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# Global Asymptotics of Orthogonal Polynomials Associated with a Generalized Freud Weight*

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(*Dedicated to Professor Philippe G. Ciarlet on the occasion of his 80th birthday*)

**Abstract** In this paper, the authors consider the asymptotic behavior of the monic polynomials orthogonal with respect to the weight function $w(x) = |x|^{2\alpha}\mathrm{e}^{-(x^4+tx^2)}$, $x \in \mathbb{R}$, where $\alpha$ is a constant larger than $-\frac{1}{2}$ and $t$ is any real number. They consider this problem in three separate cases: (i) $c > -2$, (ii) $c = -2$, and (iii) $c < -2$, where $c := tN^{-\frac{1}{2}}$ is a constant, $N = n + \alpha$ and $n$ is the degree of the polynomial. In the first two cases, the support of the associated equilibrium measure $\mu_t$ is a single interval, whereas in the third case the support of $\mu_t$ consists of two intervals. In each case, globally uniform asymptotic expansions are obtained in several regions. These regions together cover the whole complex plane. The approach is based on a modified version of the steepest descent method for Riemann-Hilbert problems introduced by Deift and Zhou (1993).

**Keywords** Orthogonal polynomials, Globally uniform asymptotics, Riemann-Hilbert problems, The second Painlevé transcendent, Theta function
**2000 MR Subject Classification** 41A60, 30E15

## 1 Introduction

In a recent paper [8], Clarkson et al. made a detailed study of the monic orthogonal polynomials $\{S_n(x;t)\}_{n=0}^{\infty}$ associated with the generalized Freud weight

$$w(x;t) = |x|^{2\alpha}\mathrm{e}^{-(x^4+tx^2)}, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\alpha > -\frac{1}{2}$ and $t \in \mathbb{R}$. In particular, they showed that these polynomials satisfy a differential-difference equation and a second-order linear differential equation. The coefficients in these equations involve the coefficient $\beta_n(t;\alpha)$ in the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\alpha)S_{n-1}(x;t), \tag{1.2}$$

where $S_{-1}(x;t) = 0$ and $S_0(x;t) = 1$. The recurrence coefficient $\beta_n(t;\alpha)$ satisfies the first discrete Painlevé equation $\mathrm{dP_I}$, and is related to solutions of the fourth Painlevé equation

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$P_{IV}$. As a follow-up, in [7] Clarkson and Jordaan investigated the asymptotic behavior of the polynomials $S_n(x;t)$, as well as the recurrence coefficient $\beta_n(t;\alpha)$, when the degree $n$, or the parameter $t$, tends to infinity. However, with regard to the polynomial $S_n(x;t)$, what they gave is a solution $\widetilde{S}_n(x)$ to an approximate equation (see [7, (4.12)]), and not an asymptotic approximation to the polynomial $S_n(x;t)$.

In the present paper, we are mainly concerned with the problem of finding asymptotic formulas for the monic polynomials $S_n(x;t)$, as $n\to\infty$, for all values of $t$, whether $t$ is fixed or varying. This problem has actually already been studied by several people in special cases. For instance, when $\alpha=0$ in (1.1) and the weight function is given in the form

$$w_N(x)=\mathrm{e}^{-N(\frac{g}{4}x^4+\frac{h}{2}x^2)},\quad g>0, \tag{1.3}$$

asymptotic formulas (as $n\to\infty$) of the associated monic orthogonal polynomials have been given by Bleher and Its in the case $h<-2\sqrt{g}$ and $\varepsilon<\frac{n}{N}<\varepsilon^{-1}$, $\varepsilon>0$. Their method makes use of a WKB formula (see [4, (7.5)]) from the differential equation theory. Without loss of generality, one can assume the constant $g$ in (1.3) to be equal to 1. To simplify the calculation, one may also take the parameter $N$ to be equal to the degree $n$. Thus, in [19] Wong and Zhang investigated the problem in the case when the weight function is given by $w(x)=\mathrm{e}^{-nV(x)}$ with

$$V(x)=\frac{x^4}{4}+\frac{h}{2}x^2$$

and $h=-2$. At about the same time, results for the same case (i.e., $h=-2$) but with a much more general $V(x)$ was obtained by Claeys et al. [5]. For fixed $t$, asymptotic formulas for the monic polynomials $S_n(x;t)$ in (1.2) can be written out directly from the results in [20]. Quite recently, in [2] and [3], Bertola and Tovbis obtained the asymptotics of the recurrence coefficients for the orthogonal polynomials associated with the weight (1.3), where $g\in\mathbb{C}$, $h=1$ and the variable $x$ is on the rays $\arg x=\theta$ (for fixed $\theta$) in the complex plane. As a follow-up of the investigations in [2] and [3], one may also consider the case when the weight function has a singularity at the origin like the one in (1.1). But, we will leave this problem to a future investigation.

In this paper, we shall study the behavior of $S_n(x;t)$ as $n\to\infty$, for all values of $t$. Our approach is completely based on the nonlinear steepest-descent method for Riemann-Hilbert problems introduced by Deift and Zhou [12], and will not make use of any asymptotic result from differential equation theory. We divide our discussion into three cases: (i) $c>-2$, (ii) $c=-2$, and (iii) $c<-2$, where $c=tN^{-\frac{1}{2}}$ is a constant and $N=n+\alpha$. As we shall show, in the first two cases, the support of the associated equilibrium measure $\mu_t$ is a single interval, whereas in the third case the support of $\mu_t$ consists of two intervals. In all three cases, we will present infinite asymptotic expansions of $S_n(z;t)$ for $z$ in various different regions. These regions together cover the whole complex $z$-plane.

## 2 Riemann-Hilbert Problem

For notational convenience, from here on we will use $\pi_n(z)$ to denote the monic orthogonal polynomials in (1.2), and write $p_n(z)=\gamma_n\pi_n(z)$ for the orthogonal polynomials with respect to the weight in (1.1). Consider the following Riemann-Hilbert problem (RHP for short) for a $2\times 2$ matrix-valued function $Y$:

($Y_\mathrm{a}$) $Y(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$;

$(Y_{\mathrm{b}})$ $Y(z)$ takes continuous boundary values $Y_+(x)$ and $Y_-(x)$ such that

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$$

for $x \in \mathbb{R} \setminus \{0\}$, where $w(x)$ is the weight function given by (1.1).

$(Y_{\mathrm{c}})$ For $z \in \mathbb{C} \setminus \mathbb{R}$,

$$Y(z) = [I + O(1/z)]\begin{pmatrix} z^{\mathbf{n}} & 0 \\ 0 & z^{-\mathbf{n}} \end{pmatrix},$$

as $z \to \infty$.

$(Y_{\mathrm{d}})$ For $z \in \mathbb{C} \setminus \mathbb{R}$,

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z|^{2\boldsymbol{\alpha}} \\ 1 & |z|^{2\boldsymbol{\alpha}} \end{pmatrix}, & \text{if } \alpha < 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } \alpha > 0, \end{cases}$$

as $z \to 0$.

The following well-known theorem is due to Fokas, Its and Kitaev [15].

**Theorem 2.1** *The unique solution to the above RHP is given by*

$$Y(z) = \begin{pmatrix} \pi_{\mathbf{n}}(z) & C(\pi_{\mathbf{n}} w)(z) \\ c_{\mathbf{n}}\pi_{\mathbf{n}-1}(z) & c_{\mathbf{n}} C(\pi_{\mathbf{n}-1} w)(z) \end{pmatrix},$$

*where* $c_{\mathbf{n}} = -2\pi \mathrm{i} \gamma_{\mathbf{n}-1}^2$ *and*

$$C[f](z) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

*is the Cauchy transform of* $f$.

To asymptotically evaluate the solution of the RHP for $Y$, we will first follow the nonlinear steepest descent method introduced by Deift and Zhou [12], and further developed in [9–11]. This method consists of a sequence of transformations

$$Y \to U \to T \to V \to S,$$

which ultimately leads to a RHP that can be solved explicitly. The transformation $Y \to U$ is just a rescaling, and the transformation $U \to T$ is a normalization process. The transformation $T \to V$ involves a factorization of the jump matrix and deformation of contours so that $V$ can be approximated by an exact solution to a RHP for $n$ large. The final transformation $V \to S$ consists of some local analysis and the construction of parametrices. Since every step of the transformation is explicit and reversible, one can obtain the asymptotics of $Y$, and hence of $\pi_{\mathbf{n}}(z)$, by a sequence of inverse transformations.

We define the first transformation $Y \to U$ by

$$U(z) := \begin{pmatrix} N^{-\frac{N}{4}} & 0 \\ 0 & N^{\frac{N}{4}} \end{pmatrix} Y(N^{\frac{1}{4}} z) \begin{pmatrix} N^{\frac{\alpha}{4}} & 0 \\ 0 & N^{-\frac{\alpha}{4}} \end{pmatrix}, \tag{2.1}$$

where $N = n + \alpha$. Then, U satisfies the following RHP:

$(U_{\mathrm{a}})$ $U(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

($U_\mathrm{b}$) $U(z)$ takes continuous boundary values $U_+(x)$ and $U_-(x)$ such that

$$U_+(x) = U_-(x) \begin{pmatrix} 1 & |x|^{2\boldsymbol{\alpha}} \mathrm{e}^{-(\mathbf{N}\mathbf{x}^4 + \mathbf{t}\mathbf{x}^2 \mathbf{N}^{\frac{1}{2}})} \\ 0 & 1 \end{pmatrix}$$

for $x \in \mathbb{R} \setminus \{0\}$,

($U_\mathrm{c}$) For $z \in \mathbb{C} \setminus \mathbb{R}$,

$$U(z) = [I + O(1/z)] \begin{pmatrix} z^{\mathbf{n}} & 0 \\ 0 & z^{-\mathbf{n}} \end{pmatrix}$$

as $z \to \infty$.

($U_\mathrm{d}$) $U(z)$ has the same behavior as $Y(z)$ for $z$ near the origin.

From Theorem 2.1 and (2.1), it is easily seen that

$$U_{11}(z) = N^{-\frac{n}{4}} \pi_{\mathbf{n}}(N^{\frac{1}{4}} z). \tag{2.2}$$

For convenience, we have scaled the variable $z$ by multiplying it by $N^{\frac{1}{4}} = (n+\alpha)^{\frac{1}{4}}$.

## 3 Normalization and Auxiliary Functions

In order to normalize the behavior of $U(z)$ at $z = \infty$, we first need to introduce some notations. Let $\alpha_{\mathbf{t}}$ be a positive real number, and denote by $\mu_{\mathbf{t}}(s)$ a probability density function supported on $[-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$, i.e., $\mu_{\mathbf{t}}(s) \geq 0$ and

$$\int_{-\boldsymbol{\alpha}_t}^{\boldsymbol{\alpha}_t} \mu_{\mathbf{t}}(s)\,\mathrm{d}s = 1. \tag{3.1}$$

The explicit formulas for $\alpha_{\mathbf{t}}$ and $\mu_{\mathbf{t}}(s)$ will be determined later.

Next, we introduce the so-called $g$-function which is the logarithmic potential of $\mu_{\mathbf{t}}(s)$; that is

$$g(z) = g_{\mathbf{t}}(z) = \int_{-\boldsymbol{\alpha}_t}^{\boldsymbol{\alpha}_t} \log(z-s)\mu_{\mathbf{t}}(s)\,\mathrm{d}s, \quad z \in \mathbb{C} \setminus (-\infty, \alpha_{\mathbf{t}}], \tag{3.2}$$

where for each $s$ we view $\log(z-s)$ as an analytic function of the variable $z$ with a branch cut along $(-\infty, s]$.

Similarly, we also define

$$\widetilde{g}(z) = \widetilde{g}_{\mathbf{t}}(z) = \int_{-\boldsymbol{\alpha}_t}^{\boldsymbol{\alpha}_t} \log(z-s)\mu_{\mathbf{t}}(s)\,\mathrm{d}s, \quad z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \infty), \tag{3.3}$$

where for each $s$ we view $\log(z-s)$ as an analytic function of the variable $z$ with a branch cut along $[s, \infty)$.

From (3.1)–(3.2), it is easy to check that the $g$-function satisfies the jump conditions

$$g_+(x) - g_-(x) = 2\pi\mathrm{i}, \quad x < -\alpha_{\mathbf{t}}, \tag{3.4}$$

and

$$g_+(x) - g_-(x) = 2\pi\mathrm{i} \int_{\mathbf{x}}^{\boldsymbol{\alpha}_t} \mu_{\mathbf{t}}(s)\,\mathrm{d}s, \quad -\alpha_{\mathbf{t}} < x < \alpha_{\mathbf{t}}. \tag{3.5}$$

On account of (3.2) and (3.4), one readily sees that $e^{ng(z)}$ can be analytically extended to $\mathbb{C} \setminus [-\alpha_t, \alpha_t]$ and

$$e^{ng(z)} = z^n[1 + O(z^{-1})] \qquad \text{as } z \to \infty. \tag{3.6}$$

By adopting the convention that $\sigma_3$ denotes the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.7}$$

we introduce the transformation

$$T(z) := e^{-\frac{1}{2} N l_N \sigma_3} U(z) e^{-(N g(z) - \frac{1}{2} N l_N - \alpha \log z)\sigma_3}, \tag{3.8}$$

where $l_N$ is a constant to be determined. A straightforward calculation shows that $T(z)$ is the unique solution of the following RHP:

($T_a$) $T(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

($T_b$) For $x \in \mathbb{R} \setminus \{0\}$,

$$T_+(x) = T_-(x) \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$\begin{aligned} J_{11}(x) &= e^{-N(g_+(x) - g_-(x)) + \alpha[(\log x)_+ - (\log x)_-]}, \\ J_{12}(x) &= e^{N(g_+(x) + g_-(x) - l_N) - (N x^4 + t x^2 N^{\frac{1}{2}})}, \\ J_{22}(x) &= e^{N(g_+(x) - g_-(x)) - \alpha[(\log x)_+ - (\log x)_-]}, \\ J_{21}(x) &= 0; \end{aligned}$$

($T_c$) $T(z)$ behaves like the identity matrix at infinity:

$$T(z) = [I + O(1/z)] \qquad \text{as } z \to \infty,$$

for $z \in \mathbb{C} \setminus \mathbb{R}$;

($T_d$) $T(z)$ has the following behavior for $z \in \mathbb{C} \setminus \mathbb{R}$ as $z \to 0$

$$T(z) = \begin{cases} O\begin{pmatrix} |z|^{\alpha} & |z|^{\alpha} \\ |z|^{\alpha} & |z|^{\alpha} \end{pmatrix}, & \text{if } \alpha < 0, \\ O\begin{pmatrix} |z|^{\alpha} & |z|^{-\alpha} \\ |z|^{\alpha} & |z|^{-\alpha} \end{pmatrix}, & \text{if } \alpha > 0. \end{cases}$$

