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Nonlinear Korn Inequalities on a Hypersurface*

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(*Dedicated to Professor Philippe G. Ciarlet on the occasion of his* 80*th birthday*)

Abstract The authors establish several estimates showing that the distance in $W^{1,p}$, $1 < p < \infty$, between two immersions from a domain of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ is bounded by the distance in L^p between two matrix fields defined in terms of the first two fundamental forms associated with each immersion. These estimates generalize previous estimates obtained by the authors and P. G. Ciarlet and weaken the assumptions on the fundamental forms at the expense of replacing them by two different matrix fields.

Keywords Differential geometry, Hypersurface, Nonlinear shell theory
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1 Introduction

The notation and terminology used in this paper is described in the next section.

In a recent article [6], the authors and P. G. Ciarlet established (in particular) the following nonlinear Korn inequalities on a surface in $\mathbb{R}^3$: Let $1 < p < \infty$, let $\omega \subset \mathbb{R}^2$ be a domain, and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ be an immersion such that the vector field $\boldsymbol{a}_3(\boldsymbol{\theta}) := (\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta})/|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|$ is also of class $\mathcal{C}^1$ over $\overline{\omega}$. Then, for each $\varepsilon > 0$, there exists a constant C_ε such that

$$\begin{aligned}&\inf_{\boldsymbol{r}\in \mathrm{Isom}_+(\mathbb{R}^3)}\{\|\boldsymbol{r}\circ\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)}+\|\boldsymbol{a}_3(\boldsymbol{r}\circ\widetilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)}\}\\&\leqslant C_\varepsilon\{\|(\widetilde{a}_{\alpha\beta})-(a_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}+\|(\widetilde{b}_{\alpha\beta})-(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\}\end{aligned}$$

and

$$\begin{aligned}&\|\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)}+\|\boldsymbol{a}_3(\widetilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)}\\&\leqslant C_\varepsilon\{\|\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)}+\|\boldsymbol{a}_3(\widetilde{\boldsymbol{\theta}})-\boldsymbol{a}_3(\boldsymbol{\theta})\|_{\boldsymbol{L}^p(\omega)}\\&\quad+\|(\widetilde{a}_{\alpha\beta})-(a_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}+\|(\widetilde{b}_{\alpha\beta})-(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\}\end{aligned}$$

for all immersions $\widetilde{\boldsymbol{\theta}} \in W^{1,2p}(\omega; \mathbb{R}^3)$ such that $\boldsymbol{a}_3(\widetilde{\boldsymbol{\theta}}) \in W^{1,2p}(\omega; \mathbb{R}^3)$ and

$$|\widetilde{R}_\alpha(y)| \geqslant \varepsilon, \quad |(\widetilde{a}_{\alpha\beta}(y))| \leqslant \frac{1}{\varepsilon} \quad \text{and} \quad |(\widetilde{a}_{\alpha\beta}(y))^{-1}| \leqslant \frac{1}{\varepsilon} \quad \text{a.e. } y \in \omega,$$

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where $\tilde{a}_{\alpha\beta}(y)$, $\tilde{b}_{\alpha\beta}(y)$ and $\tilde{R}_\alpha(y)$, respectively, denote the covariant components of the first fundamental form, the covariant components of the second fundamental form, and the principal radii of curvature, of the surface $\tilde{\boldsymbol{\theta}}(\omega)$ at the point $\tilde{\boldsymbol{\theta}}(y)$. They also showed that, if in addition

$$\tilde{\boldsymbol{\theta}}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0} \quad \text{and} \quad \boldsymbol{a}_3(\tilde{\boldsymbol{\theta}})|_{\gamma_0} = \boldsymbol{a}_3(\boldsymbol{\theta})|_{\gamma_0}$$

on some relatively open subset $\gamma_0 \neq \emptyset$ of the boundary of ω, then

$$\begin{aligned} &\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\boldsymbol{a}_3(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_3(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)} \\ &\quad \leqslant C_{\varepsilon,\gamma_0}\{\|(\tilde{a}_{\alpha\beta}) - (a_{\alpha\beta})\|_{\mathbb{L}^p(\omega)} + \|(\tilde{b}_{\alpha\beta}) - (b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\}, \end{aligned}$$

for some constant C_{ε,γ_0} (depending on ε and γ_0 in particular).

The objective of this paper is to generalize the above inequalities to hypersurfaces (submanifolds of co-dimension 1 in $\mathbb{R}^{n+1}$, $n \geqslant 2$) and to weaken the assumptions on the immersion $\tilde{\boldsymbol{\theta}}$, in particular by eliminating the restrictions in terms of the parameter ε on its principal radii of curvature and first fundamental form. This will be done at the expense of replacing in the right-hand side of the above inequalities the matrix field $(\tilde{a}_{\alpha\beta})$ by the matrix field $(\tilde{a}_{\alpha\beta})^{\frac{1}{2}}$ and the matrix field $(\tilde{b}_{\alpha\beta})$ by $(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta})$.

More specifically, we establish the following nonlinear Korn inequalities on a hypersurface in $\mathbb{R}^{n+1}$ (cf. Theorems 3.1–3.2 and 4.1–4.2): Let $1 < p < \infty$, let $\omega \subset \mathbb{R}^n$ be a domain, and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion whose unit normal vector field $\boldsymbol{a}_{n+1}(\boldsymbol{\theta}) := (\partial_1\boldsymbol{\theta} \wedge \cdots \wedge \partial_n\boldsymbol{\theta})/|\partial_1\boldsymbol{\theta} \wedge \cdots \wedge \partial_n\boldsymbol{\theta}|$ is of class $\mathcal{C}^1$ over $\overline{\omega}$. Then there exists a constant C such that

$$\begin{aligned} &\inf_{\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{R}^{n+1})} \{\|\boldsymbol{r} \circ \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\boldsymbol{a}_{n+1}(\boldsymbol{r} \circ \tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_{n+1}(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)}\} \\ &\quad \leqslant C\{\|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\} \end{aligned}$$

and

$$\begin{aligned} &\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_{n+1}(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)} \\ &\quad \leqslant C\{\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)} + \|\boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_{n+1}(\boldsymbol{\theta})\|_{\boldsymbol{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} \\ &\qquad + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\} \end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ such that $\boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$.

Furthermore, if in addition

$$\tilde{\boldsymbol{\theta}}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0} \quad \text{and} \quad \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}})|_{\gamma_0} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta})|_{\gamma_0}$$

on some relatively open subset $\gamma_0 \neq \emptyset$ of the boundary of ω, then

$$\begin{aligned} &\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) - \boldsymbol{a}_{n+1}(\boldsymbol{\theta})\|_{\boldsymbol{W}^{1,p}(\omega)} \\ &\quad \leqslant C_{\gamma_0}\{\|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}\} \end{aligned}$$

for some (other) constant C_{γ_0} (depending in particular on γ_0).

Finally, we also show that, if $\boldsymbol{\theta}$ and γ_0 are such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in any affine subspace of dimension $\leqslant (n-1)$ of $\mathbb{R}^{n+1}$, then the assumption $\boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}})|_{\gamma_0} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta})|_{\gamma_0}$ above can be dropped and the last inequality still holds, possibly with a different constant C_{γ_0}.

It is worth noticing that some of the results established in this paper, like Lemma 3.3, Theorem 3.2, Lemma 4.1, and Lemma 4.3, generalize to immersions $\boldsymbol{\theta} : \omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ previous results due to Ciarlet and Mardare [8], like Lemma 3, Theorem 2, and Lemma 4 in ibid., about immersions $\boldsymbol{\Theta} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. To see this, it suffices to particularize the immersions considered in this paper to immersions of the form $\boldsymbol{\theta} := \begin{pmatrix} \boldsymbol{\Theta} \\ 0 \end{pmatrix}$ and to notice that in this case

$$\boldsymbol{a}_{n+1}(\boldsymbol{\theta}) = (0\ 0\ \cdots\ 0\ 1)^{\mathrm{T}} \in \mathbb{R}^{n+1},$$

so that

$$b_{\alpha\beta}(\boldsymbol{\theta}) = 0 \quad \text{in } \omega.$$

Some of the results of this paper were announced in [13].

2 Preliminaries

In this article, all vector spaces are over $\mathbb{R}$. Scalars and scalar functions are denoted by normal letters, while vectors, matrices, vector fields and matrix fields are denoted by boldface letters.

For each positive integers k and l, the notations $\mathbb{M}^{l\times k}$, $\mathbb{M}^l = \mathbb{M}^{l\times l}$, $\mathbb{A}^l$, $\mathbb{S}^l$, $\mathbb{S}^l_>$, and $\mathbb{O}^l_+$, respectively, designate the space of real matrices with l rows and k columns, the space of all real square matrices of order l, the space of all antisymmetric matrices of order l, the space of all symmetric matrices of order l, the set of all positive-definite symmetric matrices of order l, and the set of all real proper orthogonal matrices of order l.

The identity matrix in $\mathbb{M}^l$ is denoted $\boldsymbol{I}$.

The Euclidean norm of a vector $\boldsymbol{v} = (v_i) \in \mathbb{R}^l$ and the Frobenius norm of a matrix $\boldsymbol{F} = (F_{ij}) \in \mathbb{M}^{l\times k}$ are denoted by

$$|\boldsymbol{v}| := \Big(\sum_{i=1}^{l} |v_i|^2\Big)^{\frac{1}{2}} \quad \text{and} \quad |\boldsymbol{F}| := \Big(\sum_{i=1}^{l}\sum_{j=1}^{k} |F_{ij}|^2\Big)^{\frac{1}{2}}.$$

Note that the above norms are invariant under rotations, in the sense that

$$|\boldsymbol{R}\boldsymbol{v}| = |\boldsymbol{v}| \quad \text{and} \quad |\boldsymbol{R}\boldsymbol{F}| = |\boldsymbol{F}| \quad \text{for all } \boldsymbol{R} \in \mathbb{O}^l_+.$$

The notation $\mathrm{Isom}_+(\mathbb{R}^l)$ designates the set of all proper isometries of $\mathbb{R}^l$, i.e.,

$$\mathrm{Isom}_+(\mathbb{R}^l) := \{\boldsymbol{r} : \mathbb{R}^l \to \mathbb{R}^l;\ \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}x \text{ for all } x \in \mathbb{R}^l,\ \boldsymbol{a} \in \mathbb{R}^l,\ \boldsymbol{R} \in \mathbb{O}^l_+\}.$$

A domain U in $\mathbb{R}^n$ is a bounded, connected, open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary, the set U being locally on the same side of its boundary (see, e.g., [2] or [14]).

Let U be an open subset of $\mathbb{R}^k$ and let $1 \leqslant p < \infty$. Given a smooth enough vector field $\boldsymbol{v} = (v_i) : U \to \mathbb{R}^l$, we let $\boldsymbol{\nabla v}(x) \in \mathbb{M}^{l\times k}$ denote the gradient matrix of the vector $\boldsymbol{v}$ at each point $x = (x_j) \in U$, i.e.,

$$\boldsymbol{\nabla v}(x) := \left(\frac{\partial v_i}{\partial x_j}(x)\right),$$

where i denotes the row index.

The usual Lebesgue and Sobolev spaces are respectively denoted by $L^p(U)$ and $W^{1,p}(U)$.

