mathbf{u}^\varepsilon) e_{qq}^\varepsilon(\mathbf{v}^\varepsilon) + 2\mu e_{ij}^\varepsilon(\mathbf{u}^\varepsilon) e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\}\mathrm{d}x^\varepsilon,$$

$$\overline{L}^\varepsilon(\mathbf{v}^\varepsilon) := \int_{\Omega^\varepsilon} \mu_0 \{M_j^\varepsilon \partial_j^\varepsilon H_i^\varepsilon v_i^\varepsilon - M_i^\varepsilon H_j^\varepsilon e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\}\mathrm{d}x^\varepsilon + \int_{\widehat{\Gamma}^\varepsilon} \frac{1}{2}\mu_0 (M_n^\varepsilon)^2 n_i^\varepsilon v_i^\varepsilon \mathrm{d}\Gamma^\varepsilon.$$

Since the bilinear form $\overline{A}^\varepsilon(\cdot,\cdot)$ is $V(\Omega^\varepsilon)$-coercive, in order to apply the Lax-Milgram's lemma, we have only to prove that the linear form $\overline{L}^\varepsilon(\cdot)$ is continuous on $V(\Omega^\varepsilon)$. For this when $1 < p < +\infty$ we denote $W_p^1(\Omega^\varepsilon)$ the Banach space of $v \in L^p(\Omega^\varepsilon)$ whose first order derivatives (in the distribution sense) also belong to $L^p(\Omega^\varepsilon)$. The continuity of the linear form $\overline{L}^\varepsilon(\cdot)$ is the object of the following lemma.

Lemma 3.1 *Let us assume*

$$M_i^\varepsilon,\ H_i^\varepsilon \in\ W_{12/5}^1(\Omega^\varepsilon). \tag{3.3}$$

Then $\overline{L}^\varepsilon(\cdot)$ *is continuous on* $V(\Omega^\varepsilon)$.

Proof (i) Since $v_i^\varepsilon \in H^1(\Omega^\varepsilon)$, thanks to the Sobolev imbedding theorem, (see e.g. [11, Chapter 2, Theorem 3.4]) we obtain that $v_i^\varepsilon \in L^6(\Omega^\varepsilon)$ and that $f_i^{m,\varepsilon} \in L^{\frac{6}{5}}(\Omega^\varepsilon)$; hence, by means of Hölder's inequality, we can infer that

$$\left| \int_{\Omega^\varepsilon} \{\mu_0 M_j^\varepsilon \partial_j^\varepsilon H_i^\varepsilon v_i^\varepsilon\}\mathrm{d}x^\varepsilon \right| \leq C\|v_i\|_{1,\Omega^\varepsilon}. \tag{3.4}$$

(ii) Since $M_i^\varepsilon, H_i^\varepsilon \in W^1_{12/5}(\Omega^\varepsilon)$, thanks to the Sobolev imbedding theorem, we get that $M_i^\varepsilon, H_i^\varepsilon \in L^{12}(\Omega^\varepsilon) \subset L^4(\Omega^\varepsilon)$ and, thus, $M_i^\varepsilon H_i^\varepsilon \in L^2(\Omega^\varepsilon)$. It then follows from Korn's inequality

$$\left| \int_{\Omega^\varepsilon} \{\mu_0 M_i^\varepsilon H_j^\varepsilon e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\} \mathrm{d}x^\varepsilon \right| \le C \| e_{ij}^\varepsilon(\mathbf{v}^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \le C\|\mathbf{v}\|_{V(\Omega^\varepsilon)}. \tag{3.5}$$