We proceed to seek for a probability density function $\mu_t(s)$ in (3.1) and a constant $l_N$ mentioned above so that $J_{12}$ in jump condition ($T_b$) becomes 1 for $x \in (-\alpha_t, \alpha_t)$. Thus, we set

$$N(g_+(x) + g_-(x) - l_N) - (N x^4 + t x^2 N^{\frac{1}{2}}) = 0 \tag{3.9}$$

for $x \in (-\alpha_t, \alpha_t)$. It follows from (3.9) by substituting $x = \alpha_t$ in (3.5) that

$$l_N = 2g(\alpha_t) - \alpha_t^4 - t\alpha_t^2 N^{-\frac{1}{2}}. \tag{3.10}$$

Differentiating (3.9) yields

$$G_+(x) + G_-(x) = \frac{i}{\pi}(4x^3 + 2txN^{-\frac{1}{2}}), \quad x \in (-\alpha_t, \alpha_t), \tag{3.11}$$

where

$$G(z) := \frac{1}{\pi \mathrm{i}} \int_{-\alpha_t}^{\alpha_t} \frac{\mu_{\mathbf{t}}(s)}{s-z}\, \mathrm{d}s, \quad z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]. \tag{3.12}$$

For convenience, we set

$$\widetilde{G}(z) := \frac{G(z)}{\sqrt{z^2 - \alpha_{\mathbf{t}}^2}}, \tag{3.13}$$

where $\sqrt{z^2 - \alpha_{\mathbf{t}}^2}$ is analytic in $\mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$ and behaves like $z$ as $z \to \infty$. Since

$$\widetilde{G}_+(x) - \widetilde{G}_-(x) = \frac{4x^3 + 2txN^{-\frac{1}{2}}}{\pi \sqrt{\alpha_{\mathbf{t}}^2 - x^2}}, \quad x \in (-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}), \tag{3.14}$$

we can solve this scalar RHP to give

$$\widetilde{G}(z) = \frac{1}{2\pi^2 \mathrm{i}} \int_{-\alpha_t}^{\alpha_t} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{\sqrt{\alpha_{\mathbf{t}}^2 - s^2}} \frac{\mathrm{d}s}{s-z},$$

or, equivalently,

$$G(z) = \frac{\sqrt{z^2 - \alpha_{\mathbf{t}}^2}}{2\pi^2 \mathrm{i}} \int_{-\alpha_t}^{\alpha_t} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{\sqrt{\alpha_{\mathbf{t}}^2 - s^2}} \frac{\mathrm{d}s}{s-z}. \tag{3.15}$$

From (3.1) and (3.12), it is easily seen that $G(z) \to 0$ and $zG(z) \to \frac{\mathrm{i}}{\pi}$ as $z \to \infty$. Hence, we have from (3.15)

$$\int_{-\alpha_t}^{\alpha_t} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{\sqrt{\alpha_{\mathbf{t}}^2 - s^2}}\, \mathrm{d}s = 0 \tag{3.16}$$

and

$$\int_{-\alpha_t}^{\alpha_t} \frac{4s^4 + 2ts^2N^{-\frac{1}{2}}}{\sqrt{\alpha_{\mathbf{t}}^2 - s^2}}\, \mathrm{d}s = 2\pi. \tag{3.17}$$

Since the integrand is an odd function, we find that (3.16) is trivially true. By a change of variable $s = \alpha_{\mathbf{t}} \sin\theta$, it follows that

$$\alpha_{\mathbf{t}} = \left( \frac{-tN^{-\frac{1}{2}} + \sqrt{t^2N^{-1} + 12}}{3} \right)^{\frac{1}{2}}. \tag{3.18}$$

We next derive an explicit formula for the probability density function $\mu_{\mathbf{t}}(s)$ in (3.1). From (3.12), one observes

$$\begin{aligned} G_+(x) &= \lim_{\varepsilon \to 0^+} G(x + \mathrm{i}\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi \mathrm{i}} \int_{-\alpha_t}^{\alpha_t} \frac{(s-x) + \mathrm{i}\varepsilon}{(s-x)^2 + \varepsilon^2} \mu_{\mathbf{t}}(s)\, \mathrm{d}s \\ &= \mu_{\mathbf{t}}(x) + \frac{\mathrm{i}}{\pi} \mathrm{P.V.} \int_{-\alpha_t}^{\alpha_t} \frac{1}{x-s} \mu_{\mathbf{t}}(s)\, \mathrm{d}s \end{aligned} \tag{3.19}$$

for $x \in (-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}})$, where P.V. denotes the Cauchy principal value (see [1, p. 518]). Therefore,

$$\mu_{\mathbf{t}}(x) = \mathrm{Re}\, G_+(x).$$

To evaluate $G(z)$ in (3.15), we first note that for any integer $m \geq 1$ and $z \in \mathbb{C} \setminus [-a, a]$ with $a > 0$, Lemma 1 in [20] gives

$$\int_{-a}^{a} \frac{s^{m}}{\sqrt{\phantom{x}}}$$

By the Plemelj formula (see [1, p. 518]), we obtain

$$g(z) - \log(z + \alpha_{\mathbf{t}}) = \frac{\sqrt{z^2 - \alpha_{\mathbf{t}}^2}}{2\pi \mathrm{i}} \int_{-\alpha_t}^{\alpha_t} \frac{H_+(x) - H_-(x)}{x - z} \, \mathrm{d}x.$$

Recall that $t > -2\sqrt{N}$ and $\alpha_{\mathbf{t}}$ in (3.18). From (3.9) it follows that

$$g(z) - \log(z + \alpha_{\mathbf{t}}) = \frac{\sqrt{z^2 - \alpha_{\mathbf{t}}^2}}{2\pi} \int_{-\alpha_t}^{\alpha_t} \frac{2\log(x + \alpha_{\mathbf{t}}) - l_{\mathbf{N}} - x^4 - tN^{-\frac{1}{2}}x^2}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}(x - z)} \, \mathrm{d}x.$$

Now, let $z \to \infty$; on account of (3.25), we have

$$2\int_{-\alpha_t}^{\alpha_t} \frac{\log(x + \alpha_{\mathbf{t}})}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}} \, \mathrm{d}x - l_{\mathbf{N}} \int_{-\alpha_t}^{\alpha_t} \frac{\mathrm{d}x}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}} - \int_{-\alpha_t}^{\alpha_t} \frac{x^4 + tN^{-\frac{1}{2}}x^2}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}} \, \mathrm{d}x = 0. \tag{3.27}$$

Let the integrals on the left-hand side of (3.27) be denoted by $\mathrm{I}_1$, $\mathrm{I}_2$ and $\mathrm{I}_3$, respectively. It is easily seen that

$$\mathrm{I}_2 = \int_{-\alpha_t}^{\alpha_t} \frac{\mathrm{d}x}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}} = \pi \tag{3.28}$$

and

$$\mathrm{I}_3 = \int_{-\alpha_t}^{\alpha_t} \frac{x^4 + tN^{-\frac{1}{2}}x^2}{\sqrt{\alpha_{\mathbf{t}}^2 - x^2}} \, \mathrm{d}x = \frac{3}{8}\pi\alpha_{\mathbf{t}}^4 + \frac{\pi}{2}\frac{t}{\sqrt{N}}\alpha_{\mathbf{t}}^2. \tag{3.29}$$

To evaluate $\mathrm{I}_1$, we note that for any $0 \le a < b < \infty$,

$$\int_{a}^{b} \frac{\log s}{\sqrt{(s - a)(b - s)}} \, \mathrm{d}s = 2\pi \log \frac{\sqrt{a} + \sqrt{b}}{2} \tag{3.30}$$

(see [18, Lemma 1]). Hence, by taking $s = x + \alpha_{\mathbf{t}}$, we obtain

$$\mathrm{I}_1 = \int_{0}^{2\alpha_t} \frac{\log s}{\sqrt{s(2\alpha_{\mathbf{t}} - s)}} \, \mathrm{d}s = \pi \log \frac{\alpha_{\mathbf{t}}}{2}. \tag{3.31}$$

Inserting (3.28)–(3.31) into (3.27) gives

$$l_{\mathbf{N}} = 2\log \frac{\alpha_{\mathbf{t}}}{2} - \frac{3}{8}\alpha_{\mathbf{t}}^4 - \frac{t}{2\sqrt{N}}\alpha_{\mathbf{t}}^2. \tag{3.32}$$

Let $\nu(z) := \sqrt{z^2 - \alpha_{\mathbf{t}}^2}(2z^2 + \alpha_{\mathbf{t}}^2 + tN^{-\frac{1}{2}})$, where $\sqrt{z^2 - \alpha_{\mathbf{t}}^2}$ is analytic in $\mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$ and behaves likes $z$ as $z \to \infty$. Clearly, $\nu_\pm(x) = \pm\pi \mathrm{i}\mu_{\mathbf{t}}(x)$ for $x \in [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$. Define

$$\phi(z) := \int_{\alpha_t}^{z} \nu(\zeta) \, \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus (-\infty, \alpha_{\mathbf{t}}], \tag{3.33}$$

where the path of integration from $\alpha_{\mathbf{t}}$ to $z$ lies entirely in the region $z \in \mathbb{C} \setminus (-\infty, \alpha_{\mathbf{t}}]$, except for the initial point $\alpha_{\mathbf{t}}$.

Similarly, we define

$$\widetilde{\phi}(z) := \int_{-\alpha_t}^{z} \nu(\zeta) \, \mathrm{d}\zeta, \quad z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \infty), \tag{3.34}$$

where the path of integration from $-\alpha_{\mathbf{t}}$ to $z$ lies entirely in the region $z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \infty)$, except for the initial point $-\alpha_{\mathbf{t}}$.

The functions $\phi(z)$ and $\widetilde{\phi}(z)$ defined above will play an important role in our argument, and some of their properties are given below. For proofs of these results, see [20].

**Proposition 3.1** *Let $\mathbb{C}_+ = \{z : \mathrm{Im}\ z > 0\}$ and $\mathbb{C}_- = \{z : \mathrm{Im}\ z < 0\}$. We have*

$$\phi(z) = \widetilde{\phi}(z) \mp \pi\mathrm{i} \tag{3.35}$$

*for $z \in \mathbb{C}_\pm$. The mapping properties of $\phi(z)$ are given by*

$$\phi(x) > 0, \quad \arg \phi(x) = 0, \quad x \in (\alpha_{\mathbf{t}}, \infty); \tag{3.36}$$

$$\phi(\alpha_{\mathbf{t}}) = 0, \quad \phi_\pm(-\alpha_{\mathbf{t}}) = \mp\pi\mathrm{i}, \quad \phi_\pm(0) = \mp\frac{\pi}{2}\mathrm{i}; \tag{3.37}$$

$$\mathrm{Re}\ \phi_\pm(x) = 0, \quad \arg \phi_\pm(x) = \pm\frac{3}{2}\pi, \quad x \in (-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}) \tag{3.38}$$

*and*

$$\phi(z) \sim \frac{z^4}{2} \tag{3.39}$$

*for $z$ large enough. Similarly, the mapping properties of $\widetilde{\phi}(z)$ are given by*

$$\widetilde{\phi}(x) > 0, \quad \arg \widetilde{\phi}(x) = 0, \quad x \in (-\infty, -\alpha_{\mathbf{t}}); \tag{3.40}$$

$$\widetilde{\phi}(-\alpha_{\mathbf{t}}) = 0, \quad \widetilde{\phi}_\pm(\alpha_{\mathbf{t}}) = \pm\pi\mathrm{i}, \quad \widetilde{\phi}_\pm(0) = \pm\frac{\pi}{2}\mathrm{i}; \tag{3.41}$$

$$\mathrm{Re}\ \widetilde{\phi}_\pm(x) = 0, \quad \arg \widetilde{\phi}_\pm(x) = \mp\frac{3}{2}\pi, \quad x \in (-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}) \tag{3.42}$$

*and*

$$\widetilde{\phi}(z) \sim \frac{z^4}{2} \tag{3.43}$$

*for $z$ large enough.*

**Proposition 3.2** *With constant $l_{\mathbf{N}}$ given in* (3.32) *and $\widetilde{l}_{\mathbf{N}}$ defined by*

$$\widetilde{l}_{\mathbf{N}} = l_{\mathbf{N}} + 2\pi\mathrm{i}, \tag{3.44}$$

*the following connection formulas between the $g$-function ($\widetilde{g}$-function) and $\phi$-function ($\widetilde{\phi}$-function) hold*

$$g(z) + \phi(z) = \frac{1}{2}\Big(z^4 + \frac{tz^2}{\sqrt{N}} + l_{\mathbf{N}}\Big), \tag{3.45}$$

$$\widetilde{g}(z) + \widetilde{\phi}(z) = \frac{1}{2}\Big(z^4 + \frac{tz^2}{\sqrt{N}} + \widetilde{l}_{\mathbf{N}}\Big). \tag{3.46}$$

*Furthermore, we have*

$$\widetilde{g}(z) = \begin{cases} g(z), & z \in \mathbb{C}_+, \\ g(z) + 2\pi\mathrm{i}, & z \in \mathbb{C}_- \end{cases} \tag{3.47}$$

*and*

$$\widetilde{\phi}(z) = \begin{cases} \phi(z) + \pi\mathrm{i}, & z \in \mathbb{C}_+, \\ \phi(z) - \pi\mathrm{i}, & z \in \mathbb{C}_-. \end{cases} \tag{3.48}$$

### 3.2 Case for $c = -2$

Now, we consider the second case $t_{\mathbf{cr}} = -2\sqrt{N}$. With

$$\alpha_{\mathbf{t}} = \alpha_{\mathbf{t}_{cr}} = \sqrt{2}, \tag{3.49}$$

we have from (3.23)

$$\mu_{\mathbf{t}}(x) = \mu_{\mathbf{t}_{cr}}(x) = \frac{2x^2\sqrt{2-x^2}}{\pi}. \tag{3.50}$$

With $\alpha_{\mathbf{t}} = \sqrt{2}$ and $t = -2\sqrt{N}$, the formula in (3.32) gives

$$l_{\mathbf{N}} = \frac{1}{2} - \log 2. \tag{3.51}$$

Let $\nu(z) := 2z^2\sqrt{z^2-2}$, where $\sqrt{z^2-2}$ is analytic in $\mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}]$ and behaves likes $z$ as $z \to \infty$. Clearly, $(\nu)_\pm(x) = \pm\pi i \mu_{\mathbf{t}}(x)$ for $x \in [-\sqrt{2}, \sqrt{2}]$. Define

$$\phi(z) := \int_{\sqrt{2}}^{\mathbf{z}} \nu(\zeta)\,d\zeta, \quad z \in \mathbb{C} \setminus (-\infty, \sqrt{2}], \tag{3.52}$$

where the path of integration from $\sqrt{2}$ to $z$ lies entirely in the region $z \in \mathbb{C} \setminus (-\infty, \sqrt{2}]$, except for the initial point $\sqrt{2}$.