The space of vector fields $\boldsymbol{v} = (v_i) : U \to \mathbb{R}^l$ with components $v_i \in L^p(U)$ is denoted by $L^p(U;\mathbb{R}^l)$ and the corresponding norm is defined by

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(U)} := \left(\int_U |\boldsymbol{v}(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

The space of vector fields $\boldsymbol{v} = (v_i) : U \to \mathbb{R}^l$ with components $v_i \in W^{1,p}(U)$ is denoted by $W^{1,p}(U;\mathbb{R}^l)$ and the corresponding norm is defined by

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(U)} := \left(\int_U (|\boldsymbol{v}(x)|^p + |\boldsymbol{\nabla v}(x)|^p)\mathrm{d}x\right)^{\frac{1}{p}}.$$

The space of matrix fields $\boldsymbol{F} = (F_{ij}) : U \to \mathbb{M}^{l\times k}$ with components $F_{ij} \in L^p(U)$ is denoted by $L^p(U;\mathbb{M}^{l\times k})$ and the corresponding norm is defined by

$$\|\boldsymbol{F}\|_{\mathbb{L}^p(U)} := \left(\int_U |\boldsymbol{F}(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

The notation $\mathcal{C}^1(\overline{U};\mathbb{R}^l)$ designates the space of all vector fields $\boldsymbol{v} \in \mathcal{C}^1(U;\mathbb{R}^l)$ that, together with their gradients $\boldsymbol{\nabla v} \in \mathcal{C}(U;\mathbb{M}^{l\times k})$, possess continuous extensions to the closure $\overline{U}$ of U. If U is a domain, one can show, by using Whitney's extension theorem (cf. [15]), that any vector field $\boldsymbol{v} \in \mathcal{C}^1(\overline{U};\mathbb{R}^n)$ possesses an extension to the space $\mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$ (cf. [7]).

In all that follows, n designates an integer $\geqslant 2$, Latin indices and exponents range in the set $\{1, 2, \cdots, n+1\}$ save when they are used for indexing sequences, Greek indices and exponents range in the set $\{1, 2, \cdots, n\}$, and the summation convention for repeated indices or exponents is used in conjunction with these rules.

Given an open subset ω of $\mathbb{R}^n$, we let $\partial_\alpha := \partial/\partial y_\alpha$, where (y_α) denotes a generic point in ω. A mapping $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}, \mathbb{R}^{n+1})$ is an immersion if the vectors $\partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent at each point $y \in \overline{\omega}$. A mapping $\boldsymbol{\theta} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$, $p \geqslant 1$, is an immersion if the vectors $\partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent at almost all point $y \in \omega$.

Given an open subset Ω of $\mathbb{R}^{n+1}$, we let $\partial_i := \partial/\partial x_i$, where (x_i) denotes a generic point in Ω. A mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^{n+1})$ is an *immersion* if the vectors $\partial_i\boldsymbol{\Theta}(x)$ are linearly independent at each point $x \in \overline{\Omega}$.

The notation $[\boldsymbol{v}_1\boldsymbol{v}_2\cdots\boldsymbol{v}_n]$ designates the matrix whose i-th column vector is $\boldsymbol{v}_i$, $i = 1, 2, \cdots, n$. Given a matrix $\boldsymbol{A}$, its component at i-th row and α-th column is denoted $(\boldsymbol{A})^i_\alpha$. Likewise, given a vector $\boldsymbol{v}$, its i-th component is denoted $(\boldsymbol{v})^i$.

Given any n vectors $\boldsymbol{v}_\alpha = (v^i_\alpha)^{n+1}_{i=1} \in \mathbb{R}^{n+1}$, $\alpha = 1, 2, \cdots, n$, the exterior product

$$\boldsymbol{w} := \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n$$

is the vector $\boldsymbol{w} = (w^i)_{i=1}^{n+1} \in \mathbb{R}^{n+1}$ whose components are defined by

$$w_i := (\mathrm{Cof}\, \boldsymbol{V})_1^i,$$

where $\boldsymbol{V}$ is a $(n+1)\times(n+1)$ square matrix whose last n column vectors are $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ (in this order) and $\mathrm{Cof}\, \boldsymbol{V}$ designates the cofactor matrix of $\boldsymbol{V}$ ($\mathrm{Cof}\, \boldsymbol{V} := (\det \boldsymbol{V})\boldsymbol{V}^{-\mathrm{T}}$ if the matrix $\boldsymbol{V}$ is invertible).

We conclude this section by enunciating two lemmas about the geometry of hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 2$, which are straightforward generalizations of similar lemmas given in [5] in the particular case $n = 2$ and $p = 2$. These lemmas show that some classical definitions and properties pertaining to hypersurfaces in $\mathbb{R}^{n+1}$ still hold under less stringent regularity assumptions than the usual ones (these definitions and properties are traditionally given and established under the assumptions that the immersions denoted $\boldsymbol{\theta}$ in Lemmas 2.1–2.2 below belong to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^{n+1})$).

Note that the functions $a_{\alpha\beta}$, $a^{\alpha\beta}$, $b_{\alpha\beta}$, b_α^σ appearing in these lemmas, respectively, denote the covariant and contravariant components of the first fundamental form and the covariant and mixed components of the second fundamental form. For such classical notions of the differential geometry of hypersurfaces, see [1, 3–4, 11–12].

The notations $(a_{\alpha\beta})$, $(a^{\alpha\beta})$, (b_α^β), and (g_{ij}) respectively designate matrices in $\mathbb{M}^n$ and $\mathbb{M}^{n+1}$ with components $a_{\alpha\beta}, a^{\alpha\beta}, b_\alpha^\beta$, and g_{ij}, the index or exponent denoted here α, or i, designating the row index.

Lemma 2.1 *Let ω be a domain in $\mathbb{R}^n$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion such that $\boldsymbol{a}_{n+1} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$, where*

$$\boldsymbol{a}_{n+1} := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n|} \quad \text{and} \quad \boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}.$$

Then the functions

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad b_{\alpha\beta} := -\partial_\alpha \boldsymbol{a}_{n+1} \cdot \boldsymbol{a}_\beta \quad \text{and} \quad b_\alpha^\sigma := a^{\beta\sigma} b_{\alpha\beta},$$

where $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$, belong to the space $\mathcal{C}^0(\overline{\omega})$. Besides,

$$b_{\alpha\beta} = b_{\beta\alpha}.$$

Lemma 2.2 *Let $1 < p < \infty$, let ω be a domain in $\mathbb{R}^n$ and let there be given an immersion $\boldsymbol{\theta} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ such that $\boldsymbol{a}_{n+1} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$, where*

$$\boldsymbol{a}_{n+1} := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n|} \quad \text{and} \quad \boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}.$$

Then the functions

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \quad \text{and} \quad b_{\alpha\beta} := -\partial_\alpha \boldsymbol{a}_{n+1} \cdot \boldsymbol{a}_\beta$$

belong to the space $L^{\frac{p}{2}}(\omega)$. Besides,

$$b_{\alpha\beta} = b_{\beta\alpha} \quad \text{a.e. in } \omega.$$

Furthermore, define the mapping $\boldsymbol{\Theta} : \omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ *by*

$$\boldsymbol{\Theta}(y, x_{n+1}) := \boldsymbol{\theta}(y) + x_{n+1}\boldsymbol{a}_{n+1}(y) \quad \text{for almost all } (y, x_{n+1}) \in \omega \times \mathbb{R}.$$

Then $\boldsymbol{\Theta} \in W^{1,p}(\omega \times]-\delta, \delta[\,; \mathbb{R}^{n+1})$ *for any* $\delta > 0$.

3 Nonlinear Korn Inequalities on a Hypersurface Without Boundary Conditions

In this section, we establish two nonlinear Korn inequalities on a hypersurface "without boundary conditions" (cf. Theorems 3.1 and 3.2); by "without boundary conditions", we mean that the values of the two immersions $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^{n+1}$ and $\tilde{\boldsymbol{\theta}} : \overline{\omega} \to \mathbb{R}^{n+1}$ on the boundary of ω are arbitrary.

The point of departure of the proofs of these nonlinear Korn inequalities on a hypersurface is the following generalization, established in [8, Lemma 2], of a geometric rigidity lemma, due to Friesecke, James and Müller [10] for $p = 2$, and later extended to $p > 1$ by Conti [9] (in [9–10], the mapping $\boldsymbol{\Theta}$ was the identity mapping).

For notational brevity, we shall drop the explicit dependence on the exponent p in the various constants found in the nonlinear Korn inequalities appearing below, as well as in their proofs.

Lemma 3.1 *Let* $1 < p < \infty$, *let* Ω *be a domain in* $\mathbb{R}^n$, *and let* $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ *be an immersion. Then there exists a constant* $C_1(\boldsymbol{\Theta})$ *such that, for all* $\tilde{\boldsymbol{\Theta}} \in W^{1,p}(\Omega; \mathbb{R}^n)$,

$$\inf_{\boldsymbol{R} \in \mathbb{O}^n_+} \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}\|_{\mathbb{L}^p(\Omega)} \leqslant C_1(\boldsymbol{\Theta}) \Big\| \inf_{\boldsymbol{R} \in \mathbb{O}^n_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}| \Big\|_{L^p(\Omega)}.$$

We now generalize the above geometric rigidity lemma to hypersurfaces $\boldsymbol{\theta}(\overline{\omega})$ in $\mathbb{R}^{n+1}$, instead of open subsets Ω of $\mathbb{R}^n$. The definition of the vector field $\boldsymbol{a}_{n+1}(\boldsymbol{\theta})$ is given in Lemmas 2.1–2.2.

Lemma 3.2 *Let* ω *be a domain in* $\mathbb{R}^n$, *let* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ *be an immersion such that* $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ *and let* $1 < p < \infty$. *Then there exists a constant* $C_2(\boldsymbol{\theta})$ *such that*

$$\begin{aligned}
&\inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} (\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}) \\
&\leqslant C_2(\boldsymbol{\theta}) \| \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} (|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}| + |\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}| + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|)\|_{L^p(\omega)}
\end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ *such that* $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$.

Proof Given a mapping $\boldsymbol{\theta}$ that satisfies all the assumptions of Lemma 3.2, define the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\omega} \times \mathbb{R}; \mathbb{R}^{n+1})$ by

$$\boldsymbol{\Theta}(y, x_{n+1}) := \boldsymbol{\theta}(y) + x_{n+1}\boldsymbol{a}_{n+1}(y) \quad \text{for all } (y, x_{n+1}) \in \overline{\omega} \times \mathbb{R}.$$

Likewise, given any mapping $\tilde{\boldsymbol{\theta}}$ that satisfies the assumptions of Lemma 3.2, define the mapping $\tilde{\boldsymbol{\Theta}} : \omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ by

$$\tilde{\boldsymbol{\Theta}}(y, x_{n+1}) := \tilde{\boldsymbol{\theta}}(y) + x_{n+1}\tilde{\boldsymbol{a}}_{n+1}(y) \quad \text{for all } (y, x_{n+1}) \in \omega \times \mathbb{R}.$$

Since $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^{n+1}$ is an immersion of class $\mathcal{C}^1$, there exists $\varepsilon = \varepsilon(\boldsymbol{\theta}) > 0$ such that the restriction, still denoted $\boldsymbol{\Theta}$ for convenience, of the mapping $\boldsymbol{\Theta}$ to the closure of set

$$\Omega = \Omega(\boldsymbol{\theta}) := \omega \times]-\varepsilon, \varepsilon[\,,$$

is an immersion of class $\mathcal{C}^1$. Since the restriction, still denoted $\tilde{\boldsymbol{\Theta}}$ for convenience, of the mapping $\tilde{\boldsymbol{\Theta}}$ to the set Ω belongs to the space $W^{1,p}(\Omega;\mathbb{R}^{n+1})$ by Lemma 2.2, all the assumptions of Lemma 3.1 are satisfied. Hence there exists a constant $c_1(\boldsymbol{\theta})$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}\|_{\mathbb{L}^p(\Omega)} \leqslant c_1(\boldsymbol{\theta}) \Big\| \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}| \Big\|_{L^p(\Omega)}$$

for any mapping $\tilde{\boldsymbol{\theta}}$ satisfying the assumptions of Lemma 3.2.