(iii) By virtue of a trace imbedding theorem (see e.g. [11, Chapter 2, Theorem 4.2]), we have that $M_i^\varepsilon|_{\partial\Omega^\varepsilon} \in L^8(\partial\Omega^\varepsilon)$. Besides, being $\mathbf{v} \in V(\Omega^\varepsilon)$, then the same trace imbedding theorem imply that $v_i|_{\partial\Omega^\varepsilon} \in L^4(\partial\Omega^\varepsilon)$, and so

$$\left| \int_{\widehat{\Gamma}^\varepsilon} \frac{1}{2}\mu_0 (M_n^\varepsilon)^2 n_i^\varepsilon v_i^\varepsilon \mathrm{d}\Gamma^\varepsilon \right| \le C\|(M_n^\varepsilon)^2\|_{L^4(\partial\Omega^\varepsilon)} \|v_i^\varepsilon\|_{L^4(\partial\Omega^\varepsilon)} \le C\|\mathbf{v}\|_{V(\Omega^\varepsilon)}. \tag{3.6}$$

Collecting (3.4)–(3.6), we obtain the desired result.

Thanks to the $V(\Omega^\varepsilon)$-coercivity of the bilinear form $\overline{A}^\varepsilon(\cdot,\cdot)$ and the continuity of the linear form $\overline{L}^\varepsilon(\cdot)$ we deduce, using the Lax-Milgram's lemma, that the variational problem (3.1) admits one and only one solution.

4 The Asymptotic Expansion

In order to perform an asymptotic analysis, we need to transform problem (3.1), posed on a variable domain Ω^ε, onto a problem posed on a fixed domain Ω (independent of ε). We suppose that the thickness of the plate h^ε depends linearly on ε, so that $h^\varepsilon = \varepsilon h$. Accordingly, we let

$$\begin{aligned} &\Omega := \omega \times (-h,h), \\ &\Gamma_0 := \gamma_0 \times (-h,h), \quad \Gamma_1 := \gamma_1 \times (-h,h), \\ &\Gamma_\pm := \omega \times \{\pm h\}, \qquad \widehat{\Gamma} := \Gamma_\pm \cup \Gamma_1, \end{aligned}$$

and we define the following change of variables (see [2]):

$$\pi^\varepsilon : x \equiv (x, x_3) \in \overline{\Omega} \mapsto x^\varepsilon \equiv (x, \varepsilon x_3) \in \overline{\Omega}^\varepsilon \quad \text{with } x = (x_\alpha).$$

By using the bijection π^ε, one has $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$. Moreover, we define the following functional spaces:

$$V(\Omega) := \{\mathbf{v} = (v_i) \in H^1(\Omega;\mathbb{R}^3);\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}, \quad W^1_{12/5}(\Omega).$$

In order to write the expression of the scaled problem, we need first to make some assumptions on the data which will define their dependences with respect to the small parameter ε. In our case, the only external data are represented by the magnetic field (B_i^ε), the magnetic intensity field (H_i^ε) and the magnetization field (M_i^ε). By virtue of the constitutive equations $B_i^\varepsilon = \mu_0\mu_r H_i^\varepsilon$ and $M_i^\varepsilon = \chi H_i^\varepsilon$, we assume that B_i^ε, H_i^ε and M_i^ε will share the same dependence on ε. The scaling of B_i^ε must reflect the fact that the magnetic field is a solenoidal field, meaning that $\partial_i^\varepsilon B_i^\varepsilon = 0$ in Ω^ε. This property must be satisfied also on the fixed domain Ω. Let us suppose that

$$B_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^q B_\alpha(x), \quad B_3^\varepsilon(x^\varepsilon) = \varepsilon^p B_3(x), \quad x \in \Omega$$

with B_i independent of ε. By applying the change of variables π^ε, we can write that the scaled divergence vanishes in Ω, so that $\varepsilon^q \partial_\alpha B_\alpha + \varepsilon^{p-1}\partial_3 B_3 = 0$ in Ω. In order to guarantee the consistency of this equation, we ask that $p = q + 1$, finding a relation between the two exponents p and q. In the sequel, we choose $q = 0$ and, hence

$$\begin{aligned} B^\varepsilon_\alpha(x^\varepsilon) &= B_\alpha(x), \quad B^\varepsilon_3(x^\varepsilon) = \varepsilon B_3(x), \quad x \in \Omega,\\ H^\varepsilon_\alpha(x^\varepsilon) &= H_\alpha(x), \quad H^\varepsilon_3(x^\varepsilon) = \varepsilon H_3(x), \quad x \in \Omega,\\ M^\varepsilon_\alpha(x^\varepsilon) &= M_\alpha(x), \quad M^\varepsilon_3(x^\varepsilon) = \varepsilon M_3(x), \quad x \in \Omega. \end{aligned}$$