Similarly, we define

$$\widetilde{\phi}(z) := \int_{-\sqrt{2}}^{\mathbf{z}} \nu(\zeta)\,d\zeta, \quad z \in \mathbb{C} \setminus (-\sqrt{2}, \infty], \tag{3.53}$$

where the path of integration from $-\sqrt{2}$ to $z$ lies entirely in the region $z \in \mathbb{C} \setminus [-\sqrt{2}, \infty)$, except for the initial point $-\sqrt{2}$.

The functions $\phi(z)$ and $\widetilde{\phi}(z)$ defined above play the same role as $\phi(z)$ and $\widetilde{\phi}(z)$ defined in (3.33)–(3.34) for case (i). The following are some of their properties.

**Proposition 3.3** *Denoting* $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ *and* $\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}$, *we have*

$$\phi(z) = \widetilde{\phi}(z) \mp \pi i \tag{3.54}$$

*for* $z \in \mathbb{C}_\pm$. *The mapping properties of* $\phi(z)$ *are given by*

$$\phi(x) > 0, \quad \arg\phi(x) = 0, \quad x \in (\sqrt{2}, \infty); \tag{3.55}$$

$$\phi(\sqrt{2}) = 0, \quad \phi_\pm(-\sqrt{2}) = \mp\pi i, \quad \phi_\pm(0) = \mp\frac{\pi}{2}i; \tag{3.56}$$

$$\operatorname{Re}\, \phi_\pm(x) = 0, \quad \arg\phi_\pm(x) = \pm\frac{3}{2}\pi, \quad x \in [0, \sqrt{2}); \tag{3.57}$$

$$\operatorname{Re}\, \phi(z) > 0, \quad \operatorname{Im}\, \phi(z) = -\frac{1}{2}\pi, \quad z \in (0, \infty i). \tag{3.58}$$

*If* $z \in \mathbb{C}_+$, *we have*

$$\phi(z) \sim -\frac{\pi}{2}i + \frac{2^{\frac{3}{2}}}{3}z^3 i \tag{3.59}$$

*as* $z \to 0$ *and*

$$\phi(z) \sim \frac{z^4}{2} \tag{3.60}$$

*for z large enough. Similarly, the mapping properties of* $\widetilde{\phi}(z)$ *are given by*

$$\widetilde{\phi}(x) > 0, \quad \arg \widetilde{\phi}(x) = 0, \quad x \in (-\infty, -\sqrt{2}); \tag{3.61}$$

$$\widetilde{\phi}(-\sqrt{2}) = 0, \quad \widetilde{\phi}_{\pm}(\sqrt{2}) = \pm \pi \mathrm{i}, \quad \widetilde{\phi}_{\pm}(0) = \pm \frac{\pi}{2}\mathrm{i}; \tag{3.62}$$

$$\mathrm{Re}\ \widetilde{\phi}_{\pm}(x) = 0, \quad \arg \widetilde{\phi}_{\pm}(x) = \mp \frac{3}{2}\pi, \quad x \in (-\sqrt{2}, 0]; \tag{3.63}$$

$$\mathrm{Re}\ \widetilde{\phi}(z) > 0, \quad \mathrm{Im}\ \widetilde{\phi}(z) = \frac{\pi}{2}, \quad z \in (0, \infty \mathrm{i}). \tag{3.64}$$

*If* $z \in \mathbb{C}_+$, *we have*

$$\widetilde{\phi}(z) \sim \frac{\pi}{2}\mathrm{i} + \frac{2^{\frac{3}{2}}}{3} z^3 \mathrm{i} \tag{3.65}$$

*as* $z \to 0$ *and*

$$\widetilde{\phi}(z) \sim \frac{z^4}{2} \tag{3.66}$$

*for z large enough.*

**Proposition 3.4** *With constant* $l_{\mathbf{N}}$ *given in* (3.51) *and* $\widetilde{l}_{\mathbf{N}}$ *defined by*

$$\widetilde{l}_{\mathbf{N}} = l_{\mathbf{N}} + 2\pi \mathrm{i}, \tag{3.67}$$

*the following connection formulas between the g-function* ($\widetilde{g}$*-function*) *and* $\phi$*-function* ($\widetilde{\phi}$*-function*) *hold*

$$g(z) + \phi(z) = \frac{1}{2}(z^4 - 2z^2 + l_{\mathbf{N}}), \tag{3.68}$$

$$\widetilde{g}(z) + \widetilde{\phi}(z) = \frac{1}{2}(z^4 - 2z^2 + \widetilde{l}_{\mathbf{N}}) \tag{3.69}$$

*and*

$$\widetilde{g}(z) = \begin{cases} g(z), & z \in \mathbb{C}_+ \\ g(z) + 2\pi \mathrm{i}, & z \in \mathbb{C}_-. \end{cases} \tag{3.70}$$

The material in this and the previous section parallels that in [19, §3].

### 3.3 Case for $c < -2$

In this third case, the equilibrium measure will be defined on two intervals, and we denote them by $E := [-b, -a] \cup [a, b]$ with $0 < a < b$. To see that there are exactly two intervals and these intervals are symmetric, we refer to [4, p. 218] and [6, p. 603]. To normalize the behavior of $U(z)$ at $z = \infty$, we define as in (3.1) a probability density function $\mu_{\mathbf{t}}(s)$ supported on $E$, i.e., $\mu_{\mathbf{t}}(s) \geq 0$ and

$$\int_{\mathbf{E}} \mu_{\mathbf{t}}(s)\, \mathrm{d}s = 1. \tag{3.71}$$

The explicit formula for $a, b$, and $\mu_{\mathbf{t}}(s)$ will be determined later.

Now, we repeat the analysis at the beginning of this section from (3.1) to (3.12) with the interval $[-\alpha_t, \alpha_t]$ replaced by $E$. With this minor change, we define

$$g(z) = \int_{-b}^{-a} \log(z-s)\mu_t(s)\,\mathrm{d}s + \int_{a}^{b} \log(z-s)\mu_t(s)\,\mathrm{d}s, \quad z \in \mathbb{C} \setminus (-\infty, b]. \tag{3.72}$$

From (3.71)–(3.72), it is easily checked that

$$g_+(x) - g_-(x) = \begin{cases} 2\pi\mathrm{i}, & x \in (-\infty, -b), \\ 2\pi\mathrm{i}\displaystyle\int_x^{-a} \mu_t(s)\,\mathrm{d}s + \pi\mathrm{i}, & x \in (-b, -a), \\ \pi\mathrm{i}, & x \in (-a, a), \\ 2\pi\mathrm{i}\displaystyle\int_x^{b} \mu_t(s)\,\mathrm{d}s, & x \in (a, b). \end{cases} \tag{3.73}$$

Also, on account of (3.72)–(3.73), one readily sees that $\mathrm{e}^{ng(z)}$ can be analytically extended to $\mathbb{C} \setminus [-b, b]$ and (3.6) holds. As in the previous cases, we introduce the transformation $T(z)$ given in (3.8), and obtain its associated RHP stated in $(T_\mathrm{a})$-$(T_\mathrm{d})$. To determine a probability density function $\mu_t(s)$ in (3.71) and a constant $l_N$ mentioned above so that the entry $J_{12}$ in jump condition $(T_\mathrm{b})$ becomes 1 for $x \in E$, we set

$$N(g_+(x) + g_-(x) - l_N) - (Nx^4 + tx^2N^{\frac{1}{2}}) = 0 \tag{3.74}$$

for $x \in E$. By substituting $x = b$ in (3.73), it follows from (3.74) that

$$l_N = 2g(b) - b^4 - tb^2N^{-\frac{1}{2}}. \tag{3.75}$$

Differentiating (3.74) yields

$$G_+(x) + G_-(x) = \frac{\mathrm{i}}{\pi}(4x^3 + 2txN^{-\frac{1}{2}}), \quad x \in E, \tag{3.76}$$

where

$$G(z) := \frac{1}{\pi\mathrm{i}} \int_E \frac{\mu_t(s)}{s-z}\,\mathrm{d}s, \quad z \in \mathbb{C} \setminus E. \tag{3.77}$$

Corresponding to (3.13), we set

$$\widetilde{G}(z) := \frac{G(z)}{\sqrt{(z^2-a^2)(z^2-b^2)}}, \tag{3.78}$$

where $\sqrt{(z^2-a^2)(z^2-b^2)}$ is analytic in $\mathbb{C} \setminus E$ and behaves like $z^2$ as $z \to \infty$ (see [9, p. 171]). Since

$$\widetilde{G}_+(x) - \widetilde{G}_-(x) = \frac{\mathrm{i}}{\pi} \frac{4x^3 + 2txN^{-\frac{1}{2}}}{(\sqrt{(x^2-a^2)(x^2-b^2)})_+}, \quad x \in E, \tag{3.79}$$

solving this scalar RHP gives

$$\widetilde{G}(z) = \frac{1}{2\pi^2} \int_E \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{(\sqrt{(s^2-a^2)(s^2-b^2)})_+} \frac{\mathrm{d}s}{s-z},$$

or equivalently

$$G(z) = \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi^2} \int_{\mathsf{E}} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{(\sqrt{(s^2-a^2)(s^2-b^2)})_+} \frac{\mathrm{d}s}{s-z}. \tag{3.80}$$

Recall (3.71) and (3.77). Since $G(z) \to 0$ and $zG(z) \to \frac{\mathrm{i}}{\pi}$ as $z \to \infty$, we have from (3.80)

$$\int_{\mathsf{E}} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{(\sqrt{(s^2-a^2)(s^2-b^2)})_+} s^{\mathsf{j}}\, \mathrm{d}s = 0 \tag{3.81}$$

for $j = 0, 1$, and

$$\int_{\mathsf{E}} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{(\sqrt{(s^2-a^2)(s^2-b^2)})_+} s^2\, \mathrm{d}s = -2\pi\mathrm{i} \tag{3.82}$$

(see [9, p. 172]). Equations (3.81)–(3.82) are equivalent to

$$\int_{\mathsf{a}}^{\mathsf{b}} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{\sqrt{(s^2-a^2)(b^2-s^2)}}\, \mathrm{d}s = 0 \tag{3.83}$$

and

$$\int_{\mathsf{a}}^{\mathsf{b}} \frac{(4s^3 + 2tsN^{-\frac{1}{2}})s^2}{\sqrt{(s^2-a^2)(b^2-s^2)}}\, \mathrm{d}s = \pi. \tag{3.84}$$

Now we have two equations with two unknowns $a$ and $b$. By a change of variable

$$s = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}},$$

we obtain after a long and complicated calculation

$$a = \Big(-\frac{tN^{-\frac{1}{2}} + 2}{2}\Big)^{\frac{1}{2}} \quad \text{and} \quad b = \Big(-\frac{tN^{-\frac{1}{2}} - 2}{2}\Big)^{\frac{1}{2}} \tag{3.85}$$

(see [4, p. 218]). We next derive an explicit formula for the probability density function $\mu_{\mathsf{t}}(s)$ in (3.71). It is well known that

$$\mu_{\mathsf{t}}(x) = \mathrm{Re}\; G_+(x).$$

To evaluate $G(z)$ in (3.80), we need Lemma 1 in [20], which states that for $b > a > 0$, we have

$$\int_{\mathsf{a}}^{\mathsf{b}} \frac{s}{\sqrt{(b-s)(s-a)}} \frac{1}{s-z}\, \mathrm{d}s = \pi\Big(1 - \frac{z}{\sqrt{(z-a)(z-b)}}\Big). \tag{3.86}$$

Recall that

$$\int_{\mathsf{a}}^{\mathsf{b}} \frac{1}{\sqrt{(b-s)(s-a)}} \frac{1}{s-z}\, \mathrm{d}s = \frac{-\pi}{\sqrt{(z-a)(z-b)}}, \tag{3.87}$$

where $\sqrt{(z-a)(z-b)}$ takes branch cut along $[a, b]$ and behaves like $z$ as $z \to \infty$. It then follows from (3.80) and (3.86)–(3.87) that

$$G(z) = \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi^2} \int_{\mathsf{E}} \frac{4s^3 + 2tsN^{-\frac{1}{2}}}{(\sqrt{(s^2-a^2)(s^2-b^2)})_+} \frac{\mathrm{d}s}{s-z}$$

$$= \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi \mathrm{i}}\Big(4z - \frac{4z^3 + 2tN^{-\frac{1}{2}}z}{\sqrt{(z^2-a^2)(z^2-b^2)}}\Big), \tag{3.88}$$

which gives

$$\mu_{\mathbf{t}}(x) = \mathrm{Re}\ G_+(x) = \frac{2|x|}{\pi}\sqrt{(x^2-a^2)(b^2-x^2)} > 0 \tag{3.89}$$

for $x \in E$, where $a, b$ are given in (3.85).