In what follows, our purpose is to find estimates of both the left-hand side and the right-hand side of the above inequality in terms of $\boldsymbol{\nabla}\boldsymbol{\theta}$ and $\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}$.

In order to find a lower bound of the left-hand side of the above inequality in terms of $\mathbb{L}^p(\omega)$-norms of $\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\nabla}\boldsymbol{\theta}$, we proceed as in the proof of Theorem 4.2 in Ciarlet, Malin, and Mardare [6]; we deduce in this fashion that there exists a constant $c_2(\boldsymbol{\theta})$ such that

$$\begin{aligned}
&\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \|\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}\|_{\mathbb{L}^p(\Omega)} \\
&\geqslant c_2(\boldsymbol{\theta})\varepsilon^{\frac{1}{p}} \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} (\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} \\
&\quad + \varepsilon\|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}),
\end{aligned}$$

where $\varepsilon = \varepsilon(\boldsymbol{\theta})$ is the constant defined above.

The next step is to find an upper bound of the $L^p(\Omega)$-norm of the

$$\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}|.$$

To this end, we first deduce that

$$\begin{aligned}
&\int_\Omega \Big(\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}| \Big)^p \mathrm{d}x = \int_\Omega \Big(\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \Big\{ \sum_i |\partial_i\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\partial_i\boldsymbol{\Theta}|^2 \Big\}^{\frac{p}{2}} \Big) \mathrm{d}x \\
&= \int_\omega \Big(\int_{-\varepsilon}^{\varepsilon} \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \Big\{ \sum_\alpha |(\partial_\alpha\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\partial_\alpha\boldsymbol{\theta}) + x_{n+1}(\partial_\alpha\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\partial_\alpha\boldsymbol{a}_{n+1})|^2 \\
&\quad + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^2 \Big\}^{\frac{p}{2}} \mathrm{d}x_{n+1} \Big) \mathrm{d}y \\
&\leqslant \int_\omega \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \Big(\int_{-\varepsilon}^{\varepsilon} \Big\{ \sum_\alpha |(\partial_\alpha\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\partial_\alpha\boldsymbol{\theta}) + x_{n+1}(\partial_\alpha\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\partial_\alpha\boldsymbol{a}_{n+1})|^2 \\
&\quad + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^2 \Big\}^{\frac{p}{2}} \mathrm{d}x_{n+1} \Big) \mathrm{d}y.
\end{aligned}$$

To estimate the integrand appearing in the right-hand side, we note that, given any vectors $\boldsymbol{v}_i \in \mathbb{R}^{n+1}$, $i = 1, \cdots, n+1$, we have

$$\Big(\sum_i |\boldsymbol{v}_i|^2\Big)^{\frac{p}{2}} \le (n+1)^{\frac{p}{2}-1} \sum_i |\boldsymbol{v}_i|^p, \quad \text{if } p \ge 2$$

(by Jensen's inequality applied to the convex function $t \in [0,\infty) \to t^{\frac{p}{2}} \in \mathbb{R}$, where $\frac{p}{2} \ge 1$) and

$$\Big(\sum_i |\boldsymbol{v}_i|^2\Big)^{\frac{p}{2}} \le \sum_i |\boldsymbol{v}_i|^p, \quad \text{if } 0 < p < 2$$

(by applying recursively the inequality $(a+t)^{\frac{p}{2}} \le a^{\frac{p}{2}} + t^{\frac{p}{2}}$ for all $a \ge 0$ and $t \ge 0$, where $0 < \frac{p}{2} < 1$). Combining these inequalities with the previous one, we next deduce that

$$\begin{aligned}
&\int_\Omega \Big(\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}|\Big)^p \mathrm{d}x \\
&\leqslant c_3 \int_\omega \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \Big\{\int_{-\varepsilon}^{\varepsilon} \Big(\sum_\alpha |(\partial_\alpha\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\partial_\alpha\boldsymbol{\theta}) + x_{n+1}(\partial_\alpha\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\partial_\alpha\boldsymbol{a}_{n+1})|^p \\
&\quad + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^p\Big) \mathrm{d}x_{n+1}\Big\} \mathrm{d}y,
\end{aligned}$$

where $c_3 = \max(1, (n+1)^{\frac{p}{2}-1})$. To further estimate the integrand appearing in the right-hand side, we note that, given any vectors $\boldsymbol{u}_\alpha \in \mathbb{R}^{n+1}$ and $\boldsymbol{v}_\alpha \in \mathbb{R}^{n+1}$, $\alpha = 1, \cdots, n$, and any $x_3 \in \mathbb{R}$, we have

$$|\boldsymbol{u}_\alpha + x_3\boldsymbol{v}_\alpha|^p \le (|\boldsymbol{u}_\alpha| + |x_3||\boldsymbol{v}_\alpha|)^p \le 2^{p-1}(|\boldsymbol{u}_\alpha|^p + |x_3||\boldsymbol{v}_\alpha|^p), \quad \text{if } p \ge 1$$

(again by Jensen's inequality). Using this inequality to estimate the right-hand side of the previous one, we finally deduce that

$$\begin{aligned}
&\int_\Omega \Big(\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}|\Big)^p \mathrm{d}x \\
&\leqslant c_3 \int_\omega \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} \Big\{2^p\varepsilon \sum_\alpha |\partial_\alpha\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\partial_\alpha\boldsymbol{\theta}|^p + 2^p\frac{\varepsilon^{p+1}}{p+1}\sum_\alpha |\partial_\alpha\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\partial_\alpha\boldsymbol{a}_{n+1}|^p \\
&\quad + 2\varepsilon|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^p\Big\} \mathrm{d}y.
\end{aligned}$$

Hence there exists a constant $c_4(\boldsymbol{\theta})$ such that

$$\begin{aligned}
&\Big\|\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} |\boldsymbol{\nabla}\tilde{\boldsymbol{\Theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Theta}|\Big\|_{L^p(\Omega)} \\
&\leqslant c_4(\boldsymbol{\theta})\varepsilon^{\frac{1}{p}} \Big\|\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+} (|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}| + \varepsilon|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}| + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|)\Big\|_{L^p(\omega)}.
\end{aligned}$$

Consequently, the announced inequality follows with

$$C_2(\boldsymbol{\theta}) = \frac{c_1(\boldsymbol{\theta})c_4(\boldsymbol{\theta})\max(1,\varepsilon)}{c_2(\boldsymbol{\theta})\min(1,\varepsilon)}.$$

We are now in a position to establish our first nonlinear Korn inequality on a hypersurface without boundary conditions.

Theorem 3.1 *Let ω be a domain in $\mathbb{R}^n$, let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion such that $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ and let $1 < p < \infty$. Then there exists a constant $C_3(\boldsymbol{\theta})$ such that*

$$\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}(\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)})$$
$$\leqslant C_3(\boldsymbol{\theta})(\|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)})$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$.

Proof The proof consists in finding an upper bound of the right-hand side of the inequality of Lemma 3.2 featuring the fundamental forms of $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ instead of $\boldsymbol{\nabla}\boldsymbol{\theta}$, $\boldsymbol{\nabla}\boldsymbol{a}_{n+1}$, $\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1}$. So, we begin by expressing each term $|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}|$, $|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}|$, $|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|$ in terms of $(a_{\alpha\beta})$, $(b_{\alpha\beta})$, $(\tilde{a}_{\alpha\beta})$, $(\tilde{b}_{\alpha\beta})$.

Let $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ satisfy the assumptions of the theorem. Then

$$\boldsymbol{F} := [\boldsymbol{\nabla}\boldsymbol{\theta} \quad \boldsymbol{a}_{n+1}] \in \mathcal{C}^0(\overline{\omega}; \mathbb{M}^{n+1}) \quad \text{and} \quad \tilde{\boldsymbol{F}} := [\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} \quad \tilde{\boldsymbol{a}}_{n+1}] \in L^p(\omega; \mathbb{M}^{n+1}),$$

which in turn imply that

$$\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \begin{pmatrix} (a_{\alpha\beta}) & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} \in \mathcal{C}^0(\overline{\omega}; \mathbb{S}^{n+1}_>) \text{ and } \tilde{\boldsymbol{F}}^{\mathrm{T}}\tilde{\boldsymbol{F}} = \begin{pmatrix} (\tilde{a}_{\alpha\beta}) & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} \in L^{\frac{p}{2}}(\omega; \mathbb{S}^{n+1}_>).$$

The matrix polar decomposition theorem shows that

$$\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{U} \quad \text{in } \overline{\omega} \quad \text{and} \quad \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{Q}}\tilde{\boldsymbol{U}} \quad \text{a.e. in } \omega,$$

where

$$\boldsymbol{U} := (\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F})^{\frac{1}{2}} \in \mathcal{C}^0(\overline{\omega}; \mathbb{S}^{n+1}_>) \quad \text{and} \quad \boldsymbol{Q} := \boldsymbol{F}\boldsymbol{U}^{-1} \in \mathcal{C}^0(\overline{\omega}; \mathbb{O}^{n+1}_+),$$
$$\tilde{\boldsymbol{U}} := (\tilde{\boldsymbol{F}}^{\mathrm{T}}\tilde{\boldsymbol{F}})^{\frac{1}{2}} \in L^p(\omega; \mathbb{S}^{n+1}_>) \quad \text{and} \quad \tilde{\boldsymbol{Q}} := \tilde{\boldsymbol{F}}\tilde{\boldsymbol{U}}^{-1} \in L^\infty(\omega; \mathbb{O}^{n+1}_+).$$

Hence

$$|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}|^2 + |\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^2 = |\tilde{\boldsymbol{F}} - \boldsymbol{R}\boldsymbol{F}|^2 = |\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{U}} - \boldsymbol{R}\boldsymbol{Q}\boldsymbol{U}|^2 = |\tilde{\boldsymbol{U}} - \tilde{\boldsymbol{Q}}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{Q}\boldsymbol{U}|^2.$$

Next, combining the Weingarten equations

$$\boldsymbol{\nabla}\boldsymbol{a}_{n+1} = -(\boldsymbol{\nabla}\boldsymbol{\theta})\boldsymbol{S}, \quad \text{where } \boldsymbol{S} := (b^\sigma_\alpha) = (a_{\alpha\beta})^{-1}(b_{\alpha\beta}),$$
$$\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} = -(\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}})\tilde{\boldsymbol{S}}, \quad \text{where } \tilde{\boldsymbol{S}} := (\tilde{b}^\sigma_\alpha) = (\tilde{a}_{\alpha\beta})^{-1}(\tilde{b}_{\alpha\beta})$$

with the relations

$$\boldsymbol{\nabla}\boldsymbol{\theta} = \boldsymbol{Q}\begin{pmatrix} (a_{\alpha\beta})^{\frac{1}{2}} \\ \boldsymbol{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{Q}}\begin{pmatrix} (\tilde{a}_{\alpha\beta})^{\frac{1}{2}} \\ \boldsymbol{0} \end{pmatrix}$$