With the unknown displacement field $\mathbf{u}^\varepsilon$, we associate the scaled unknown displacement field $\mathbf{u}(\varepsilon)$ defined by

$$\begin{aligned} u^\varepsilon_\alpha(x^\varepsilon) &= u_\alpha(\varepsilon)(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\\ u^\varepsilon_3(x^\varepsilon) &= \frac{1}{\varepsilon} u_3(\varepsilon)(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \end{aligned}$$

We likewise associate with any test function $\mathbf{v}^\varepsilon$, the scaled test function $\mathbf{v}$, defined by the scalings:

$$\begin{aligned} v^\varepsilon_\alpha(x^\varepsilon) &= v_\alpha(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,\\ v^\varepsilon_3(x^\varepsilon) &= \frac{1}{\varepsilon} v_3(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \end{aligned}$$

According to the previous hypothesis, problem (3.1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled variational problem:

$$\begin{aligned} &\text{Find } \mathbf{u}(\varepsilon) = (u_i(\varepsilon)) \in V(\Omega) \quad \text{such that}\\ &A(\mathbf{u}(\varepsilon), \mathbf{v}) = L(\mathbf{v}) \quad \text{for all } \mathbf{v} = (v_i) \in V(\Omega), \end{aligned} \tag{4.1}$$

where the scaled bilinear form $A(\cdot,\cdot)$ and the scaled linear form $L(\cdot)$ are, respectively, defined by

$$A(\mathbf{u}(\varepsilon), \mathbf{v}) := \frac{1}{\varepsilon^4} a_{-4}(\mathbf{u}(\varepsilon), \mathbf{v}) + \frac{1}{\varepsilon^2} a_{-2}(\mathbf{u}(\varepsilon), \mathbf{v}) + a_0(\mathbf{u}(\varepsilon), \mathbf{v})$$

with

$$\begin{aligned} a_{-4}(\mathbf{u}(\varepsilon), \mathbf{v}) &:= \int_\Omega (\lambda + 2\mu) e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) \mathrm{d}x,\\ a_{-2}(\mathbf{u}(\varepsilon), \mathbf{v}) &:= \int_\Omega \{4\mu e_{\alpha 3}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) + \lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) + \lambda e_{33}(\mathbf{u}(\varepsilon)) e_{\sigma\sigma}(\mathbf{v})\} \mathrm{d}x,\\ a_0(\mathbf{u}(\varepsilon), \mathbf{v}) &:= \int_\Omega \{\lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) e_{\tau\tau}(\mathbf{v}) + 2\mu e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + \mu_o M_i H_j e_{ij}(\mathbf{v})\} \mathrm{d}x,\\ L(\mathbf{v}) &:= \int_\Omega \mu_0 M_j \partial_j H_i v_i \mathrm{d}x + \frac{1}{2} \int_{\Gamma_+} \mu_0 (M_3^+)^2 v_3^+ \mathrm{d}\Gamma - \frac{1}{2} \int_{\Gamma_-} \mu_0 (M_3^-)^2 v_3^- \mathrm{d}\Gamma\\ &\quad + \frac{1}{2} \int_{\Gamma_1} \mu_0 (M_\beta n_\beta)^2 n_\alpha v_\alpha \mathrm{d}\Gamma, \end{aligned}$$

where $\phi^\pm := \phi(x, \pm h)$ denotes the restriction of ϕ on $\Gamma_\pm$. Since $H_i, M_i \in W^1_{12/5}(\Omega)$, thanks to Lemma 3.1 and Lax-Milgram's lemma, we can prove that the scaled problem admits one and only one solution.