Next, we proceed to calculate $l_{\mathbf{N}}$ in (3.75). By (3.71)–(3.72), the function $g(z) - \frac{1}{2}(\log(z+b) + \log(z-b))$ is analytic in $\mathbb{C} \setminus E$ and

$$g(z) - \frac{1}{2}(\log(z+b) + \log(z-b)) \to 0 \quad \text{as } z \to \infty. \tag{3.90}$$

Let us define

$$H(z) := \frac{g(z) - \frac{1}{2}(\log(z+b) + \log(z-b))}{\sqrt{(z^2-a^2)(z^2-b^2)}}, \tag{3.91}$$

where $\sqrt{(z^2-a^2)(z^2-b^2)}$ is analytic in $\mathbb{C} \setminus E$ and behaves like $z^2$ as $z \to \infty$. It can be readily verified that for $x \in E$,

$$H_+(x) - H_-(x) = \frac{g_+(x) + g_-(x) - \log(b^2-x^2)}{(\sqrt{(x^2-a^2)(x^2-b^2)})_+}.$$

By the Plemelj formula (see [1, p. 518]), we obtain

$$g(z) - \frac{1}{2}(\log(z+b) + \log(z-b)) = \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi \mathrm{i}}\int_{\mathbf{E}} \frac{H_+(x) - H_-(x)}{x-z}\,\mathrm{d}x.$$

From (3.74), it follows

$$\begin{aligned} &g(z) - \frac{1}{2}(\log(z+b) + \log(z-b)) \\ &= \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi \mathrm{i}}\int_{\mathbf{E}} \frac{l_{\mathbf{N}} + x^4 + tN^{-\frac{1}{2}}x^2 - \log(b^2-x^2)}{(\sqrt{(x^2-a^2)(x^2-b^2)})_+(x-z)}\mathrm{d}x, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &g(z) - \frac{1}{2}(\log(z+b) + \log(z-b)) \\ &= \frac{\sqrt{(z^2-a^2)(z^2-b^2)}}{2\pi \mathrm{i}}\Big(\int_{\mathbf{a}}^{\mathbf{b}} \frac{l_{\mathbf{N}} + x^4 + tN^{-\frac{1}{2}}x^2}{\mathrm{i}\sqrt{(x^2-a^2)(b^2-x^2)}}\frac{2x}{x^2-z^2}\,\mathrm{d}x \\ &\quad - 2\int_{\mathbf{a}}^{\mathbf{b}} \frac{x\log(b^2-x^2)}{\mathrm{i}\sqrt{(x^2-a^2)(b^2-x^2)}}\frac{\mathrm{d}x}{x^2-z^2}\Big). \end{aligned}$$

Now, let $z \to \infty$; on account of (3.90), we have

$$\begin{aligned} &\int_{\mathbf{a}}^{\mathbf{b}} \frac{2xl_{\mathbf{N}}}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,\mathrm{d}x + \int_{\mathbf{a}}^{\mathbf{b}} \frac{(x^4 + tN^{-\frac{1}{2}}x^2)2x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,\mathrm{d}x \\ &- 2\int_{\mathbf{a}}^{\mathbf{b}} \frac{x\log(b^2-x^2)}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,\mathrm{d}x = 0. \end{aligned} \tag{3.92}$$

Let the integrals on the left-hand side of (3.92) be denoted by $\mathrm{I}_1$, $\mathrm{I}_2$ and $\mathrm{I}_3$, respectively. As in Subsections 3.1–3.2, we have

$$\mathrm{I}_1 = \int_a^b \frac{2x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,\mathrm{d}x = \pi, \tag{3.93}$$

$$\mathrm{I}_2 = \int_a^b \frac{(x^4 + tN^{-\frac{1}{2}}x^2)}{\sqrt{(x^2-a^2)(b^2-x^2)}} 2x\,\mathrm{d}x = \frac{\pi}{2} - \frac{\pi}{4}t^2N^{-1} \tag{3.94}$$

and

$$\int_a^b \frac{\log s}{\sqrt{(s-a)(b-s)}}\,\mathrm{d}s = 2\pi \log \frac{\sqrt{a}+\sqrt{b}}{2}.$$

Taking $s = b^2 - x^2$, we obtain

$$\mathrm{I}_3 = \int_0^{b^2-a^2} \frac{\log s}{\sqrt{s(b^2-a^2-s)}}\,\mathrm{d}s = \pi \log \frac{1}{2}. \tag{3.95}$$

Inserting (3.93)–(3.95) into (3.92) gives

$$l_N = -\log 2 - \frac{1}{2} + \frac{t^2N^{-1}}{4}. \tag{3.96}$$

As before, let $\nu(z) := 2z\sqrt{(z^2-a^2)(z^2-b^2)}$, where $\sqrt{(z^2-a^2)(z^2-b^2)}$ is analytic in $\mathbb{C}\setminus E$ and behaves likes $z^2$ as $z \to \infty$. Clearly, $\nu_\pm(x) = \pm\pi\mathrm{i}\mu_t(x)$ for $x \in E$. Define

$$\phi(z) := \int_b^z \nu(\zeta)\,\mathrm{d}\zeta, \quad z \in \mathbb{C}\setminus(-\infty, b], \tag{3.97}$$

where the path of integration from $b$ to $z$ lies entirely in the region $z \in \mathbb{C}\setminus(-\infty, b]$, except for the initial point $b$. Similarly, define

$$\widetilde{\phi}(z) := \int_{-b}^z \nu(\zeta)\,\mathrm{d}\zeta, \quad z \in \mathbb{C}\setminus[-b, \infty), \tag{3.98}$$

where the path of integration from $-b$ to $z$ lies entirely in the region $z \in \mathbb{C}\setminus[-b,\infty)$, except for the initial point $-b$. Moreover, we define

$$\overline{\phi}(z) := \int_a^z \nu(\zeta)\,\mathrm{d}\zeta, \quad z \in \mathbb{C}\setminus(-\infty,-a)\cup(a,\infty), \tag{3.99}$$

where the path of integration from $a$ to $z$ lies entirely in the region $z \in \mathbb{C}\setminus(-\infty,-a)\cup(a,\infty)$, except for the initial point $a$. The following two results correspond to Propositions 3.1–3.2 for case (i), and to Propositions 3.3–3.4 for case (ii).

**Proposition 3.5** *The mapping properties of $\phi(z)$ are given by*

$$\phi(x) > 0, \quad \arg\phi(x) = 0, \quad x \in (b,\infty); \tag{3.100}$$

$$\phi(b) = 0, \quad \phi_\pm(a) = \mp\frac{\pi}{2}\mathrm{i}, \quad \phi_\pm(-b) = \mp\pi\mathrm{i}; \tag{3.101}$$

$$\mathrm{Re}\ \phi_\pm(x) = 0, \quad \arg\phi_\pm(x) = \pm\frac{3}{2}\pi, \quad x \in (a,b) \tag{3.102}$$

*and*

$$\phi(z) \sim \frac{z^4}{2} \tag{3.103}$$

*for z large enough. Similarly, the mapping properties of* $\widetilde{\phi}(z)$ *are given by*

$$\widetilde{\phi}(x) > 0, \quad \arg\widetilde{\phi}(x) = 0, \quad x \in (-\infty, -b); \tag{3.104}$$

$$\widetilde{\phi}(-b) = 0, \quad \widetilde{\phi}_{\pm}(-a) = \pm\frac{\pi}{2}\mathrm{i}, \quad \widetilde{\phi}_{\pm}(b) = \pm\pi\mathrm{i}; \tag{3.105}$$

$$\mathrm{Re}\, \widetilde{\phi}_{\pm}(x) = 0, \quad \arg\widetilde{\phi}_{\pm}(x) = \mp\frac{3}{2}\pi, \quad x \in (-b, -a) \tag{3.106}$$

*and*

$$\widetilde{\phi}(z) \sim \frac{z^4}{2} \tag{3.107}$$

*for z large enough. Moreover, the mapping properties of* $\overline{\phi}(z)$ *are given by*

$$\overline{\phi}(x) > 0, \quad \arg\overline{\phi}(x) = 0, \quad x \in (-a, a); \tag{3.108}$$

$$\overline{\phi}(\pm a) = 0, \quad \overline{\phi}_{\pm}(-b) = \mp\frac{\pi}{2}\mathrm{i}, \quad \overline{\phi}_{\pm}(b) = \pm\frac{\pi}{2}\mathrm{i} \tag{3.109}$$

*and*

$$\overline{\phi}(z) \sim \frac{z^4}{2} \tag{3.110}$$

*for z large enough.*

**Proposition 3.6** *The following connection formula holds between the g-function and* $\phi$*-function*

$$\frac{1}{2}\Big(z^4 + \frac{tz^2}{\sqrt{N}} + l_{\mathbf{N}}\Big) = g(z) + \phi(z). \tag{3.111}$$

*Furthermore, we have for* $z \in \mathbb{C}_{\pm}$,

$$\phi(z) = \begin{cases} \widetilde{\phi}(z) \mp \pi\mathrm{i}, \\ \overline{\phi}(z) \mp \frac{1}{2}\pi\mathrm{i}. \end{cases} \tag{3.112}$$

## 4 Contour Deformation

In this section, we consider the problem of contour deformation in three separate cases. We give more details in the first case; for the other two cases, we only present the conclusions since the results can be obtained in the same manner as in the first case.

### 4.1 Case for $c > -2$

With the properties of $\phi(z)$ and $\widetilde{\phi}(z)$ established in Proposition 3.1, the jump matrix for $T(z)$ in condition $(T_{\mathrm{b}})$ can be written as follows:

$$T_+(x) = T_-(x)\begin{pmatrix} e^{2N\phi_+(x)} & 1 \\ 0 & e^{2N\phi_-(x)} \end{pmatrix} \tag{4.1}$$

for $x \in (0, \alpha_{\mathbf{t}})$,

$$T_+(x) = T_-(x)\begin{pmatrix} e^{2N\widetilde{\phi}_+(x)} & 1 \\ 0 & e^{2N\widetilde{\phi}_-(x)} \end{pmatrix} \tag{4.2}$$

for $x \in (-\alpha_{\mathbf{t}}, 0)$,

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & \mathrm{e}^{-2\mathbf{N}\boldsymbol{\varphi}(\mathbf{x})} \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

for $x \in (\alpha_{\mathbf{t}}, \infty)$,

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & \mathrm{e}^{-2\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{x})} \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

for $x \in (-\infty, -\alpha_{\mathbf{t}})$.

Note that for the jump matrix on $(0, \alpha_{\mathbf{t}})$, we have the following factorization:

$$\begin{pmatrix} \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}_+(\mathbf{x})} & 1 \\ 0 & \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}_-(\mathbf{x})} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}_-(\mathbf{x})} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}_+(\mathbf{x})} & 1 \end{pmatrix}. \tag{4.5}$$

The first and the third matrices on the right-hand side of (4.5) have the analytic continuation

$$\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & 1 \end{pmatrix}$$

on both sides of $(0, \alpha_{\mathbf{t}})$. When $x \in (-\alpha_{\mathbf{t}}, 0)$, we have a similar result with $\phi(x)$ replaced by $\widetilde{\phi}(x)$. Based on the factorization in (4.5), we transform the RHP for $T(z)$ into a RHP for $V(z)$ given below, by opening a lens around $(-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}})$ going through the origin (see Figure 1). The precise shape of the lens will be specified later; for now we choose it to be contained in the region of analyticity of $\phi(z)$ and $\widetilde{\phi}(z)$.

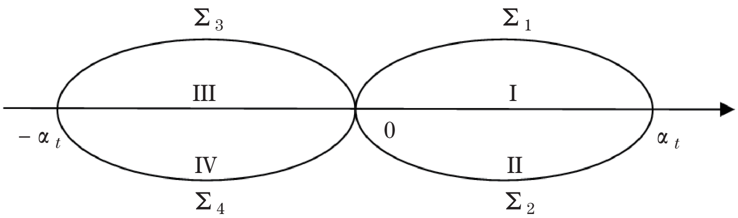


Figure 1 The lens-shaped contour Σ going through the origin.

Let $\Sigma = \bigcup_{\mathbf{i}=1}^{4} \Sigma_{\mathbf{i}}$ denote the lens-shaped contour shown in Figure 1. The second transformation $T \to V$ is then defined by

$$V(z) := \begin{cases} T(z) & \text{for } z \text{ outside the lens-shaped region,} \\ T(z)\begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & 1 \end{pmatrix} & \text{for } z \in \mathrm{I}, \\ T(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & 1 \end{pmatrix} & \text{for } z \in \mathrm{II}, \\ T(z)\begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{2\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & 1 \end{pmatrix} & \text{for } z \in \mathrm{III}, \\ T(z)\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{2\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & 1 \end{pmatrix} & \text{for } z \in \mathrm{IV}, \end{cases} \tag{4.6}$$

where regions I, II, III, IV are also shown in Figure 1. Furthermore, we define the jump matrices

$$J_{\mathbf{V}}(z) = \begin{pmatrix} 1 & 0 \\ e^{2\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & 1 \end{pmatrix}, \qquad z \in \Sigma_1 \cup \Sigma_2, \tag{4.7}$$

$$J_{\mathbf{V}}(z) = \begin{pmatrix} 1 & 0 \\ e^{2\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & 1 \end{pmatrix}, \qquad z \in \Sigma_3 \cup \Sigma_4, \tag{4.8}$$

$$J_{\mathbf{V}}(x) = \begin{pmatrix} 1 & e^{-2\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{x})} \\ 0 & 1 \end{pmatrix}, \qquad x < -\alpha_{\mathbf{t}}, \tag{4.9}$$

$$J_{\mathbf{V}}(x) = \begin{pmatrix} 1 & e^{-2\mathbf{N}\boldsymbol{\varphi}(\mathbf{x})} \\ 0 & 1 \end{pmatrix}, \qquad x > \alpha_{\mathbf{t}}, \tag{4.10}$$

$$J_{\mathbf{V}}(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad x \in (-\alpha_{\mathbf{t}}, 0) \cup (0, \alpha_{\mathbf{t}}). \tag{4.11}$$

It is readily verified that $V$ satisfies the conditions of the following RHP:

($V_{\mathrm{a}}$) $V(z)$ is analytic in $\mathbb{C} \setminus (\Sigma \cup \mathbb{R})$;

($V_{\mathrm{b}}$) for $x \in \Sigma \cup \mathbb{R}$,

$$V_+(x) = V_-(x) J_{\mathbf{V}}(x);$$

($V_{\mathrm{c}}$) $V(z)$ behaves like the identity matrix at infinity:

$$V(z) = [I + O(1/z)] \quad \text{as } z \to \infty,$$

for $z \in \mathbb{C} \setminus (\Sigma \cup \mathbb{R})$;

($V_{\mathrm{d}}$) if $\alpha < 0$,

$$V(z) = O\begin{pmatrix} |z|^{\boldsymbol{\alpha}} & |z|^{\boldsymbol{\alpha}} \\ |z|^{\boldsymbol{\alpha}} & |z|^{\boldsymbol{\alpha}} \end{pmatrix} \quad \text{as } z \to 0,\ z \in \mathbb{C} \setminus \Sigma \cup \mathbb{R};$$

if $\alpha > 0$,

$$V(z) = \begin{cases} O\begin{pmatrix} |z|^{\boldsymbol{\alpha}} & |z|^{-\boldsymbol{\alpha}} \\ |z|^{\boldsymbol{\alpha}} & |z|^{-\boldsymbol{\alpha}} \end{pmatrix} & \text{as } z \to 0 \text{ for } z \text{ outside the lens regions,} \\ O\begin{pmatrix} |z|^{-\boldsymbol{\alpha}} & |z|^{-\boldsymbol{\alpha}} \\ |z|^{-\boldsymbol{\alpha}} & |z|^{-\boldsymbol{\alpha}} \end{pmatrix} & \text{as } z \to 0 \text{ for } z \text{ inside the lens regions.} \end{cases}$$

From Proposition 3.1 and the behavior of $\phi(z)$ in (3.33) near $z = \alpha_{\mathbf{t}}$, the lens-shaped regions can be chosen sufficiently small so that

$$\operatorname{Re} \phi(z) < 0 \tag{4.12}$$

for $z \in \mathrm{I} \cup \mathrm{II}$. Similarly, it can be shown that

$$\operatorname{Re} \widetilde{\phi}(z) < 0 \tag{4.13}$$

for $z \in \mathrm{III} \cup \mathrm{IV}$. These together with (3.36) and (3.40) imply that the jump matrix $J_{\mathbf{V}}(z)$ tends exponentially to the identity matrix as $n \to \infty$, for $z \in (-\infty, -\alpha_{\mathbf{t}}) \cup (\alpha_{\mathbf{t}}, \infty) \cup \Sigma$. When

$z \in (-\alpha_{\mathbf{t}}, 0) \cup (0, \alpha_{\mathbf{t}})$, $J_{\mathbf{V}}(z)$ is the constant matrix given by (4.11). It is therefore natural to suggest that for large $n$, the solution of the RHP for $V(z)$ may behave asymptotically like the solution of the following RHP for $V_\infty(z)$:

$(V_{\infty,\mathrm{a}})$ $V_\infty(z)$ is analytic in $\mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$;

$(V_{\infty,\mathrm{b}})$ for $x \in (-\alpha_{\mathbf{t}}, 0) \cup (0, \alpha_{\mathbf{t}})$,

$$(V_\infty)_+(x) = (V_\infty)_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$(V_{\infty,\mathrm{c}})$ $V_\infty(z)$ behaves like the identity matrix at infinity:

$$V_\infty(z) = [I + O(1/z)] \quad \text{as } z \to \infty,$$

for $z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$.