(deduced from the above expressions of $\boldsymbol{F}$, $\boldsymbol{U}$, $\widetilde{\boldsymbol{F}}$ and $\widetilde{\boldsymbol{U}}$) implies that

$$
\begin{aligned}
|\boldsymbol{\nabla}\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}| &= \left| \widetilde{\boldsymbol{Q}} \begin{pmatrix} (\widetilde{a}_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \widetilde{\boldsymbol{S}} - \boldsymbol{R}\boldsymbol{Q} \begin{pmatrix} (a_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \boldsymbol{S} \right| \\
&= \left| \begin{pmatrix} (\widetilde{a}_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \widetilde{\boldsymbol{S}} - \widetilde{\boldsymbol{Q}}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{Q} \begin{pmatrix} (a_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \boldsymbol{S} \right|.
\end{aligned}
$$

Then we infer from the above relations that

$$
\begin{aligned}
&\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_{+}} (|\boldsymbol{\nabla}\widetilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}| + |\boldsymbol{\nabla}\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}| + |\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|) \\
&\leqslant \sqrt{3} \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_{+}} (|\boldsymbol{\nabla}\widetilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}|^2 + |\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|^2 + |\boldsymbol{\nabla}\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}|^2)^{\frac{1}{2}} \\
&= \sqrt{3} \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_{+}} \left(|\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{Q}}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{Q}\boldsymbol{U}|^2 + \left| \begin{pmatrix} (\widetilde{a}_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \widetilde{\boldsymbol{S}} - \widetilde{\boldsymbol{Q}}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{Q} \begin{pmatrix} (a_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \boldsymbol{S} \right|^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_{+}} (|\boldsymbol{\nabla}\widetilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}| + |\boldsymbol{\nabla}\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}| + |\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}|) \\
&\leqslant \sqrt{3} \left(|\widetilde{\boldsymbol{U}} - \boldsymbol{U}|^2 + \left| \begin{pmatrix} (\widetilde{a}_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \widetilde{\boldsymbol{S}} - \begin{pmatrix} (a_{\alpha\beta})^{\frac{1}{2}} \\ \mathbf{0} \end{pmatrix} \boldsymbol{S} \right|^2 \right)^{\frac{1}{2}} \\
&\leqslant \sqrt{3}(|(\widetilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}| + |(\widetilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\widetilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})|).
\end{aligned}
$$

Then the announced inequality is obtained with $C_3(\boldsymbol{\theta}) = \sqrt{3}\, C_2(\boldsymbol{\theta})$ by using the above inequality in the right-hand side of the inequality of Lemma 3.2.

In the remaining of this section, we establish our second nonlinear Korn inequality on a hypersurface without boundary conditions. It shows that the infimum in the left-hand side of the nonlinear Korn inequality of Theorem 3.1 can be dropped if a weaker norm of $(\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta})$ and $(\widetilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1})$ is added to its right-hand side.

The key for proving such a nonlinear Korn inequality is the following lemma.

Lemma 3.3 *Let ω be a domain in $\mathbb{R}^n$, let $1 < p < \infty$, let $\boldsymbol{\theta} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ and let $\mathbb{U} \subset \mathbb{M}^{n+1}$ be any non-empty set. Then there exists a constant $C_4(\boldsymbol{\theta})$ such that, for all $\widetilde{\boldsymbol{\theta}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$,*

$$
\|\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} \leqslant C_4(\boldsymbol{\theta}) \left(\|\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)} + \inf_{\boldsymbol{R}\in\mathbb{U}} \|\boldsymbol{\nabla}\widetilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} \right).
$$

Proof Assume on the contrary that, for each $k \in \mathbb{N}^*$, there exists a vector field $\widetilde{\boldsymbol{\theta}}_k \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ such that

$$
\|\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} > k \left(\|\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)} + \inf_{\boldsymbol{R}\in\mathbb{U}} \|\boldsymbol{\nabla}\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} \right).
$$

Let

$$
\eta_k := \|\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} \quad \text{and} \quad \boldsymbol{u}_k := \frac{1}{\eta_k}(\widetilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}).
$$

Then, for each $k \geqslant 1$,

$$\begin{aligned}&\eta_k > 0, \quad \|\boldsymbol{u}_k\|_{\boldsymbol{W}^{1,p}(\omega)} = 1,\\ &\|\boldsymbol{u}_k\|_{\boldsymbol{L}^p(\omega)} + \inf_{\boldsymbol{R}\in\mathbb{U}} \Big\| \frac{1}{\eta_k}(\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}_k - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}) \Big\|_{\mathbb{L}^p(\omega)} < \frac{1}{k}.\end{aligned}$$

Since the space $W^{1,p}(\omega;\mathbb{R}^{n+1})$ $(1 < p < \infty)$ is reflexive and its inclusion in $L^p(\omega;\mathbb{R}^{n+1})$ is compact, it follows that there exists a subsequence (l) of (k) such that, as $l \to \infty$,

$$\boldsymbol{u}_l \to \boldsymbol{u} \quad \text{in } L^p(\omega;\mathbb{R}^{n+1}) \quad \text{and} \quad \boldsymbol{\nabla}\boldsymbol{u}_l \rightharpoonup \boldsymbol{\nabla}\boldsymbol{u} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

Then the above inequality implies that $\boldsymbol{u} = \boldsymbol{0}$ in ω and that there exists a sequence $(\boldsymbol{R}_l) \subset \mathbb{U}$ such that, as $l \to \infty$,

$$\boldsymbol{\nabla}\boldsymbol{u}_l + \frac{1}{\eta_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla}\boldsymbol{\theta} \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

The above relation combined with the weak convergence $\boldsymbol{\nabla}\boldsymbol{u}_l \rightharpoonup \boldsymbol{0}$ in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$ implies that

$$\boldsymbol{F}_l := \frac{1}{\eta_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla}\boldsymbol{\theta} \rightharpoonup \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

Since the sequence $(\boldsymbol{F}_l)$ belongs to a subspace of $L^p(\omega;\mathbb{M}^{(n+1)\times n})$ of finite dimension, namely

$$\{\boldsymbol{F}\boldsymbol{\nabla}\boldsymbol{\theta};\ \boldsymbol{F} \in \mathbb{M}^{n+1}\},$$

it follows that

$$\boldsymbol{F}_l \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

Hence

$$\boldsymbol{\nabla}\boldsymbol{u}_l \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

Since in addition $\boldsymbol{u}_l \to \boldsymbol{0}$ in $L^p(\omega;\mathbb{R}^{n+1})$, it follows that

$$\boldsymbol{u}_l \to \boldsymbol{0} \quad \text{in } W^{1,p}(\omega;\mathbb{R}^{n+1}) \quad \text{as } l \to \infty.$$

This contradicts that $\|\boldsymbol{u}_l\|_{\boldsymbol{W}^{1,p}(\omega)} = 1$ for all $l \geqslant 1$. Therefore, the announced inequality holds.

We are now in a position to establish our second nonlinear Korn inequality on a hypersurface without boundary conditions.

Theorem 3.2 *Let ω be a domain in $\mathbb{R}^n$, let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^{n+1})$ be an immersion such that $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^{n+1})$, and let $1 < p < \infty$. Then there exists a constant $C_5(\boldsymbol{\theta})$ such that*

$$\begin{aligned}&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}\\ &\leqslant C_5(\boldsymbol{\theta})(\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}\\ &\quad + \|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)})\end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega;\mathbb{R}^{n+1})$.

Proof The proof is an immediate consequence of Theorem 3.1 combined with the inequality of Lemma 3.3 with $\mathbb{U} = \mathbb{O}^{n+1}_+$ (applied twice, to estimate both $\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)}$ and $\|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}$).

4 Nonlinear Korn Inequality on a Hypersurface with Boundary Conditions

In this section, we establish a nonlinear Korn inequality on a hypersurface for mappings subjected to specific boundary conditions (cf. Theorems 4.1–4.2 below). We consider two types of boundary conditions, one corresponding to the case where the hypersurface is kept fixed on a portion of its boundary, and one corresponding to the case where both the hypersurface and its positively-oriented unit normal vector field are kept fixed on a portion of the boundary of the hypersurface.

We begin by showing that it is possible to drop the term $\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)}$ in the right-hand side of the inequality of Lemma 3.3 by imposing a boundary condition on a portion γ_0 of $\partial\omega$. Note however the stronger assumptions of Lemma 4.1 compared with those of Lemma 3.3: The extra regularity for $\boldsymbol{\theta}$ and the particular choice $\mathbb{U} = \mathbb{O}^{n+1}_+$.

Lemma 4.1 *Let ω be a domain in $\mathbb{R}^n$, let $1 < p < \infty$, let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion, and let γ_0 be a relatively open subset of $\partial\omega$ such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in any affine subspace of $\mathbb{R}^{n+1}$ of dimension $\leqslant (n-1)$. Then there exists a constant $C_6(\gamma_0, \boldsymbol{\theta})$ such that*

$$\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} \leqslant C_6(\gamma_0, \boldsymbol{\theta}) \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} \|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)}$$

for all $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ that satisfies $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0.

Proof Assume on the contrary that for all $k \in \mathbb{N}^*$, there exists $\tilde{\boldsymbol{\theta}}_k \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ satisfying $\tilde{\boldsymbol{\theta}}_k = \boldsymbol{\theta}$ on γ_0 such that

$$\|\tilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} > k \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} \|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}_k - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)}.$$

It follows that $\|\tilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} > 0$ and there exists $\boldsymbol{R}_k \in \mathbb{O}^{n+1}_+$ such that $\|\tilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} > k\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}_k - \boldsymbol{R}_k\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)}$.

Let

$$\eta_k := \|\tilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} \quad \text{and} \quad \boldsymbol{u}_k := \frac{1}{\eta_k}(\tilde{\boldsymbol{\theta}}_k - \boldsymbol{\theta}).$$

Then, for all $k \in \mathbb{N}^*$,

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}^{1,p}(\omega)} = 1 \quad \text{and} \quad \frac{1}{k} > \Big\|\boldsymbol{\nabla}\boldsymbol{u}_k + \frac{1}{\eta_k}(\boldsymbol{I} - \boldsymbol{R}_k)\boldsymbol{\nabla}\boldsymbol{\theta}\Big\|_{\mathbb{L}^p(\omega)}.$$

Consequently, there exist $\boldsymbol{u} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ and a subsequence (l) of (k) such that, as $l \to \infty$,

$$\begin{aligned}
&\|\boldsymbol{u}_l\|_{\boldsymbol{W}^{1,p}(\omega)} = 1, \quad \boldsymbol{u}_l \to \boldsymbol{u} \quad \text{in } L^p(\omega; \mathbb{R}^{n+1}),\\
&\boldsymbol{\nabla}\boldsymbol{u}_l \rightharpoonup \boldsymbol{\nabla}\boldsymbol{u} \quad \text{and} \quad \boldsymbol{\nabla}\boldsymbol{u}_l + \frac{1}{\eta_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla}\boldsymbol{\theta} \to \boldsymbol{0} \quad \text{in } L^p(\omega; \mathbb{M}^{(n+1)\times n}).
\end{aligned}$$

As in the proof of Lemma 3.3, the last two convergences together imply that

$$\boldsymbol{\nabla}\boldsymbol{u}_l \to \boldsymbol{\nabla}\boldsymbol{u} \quad \text{and} \quad \frac{1}{\eta_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla}\boldsymbol{\theta} \to -\boldsymbol{\nabla}\boldsymbol{u} \quad \text{in } L^p(\omega; \mathbb{M}^{(n+1)\times n}).$$

Since then $\boldsymbol{u}_l \to \boldsymbol{u}$ in $W^{1,p}(\omega;\mathbb{R}^{n+1})$, we deduce that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\omega)} = 1$.