We are now in position to perform an asymptotic analysis of the scaled problem (4.1). Since the scaled problem (4.1) has a polynomial structure with respect to the small parameter ε, we can look for the solution of the problem as a formal series of powers of ε:

$$\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon^2 \mathbf{u}^2 + \varepsilon^4 \mathbf{u}^4 + \cdots . \tag{4.2}$$

Hence, by substituting expressions (4.2) in (4.1) and by identifying the terms with identical power of ε, we can write the following sequence of variational subproblems:

$$\begin{aligned}
&\mathcal{P}_{-4} : a_{-4}(\mathbf{u}^0, \mathbf{v}) = 0,\\
&\mathcal{P}_{-2} : a_{-4}(\mathbf{u}^2, \mathbf{v}) + a_{-2}(\mathbf{u}^0, \mathbf{v}) = 0,\\
&\mathcal{P}_0 : a_{-4}(\mathbf{u}^4, \mathbf{v}) + a_{-2}(\mathbf{u}^2, \mathbf{v}) + a_0(\mathbf{u}^0, \mathbf{v}) = L(\mathbf{v}),\\
&\quad \vdots
\end{aligned} \tag{4.3}$$

By solving the above variational problems, we can characterize the leading term of the asymptotic expansion $\mathbf{u}^0$, the so-called the limit displacement field, and its associated limit problem.

5 The Limit Problem

We define the usual functional space of Kirchhoff-Love admissible displacements:

$$V_{KL}(\Omega) := \{\mathbf{v} \in V(\Omega);\ e_{i3}(\mathbf{v}) = 0\}$$

and

$$\begin{aligned}
V_H(\omega) &:= \{\mathbf{v}_H = (v_\alpha) \in H^1(\omega; \mathbb{R}^2);\ \mathbf{v}_H = \mathbf{0} \text{ on } \gamma_0\},\\
V_3(\omega) &:= \{v_3 \in H^2(\omega);\ v_3 = 0 \text{ and } \partial_\nu v_3 = 0 \text{ on } \gamma_0\},
\end{aligned}$$

where $\boldsymbol{\nu} = (\nu_\alpha)$ is the outer unit normal vector to γ.

Theorem 5.1 (a) *The leading term* $\mathbf{u}^0$ *of the asymptotic expansion* (4.2) *satisfies the following variational problem:*

$$\begin{aligned}
&\textit{Find } \mathbf{u}^0 \in V_{KL}(\Omega) \quad \textit{such that}\\
&\mathcal{A}(\mathbf{u}^0, \mathbf{v}) = L(\mathbf{v}) \qquad \textit{for all } \mathbf{v} \in V_{KL}(\Omega),
\end{aligned} \tag{5.1}$$

where

$$\mathcal{A}(\mathbf{u}^0, \mathbf{v}) := \int_\Omega \frac{2\mu\lambda}{\lambda + 2\mu} e_{\sigma\sigma}(\mathbf{u}^0) e_{\tau\tau}(\mathbf{v}) + 2\mu e_{\alpha\beta}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + \mu_0 M_\alpha H_\beta e_{\alpha\beta}(\mathbf{v})\ \mathrm{d}x. \tag{5.2}$$

(b) *The sequence* $\{\mathbf{u}(\varepsilon)\}_{\varepsilon>0}$ *strongly converges in* $H^1(\Omega; \mathbb{R}^3)$ *to* $\mathbf{u}^0$*, the solution of the limit problem* (5.1).

Proof The proof is straightforward, following the approach by [2].