The construction of $V_\infty(z)$ was done in [9–11], and we have

$$V_\infty(z) = \begin{pmatrix} \dfrac{\beta(z) + \beta(z)^{-1}}{2} & \dfrac{\beta(z) - \beta(z)^{-1}}{2\mathrm{i}} \\ \dfrac{\beta(z) - \beta(z)^{-1}}{-2\mathrm{i}} & \dfrac{\beta(z) + \beta(z)^{-1}}{2} \end{pmatrix}, \tag{4.14}$$

where

$$\beta(z) = \left(\frac{z - \alpha_{\mathbf{t}}}{z + \alpha_{\mathbf{t}}}\right)^{\frac{1}{4}} \tag{4.15}$$

with a branch cut along $[-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]$ and $\beta(z) \to 1$ as $z \to \infty$. It is worthwhile to point out that $V_\infty(z)$ has the factorization

$$V_\infty(z) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \beta(z)^{-\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ i & \mathrm{i} \end{pmatrix} \beta(z)^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -\mathrm{i} \\ -1 & -\mathrm{i} \end{pmatrix}, \tag{4.16}$$

where $\sigma_3$ is the Pauli matrix given in (3.7). By (4.6) and (4.12)–(4.13), it is clear that $T \sim V$ as $n \to \infty$. Hence, it follows that

$$T(z) \sim V_\infty(z) \quad \text{for } z \in \mathbb{C} \setminus [-\alpha_{\mathbf{t}}, \alpha_{\mathbf{t}}]. \tag{4.17}$$

On account of (3.8), we can work backwards to get

$$U(z) \sim \mathrm{e}^{\frac{1}{2}\mathbf{N}\,\mathbf{l}_N \boldsymbol{\sigma}_3} V_\infty(z) \mathrm{e}^{[\mathbf{N}\,(\mathbf{g(z)} - \frac{1}{2}\mathbf{l}_N) - \boldsymbol{\alpha} \log \mathbf{z}]\boldsymbol{\sigma}_3}, \tag{4.18}$$

where $l_{\mathbf{N}}$ is given in (3.32).

Let $L(z)$ represent the right-hand side of (4.18). By (3.47), it is easily verified that

$$g(z) - \log z + \pi\mathrm{i} = \widetilde{g}(z) - \log(-z) \tag{4.19}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $\log(-z)$ is analytic in $z \in \mathbb{C} \setminus [0, \infty)$. Hence, we obtain from (3.44), (3.47) and (4.19) that

$$\begin{aligned} L(z) &= \mathrm{e}^{\frac{1}{2} N l_N \sigma_3} V_\infty(z) \mathrm{e}^{[ng(z)+\alpha(g(z)-\log z)]\sigma_3} \mathrm{e}^{-\frac{1}{2} N l_N \sigma_3} \\ &= \mathrm{e}^{-\alpha\pi \mathrm{i}\sigma_3} \mathrm{e}^{\frac{1}{2} N \widetilde{l}_N \sigma_3} V_\infty(z) \mathrm{e}^{n\widetilde{g}(z)\sigma_3} \mathrm{e}^{\alpha(g(z)-\log z+\pi \mathrm{i})\sigma_3} \mathrm{e}^{-\frac{1}{2} N \widetilde{l}_N \sigma_3} \\ &= \mathrm{e}^{-\alpha\pi \mathrm{i}\sigma_3} \mathrm{e}^{\frac{1}{2} N \widetilde{l}_N \sigma_3} V_\infty(z) \mathrm{e}^{[N(\widetilde{g}(z)-\frac{1}{2}\widetilde{l}_N)-\alpha\log(-z)]\sigma_3} \end{aligned} \tag{4.20}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$.

## 4.2 Case for $c = -2$

Here, we follow the arguments given in Subsection 4.1, and transform the RHP for $T(z)$ into the RHP for $V(z)$ by opening a lens passing through $-\sqrt{2}, 0$ and $\sqrt{2}$ as shown in Figure 1. By the same reasoning given in Subsection 4.1, it is natural to suggest that for large $n$, the solution of the RHP for $V(z)$ should behave asymptotically like the solution of the following RHP for $V_\infty(z)$:

($V_{\infty,\mathrm{a}}$) $V_\infty(z)$ is analytic in $\mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}]$;

($V_{\infty,\mathrm{b}}$) for $x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$,

$$(V_\infty)_+(x) = (V_\infty)_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

($V_{\infty,\mathrm{c}}$) $V_\infty(z)$ behaves like the identity matrix at infinity:

$$V_\infty(z) = [I + O(1/z)] \quad \text{as } z \to \infty.$$

This problem can be solved explicitly, and its solution is given by

$$V_\infty(z) = \begin{pmatrix} \dfrac{\beta(z)+\beta(z)^{-1}}{2} & \dfrac{\beta(z)-\beta(z)^{-1}}{2\mathrm{i}} \\ \dfrac{\beta(z)-\beta(z)^{-1}}{-2\mathrm{i}} & \dfrac{\beta(z)+\beta(z)^{-1}}{2} \end{pmatrix}, \tag{4.21}$$

where

$$\beta(z) = \left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)^{\frac{1}{4}} \tag{4.22}$$

with a branch cut along $[-\sqrt{2}, \sqrt{2}]$ and $\beta(z) \to 1$ as $z \to \infty$ (see (4.14)–(4.15)). Again, we note that $V_\infty(z)$ has the factorization given in (4.16).

## 4.3 Case for $c < -2$

With the properties of $\phi(z)$, $\widetilde{\phi}(z)$ and $\overline{\phi}(z)$ established in Proposition 3.6, the jump matrix for $T(z)$ in condition ($T_\mathrm{b}$) can be written as follows:

$$T_+(x) = T_-(x) \begin{pmatrix} \mathrm{e}^{2N\varphi_+(x)} & 1 \\ 0 & \mathrm{e}^{2N\varphi_-(x)} \end{pmatrix} \tag{4.23}$$

for $x \in (a, b)$,

$$T_+(x) = T_-(x) \begin{pmatrix} e^{2N\widetilde{\varphi}_+(x)} & 1 \\ 0 & e^{2N\widetilde{\varphi}_-(x)} \end{pmatrix} \tag{4.24}$$

for $x \in (-b, -a)$,

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & e^{-2N\varphi(x)} \\ 0 & 1 \end{pmatrix} \tag{4.25}$$

for $x \in (b, \infty)$,

$$T_+(x) = T_-(x) \begin{pmatrix} 1 & e^{-2N\widetilde{\varphi}(x)} \\ 0 & 1 \end{pmatrix} \tag{4.26}$$

for $x \in (-\infty, -b)$,

$$T_+(x) = T_-(x) \begin{pmatrix} e^{n\pi i + \alpha\pi i} & e^{-2N\overline{\varphi}(x)} \\ 0 & e^{-n\pi i - \alpha\pi i} \end{pmatrix} \tag{4.27}$$

for $x \in (-a, 0)$,

$$T_+(x) = T_-(x) \begin{pmatrix} e^{-n\pi i - \alpha\pi i} & e^{-2N\overline{\varphi}(x)} \\ 0 & e^{n\pi i + \alpha\pi i} \end{pmatrix} \tag{4.28}$$

for $x \in (0, a)$.

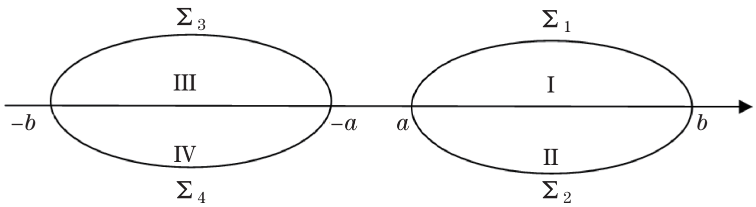


Figure 2 The lens-shaped contour $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_4$.

From (4.5), the transformation $T \to V$ is then defined by

$$V(z) := \begin{cases} T(z) & \text{for } z \text{ outside the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{2N\varphi(z)} & 1 \end{pmatrix} & \text{for } z \in \text{I}, \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{2N\varphi(z)} & 1 \end{pmatrix} & \text{for } z \in \text{II}, \\ T(z) \begin{pmatrix} 1 & 0 \\ -e^{2N\widetilde{\varphi}(z)} & 1 \end{pmatrix} & \text{for } z \in \text{III}, \\ T(z) \begin{pmatrix} 1 & 0 \\ e^{2N\widetilde{\varphi}(z)} & 1 \end{pmatrix} & \text{for } z \in \text{IV}. \end{cases} \tag{4.29}$$

Furthermore, we define the jump matrix

$$J_{\mathbf{V}}(z) := \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2N\varphi(z)} & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ e^{2N\widetilde{\varphi}(z)} & 1 \end{pmatrix}, & z \in \Sigma_3 \cup \Sigma_4, \\ \begin{pmatrix} 1 & e^{-2N\widetilde{\varphi}(x)} \\ 0 & 1 \end{pmatrix}, & x < -b, \\ \begin{pmatrix} 1 & e^{-2N\varphi(x)} \\ 0 & 1 \end{pmatrix}, & x > b, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & x \in (-b,-a) \cup (a,b), \\ \begin{pmatrix} e^{n\pi i + \alpha\pi i} & e^{-2N\overline{\varphi}(x)} \\ 0 & e^{-n\pi i - \alpha\pi i} \end{pmatrix}, & x \in (-a,0), \\ \begin{pmatrix} e^{-n\pi i - \alpha\pi i} & e^{-2N\overline{\varphi}(x)} \\ 0 & e^{n\pi i + \alpha\pi i} \end{pmatrix}, & x \in (0,a). \end{cases} \tag{4.30}$$

Since $\overline{\phi}(x) > 0$ for $x \in (-a,a)$, $\widetilde{\phi}(x) > 0$ for $x < -b$ and $\phi(x) > 0$ for $x > b$, as in the previous cases we expect the solution of the RHP for $V(z)$ to behave asymptotically like the solution of the following RHP for $V_\infty(z)$:

($V_{\infty,\mathrm{a}}$) $V_\infty(z)$ is analytic in $\mathbb{C} \setminus [-b,b]$;

($V_{\infty,\mathrm{b}}$) for $x \in E$,

$$(V_\infty)_+(x) = (V_\infty)_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for $x \in (-a,0)$,

$$(V_\infty)_+(x) = (V_\infty)_-(x) \begin{pmatrix} e^{n\pi i + \alpha\pi i} & 0 \\ 0 & e^{-n\pi i - \alpha\pi i} \end{pmatrix},$$

and for $x \in (0,a)$,

$$(V_\infty)_+(x) = (V_\infty)_-(x) \begin{pmatrix} e^{-n\pi i - \alpha\pi i} & 0 \\ 0 & e^{n\pi i + \alpha\pi i} \end{pmatrix};$$

($V_{\infty,\mathrm{c}}$) for $z \in \mathbb{C} \setminus E$,

$$V_\infty(z) = [I + O(1/z)] \quad \text{as } z \to \infty.$$

We construct $V_\infty(z)$ in terms of the Szegö function $D$ associated with $e^{\alpha\pi i}$ on $E$, which is an analytic function on $\mathbb{C}\setminus[-b,b]$ satisfying $D_+(x)D_-(x) = 1$ for $x \in E$, and $D_+(x) = e^{-\pi i\alpha}D_-(x)$ for $x \in [-a,0]$ and $D_+(x) = e^{\pi i\alpha}D_-(x)$ for $x \in [0,a]$. We seek the function $D(z)$ in the form $D(z) = \exp(\widetilde{D})$. Then the problem is reduced to constructing a scalar function $\widetilde{D}(z)$, which is an analytic function on $\mathbb{C}\setminus E$ satisfying $\widetilde{D}_+(x)+\widetilde{D}_-(x) = 0$ for $x \in E$, and $\widetilde{D}_+(x) = \widetilde{D}_-(x)-\pi i\alpha$ for $x \in (-a,0)$ and $\widetilde{D}_+(x) = \widetilde{D}_-(x) + \pi i\alpha$ for $x \in (0,a)$. We can solve this scalar RHP by the Plemelj formula to yield

$$\widetilde{D}(z) = -\frac{\alpha\sqrt{(z^2-a^2)(z^2-b^2)}}{2}\Big(\int_0^{a^2} \frac{1}{\sqrt{(a^2-x)(b^2-x)}}\frac{dx}{x-z^2}\Big).$$

This gives

$$D(z) = \exp\Big[-\frac{\alpha\sqrt{(z^2-a^2)(z^2-b^2)}}{2}\Big(\int_0^{a^2}\frac{1}{\sqrt{(a^2-x)(b^2-x)}}\frac{dx}{x-z^2}\Big)\Big] \tag{4.31}$$

and

$$D_\infty = \lim_{z\to\infty} D(z) = \Big(\frac{b+a}{b-a}\Big)^{\frac{\alpha}{2}}. \tag{4.32}$$