We now prove that $\boldsymbol{u} = \boldsymbol{0}$, which will end the proof since it contradicts that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\omega)} = 1$. Let $\eta \in [0,\infty]$ and let $\boldsymbol{R} \in \mathbb{O}^{n+1}_+$ be such that, for some subsequence (m) of (l),

$$\eta_m \to \eta \quad \text{and} \quad \boldsymbol{R}_m \to \boldsymbol{R} \quad \text{as } m \to \infty.$$

We distinguish three cases: $\eta = \infty$, $0 < \eta < \infty$ and $\eta = 0$.

First, if $\eta = \infty$, then $\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{\nabla\theta} \to \boldsymbol{0}$ in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$. Hence $-\boldsymbol{\nabla u} = \boldsymbol{0}$ in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$. But $\boldsymbol{u} = \boldsymbol{0}$ on γ_0 since $\boldsymbol{u}_l = \boldsymbol{0}$ on γ_0 and $\boldsymbol{u}_l \to \boldsymbol{u}$ in $W^{1,p}(\omega;\mathbb{R}^{n+1})$. Therefore $\boldsymbol{u} = \boldsymbol{0}$.

Second, if $0 < \eta < \infty$, then $\frac{1}{\eta}(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{\nabla\theta} = -\boldsymbol{\nabla u}$ in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$. Hence there exists $\boldsymbol{a} \in \mathbb{R}^{n+1}$ such that $\boldsymbol{u} = \boldsymbol{a} + \frac{1}{\eta}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\theta}$ in $W^{1,p}(\omega;\mathbb{R}^{n+1})$. Since $\tilde{\boldsymbol{\theta}}_k = \boldsymbol{\theta}$ on γ_0 for all $k \in \mathbb{N}^*$, and since $\boldsymbol{u}_l = \frac{1}{\eta_l}(\tilde{\boldsymbol{\theta}}_l - \boldsymbol{\theta}) \to \boldsymbol{u}$ in $W^{1,p}(\omega;\mathbb{R}^{n+1})$, we have that $\boldsymbol{u} = \boldsymbol{0}$ on γ_0. Therefore $\boldsymbol{a} + \frac{1}{\eta}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\theta} = \boldsymbol{0}$ on γ_0. Consequently,

$$\eta\boldsymbol{a} + (\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\theta}(y) = \boldsymbol{0} \quad \text{for all } y \in \gamma_0.$$

This means that the isometry $\boldsymbol{r} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\boldsymbol{r}(x) := \eta\boldsymbol{a} + \boldsymbol{R}x$ for all $x \in \mathbb{R}^{n+1}$, satisfies $\boldsymbol{r}(x) = x$ for all $x \in \boldsymbol{\theta}(\gamma_0)$. Since the set of all fixed points of an isometry of $\mathbb{R}^{n+1}$ is either $\mathbb{R}^{n+1}$ (if the isometry is the identity mapping) or an affine subspace of $\mathbb{R}^{n+1}$ of dimension $\leqslant (n-1)$ (otherwise), the assumption on $\boldsymbol{\theta}(\gamma_0)$ of Lemma 4.1 implies that $\boldsymbol{r}(x) = x$ for all $x \in \mathbb{R}^{n+1}$. Therefore, $\eta\boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{R} = \boldsymbol{I}$. Since $\eta > 0$, this next implies that $\boldsymbol{u} = \boldsymbol{0}$.

Finally, assume that $\eta = 0$. Then the convergence

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{\nabla\theta} \to -\boldsymbol{\nabla u}$$

in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$ implies that each component of the matrix $(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{\nabla\theta}$ vanishes almost everywhere in ω. Hence the matrix $(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{\nabla\theta}(y)$ coincides with the zero matrix $\mathbb{M}^{(n+1)\times n}$ for almost all $y \in \omega$. This means that

$$\boldsymbol{R}\boldsymbol{v} = \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in S := \{\partial_1\boldsymbol{\theta}(y), \cdots, \partial_n\boldsymbol{\theta}(y);\ y \in \omega \setminus N\}$$

for some negligible subset N of ω. Since the set S contains (at least) n linearly independent vectors (recall that $\boldsymbol{\theta}$ is an immersion by assumption), one has $\boldsymbol{R} = \boldsymbol{I}$.

Furthermore, there exists a subsequence of $\left(\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{\nabla\theta}\right)_m$, still indexed by m for simplicity, that converges almost everywhere in ω. Since $\boldsymbol{\theta}$ is an immersion, there exist n linearly independent vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ (for example, $\partial_1\boldsymbol{\theta}(y), \cdots, \partial_n\boldsymbol{\theta}(y)$ at some point $y \in \omega$) such that, for all $\alpha \in \{1, \cdots, n\}$, as $m \to \infty$,

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{v}_\alpha \to \boldsymbol{w}_\alpha \quad \text{in } \mathbb{R}^{n+1} \quad \text{for some } \boldsymbol{w}_\alpha \in \mathbb{R}^{n+1}.$$

It follows that, for each $\boldsymbol{v} \in E := \text{span}\,\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$,

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{v} \quad \text{converges in } \mathbb{R}^{n+1}.$$

Hence, given any orthonormal basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ in E, for all $\alpha \in \{1, 2, \cdots, n\}$,

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{e}_\alpha \quad \text{converges in } \mathbb{R}^{n+1}.$$

In order to prove that the sequence $\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)$ converges in $\mathbb{M}^{n+1}$, it suffices to prove that the sequence $\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{e}_{n+1}$, where $\boldsymbol{e}_{n+1} \in E^\perp$ is the unique unit vector such that $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ is a positive basis in $\mathbb{R}^{n+1}$, converges in $\mathbb{R}^{n+1}$. Let

$$\boldsymbol{r}_i^m := \boldsymbol{R}_m \boldsymbol{e}_i \in \mathbb{R}^{n+1} \quad \text{for all } i \in \{1, 2, \cdots, n+1\}.$$

Then, for all $\alpha \in \{1, 2, \cdots, n\}$, there exists $\boldsymbol{f}_\alpha \in \mathbb{R}^{n+1}$ such that

$$\frac{1}{\eta_m}(\boldsymbol{e}_\alpha - \boldsymbol{r}_\alpha^m) \to \boldsymbol{f}_\alpha \quad \text{in } \mathbb{R}^{n+1}.$$

Since $\boldsymbol{R}_m \in \mathbb{O}_+^{n+1}$, one has $\boldsymbol{r}_{n+1}^m = \boldsymbol{r}_1^m \wedge \boldsymbol{r}_2^m \wedge \cdots \wedge \boldsymbol{r}_n^m$. Consequently,

$$\begin{aligned}
&\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{e}_{n+1} \\
&= \frac{1}{\eta_m}(\boldsymbol{e}_{n+1} - \boldsymbol{r}_{n+1}^m) = \frac{1}{\eta_m}(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n - \boldsymbol{r}_1^m \wedge \cdots \wedge \boldsymbol{r}_n^m) \\
&= \frac{1}{\eta_m}(\boldsymbol{e}_1 - \boldsymbol{r}_1^m) \wedge \boldsymbol{e}_2 \wedge \cdots \wedge \boldsymbol{e}_n + \frac{1}{\eta_m}\boldsymbol{r}_1^m \wedge (\boldsymbol{e}_2 - \boldsymbol{r}_2^m) \wedge \boldsymbol{e}_3 \wedge \cdots \wedge \boldsymbol{e}_n \\
&\quad + \cdots + \frac{1}{\eta_m}\boldsymbol{r}_1^m \wedge \cdots \wedge \boldsymbol{r}_{n-1}^m(\boldsymbol{e}_n - \boldsymbol{r}_n^m).
\end{aligned}$$

Therefore, since $\boldsymbol{r}_\alpha^m = \boldsymbol{R}_m \boldsymbol{e}_\alpha \to \boldsymbol{I}\boldsymbol{e}_\alpha = \boldsymbol{e}_\alpha$ as $m \to \infty$, we have

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{e}_{n+1} \to \boldsymbol{f}_1 \wedge \boldsymbol{e}_2 \wedge \cdots \wedge \boldsymbol{e}_n + \boldsymbol{e}_1 \wedge \boldsymbol{f}_2 \wedge \boldsymbol{e}_3 \wedge \cdots \wedge \boldsymbol{e}_n + \cdots + \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{n-1} \wedge \boldsymbol{f}_n.$$

The above convergences show that there exists a matrix $\boldsymbol{A} \in \mathbb{M}^{n+1}$ such that, on the one hand,

$$\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m) \to \boldsymbol{A} \quad \text{as } m \to \infty.$$

On the other hand, the relation $\boldsymbol{R}_m^{\mathrm{T}}\boldsymbol{R}_m = \boldsymbol{I}$ implies that, for all m,

$$\boldsymbol{R}_m^{\mathrm{T}}\left(\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\right) = \frac{1}{\eta_m}(\boldsymbol{R}_m^{\mathrm{T}} - \boldsymbol{I}) = -\left(\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\right)^{\mathrm{T}}.$$

Letting $m \to \infty$ in the above equality (recall that $\boldsymbol{R}_m \to \boldsymbol{I}$ as $m \to \infty$) we get $\boldsymbol{I}^{\mathrm{T}}\boldsymbol{A} = -\boldsymbol{A}^{\mathrm{T}}$, which means that $\boldsymbol{A}$ is antisymmetric.