Let us focus our attention on the expression of the magnetic force work $L(\cdot)$ of the limit problem. By choosing a test function $\mathbf{v} \in V_{KL}(\Omega)$, namely $v_\alpha(x, x_3) := \eta_\alpha(x) - x_3 \partial_\alpha \eta_3(x)$ and $v_3(x, x_3) := \eta_3(x)$, with $\eta_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$, after an integration along x_3 and by applying the Gauss-Green's formula, we get

$$\begin{aligned}
L(\boldsymbol{\eta}) = &\int_\Omega \mu_0 \{(M_3 \partial_3 H_3 + M_\alpha \partial_\alpha H_3)\eta_3 + (M_\beta \partial_\beta H_\alpha + M_3 \partial_3 H_\alpha)(\eta_\alpha - x_3 \partial_\alpha \eta_3)\} \mathrm{d}x\\
&+ \frac{1}{2} \int_{\Gamma_+} \mu_0 (M_3^+)^2 \eta_3 \mathrm{d}\Gamma - \frac{1}{2} \int_{\Gamma_-} \mu_0 (M_3^-)^2 \eta_3 \mathrm{d}\Gamma + \frac{1}{2} \int_{\Gamma_1} \mu_0 (M_\beta n_\beta)^2 n_\alpha (\eta_\alpha - x_3 \partial_\alpha \eta_3) \mathrm{d}\Gamma\\
= &\int_\omega f_i^m \eta_i \mathrm{d}x + \int_{\gamma_1} g_i^m \eta_i \mathrm{d}x + \int_{\gamma_1} h_3^m \partial_\nu \eta_3 \mathrm{d}x,
\end{aligned}$$

where the reduced magnetic forces f_i^m, g_i^m and h_3^m have the following form:

$$\begin{aligned}
f_\alpha^m &:= \mu_0\chi\langle H_i\partial_i H_\alpha\rangle = \mu_0\langle f_\alpha^m\rangle,\\
f_3^m &:= \mu_o\chi \;\; \frac{1}{2}(\chi+1)((H_3^+)^2-(H_3^-)^2) + \langle H_\beta\partial_\beta H_3\rangle + \partial_\alpha\langle\!\langle H_i\partial_i H_\alpha\rangle\!\rangle\\
&= \mu_0\chi \;\; \frac{1}{2}(\chi+1)((H_3^+)^2-(H_3^-)^2) + \langle H_\beta\partial_\beta H_3\rangle \;\; + \mu_0\langle\!\langle\partial_\alpha f_\alpha^m\rangle\!\rangle,\\
g_\alpha^m &:= \frac{1}{2}\mu_0\chi^2\langle H_\nu^2\rangle\nu_\alpha,\\
g_3^m &:= -\mu_0\chi\langle\!\langle H_i\partial_i H_\alpha\rangle\!\rangle\nu_\alpha = -\mu_0\langle\!\langle f_\alpha^m\rangle\!\rangle\nu_\alpha,\\
h_3^m &:= -\frac{1}{2}\mu_0\chi^2\langle\!\langle H_\nu^2\rangle\!\rangle,
\end{aligned}$$

where $H_\nu := H_\alpha\nu_\alpha$, and

$$\langle\phi\rangle(x) := \int_{-h}^{h}\phi(x,x_3)\mathrm{d}x_3, \quad \langle\!\langle\phi\rangle\!\rangle(x) := \int_{-h}^{h}x_3\phi(x,x_3)\mathrm{d}x_3.$$

It easy to verify that if the induced magnetic intensity field is normal to the middle plane of the plate, with $H_\alpha = 0$, the form of the limit magnetic force acting on a plate depends just on the jump of the square of magnetic intensities evaluated at the top and bottom faces of the plate. Indeed, since $f_\alpha^m = g_\alpha^m = h_3^m = 0$, one has

$$f_3^m := \frac{1}{2}\mu_0\chi(1+\chi)\{(H_3^+)^2-(H_3^-)^2\} \approx \frac{1}{2}\mu_0\chi^2\{(H_3^+)^2-(H_3^-)^2\} \tag{5.3}$$

being χ very large for soft ferromagnetic materials. Equation (5.3) is analogue to the one presented in [16] and it can be considered as a mathematical justification of the magnetic force acting on a plate, which is usually employed in magnetic instability problems.