Therefore, the RHP for $V_\infty(z)$ can be transformed into the RHP for $\widetilde{V}_\infty(z)$ with the jump matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $E$, via the formula

$$\widetilde{V}_\infty(z) = D_\infty^{-\sigma_3} V_\infty(z) D(z)^{\sigma_3}.$$

Thus, we have the RHP for $\widetilde{V}_\infty(z)$:

($\widetilde{V}_{\infty,\mathrm{a}}$) $\widetilde{V}_\infty(z)$ is analytic in $\mathbb{C}\setminus[-b,b]$;

($\widetilde{V}_{\infty,\mathrm{b}}$) for $x \in E$,

$$(\widetilde{V}_\infty)_+(x) = (\widetilde{V}_\infty)_-(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and for $x \in (-a,a)\setminus\{0\}$,

$$(\widetilde{V}_\infty)_+(x) = (\widetilde{V}_\infty)_-(x)\begin{pmatrix} e^{-n\pi i} & 0 \\ 0 & e^{n\pi i} \end{pmatrix};$$

($\widetilde{V}_{\infty,\mathrm{c}}$) for $z \in \mathbb{C}\setminus E$,

$$\widetilde{V}_\infty(z) = [I + O(1/z)] \quad \text{as } z\to\infty.$$

The RHP for $\widetilde{V}_\infty$ can be solved explicitly by

$$\widetilde{V}_\infty(z) = \begin{pmatrix} \dfrac{\widetilde{\gamma}(z)+\widetilde{\gamma}(z)^{-1}}{2} & \dfrac{\widetilde{\gamma}(z)-\widetilde{\gamma}(z)^{-1}}{-2i} \\ \dfrac{\widetilde{\gamma}(z)-\widetilde{\gamma}(z)^{-1}}{2i} & \dfrac{\widetilde{\gamma}(z)+\widetilde{\gamma}(z)^{-1}}{2} \end{pmatrix}, \tag{4.33}$$

where

$$\widetilde{\gamma}(z) = \Big(\frac{z+b}{z-b}\Big)^{\frac14}\Big(\frac{z-a}{z+a}\Big)^{\frac{(-1)^n}{4}} \tag{4.34}$$

with a branch cut along $E$ and $\widetilde{\gamma}(z)\to 1$ as $z\to\infty$. (This can actually be derived by using an elliptic theta function (see [16, pp. 15–16]). It is worthwhile to note that $\widetilde{V}_\infty(z)$ can be written as

$$\widetilde{V}_\infty(z) = \frac12\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\widetilde{\gamma}(z)^{\sigma_3}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \frac12\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix}\widetilde{\gamma}(z)^{-\sigma_3}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}, \tag{4.35}$$

where $\sigma_3$ is the Pauli matrix given in (3.7). Since $T \sim V$ as $n\to\infty$, it follows that

$$T(z) \sim D_\infty^{\sigma_3}\widetilde{V}_\infty(z)D(z)^{-\sigma_3} \quad \text{for } z\in\mathbb{C}\setminus E \tag{4.36}$$

(see (4.17)). On account of (3.8), we can work backwards to get

$$U(z) \sim e^{\frac12 N l_N \sigma_3} D_\infty^{\sigma_3}\widetilde{V}_\infty(z)D(z)^{-\sigma_3}e^{N(g(z)-\frac12 l_N - \alpha\log z)\sigma_3}, \tag{4.37}$$

where $l_N$ is given in (3.96).

# 5 Construction of the Parametrices

In this section, we will construct an approximation $U^*(z)$ to the solution of the RHP for $U(z)$ for large $n$. We again divide our discussion into three cases. For the first case, we bring in Bessel functions in the formation of a parametrix in a region containing the origin. For the second case, we introduce solutions associated with Painlevé II equation and construct the model RHP in the formation of a parametrix in a region containing the origin. For the third case, no special attention is required in the neighborhood of the origin, since it now lies outside the support of the equilibrium measure.

For regions outside the origin, we make use of Airy functions in all three cases. We give more details for the first case, and present only the conclusions for the two other cases.

## 5.1 Case for $c > -2$

Due to the singularity of $|x|^{2\boldsymbol{\alpha}}$ in the weight function, special attention must be paid to the neighborhood of the origin, which we will discuss first. The arguments in this subsection parallel those in [20].

### 5.1.1 Parametrix in the neighborhood of the origin

Let $U_{\boldsymbol{\delta}}$ be a small disk with center at 0 and radius $\delta > 0$. We seek a $2\times 2$ matrix-valued function $P(z)$ in $U_{\boldsymbol{\delta}}$, which has the same jumps as $U(z)$ and matches the behavior of $U(z)$ on the boundary $\partial U_{\boldsymbol{\delta}}$ of the disk (see (4.18)). That is, we wish to find a $2\times 2$ matrix-valued function that satisfies the following RHP:

($P_{\mathrm{a}}$) $P(z)$ is analytic in $U_{\boldsymbol{\delta}}\setminus\mathbb{R}$;

($P_{\mathrm{b}}$) for $x\in(-\delta,\delta)\setminus\{0\}$,

$$P_+(x) = P_-(x)\begin{pmatrix} 1 & |x|^{2\boldsymbol{\alpha}}\mathrm{e}^{-(\mathbf{N}\mathbf{x}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{x}^2)} \\ 0 & 1\end{pmatrix};$$

($P_{\mathrm{c}}$) for $z\in\partial U_{\boldsymbol{\delta}}\setminus\mathbb{R}$, $P(z)$ satisfies the matching condition

$$P(z)\sim \mathrm{e}^{\frac{1}{2}\mathbf{N}\mathbf{l}_N\boldsymbol{\sigma}_3}V_\infty(z)\mathrm{e}^{[\mathbf{N}(\mathbf{g}(\mathbf{z})-\frac{1}{2}\mathbf{l}_N)-\boldsymbol{\alpha}\log\mathbf{z}]\boldsymbol{\sigma}_3}$$

as $n\to\infty$.

Denote by

$$\widetilde{w}(z) := z^{2\boldsymbol{\alpha}}\mathrm{e}^{-(\mathbf{N}\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)} \tag{5.1}$$

the analytic continuation of the function $|x|^{2\boldsymbol{\alpha}}\mathrm{e}^{-(\mathbf{N}\mathbf{x}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{x}^2)}$ to the whole complex plane with a branch cut along the negative real-axis. It is easily seen that the jump matrix in ($P_{\mathrm{b}}$) has the following factorization:

$$\begin{pmatrix} 1 & |x|^{2\boldsymbol{\alpha}}\mathrm{e}^{-(\mathbf{N}\mathbf{x}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{x}^2)} \\ 0 & 1\end{pmatrix} = \begin{cases} \widetilde{w}(x)^{\frac{\sigma_3}{2}}\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\widetilde{w}(x)^{-\frac{\sigma_3}{2}} & \text{for } x>0,\\ \widetilde{w}_-(x)^{\frac{\sigma_3}{2}}\begin{pmatrix} \mathrm{e}^{2\boldsymbol{\pi}\boldsymbol{\alpha}\mathrm{i}} & 1\\ 0 & \mathrm{e}^{-2\boldsymbol{\pi}\boldsymbol{\alpha}\mathrm{i}}\end{pmatrix}\widetilde{w}_+(x)^{-\frac{\sigma_3}{2}} & \text{for } x<0.\end{cases} \tag{5.2}$$

In order to solve the RHP for $P(z)$, we first transform it into the RHP for

$$\widetilde{P}(z) := P(z)\widetilde{w}(z)^{\frac{\sigma_3}{2}} = P(z)z^{\boldsymbol{\alpha}\boldsymbol{\sigma}_3}\mathrm{e}^{-\frac{\sigma_3}{2}(\mathbf{N}\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)}. \tag{5.3}$$

Clearly, the function on the right-hand side of $P(z)$ in (5.3) is the analytic continuation of the function $|x|^{\alpha}\mathrm{e}^{-\frac{Nx^4+t\sqrt{N}x^2}{2}}$ to the whole complex plane with a branch cut along the negative real-axis. Now, let us consider the RHP for $\widetilde{P}$:

($\widetilde{P}_{\mathrm{a}}$) $\widetilde{P}(z)$ is analytic in $U_{\delta}\setminus\mathbb{R}$;

($\widetilde{P}_{\mathrm{b}}$) for $x\in(-\delta,\delta)\setminus\{0\}$,

$$\widetilde{P}_{+}(x)=\widetilde{P}_{-}(x)\begin{cases}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} & \text{for } x\in(0,\delta),\\ \begin{pmatrix}\mathrm{e}^{2\pi\alpha\mathrm{i}} & 1\\ 0 & \mathrm{e}^{-2\pi\alpha\mathrm{i}}\end{pmatrix} & \text{for } x\in(-\delta,0);\end{cases}$$

($\widetilde{P}_{\mathrm{c}}$) $\widetilde{P}(z)$ satisfies the matching condition

$$\widetilde{P}(z)\sim\mathrm{e}^{\frac{1}{2}Nl_N\sigma_3}V_{\infty}(z)\mathrm{e}^{-N\varphi(z)\sigma_3}$$

as $n\to\infty$ (see (3.45) and (4.18)).

To solve the above RHP for $\widetilde{P}(z)$, we first consider the matrix-valued function

$$\Phi_{\alpha}(\zeta)=\begin{cases}\zeta^{\frac{1}{2}}\begin{pmatrix}\sqrt{\pi}I_{\alpha+\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}) & -\dfrac{K_{\alpha+\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}})}{\sqrt{\pi}}\\ -\mathrm{i}\sqrt{\pi}I_{\alpha-\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}) & -\dfrac{\mathrm{i}K_{\alpha-\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}})}{\sqrt{\pi}}\end{pmatrix}\mathrm{e}^{\frac{1}{2}\pi\alpha\mathrm{i}\sigma_3}, & \operatorname{Im}\zeta>0,\\ \zeta^{\frac{1}{2}}\begin{pmatrix}-\mathrm{i}\sqrt{\pi}I_{\alpha+\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}) & -\dfrac{\mathrm{i}K_{\alpha+\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}})}{\sqrt{\pi}}\\ \sqrt{\pi}I_{\alpha-\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}) & -\dfrac{K_{\alpha-\frac{1}{2}}(\zeta\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}})}{\sqrt{\pi}}\end{pmatrix}\mathrm{e}^{-\frac{1}{2}\pi\alpha\mathrm{i}\sigma_3}, & \operatorname{Im}\zeta<0,\end{cases}\tag{5.4}$$

where $\zeta^{\frac{1}{2}}$ takes the branch cut along the negative real-axis, and $I_{\nu}$, $K_{\nu}$ are modified Bessel functions defined in the complex plane with cut along the negative real-axis. Recall that

$$\begin{aligned}I_{\nu}(z\mathrm{e}^{m\pi\mathrm{i}})&=\mathrm{e}^{m\nu\pi\mathrm{i}}I_{\nu}(z),\\ K_{\nu}(z\mathrm{e}^{m\pi\mathrm{i}})&=\mathrm{e}^{-m\nu\pi\mathrm{i}}K_{\nu}(z)-\pi\mathrm{i}\sin(m\nu\pi)\csc(\nu\pi)I_{\nu}(z),\end{aligned}$$

where $m$ is an integer. It is easily shown that

$$(\Phi_{\alpha})_{+}(\zeta)=(\Phi_{\alpha})_{-}(\zeta)\begin{cases}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} & \text{for } \zeta>0,\\ \begin{pmatrix}\mathrm{e}^{2\pi\alpha\mathrm{i}} & 1\\ 0 & \mathrm{e}^{-2\pi\alpha\mathrm{i}}\end{pmatrix} & \text{for } \zeta<0.\end{cases}\tag{5.5}$$

Furthermore, from the well-known formulas

$$I_{\nu}(z)\sim\frac{1}{\sqrt{2\pi z}}\mathrm{e}^{z},\quad K_{\nu}(z)\sim\sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\tag{5.6}$$

as $z \to \infty$ in $|\arg z| < \frac{\pi}{2}$, we have

$$\Phi_{\alpha}(\zeta) \sim \begin{cases} \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \mathrm{e}^{(\frac{\pi}{4}\mathrm{i} - \zeta \mathrm{i} + \frac{1}{2}\pi\alpha \mathrm{i})\sigma_3}, & \mathrm{Im}\, \zeta > 0 \\ \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \mathrm{e}^{(\frac{\pi}{4}\mathrm{i} - \zeta \mathrm{i} + \frac{1}{2}\pi\alpha \mathrm{i})\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \mathrm{Im}\, \zeta < 0 \end{cases} \tag{5.7}$$

as $\zeta \to \infty$. Next, we introduce the function

$$\eta_N(z) := \begin{cases} -\mathrm{i}\phi(z) + \mathrm{i}\phi_+(0), & \mathrm{Im}\, z > 0, \\ \mathrm{i}\phi(z) + \mathrm{i}\phi_+(0), & \mathrm{Im}\, z < 0, \end{cases} \tag{5.8}$$

which is analytic in a neighborhood of the origin by Proposition 3.1. In fact, it follows from (3.23) and (3.33) that

$$\eta_N(z) = \alpha_t(\alpha_t^2 + tN^{-\frac{1}{2}})z + O(z^2) \tag{5.9}$$

as $z \to 0$. A comparison of conditions $(\widetilde{P}_{\mathrm{b}})$ and $(\widetilde{P}_{\mathrm{c}})$ for $\widetilde{P}(z)$ with (5.5) and (5.7) shows that

$$\widetilde{P}(z) = \frac{1}{\sqrt{2}} \mathrm{e}^{\frac{1}{2}N l_N \sigma_3} E_0(z) \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \Phi_{\alpha}[N\eta_N(z)], \tag{5.10}$$

where

$$E_0(z) = V_\infty(z) \mathrm{e}^{-(\frac{1}{2}\pi\alpha \mathrm{i} + N\phi_+(0) + \frac{1}{4}\pi \mathrm{i})\sigma_3} \tag{5.11}$$

for $z \in \mathbb{C}_+$ and

$$E_0(z) = V_\infty(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{e}^{-(\frac{1}{2}\pi\alpha \mathrm{i} + N\phi_+(0) + \frac{1}{4}\pi \mathrm{i})\sigma_3} \tag{5.12}$$

for $z \in \mathbb{C}_-$. The jump condition $(V_{\infty,\mathrm{b}})$ for $(V_\infty)$ implies that $E_0(z)$ is actually analytic in $U_\delta$. Indeed, with the aid of (4.14) and (5.11)–(5.12), we have the explicit formula

$$E_0(z) = \frac{1}{2} \begin{pmatrix} b(z) - \mathrm{i}b(z)^{-1} & b(z) + ib(z)^{-1} \\ \mathrm{i}b(z) - b(z)^{-1} & \mathrm{i}b(z) + b(z)^{-1} \end{pmatrix} \mathrm{e}^{-(\frac{1}{2}\pi\alpha \mathrm{i} + N\phi_+(0))\sigma_3}, \tag{5.13}$$

where

$$b(z) = \left(\frac{\alpha_t - z}{\alpha_t + z}\right)^{\frac{1}{4}}$$

is analytic in $\mathbb{C} \setminus (-\infty, -\alpha_t] \cup [\alpha_t, \infty)$, where $\alpha_t$ is shown in (3.18). Finally, a combination of (5.3) and (5.10) gives

$$P(z) = \frac{1}{\sqrt{2}} \mathrm{e}^{\frac{1}{2}N l_N \sigma_3} E_0(z) \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \Phi_{\alpha}[N\eta_N(z)] \mathrm{e}^{\frac{\sigma_3}{2}(Nz^4 + tz^2 N^{\frac{1}{2}})} z^{-\alpha\sigma_3}. \tag{5.14}$$