To summarize, we proved that, if $\eta_m \to \eta = 0$ as $m \to \infty$, then, even if it means extracting a subsequence,

$$\begin{aligned}
&\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m)\boldsymbol{\nabla\theta} \to -\boldsymbol{\nabla u} \quad \text{in } L^p(\omega; \mathbb{M}^{(n+1)\times n}) \quad \text{as } m \to \infty, \\
&\frac{1}{\eta_m}(\boldsymbol{I} - \boldsymbol{R}_m) \to \boldsymbol{A} \in \mathbb{A}^{n+1} \quad \text{as } m \to \infty.
\end{aligned}$$

Hence $-\boldsymbol{\nabla u} = \boldsymbol{A}\boldsymbol{\nabla\theta}$ almost everywhere in ω, which in turn implies that there exists $\boldsymbol{a} \in \mathbb{R}^{n+1}$ such that

$$\boldsymbol{u}(y) = \boldsymbol{a} - \boldsymbol{A}\boldsymbol{\theta}(y) \quad \text{for almost all } y \in \omega.$$

Besides, the trace of $\boldsymbol{u}$ on γ_0 vanishes (since $\boldsymbol{u}_l \to \boldsymbol{u}$ in $W^{1,p}(\omega;\mathbb{R}^{n+1})$ as $l \to \infty$ and $\boldsymbol{u}_l := \frac{1}{\eta_l}(\tilde{\boldsymbol{\theta}}_l - \boldsymbol{\theta})$ vanishes on γ_0). Therefore the infinitesimal isometry $\boldsymbol{\mu} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\boldsymbol{\mu}(x) = \boldsymbol{a} - \boldsymbol{A}x$ for all $x \in \mathbb{R}^{n+1}$ vanishes on the set $\boldsymbol{\theta}(\gamma_0)$. In other words,

$$\boldsymbol{\theta}(\gamma_0) \subset \operatorname{Ker}\boldsymbol{\mu} := \{x \in \mathbb{R}^{n+1};\ \boldsymbol{A}x = \boldsymbol{a}\}.$$

Then the assumption on $\boldsymbol{\theta}(\gamma_0)$ of Lemma 4.1 implies that $\boldsymbol{u} = \boldsymbol{0}$, since the set $\operatorname{Ker}\boldsymbol{\mu}$ is either $\mathbb{R}^{n+1}$ (if $\boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{A} = \boldsymbol{0}$) or an affine subspace of $\mathbb{R}^{n+1}$ of dimension $\leqslant (n-1)$ (otherwise). To prove the last assertion, one notices on the one hand that either $\operatorname{Ker}\boldsymbol{\mu} = \emptyset$ or $\operatorname{Ker}\boldsymbol{\mu} = x_0 + \operatorname{Ker}\boldsymbol{A}$, where x_0 is a particular solution of $\boldsymbol{A}x = \boldsymbol{a}$; but on the other hand $\operatorname{Ker}\boldsymbol{A} \oplus \operatorname{Im}\boldsymbol{A} = \mathbb{R}^{n+1}$, so $\dim(\operatorname{Ker}\boldsymbol{A}) = (n+1) - \operatorname{Im}\boldsymbol{A}$, and $\dim(\operatorname{Im}\boldsymbol{A})$ is either 0 (if $\boldsymbol{A} = \boldsymbol{0}$) or $\dim(\operatorname{Im}\boldsymbol{A}) \geqslant 2$ if $\boldsymbol{A} \neq \boldsymbol{0}$.

In order to prove our first nonlinear Korn inequality of this section (cf. Theorem 4.1 below), where $\tilde{\boldsymbol{a}}_{n+1}$ and $\boldsymbol{a}_{n+1}$ do not necessarily coincide on γ_0, we need to supplement the estimate of $\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)}$ provided by Lemma 4.1 by an estimate of $\|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}$. This is the object of the following lemma.

Lemma 4.2 *Let $\omega, p, \boldsymbol{\theta}$ and γ_0 satisfy the assumptions of Lemma* 4.1. *Assume in addition that $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta})$ belongs to $\mathcal{C}^1(\overline{\omega};\mathbb{R}^{n+1})$. Then there exists a constant $C_7(\gamma_0,\boldsymbol{\theta})$ such that*

$$\begin{aligned}
&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}\\
&\leqslant C_7(\gamma_0,\boldsymbol{\theta}) \inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}(\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla\theta}\|_{\mathbb{L}^p(\omega)}\\
&\quad + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)})
\end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ and $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0.

Proof Assume that this assertion is false. Then for all $k \in \mathbb{N}^*$, there exist $\tilde{\boldsymbol{\theta}}^k \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ and $\boldsymbol{R}_k \in \mathbb{O}^{n+1}_+$ such that $\tilde{\boldsymbol{a}}^k_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}^k) \in W^{1,p}(\omega;\mathbb{R}^{n+1})$, $\tilde{\boldsymbol{\theta}}^k = \boldsymbol{\theta}$ on γ_0 and

$$\begin{aligned}
&\|\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}\\
&> k(\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}^k - \boldsymbol{R}_k\boldsymbol{\nabla\theta}\|_{\mathbb{L}^p(\omega)} + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}_k\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}_k\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}).
\end{aligned}$$

Let

$$\begin{aligned}
&\mu_k := \|\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)},\\
&\tilde{\boldsymbol{\xi}}_k := \frac{1}{\mu_k}(\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}), \quad \tilde{\boldsymbol{\eta}}_k := \frac{1}{\mu_k}(\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}).
\end{aligned}$$

Then $\|\tilde{\boldsymbol{\xi}}_k\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}_k\|_{\boldsymbol{W}^{1,p}(\omega)} = 1$ for all $k \in \mathbb{N}^*$ and, by using the same arguments as in the proof of Lemma 4.1, there exists a subsequence (l) of (k) such that, for some $\tilde{\boldsymbol{\xi}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$,

$\tilde{\boldsymbol{\eta}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$, $\mu \in [0,\infty]$ and $\boldsymbol{R} \in \mathbb{O}^{n+1}_{+}$,

$$\begin{aligned} &\tilde{\boldsymbol{\xi}}_l \to \tilde{\boldsymbol{\xi}} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_l \to \tilde{\boldsymbol{\eta}} \quad \text{in } W^{1,p}(\omega;\mathbb{R}^{n+1}),\\ &\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{\nabla\theta} \to -\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}),\\ &\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{a}_{n+1} \to -\tilde{\boldsymbol{\eta}} \quad \text{in } L^p(\omega;\mathbb{R}^{n+1}). \end{aligned}$$

On the other hand, Lemma 4.1 shows that, for some constant $C_6(\gamma_0,\boldsymbol{\theta})$, $\|\tilde{\boldsymbol{\theta}}_l - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} \leqslant C_6(\gamma_0,\boldsymbol{\theta})\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}_l - \boldsymbol{R}_l\boldsymbol{\nabla\theta}\|_{\mathbb{L}^p(\omega)}$, which in turn implies that

$$\|\tilde{\boldsymbol{\xi}}_l\|_{\boldsymbol{W}^{1,p}(\omega)} \leqslant C_6(\gamma_0,\boldsymbol{\theta})\Big\|\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}}_l + \frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{\nabla\theta}\Big\|_{\mathbb{L}^p(\omega)}.$$

Hence $\|\tilde{\boldsymbol{\xi}}_l\|_{\boldsymbol{W}^{1,p}(\omega)} \to 0$ as $l \to \infty$, so that $\tilde{\boldsymbol{\xi}} = \boldsymbol{0}$. Therefore, as $l \to \infty$,

$$\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{\nabla\theta} \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}).$$

Since $\boldsymbol{\theta}$ is an immersion, the above relation implies (as in the proof of Lemma 4.1) that, as $l \to \infty$,

$$\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l) \to \boldsymbol{0} \quad \text{in } \mathbb{M}^{n+1}.$$

Therefore, $\tilde{\boldsymbol{\eta}} = -\lim_{l\to\infty}\Big(\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{a}_{n+1}\Big) = \boldsymbol{0}$. Hence

$$0 = \|\tilde{\boldsymbol{\xi}}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}\|_{\boldsymbol{W}^{1,p}(\omega)} = \lim_{l\to\infty}(\|\tilde{\boldsymbol{\xi}}_l\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}_l\|_{\boldsymbol{W}^{1,p}(\omega)}) = 1,$$

a contradiction.

We are now in a position to establish our first nonlinear Korn inequality on a hypersurface with boundary conditions: It is similar to the one established in Theorem 3.2, but without the terms $\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{L}^p(\omega)}$ and $\|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}$ in its right-hand side.

Theorem 4.1 *Let ω be a domain in $\mathbb{R}^n$, let $1 < p < \infty$, let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^{n+1})$ be an immersion such that $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^{n+1})$, and let γ_0 be a relatively open subset of $\partial\omega$ such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in any affine subspace of $\mathbb{R}^{n+1}$ of dimension $\leqslant (n-1)$. Then there exists a constant $C_8(\gamma_0,\boldsymbol{\theta})$ such that*

$$\begin{aligned} &\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}\\ &\leqslant C_8(\gamma_0,\boldsymbol{\theta})(\|(\tilde{a}_{\alpha\beta})^{\frac12} - (a_{\alpha\beta})^{\frac12}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac12}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac12}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)}) \end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega;\mathbb{R}^{n+1})$ and $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0.

Proof This inequality is an immediate consequence of Theorem 3.1 and Lemma 4.2.

In the remaining of this section, we show that the Korn inequality on a hypersurface of Theorem 4.1 holds different set of assumptions. Specifically, it shows that the assumption on $\boldsymbol{\theta}(\gamma_0)$ can be dropped provided the mappings $\tilde{\boldsymbol{\theta}}$ satisfy the additional boundary condition $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}$ on γ_0.

To do so, we first need to prove the next lemma, which removes the restriction on γ_0 imposed in Lemma 4.2 at the expense of adding boundary conditions for the normal vector fields $\boldsymbol{a}_{n+1}$ and $\tilde{\boldsymbol{a}}_{n+1}$ to the hypersurfaces $\boldsymbol{\theta}(\omega)$ and $\tilde{\boldsymbol{\theta}}(\omega)$.

Lemma 4.3 *Let ω be a domain in $\mathbb{R}^n$, let $1 < p < \infty$, let γ_0 be any non-empty relatively open subset of $\partial\omega$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion that satisfies $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$. Then there exists a constant $C_9(\gamma_0, \boldsymbol{\theta})$ such that*

$$\begin{aligned}
&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)} \\
&\leqslant C_9(\gamma_0, \boldsymbol{\theta}) \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} (\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} \\
&\quad + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)})
\end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0, and $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}$ on γ_0.

Proof Assume that such a constant does not exist. Then for all $k \in \mathbb{N}^*$, there exists $\tilde{\boldsymbol{\theta}}^k \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ satisfying $\tilde{\boldsymbol{a}}^k_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}^k) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$, $\tilde{\boldsymbol{\theta}}^k = \boldsymbol{\theta}$ on γ_0 and $\tilde{\boldsymbol{a}}^k_{n+1} = \boldsymbol{a}_{n+1}$ on γ_0, such that

$$\begin{aligned}
&\|\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)} \\
&> k \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} (\|\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}^k - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\theta}\|_{\mathbb{L}^p(\omega)} + \|\boldsymbol{\nabla}\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{a}_{n+1}\|_{\mathbb{L}^p(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}\boldsymbol{a}_{n+1}\|_{\boldsymbol{L}^p(\omega)}).
\end{aligned}$$

Let

$$\mu_k := \|\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)}.$$

The previous inequality shows that, for each $k \in \mathbb{N}^*$, $\mu_k > 0$ and there exists $\boldsymbol{R}_k \in \mathbb{O}^{n+1}_+$ such that

$$\begin{aligned}
&\Big\|\frac{1}{\mu_k}(\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}_k\boldsymbol{a}_{n+1})\Big\|_{\boldsymbol{L}^p(\omega)} + \Big\|\boldsymbol{\nabla}\Big(\frac{1}{\mu_k}(\tilde{\boldsymbol{\theta}}^k - \boldsymbol{R}_k\boldsymbol{\theta})\Big)\Big\|_{\mathbb{L}^p(\omega)} \\
&+ \Big\|\boldsymbol{\nabla}\Big(\frac{1}{\mu_k}(\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{R}_k\boldsymbol{a}_{n+1})\Big)\Big\|_{\mathbb{L}^p(\omega)} < \frac{1}{k}.
\end{aligned}$$

Let

$$\tilde{\boldsymbol{\xi}}_k := \frac{1}{\mu_k}(\tilde{\boldsymbol{\theta}}^k - \boldsymbol{\theta}) \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_k := \frac{1}{\mu_k}(\tilde{\boldsymbol{a}}^k_{n+1} - \boldsymbol{a}_{n+1}).$$

Clearly,

$$\|\tilde{\boldsymbol{\xi}}_k\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}_k\|_{\boldsymbol{W}^{1,p}(\omega)} = 1 \quad \text{for all } k \in \mathbb{N}^*.$$