The limit problem (5.1) can be decoupled into a membrane and a bending problem, by virtue of the Kirchhoff-Love limit displacement field. The membrane problem reads as follows:

$$\begin{cases}\text{Find } \mathbf{u}_H^0 = (u_\alpha^0) \in V_H(\omega) & \text{such that}\\[2ex] \displaystyle\int_\omega n_{\alpha\beta}(\mathbf{u}_H^0)e_{\alpha\beta}(\boldsymbol{\eta}_H)\mathrm{d}x = \int_\omega f_\alpha^m\eta_\alpha\mathrm{d}x + \int_{\gamma_1} g_\alpha^m\eta_\alpha\mathrm{d}x & \text{for all } \boldsymbol{\eta}_H = (\eta_\alpha) \in V_H(\omega),\end{cases}$$

where

$$n_{\alpha\beta}(\mathbf{u}_H^0) := \frac{4h\lambda\mu}{\lambda+2\mu}e_{\sigma\sigma}(\mathbf{u}_H^0)\delta_{\alpha\beta} + 4h\mu e_{\alpha\beta}(\mathbf{u}_H^0) + \mu_0\langle M_\alpha H_\beta\rangle$$

represents the ferromagnetic membrane stress tensor. After an integration by parts, we find that the membrane displacements u_α^0 solve the following membrane differential problem:

$$\begin{cases}\text{Field equation:} & \\ -\partial_\beta n_{\alpha\beta} = f_\alpha^m & \text{in } \omega.\\ \text{Boundary conditions:} & \\ n_{\alpha\beta}\nu_\beta = g_\alpha^m & \text{on } \gamma_1,\\ u_\alpha = 0 & \text{on } \gamma_0.\end{cases}$$

The bending problem takes the following form:

$$\begin{cases}\text{Find } u_3^0 \in V_3(\omega) & \text{such that}\\[2ex] \displaystyle\int_\omega m_{\alpha\beta}(u_3^0)\partial_{\alpha\beta}\eta_3\mathrm{d}x = \int_\omega f_3^m\eta_3\mathrm{d}x + \int_{\gamma_1} g_3^m\eta_3\mathrm{d}x + \int_{\gamma_1} h_3^m\partial_\nu\eta_3\mathrm{d}x & \text{for all } \eta_3 \in V_3(\omega),\end{cases}$$

where

$$m_{\alpha\beta}(u_3^0) := \frac{4h^3\lambda\mu}{3(\lambda+2\mu)}\Delta u_3^0\delta_{\alpha\beta} + \frac{4h^3\mu}{3}\partial_{\alpha\beta}u_3^0 - \mu_0\langle\!\langle M_\alpha H_\beta\rangle\!\rangle, \quad \Delta := \partial_{\sigma\sigma}$$

represents the ferromagnetic moment stress tensor. After an integration by parts, we find that the transversal displacement u_3^0 solves the following bending differential problem:

$$\begin{cases} \text{Field equation:} & \\ \partial_{\alpha\beta}m_{\alpha\beta} = \dfrac{2h^3}{3}\dfrac{\lambda+\mu}{\lambda+2\mu}\Delta\Delta u_3^0 - \mu_0\partial_{\alpha\beta}\langle\!\langle M_\alpha H_\beta\rangle\!\rangle = f_3^m & \text{in } \omega. \\ \text{Boundary conditions:} & \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = g_3^m & \text{on } \gamma_1, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = h_3^m & \text{on } \gamma_1, \\ u_3 = \partial_\nu u_3 = 0 & \text{on } \gamma_0, \end{cases} \tag{5.4}$$

where $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ represents the unit tangent vector to γ.

Considering the case of an induced magnetic intensity field, normal to the middle plane of the plate, with $H_\alpha = 0$, $n_{\alpha\beta}$ and $m_{\alpha\beta}$ reduce to the classical elastic membrane stress tensor and moment stress tensor. Besides, since $f_\alpha^m = g_\alpha^m = 0$, we can infer that the membrane problem admits the only zero solution, so that $n_{\alpha\beta} = 0$, and thus, in this case, the plate equilibrium problem takes just into account the bending behavior.

6 Concluding Remarks

In this work we derive a model of a soft ferromagnetic isotropic linear otccb df

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