### 5.1.2 Parametrices outside $U_\delta$

For $z$ outside the $\delta$-neighborhood $U_\delta$ of the origin, we will construct the parametrices by using Airy functions and elementary functions. To facilitate the following discussions, we divide the complex plane into four parts: $U_\delta$ and $\Omega_i$, $i = 1, 2, 3$, by the contours $\Gamma := \bigcup_{i=1}^{8} \Gamma_i$ (see Figure

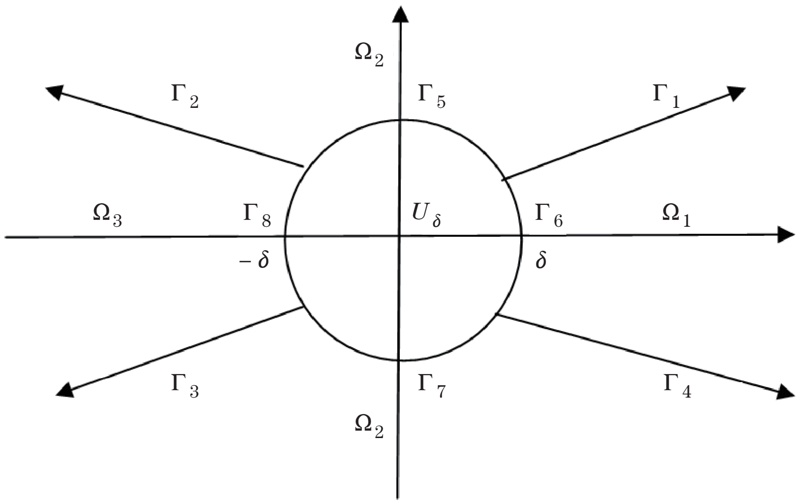


Figure 3 Contour $\Gamma_i,\ i = 1, \cdots, 8$ and the domains $\Omega_1, \Omega_2, \Omega_3$.

3). Note that $\partial U_{\boldsymbol{\delta}} = \bigcup_{\mathbf{i}=5}^{8} \Gamma_{\mathbf{i}}$, and $\Gamma_3 \cup \Gamma_4$ is the complex conjugation of $\Gamma_1 \cup \Gamma_2$. The curve $\Gamma_1$ is chosen so that the function $\phi(z)$ is one-to-one in $\Omega_1 \cap \mathbb{C}_+$ and satisfies $0 < \arg\ \phi(z) < \frac{3}{2}\pi$ for $z \in \Omega_1 \cap \mathbb{C}_+$. Similarly, we choose $\Gamma_2$ such that $-\frac{3}{2}\pi < \arg\ \widetilde{\phi}(z) < 0$ for $z \in \Omega_3 \cap \mathbb{C}_+$.

In view of (4.16), we can rewrite (4.18) as

$$U(z) \sim \frac{1}{2} e^{\frac{1}{2} \mathbf{N} \mathbf{l}_N \boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \beta(z)^{-\boldsymbol{\sigma}_3} \begin{pmatrix} e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix} e^{[\mathbf{N}(\mathbf{g}(\mathbf{z}) + \boldsymbol{\varphi}(\mathbf{z}) - \frac{1}{2} \mathbf{l}_N) - \boldsymbol{\alpha} \log \mathbf{z}] \boldsymbol{\sigma}_3}$$

and from (3.45) we obtain

$$U(z) \sim \frac{1}{2} e^{\frac{1}{2} \mathbf{N} \mathbf{l}_N \boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \beta(z)^{-\boldsymbol{\sigma}_3} \begin{pmatrix} e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix} e^{[\frac{1}{2} \mathbf{N} \mathbf{z}^4 + \mathbf{t} \sqrt{\mathbf{N}} \mathbf{z}^2 - \boldsymbol{\alpha} \log \mathbf{z}] \boldsymbol{\sigma}_3}. \tag{5.15}$$

To find an approximation to $U(z)$, we first look for a matrix which is asymptotic to

$$\begin{pmatrix} e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i} e^{\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix}. \tag{5.16}$$

From Proposition 3.1, it is clear that the function defined by

$$\xi_{\mathbf{N}} = f_{\mathbf{N}}(z) = \Big(\frac{3}{2} \phi(z)\Big)^{\frac{2}{3}} \tag{5.17}$$

is analytic in $\mathbb{C} \setminus (-\infty, -\alpha_{\mathbf{t}}]$, where $\phi(z)$ is defined in (3.33) and depends on $N$. In particular, by the construction of $\Omega_1$, we have for $z \in \Omega_1 \cap \mathbb{C}_+$,

$$0 < \arg f_{\mathbf{N}}(z) < \pi. \tag{5.18}$$

Also, for $z \neq \alpha_{\mathbf{t}}$, $N^{\frac{2}{3}} \xi_{\mathbf{N}} = N^{\frac{2}{3}} f_{\mathbf{N}}(z) \to \infty$ as $n \to \infty$. From the asymptotic behavior of the Airy function (see [17, p. 392]), we have

$$\mathrm{Ai}(N^{\frac{2}{3}} \xi_{\mathbf{N}}) \sim \frac{N^{-\frac{1}{6}}}{2\sqrt{\pi}} \xi_{\mathbf{N}}^{-\frac{1}{4}} e^{-\frac{2}{3} \mathbf{N} \boldsymbol{\xi}_N^{\frac{3}{2}}} = \frac{N^{-\frac{1}{6}}}{2\sqrt{\pi}} (f_{\mathbf{N}}(z))^{-\frac{1}{4}} e^{-\mathbf{N} \boldsymbol{\varphi}(\mathbf{z})},$$

$$\mathrm{Ai}'(N^{\frac{2}{3}}\xi_N) \sim -\frac{N^{\frac{1}{6}}}{2\sqrt{\pi}}\xi_N^{\frac{1}{4}}\mathrm{e}^{-\frac{2}{3}N\xi_N^{\frac{3}{2}}} = -\frac{N^{\frac{1}{6}}}{2\sqrt{\pi}}(f_N(z))^{\frac{1}{4}}\mathrm{e}^{-N\varphi(z)},$$

$$\mathrm{Ai}(N^{\frac{2}{3}}w^2\xi_N) = \mathrm{Ai}(N^{\frac{2}{3}}w^{-1}\xi_N) \sim \frac{N^{-\frac{1}{6}}}{2\sqrt{\pi}}\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}(f_N(z))^{-\frac{1}{4}}\mathrm{e}^{N\varphi(z)},$$

$$\mathrm{Ai}'(N^{\frac{2}{3}}w^2\xi_N) = \mathrm{Ai}'(N^{\frac{2}{3}}w^{-1}\xi_N) \sim -\frac{N^{\frac{1}{6}}}{2\sqrt{\pi}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}(f_N(z))^{\frac{1}{4}}\mathrm{e}^{N\varphi(z)},$$

where $w = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$. It is then immediate that for $z \in \Omega_1 \cap \mathbb{C}_+$,

$$\begin{pmatrix} \mathrm{e}^{-N\varphi(z)} & \mathrm{i}\mathrm{e}^{N\varphi(z)} \\ -\mathrm{e}^{-N\varphi(z)} & \mathrm{i}\mathrm{e}^{N\varphi(z)} \end{pmatrix} \sim 2\sqrt{\pi}(N^{\frac{2}{3}}f_N(z))^{\frac{1}{4}\sigma_3} \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\xi_N) & -w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\xi_N) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_N) & -w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\xi_N) \end{pmatrix}. \tag{5.19}$$

Similarly, for $z \in \Omega_1 \cap \mathbb{C}_-$, we also have $-\pi < \arg f_N(z) < 0$ and

$$\begin{pmatrix} \mathrm{e}^{-N\varphi(z)} & \mathrm{i}\mathrm{e}^{N\varphi(z)} \\ -\mathrm{e}^{-N\varphi(z)} & \mathrm{i}\mathrm{e}^{N\varphi(z)} \end{pmatrix} \sim 2\sqrt{\pi}(N^{\frac{2}{3}}f_N(z))^{\frac{1}{4}\sigma_3} \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\xi_N) & w\mathrm{Ai}(N^{\frac{2}{3}}w\xi_N) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_N) & w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\xi_N) \end{pmatrix}. \tag{5.20}$$

For $z \in \Omega_3$, there is a result corresponding to (5.15) with $\phi(z)$ and $l_N$ replaced by $\widetilde{\phi}(z)$ and $\widetilde{l}_N$, respectively. Indeed, it follows from (3.46), (4.16) and (4.20) that

$$U(z) \sim \frac{1}{2}\mathrm{e}^{-\alpha\pi\mathrm{i}\sigma_3}\mathrm{e}^{\frac{1}{2}N\widetilde{l}_N\sigma_3}\begin{pmatrix} 1 & -1 \\ \mathrm{i} & \mathrm{i} \end{pmatrix}\beta(z)^{\sigma_3}\begin{pmatrix} \mathrm{e}^{-N\widetilde{\varphi}(z)} & -\mathrm{i}\mathrm{e}^{N\widetilde{\varphi}(z)} \\ -\mathrm{e}^{-N\widetilde{\varphi}(z)} & -\mathrm{i}\mathrm{e}^{N\widetilde{\varphi}(z)} \end{pmatrix}$$
$$\times\, \mathrm{e}^{[\frac{1}{2}Nz^4 + t\sqrt{N}z^2 - \alpha\log(-z)]\sigma_3}. \tag{5.21}$$

Let

$$\widetilde{\xi}_N = \widetilde{f}_N(z) = \left(\frac{3}{2}\widetilde{\phi}(z)\right)^{\frac{2}{3}}, \tag{5.22}$$

which is analytic in $\mathbb{C} \setminus [\alpha_t, \infty)$. Also, note that

$$|\arg \widetilde{f}_N(z)| < \pi \tag{5.23}$$

for $z \in \Omega_3$. Hence, as before, it can be shown that the matrix

$$\begin{pmatrix} \mathrm{e}^{-N\widetilde{\varphi}(z)} & -\mathrm{i}\mathrm{e}^{N\widetilde{\varphi}(z)} \\ -\mathrm{e}^{-N\widetilde{\varphi}(z)} & -\mathrm{i}\mathrm{e}^{N\widetilde{\varphi}(z)} \end{pmatrix}$$

is the leading term in the asymptotic expansion of the matrices

$$2\sqrt{\pi}(N^{\frac{2}{3}}\widetilde{f}_N(z))^{\frac{1}{4}\sigma_3}\begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_N) & -w\mathrm{Ai}(N^{\frac{2}{3}}w\widetilde{\xi}_N) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_N) & -w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\widetilde{\xi}_N) \end{pmatrix} \tag{5.24}$$

for $z \in \Omega_3 \in \mathbb{C}_+$, and

$$2\sqrt{\pi}(N^{\frac{2}{3}}\widetilde{f}_N(z))^{\frac{1}{4}\sigma_3}\begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_N) & w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\widetilde{\xi}_N) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_N) & w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\widetilde{\xi}_N) \end{pmatrix} \tag{5.25}$$

for $z \in \Omega_3 \in \mathbb{C}_-$. Define the matrix function

$$Q(z) := \begin{cases} \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}}) \end{pmatrix}, & z \in \Omega_1 \cap \mathbb{C}_+, \\ \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w\mathrm{Ai}(N^{\frac{2}{3}}w\xi_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\xi_{\mathbf{N}}) \end{pmatrix}, & z \in \Omega_1 \cap \mathbb{C}_-, \\ \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w\mathrm{Ai}(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}}) \end{pmatrix}, & z \in \Omega_3 \cap \mathbb{C}_+, \\ \begin{pmatrix} \mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}}) \end{pmatrix}, & z \in \Omega_3 \cap \mathbb{C}_-. \end{cases} \tag{5.26}$$

An appeal to the formulas

$$\mathrm{Ai}(z) + w\mathrm{Ai}(wz) + w^2\mathrm{Ai}(w^2z) = 0$$

and

$$\mathrm{Ai}'(z) + w^2\mathrm{Ai}'(wz) + w\mathrm{Ai}'(w^2z) = 0$$

shows that $Q(z)$ satisfies

$$Q_+(x) = Q_-(x)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \setminus [-\delta, \delta]. \tag{5.27}$$