Since the space $W^{1,p}(\omega; \mathbb{R}^{n+1})$ is reflexive (recall that $1 < p < \infty$) and its inclusion in $L^p(\omega; \mathbb{R}^{n+1})$ is compact, the above inequality implies that there exist $\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$, $\mu \in [0, \infty]$, $\boldsymbol{R} \in \mathbb{O}^{n+1}_+$ and a subsequence (l) of (k) such that, as $l \to \infty$,

$$\tilde{\boldsymbol{\xi}}_l \to \tilde{\boldsymbol{\xi}} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_l \to \tilde{\boldsymbol{\eta}} \qquad \text{in } L^p(\omega; \mathbb{R}^{n+1}),$$

$$\begin{aligned}
&\nabla\tilde{\boldsymbol{\xi}}_l \rightharpoonup \nabla\tilde{\boldsymbol{\xi}} \quad \text{and} \quad \nabla\tilde{\boldsymbol{\eta}}_l \rightharpoonup \nabla\tilde{\boldsymbol{\eta}} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}),\\
&\mu_l \to \mu \quad \text{in } \mathbb{R} \quad \text{and} \quad \boldsymbol{R}_l \to \boldsymbol{R} \quad \text{in } \mathbb{M}^{n+1}.
\end{aligned}$$

Since in addition, as $k \to \infty$,

$$\nabla\tilde{\boldsymbol{\xi}}_k + \frac{1}{\mu_k}(\boldsymbol{I}-\boldsymbol{R}_k)\nabla\boldsymbol{\theta} \to \boldsymbol{0} \quad \text{and} \quad \nabla\tilde{\boldsymbol{\eta}}_k + \frac{1}{\mu_k}(\boldsymbol{I}-\boldsymbol{R}_k)\nabla\boldsymbol{a}_{n+1} \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}),$$

the sequences $(\nabla\tilde{\boldsymbol{\xi}}_k)$ and $(\nabla\tilde{\boldsymbol{\eta}}_l)$ converge strongly in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$ (cf. the proof of Lemma 3.3). Hence

$$\begin{aligned}
&\tilde{\boldsymbol{\xi}}_l \to \tilde{\boldsymbol{\xi}} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_l \to \tilde{\boldsymbol{\eta}} && \text{in } W^{1,p}(\omega;\mathbb{R}^{n+1}),\\
&\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\nabla\boldsymbol{a}_{n+1} \to -\nabla\tilde{\boldsymbol{\eta}} && \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}),\\
&\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\nabla\boldsymbol{\theta} \to -\nabla\tilde{\boldsymbol{\xi}} && \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}),\\
&\tilde{\boldsymbol{\eta}}_l + \frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{a}_{n+1} \to \boldsymbol{0} && \text{in } L^p(\omega;\mathbb{R}^{n+1}).
\end{aligned}$$

In what follows, we will show that the last two convergences imply that

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{0} \quad \text{and} \quad \tilde{\boldsymbol{\eta}} = \boldsymbol{0} \quad \text{a.e. in } \omega,$$

which will yield a contradiction with

$$\|\tilde{\boldsymbol{\xi}}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}\|_{\boldsymbol{W}^{1,p}(\omega)} = \lim_{l\to\infty}(\|\tilde{\boldsymbol{\xi}}_l\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{\eta}}_l\|_{\boldsymbol{W}^{1,p}(\omega)}) = 1.$$

To this end, we distinguish three cases: $\mu = \infty$, $0 < \mu < \infty$, $\mu = 0$.

First, assume that $\mu = \infty$. Then $\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l) \to \boldsymbol{0}$ in $\mathbb{M}^{n+1}$, which combined with the last two convergences above implies that

$$\nabla\tilde{\boldsymbol{\xi}} = \boldsymbol{0} \quad \text{and} \quad \tilde{\boldsymbol{\eta}} = \boldsymbol{0} \quad \text{a.e. in } \omega.$$

Since in addition $\tilde{\boldsymbol{\xi}} = \boldsymbol{0}$ on γ_0, $\tilde{\boldsymbol{\xi}}$ also vanishes a.e. in ω.

Second, assume that $0 < \mu < \infty$. Then the convergences

$$\frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\nabla\boldsymbol{\theta} \to -\nabla\tilde{\boldsymbol{\xi}} \quad \text{in } L^p(\omega;\mathbb{M}^{(n+1)\times n}) \quad \text{and} \quad \tilde{\boldsymbol{\eta}}_l + \frac{1}{\mu_l}(\boldsymbol{I}-\boldsymbol{R}_l)\boldsymbol{a}_{n+1} \to \boldsymbol{0} \quad \text{in } L^p(\omega;\mathbb{R}^{n+1})$$

imply that

$$\frac{1}{\mu}(\boldsymbol{I}-\boldsymbol{R})\nabla\boldsymbol{\theta} = -\nabla\tilde{\boldsymbol{\xi}} \quad \text{and} \quad \frac{1}{\mu}(\boldsymbol{I}-\boldsymbol{R})\boldsymbol{a}_{n+1} = -\tilde{\boldsymbol{\eta}}.$$

Hence there exists $\boldsymbol{a} \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned}
&\tilde{\boldsymbol{\xi}}(y) = \boldsymbol{a} + \frac{1}{\mu}(\boldsymbol{R}-\boldsymbol{I})\boldsymbol{\theta}(y) \quad \text{for almost all } y \in \omega,\\
&\tilde{\boldsymbol{\eta}}(y) = \frac{1}{\mu}(\boldsymbol{R}-\boldsymbol{I})\boldsymbol{a}_{n+1}(y) \quad \text{for almost all } y \in \omega.
\end{aligned}$$

Since $\boldsymbol{\xi} = \mathbf{0}$ on γ_0 and $\boldsymbol{\eta} = \mathbf{0}$ on γ_0, the above relations imply that there exists $y_* \in \gamma_0$ such that

$$\boldsymbol{a} + \frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\tau}_\alpha(y_*) = \mathbf{0} \quad \text{and} \quad \frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{a}_{n+1}(y_*) = \mathbf{0},$$

where $\boldsymbol{\tau}_\alpha(y_*)$, $\alpha = 1, 2, \cdots, n-1$, denote a $(n-1)$-tuple of linearly independent vectors in $\mathbb{R}^{n+1}$ that are orthogonal to $\boldsymbol{a}_{n+1}(y_*)$.

More specifically, since ω is a domain in $\mathbb{R}^n$, $\partial\omega$ is locally the graph of a Lipschitz function. In particular, there exist $1 \leqslant j \leqslant n$ and a Lipschitz function $\psi_j : U \to \mathbb{R}$, where U is an open set of $\mathbb{R}^{n-1}$, and an open ball V in $\mathbb{R}^n$, such that

$$V \cap \gamma_0 = \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n;\ y_j = \psi_j(y');\ y' := (y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n) \in U\}.$$

It is well-known that ψ_j is differentiable at almost all points $y' \in U$. Define the mapping $\boldsymbol{\psi} : U \to \mathbb{R}^n$ by letting

$$\boldsymbol{\psi}(y') := \begin{pmatrix} y_1 \\ \vdots \\ y_{j-1} \\ \psi_j(y') \\ y_{j+1} \\ \vdots \\ y_n \end{pmatrix} \quad \text{for all } y' \in U.$$

Let y'_* denote any point where ψ_j is differentiable and let $y_* := \boldsymbol{\psi}(y'_*)$. Then the $(n-1)$ vectors $\partial_\alpha \boldsymbol{\psi}(y'_*)$, $\alpha \in \{1, \cdots, n\} \setminus \{j\}$ are well defined, and they are linearly independent.

Since

$$\boldsymbol{a} + \frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\theta}(\boldsymbol{\psi}(y')) = \mathbf{0} \quad \text{for all } y' \in U,$$

we have in particular that

$$\begin{aligned} &\frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\partial_\alpha(\boldsymbol{\theta} \circ \boldsymbol{\psi})(y'_*) = \mathbf{0}, \quad \alpha \in \{1, \cdots, n\} \setminus \{j\}, \\ \Leftrightarrow\ &\frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\nabla\theta}(y_*)\partial_\alpha \boldsymbol{\psi}(y'_*) = \mathbf{0}, \quad \alpha \in \{1, \cdots, n\} \setminus \{j\}, \\ \Leftrightarrow\ &\frac{1}{\mu}(\boldsymbol{R} - \boldsymbol{I})\boldsymbol{\tau}_\alpha(y_*) = \mathbf{0}, \quad \alpha = 1, 2, \cdots, n-1, \end{aligned}$$

where

$$\begin{aligned} &\boldsymbol{\tau}_1(y_*) := \boldsymbol{\nabla\theta}(y_*)\partial_1\boldsymbol{\psi}(y'_*), \quad \cdots, \quad \boldsymbol{\tau}_{j-1}(y_*) := \boldsymbol{\nabla\theta}(y_*)\partial_{j-1}\boldsymbol{\psi}(y'_*), \\ &\boldsymbol{\tau}_j(y_*) := \boldsymbol{\nabla\theta}(y_*)\partial_{j+1}\boldsymbol{\psi}(y'_*), \quad \cdots, \quad \boldsymbol{\tau}_{n-1}(y_*) := \boldsymbol{\nabla\theta}(y_*)\partial_n\boldsymbol{\psi}(y'_*). \end{aligned}$$

Note that the vectors $\boldsymbol{\tau}_1(y_*), \cdots, \boldsymbol{\tau}_{n-1}(y_*)$ are linearly independent since $\boldsymbol{\theta}$ is an immersion at y_*, so the vectors $\partial_1\boldsymbol{\theta}(y_*), \cdots, \partial_n\boldsymbol{\theta}(y_*)$ are linearly independent. Indeed, since

$$[\boldsymbol{\tau}_1(y_*) \cdots \boldsymbol{\tau}_{n-1}(y_*)] = \boldsymbol{\nabla\theta}(y_*)[\partial_1\boldsymbol{\psi}(y'_*) \cdots \partial_{j-1}\boldsymbol{\psi}(y'_*)\, \partial_{j+1}\boldsymbol{\psi}(y'_*) \cdots \partial_n\boldsymbol{\psi}(y'_*)],$$

the rank of the matrix in the left-hand side is $(n-1)$.

Note also that the vectors $\boldsymbol{\tau}_\alpha(y_*)$ are orthogonal to $\boldsymbol{a}_{n+1}(y_*)$ since

$$\boldsymbol{\tau}_1(y_*) = \sum_{\beta=1}^{n} [\partial_1\boldsymbol{\psi}(y'_*)]^\beta \partial_\beta\boldsymbol{\theta}(y_*), \cdots, \boldsymbol{\tau}_{n-1}(y_*) = \sum_{\beta=1}^{n} [\partial_n\boldsymbol{\psi}(y'_*)]^\beta \partial_\beta\boldsymbol{\theta}(y_*),$$

and $\partial_\beta\boldsymbol{\theta}(y_*) \cdot \boldsymbol{a}_{n+1}(y_*) = 0$ for all $\beta \in \{1, \cdots, n\}$ by the definition of $\boldsymbol{a}_{n+1}(y_*)$.

So we just proved that

$$\boldsymbol{R}\boldsymbol{v} = \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in E := \text{span}\{\boldsymbol{\tau}_1(y_*), \cdots, \boldsymbol{\tau}_{n-1}(y_*), \boldsymbol{a}_{n+1}(y_*)\}.$$

Since $\dim E = n$ and $\boldsymbol{R} \in \mathbb{O}^{n+1}_+$, it follows that $\boldsymbol{R} = \boldsymbol{I}$, because, if $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ is an orthogonal basis in $\mathbb{R}^{n+1}$ such that $E = \text{span}\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ and $\det\,[\boldsymbol{e}_1 \cdots \boldsymbol{e}_{n+1}] > 0$, then, for all $\alpha = 1, \cdots, n$,

$$\boldsymbol{R}\boldsymbol{e}_{n+1} \cdot \boldsymbol{e}_\alpha = \boldsymbol{e}_{n+1} \cdot \boldsymbol{R}^{\mathrm{T}}\boldsymbol{e}_\alpha = \boldsymbol{e}_{n+1} \cdot \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{R}\boldsymbol{e}_\alpha) = \boldsymbol{e}_{n+1} \cdot \boldsymbol{e}_\alpha = 0,$$

so that $\boldsymbol{R}\boldsymbol{e}_{n+1} \perp E$. Therefore, either $\boldsymbol{R}\boldsymbol{e}_{n+1} = \boldsymbol{e}_{n+1}$, or $\boldsymbol{R}\boldsymbol{e}_{n+1} = -\boldsymbol{e}_{n+1}$. But, $\boldsymbol{R}\boldsymbol{e}_{n+1} = -\boldsymbol{e}_{n+1}$ implies that

$$-1 = \det\,[\boldsymbol{e}_1 \cdots \boldsymbol{e}_n\ (-\boldsymbol{e}_{n+1})] = \det\,(\boldsymbol{R}[\boldsymbol{e}_1 \cdots \boldsymbol{e}_{n+1}]) = 1,$$

a contradiction. Consequently, $\boldsymbol{R} = \boldsymbol{I}$ and so

$$\tilde{\boldsymbol{\xi}}(y) = \boldsymbol{a} \quad \text{and} \quad \tilde{\boldsymbol{\eta}}(y) = \boldsymbol{0} \quad \text{a. e. } y \in \omega.$$

Besides, $\tilde{\boldsymbol{\xi}} = \boldsymbol{0}$ on γ_0, so that $\boldsymbol{a} = \boldsymbol{0}$.

Third, assume that $\mu = 0$. We then infer from the convergence

$$\frac{1}{\mu_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta} \to -\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}} \quad \text{in } L^p(\omega; \mathbb{M}^{(n+1)\times n}) \quad \text{as } l \to \infty,$$

which implies in particular that

$$\int_\omega \Big(\frac{1}{\mu_l}\Big)^p |(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta}|^p \mathrm{d}y \to \int_\omega |\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}}|^p \mathrm{d}y \quad \text{as } l \to \infty,$$

that the limit $\boldsymbol{R} = \lim_{l\to\infty} \boldsymbol{R}_l$ satisfies $\boldsymbol{R} = \boldsymbol{I}$. Indeed, if on the contrary $\boldsymbol{R} \neq \boldsymbol{I}$, then

$$|(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta}(y)| \geqslant |(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{\nabla\theta}(y)| - |\boldsymbol{R}_l - \boldsymbol{R}||\boldsymbol{\nabla\theta}(y)| \quad \text{for all } l.$$

Furthermore, let $y^* \in \omega$ and let $\varepsilon := |(\boldsymbol{I} - \boldsymbol{R})\boldsymbol{\nabla\theta}(y^*)|$. Then $\varepsilon > 0$ (since $\text{rank}\,\boldsymbol{\nabla\theta}(y^*) = n$ and $\dim\ \text{Ker}(\boldsymbol{I} - \boldsymbol{R}) \leqslant n - 1$). Since $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ and $\boldsymbol{R}_l \to \boldsymbol{R}$ as $l \to \infty$, there exists $\delta > 0$ and $l_0 \in \mathbb{N}^*$ such that

$$|(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta}(y)| \geqslant \frac{\varepsilon}{2} \quad \text{and} \quad |\boldsymbol{R}_l - \boldsymbol{R}||\boldsymbol{\nabla\theta}(y)| \leqslant \frac{\varepsilon}{4}$$

for all $y \in B(y^*, \delta)$ and all $l \geqslant l_0$. Therefore, for all $l \geqslant l_0$,

$$\int_\omega \Big(\frac{1}{\mu_l}|(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta}|\Big)^p \mathrm{d}y \geqslant \int_{B(y^*,\delta)} \Big(\frac{1}{\mu_l}|(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta}|\Big)^p \mathrm{d}y \geqslant \int_{B(y^*,\delta)} \Big(\frac{1}{\mu_l}\frac{\varepsilon}{4}\Big)^p \mathrm{d}y,$$

which implies that $\int_\omega |\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}}|^p \mathrm{d}y = +\infty$, a contradiction.

Next, using that the column vector fields of $\boldsymbol{\nabla\theta} \in \mathcal{C}^0(\overline{\omega};\mathbb{R}^{n+1})$ are linearly independent at each point of $\overline{\omega}$, we deduce (as in the proof of Lemma 3.3) from the convergence $\frac{1}{\mu_l}(\boldsymbol{I} - \boldsymbol{R}_l)\boldsymbol{\nabla\theta} \to -\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}}$ in $L^p(\omega;\mathbb{M}^{(n+1)\times n})$ that there exists an antisymmetric matrix $\boldsymbol{A} \in \mathbb{A}^{n+1}$ and a subsequence (m) of (l) such that

$$\frac{1}{\mu_m}(\boldsymbol{I} - \boldsymbol{R}_m) \to \boldsymbol{A} \quad \text{in } \mathbb{M}^{n+1} \quad \text{as } m \to \infty.$$

Hence

$$-\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\nabla\theta} \quad \text{a.e. in } \omega,$$

which in turn implies that there exists $\boldsymbol{a} \in \mathbb{R}^{n+1}$ such that

$$\tilde{\boldsymbol{\xi}}(y) = \boldsymbol{a} - \boldsymbol{A}\boldsymbol{\nabla\theta}(y) \quad \text{for almost all } y \in \omega.$$

Besides, since

$$\tilde{\boldsymbol{\eta}} = \lim_{m\to\infty} \tilde{\boldsymbol{\eta}}_m = \lim_{m\to\infty} \frac{1}{\mu_m}(\boldsymbol{R}_m - \boldsymbol{I})\boldsymbol{a}_{n+1} = -\boldsymbol{A}\boldsymbol{a}_{n+1} \quad \text{in } L^p(\omega;\mathbb{R}^{n+1}),$$

we have

$$\tilde{\boldsymbol{\eta}}(y) = -\boldsymbol{A}\boldsymbol{a}_{n+1}(y) \quad \text{for almost all } y \in \omega.$$

Therefore, using that $\tilde{\boldsymbol{\xi}} = \boldsymbol{0}$ on γ_0 and $\tilde{\boldsymbol{\eta}} = \boldsymbol{0}$ on γ_0, the same argument as that used in the case $0 < \mu < \infty$ shows that there exists $y_* \in \gamma_0$ such that

$$-\boldsymbol{A}\boldsymbol{\tau}_\alpha(y_*) = \boldsymbol{0} \quad \text{and} \quad -\boldsymbol{A}\boldsymbol{a}_{n+1}(y_*) = \boldsymbol{0}$$

for some vectors $\boldsymbol{\tau}_\alpha(y_*) \in \mathbb{R}^{n+1}$, $\alpha = 1, 2, \cdots, n-1$, that are linearly independent and orthogonal to $\boldsymbol{a}_{n+1}(y_*)$. Thus the antisymmetric matrix $\boldsymbol{A} \in \mathbb{A}^{n+1}$ satisfies

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0} \quad \text{for all } \boldsymbol{v} \in E,$$

where E denotes the subspace of dimension n of $\mathbb{R}^{n+1}$ spanned by the vectors $\boldsymbol{\tau}_1(y_*), \cdots, \boldsymbol{\tau}_{n-1}(y_*)$, $\boldsymbol{a}_{n+1}(y_*)$. Hence $\boldsymbol{A} = \boldsymbol{0}$. To see this, let $\boldsymbol{w} \in E^\perp$, $\boldsymbol{w} \neq \boldsymbol{0}$. Then $\mathbb{R}^{n+1} = E \oplus (\mathbb{R}\boldsymbol{w})$ and

$$\begin{aligned} \boldsymbol{A}\boldsymbol{w} \cdot \boldsymbol{v} &= \boldsymbol{w} \cdot \boldsymbol{A}^{\mathrm{T}}\boldsymbol{v} = -\boldsymbol{w} \cdot \boldsymbol{A}\boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in E, \\ \boldsymbol{A}\boldsymbol{w} \cdot \boldsymbol{w} &= \boldsymbol{w} \cdot \boldsymbol{A}^{\mathrm{T}}\boldsymbol{w} = -\boldsymbol{w} \cdot \boldsymbol{A}\boldsymbol{w} = 0. \end{aligned}$$

Hence, $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \mathbb{R}^{n+1}$, which means that $\boldsymbol{A} = \boldsymbol{0}$.

Finally, since $\tilde{\boldsymbol{\xi}}(y) = \boldsymbol{a} - \boldsymbol{A}\boldsymbol{\nabla\theta}(y) = \boldsymbol{a}$ for almost all $y \in \omega$ and since $\tilde{\boldsymbol{\xi}} = \boldsymbol{0}$ on γ_0, the vector $\boldsymbol{a}$ vanishes. Therefore,

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{0} \quad \text{and} \quad \tilde{\boldsymbol{\eta}} = -\boldsymbol{A}\boldsymbol{a}_{n+1} = \boldsymbol{0} \quad \text{a.e. in } \omega.$$

We are now in a position to establish our second nonlinear Korn inequality on a hypersurface with boundary conditions.

Theorem 4.2 *Let ω be a domain in $\mathbb{R}^n$, let $1 < p < \infty$, let γ_0 be any non-empty relatively open subset of $\partial\omega$, and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$ be an immersion such that $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}(\boldsymbol{\theta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^{n+1})$. Then there exists a constant $C_{10}(\gamma_0, \boldsymbol{\theta})$ such that*

$$\begin{aligned}&\|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_{\boldsymbol{W}^{1,p}(\omega)} + \|\tilde{\boldsymbol{a}}_{n+1} - \boldsymbol{a}_{n+1}\|_{\boldsymbol{W}^{1,p}(\omega)} \\ &\leqslant C_{10}(\gamma_0, \boldsymbol{\theta})(\|(\tilde{a}_{\alpha\beta})^{\frac{1}{2}} - (a_{\alpha\beta})^{\frac{1}{2}}\|_{\mathbb{L}^p(\omega)} + \|(\tilde{a}_{\alpha\beta})^{-\frac{1}{2}}(\tilde{b}_{\alpha\beta}) - (a_{\alpha\beta})^{-\frac{1}{2}}(b_{\alpha\beta})\|_{\mathbb{L}^p(\omega)})\end{aligned}$$

for all immersions $\tilde{\boldsymbol{\theta}} \in W^{1,p}(\omega; \mathbb{R}^{n+1})$ that satisfy $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}(\tilde{\boldsymbol{\theta}}) \in W^{1,p}(\omega; \mathbb{R}^{n+1})$, $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ on γ_0 and $\tilde{\boldsymbol{a}}_{n+1} = \boldsymbol{a}_{n+1}$ on γ_0.

Proof It suffices to estimate the left-hand side in Theorem 3.1 by applying Lemma 4.3.

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