The above heuristic argument now suggests that $U(z)$ is asymptotically approximated by

$$P^*(z) := \begin{cases} E(z)M(z), & z \in \Omega_1, \\ \widetilde{E}(z)\widetilde{M}(z), & z \in \Omega_3, \\ \mathrm{e}^{\frac{1}{2}\mathbf{N}\,\mathbf{l}_N\boldsymbol{\sigma}_3}V_\infty(z)\mathrm{e}^{(\mathbf{N}(\mathbf{g}(\mathbf{z})-\frac{1}{2}\mathbf{l}_N)-\boldsymbol{\alpha}\log\mathbf{z})\boldsymbol{\sigma}_3}, & z \in \Omega_2, \end{cases} \tag{5.28}$$

where

$$\begin{aligned} E(z) &:= \sqrt{\pi}\mathrm{e}^{\frac{1}{2}\mathbf{N}\,\mathbf{l}_N\boldsymbol{\sigma}_3}\begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix}\left(\frac{N^{\frac{1}{6}}\xi_{\mathbf{N}}^{\frac{1}{4}}(z)}{\beta(z)}\right)^{\boldsymbol{\sigma}_3}, \\ M(z) &:= Q(z)\mathrm{e}^{[\frac{1}{2}(\mathbf{N}\,\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)-\boldsymbol{\alpha}\log\mathbf{z}]\boldsymbol{\sigma}_3} \end{aligned} \tag{5.29}$$

for $z \in \Omega_1$, and

$$\begin{aligned} \widetilde{E}(z) &:= \sqrt{\pi}\mathrm{e}^{-\boldsymbol{\alpha}\boldsymbol{\pi}\mathrm{i}\boldsymbol{\sigma}_3}\mathrm{e}^{\frac{1}{2}\mathbf{N}\,\widetilde{\mathbf{l}}_N\boldsymbol{\sigma}_3}\begin{pmatrix} 1 & -1 \\ \mathrm{i} & \mathrm{i} \end{pmatrix}(N^{\frac{1}{6}}\widetilde{\xi}_{\mathbf{N}}^{\frac{1}{4}}(z)\beta(z))^{\boldsymbol{\sigma}_3}, \\ \widetilde{M}(z) &:= Q(z)\mathrm{e}^{(\frac{1}{2}(\mathbf{N}\,\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)-\boldsymbol{\alpha}\log(-\mathbf{z}))\boldsymbol{\sigma}_3} \end{aligned} \tag{5.30}$$

for $z \in \Omega_3$. Note that the functions $\exp\{\frac{1}{2}(Nz^4+t\sqrt{N}z^2)-\alpha\log z\}$ and $\exp\{\frac{1}{2}(Nz^4+t\sqrt{N}z^2)-\alpha\log(-z)\}$ are analytic continuations of

$$(|x|^{2\boldsymbol{\alpha}}\mathrm{e}^{-(\mathbf{N}\,\mathbf{x}^4+\mathbf{t}\mathbf{N}^{\frac{1}{2}}\mathbf{x}^2)})^{-\frac{1}{2}} = |x|^{-\boldsymbol{\alpha}}\mathrm{e}^{\frac{1}{2}(\mathbf{N}\,\mathbf{z}^4+\mathbf{t}\mathbf{N}^{\frac{1}{2}}\mathbf{z}^2)}$$

to the cut planes $\mathbb{C} \setminus (-\infty, 0]$ and $\mathbb{C} \setminus [0, \infty)$, respectively. Hence, from (5.27) it follows that

$$M_+(x) = M_-(z)\begin{pmatrix} 1 & |x|^{2\boldsymbol{\alpha}}\mathrm{e}^{-\mathbf{N}\,\mathbf{x}^4-\mathbf{t}\mathbf{N}^{\frac{1}{2}}\mathbf{x}^2} \\ 0 & 1 \end{pmatrix}, \quad x \in (\delta, \infty), \tag{5.31}$$

$$\widetilde{M}_+(x) = \widetilde{M}_-(z) \begin{pmatrix} 1 & |x|^{2\alpha} e^{-Nx^4 - tN^{\frac{1}{2}}x^2} \\ 0 & 1 \end{pmatrix}, \quad x \in (-\infty, -\delta). \tag{5.32}$$

On the other hand, since $\frac{\xi_N^{\frac{1}{4}}(z)}{\beta(z)}$ and $\frac{\widetilde{\xi}_N^{\frac{1}{4}}(z)}{\beta(z)}$ are analytic in $\mathbb{C} \setminus (-\infty, -\alpha_t]$ and $\mathbb{C} \setminus [\alpha_t, \infty)$, respectively, one easily sees that $E_+(x) = E_-(x)$ for $x \in (\delta, \infty)$ and $\widetilde{E}_+(x) = \widetilde{E}_-(x)$ for $x \in (-\infty, -\delta)$. From (5.28) and (5.31), it can be shown that

$$P_+^*(x) = P_-^*(x) \begin{pmatrix} 1 & |x|^{2\alpha} e^{-(Nx^4 + tN^{\frac{1}{2}}x^2)} \\ 0 & 1 \end{pmatrix} \tag{5.33}$$

for all $x \in \mathbb{R} \setminus [-\delta, \delta]$. Furthermore, $P^*(z)$ has the same large $z$ behavior as $U(z)$ shown in $(U_c)$ (see [20, (5.38) and (5.39)]).

In summary, it is now natural to suggest that $U(z)$ is asymptotically approximated by

$$U^*(z) = \begin{cases} P(z), & z \in U_\delta, \\ P^*(z), & z \in \mathbb{C} \setminus U_\delta. \end{cases} \tag{5.34}$$

### 5.2 Case for $c = -2$

In this subsection, we first introduce the $\Psi_\alpha$ function associated with Painlevé II equation and then construct the model RHP to be used in the formation of a parametrix in a region containing the origin.

#### 5.2.1 $\Psi_\alpha$ functions for Painlevé II equation and model RHP

Let $\Psi_\alpha(\zeta, s)$ be a $2 \times 2$ complex-valued matrix function. We consider the following linear differential equations:

$$\frac{\partial \Psi_\alpha}{\partial \zeta} = \begin{pmatrix} -4i\zeta^2 - i(s + 2u^2) & 4\zeta u + 2iv + \dfrac{\alpha}{\zeta} \\ 4\zeta u - 2iv + \dfrac{\alpha}{\zeta} & 4i\zeta^2 + i(s + 2u^2) \end{pmatrix} \Psi_\alpha \tag{5.35}$$

and

$$\frac{\partial \Psi_\alpha}{\partial s} = \begin{pmatrix} -i\zeta & u \\ u & i\zeta \end{pmatrix} \Psi_\alpha, \tag{5.36}$$

where $u$ and $v$ are functions of $s$. The compatibility condition for $\Psi_\alpha$ shows that $u(s)$ should satisfy the Painlevé equation

$$u'' = su + 2u^3 - \alpha \tag{5.37}$$

and $v(s) = u'(s)$. Now, we focus on equation (5.35) and view $u, v, s$ as parameters. In each sector

$$S_k := \left\{ \zeta \in \mathbb{C} \;\middle|\; (k-2)\frac{\pi}{3} < \arg\zeta < k\frac{\pi}{3} \right\}$$

for $k = 1, \cdots, 6$, there exists a unique solution $\Psi_{\alpha,k}$ of (5.35) such that

$$\Psi_{\alpha,k}(\zeta, s) \sim \left( I + \sum_{j=1}^{\infty} \frac{\Phi_j}{\zeta^j} \right) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} = \left( I + O\left(\frac{1}{\zeta}\right) \right) e^{-i(\frac{4}{3}\zeta^3 + s\zeta)\sigma_3} \tag{5.38}$$

as $\zeta \to \infty$ in each section $S_{\mathbf{k}}$.

Since $\Psi_{\boldsymbol{\alpha},\mathbf{k}}(\zeta, s)$ are solutions of the same linear differential equation, they are related by the so-called Stokes matrices $A_{\mathbf{k}}$, i.e.,

$$\Psi_{\boldsymbol{\alpha},\mathbf{k}+1}(\zeta, s) = \Psi_{\boldsymbol{\alpha},\mathbf{k}}(\zeta, s)A_{\mathbf{k}}, \quad k = 1, 2, \cdots, 5$$

and

$$\Psi_{\boldsymbol{\alpha},1}(\zeta, s) = \Psi_{\boldsymbol{\alpha},6}(\zeta, s)A_6.$$

Here, each $A_{\mathbf{k}}$ is a triangular matrix of the form

$$A_{\mathbf{k}} = \begin{pmatrix} 1 & 0 \\ a_{\mathbf{k}} & 1 \end{pmatrix} \quad \text{for } k \text{ odd}$$

and

$$A_{\mathbf{k}} = \begin{pmatrix} 1 & a_{\mathbf{k}} \\ 0 & 1 \end{pmatrix} \quad \text{for } k \text{ even},$$

where $a_{\mathbf{k}}$, $k = 1, 2, \cdots, 6$, are complex numbers satisfying

$$a_{\mathbf{k}+3} = a_{\mathbf{k}}, \quad k = 1, 2, 3, \quad \text{and} \quad a_1 + a_2 + a_3 + a_1 a_2 a_3 = -2\mathrm{i} \sin \pi\alpha. \tag{5.39}$$

The above properties can be found in [14, p. 164]. The entries $a_{\mathbf{k}}$, which are also called Stokes multipliers, are in general dependent on the parameters $s, u$ and $v$. However, Flaschka and Newell [13] showed that if $u(s)$ is a solution of Painlevé II equation (5.37) and $v(s) = u'(s)$, then $a_{\mathbf{k}}$ are constants. In particular, for the Hastings-Mcleod solution of Painlevé II equation, which satisfies the boundary conditions

$$u(s) \sim \frac{\alpha}{s} \quad \text{as } s \to \infty$$

and

$$u(s) \sim \sqrt{\frac{-s}{2}} \quad \text{as } s \to -\infty,$$

the choice is

$$a_1 = \mathrm{e}^{-\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}}, \quad a_2 = 0, \quad a_3 = -\mathrm{e}^{\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}}$$

(see [6, p. 604]). If we set

$$\widetilde{\Psi}(\zeta, s) := \begin{cases} \Psi_3(\zeta, s) & \text{for } \operatorname{Im} \zeta > 0, \\ \Psi_5(\zeta, s) & \text{for } \operatorname{Im} \zeta < 0, \end{cases} \tag{5.40}$$

then it follows from above equations that $\widetilde{\Psi}(\zeta, s)$ satisfies the following RHP:

($\widetilde{\Psi}_{\mathrm{a}}$) $\widetilde{\Psi}(\zeta, s)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

($\widetilde{\Psi}_{\mathrm{b}}$) for $\zeta \in \mathbb{R}$,

$$\widetilde{\Psi}_+(\zeta, s) = \widetilde{\Psi}_-(\zeta, s) \begin{cases} \begin{pmatrix} 0 & -\mathrm{e}^{-\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}} \\ \mathrm{e}^{\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}} & 1 \end{pmatrix} & \text{for } \zeta < 0, \\ \begin{pmatrix} 0 & -\mathrm{e}^{\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}} \\ \mathrm{e}^{-\boldsymbol{\pi}\mathrm{i}\boldsymbol{\alpha}} & 1 \end{pmatrix} & \text{for } \zeta > 0; \end{cases}$$

($\widetilde{\Psi}_{\mathrm{c}}$) as $\zeta \to \infty$,

$$\widetilde{\Psi}(\zeta, s) = (I + O(\zeta^{-1}))\mathrm{e}^{-\mathrm{i}(\frac{4}{3}\boldsymbol{\zeta}^3 + \mathbf{s}\boldsymbol{\zeta})\boldsymbol{\sigma}_3}.$$

Since all solutions to the second Painlevé equation are meromorphic functions with infinite number of poles, the RHP for $\widetilde{\Psi}$ shown above is solvable if and only if $s$ does not belong to the set of poles of $u(s)$.

With the above preparation, we are now ready to formulate a model RHP for our later use. Let $\theta \in \mathbb{R}$, and define

$$\overline{\Psi}(\zeta; s, \theta) := \begin{cases} e^{i\theta\sigma_3}\widetilde{\Psi}(\zeta, s)e^{i(\frac{4}{3}\zeta^3+s\zeta-\theta)\sigma_3} & \text{for } \operatorname{Im}\zeta > 0, \\ e^{i\theta\sigma_3}\widetilde{\Psi}(\zeta, s)e^{i(\frac{4}{3}\zeta^3+s\zeta-\theta)\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{for } \operatorname{Im}\zeta < 0. \end{cases} \tag{5.41}$$

It is then easily seen that $\overline{\Psi}(\zeta; s, \theta)$ is a solution of the following RHP:

($\overline{\Psi}_\text{a}$) $\overline{\Psi}(\zeta; s, \theta)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$;

($\overline{\Psi}_\text{b}$) for $\zeta \in \mathbb{R}$,

$$\overline{\Psi}_+(\zeta; s, \theta) = \overline{\Psi}_-(\zeta; s, \theta)\begin{cases} \begin{pmatrix} e^{\pi i\alpha}e^{2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \\ 0 & e^{-\pi i\alpha}e^{-2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} \end{pmatrix} & \text{for } \zeta < 0, \\ \begin{pmatrix} e^{-\pi i\alpha}e^{2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} & 1 \\ 0 & e^{\pi i\alpha}e^{-2i(\frac{4}{3}\zeta^3+s\zeta-\theta)} \end{pmatrix} & \text{for } \zeta > 0, \end{cases}$$

where the real line is oriented from $-\infty$ to $\infty$, $\phi$ and $\widetilde{\phi}$ are defined in (3.52) and (3.53), respectively;

($\overline{\Psi}_\text{c}$) as $\zeta \to \infty$,

$$\overline{\Psi}(\zeta; s, \theta) = \begin{cases} I + O(\zeta^{-1}), & \text{if } \operatorname{Im}\zeta > 0, \\ (I + O(\zeta^{-1}))\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } \operatorname{Im}\zeta < 0. \end{cases}$$

In the following subsection, we will show how $\overline{\Psi}(\zeta; s, \theta)$ is involved in the construction of a parametrix in the neighborhood of $z = 0$.

### 5.2.2 Parametrix in the neighborhood of the origin

Let $U_\delta$ be a domain in the complex plane containing the origin, the size of which will be determined later. We first wish to find a matrix-valued function $Q(z)$ that satisfies the following RHP:

($Q_\text{a}$) $Q(z)$ is analytic in $U_\delta \setminus \mathbb{R}$;

($Q_\text{b}$) for $x \in U_\delta \cap \mathbb{R}$,

$$Q_+(x) = Q_-(x)\begin{cases} \begin{pmatrix} e^{2N\varphi_+} & 1 \\ 0 & e^{2N\varphi_-} \end{pmatrix} & \text{for } x > 0, \\ \begin{pmatrix} e^{2N\widetilde{\varphi}_+} & 1 \\ 0 & e^{2N\widetilde{\varphi}_-} \end{pmatrix} & \text{for } x < 0, \end{cases}$$

where $\phi$ and $\widetilde{\phi}$ are defined in (3.52) and (3.53).

($Q_\text{c}$) $Q(z)$ satisfies the matching condition

$$Q(z) \sim V_\infty(z)$$

as $n \to \infty$ for $z \in U_{\boldsymbol{\delta}} \setminus \mathbb{R}$, where $V_\infty$ is the matrix given in (4.21)–(4.22).

A comparison of conditions $(\overline{\Psi}_{\mathrm{b}})$ and $(Q_{\mathrm{b}})$ invokes us to introduce the mapping

$$\mathrm{i}\Big(\frac{4}{3}\zeta^3 + s\zeta - \theta - \frac{\pi}{2}\alpha\Big) = \begin{cases} N\phi(z) & \text{for } \operatorname{Im} z > 0, \\ -N\phi(z) & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{5.42}$$

By Proposition 3.3, we see that the right-hand side of (5.42) is analytic in

$$\mathbb{C} \setminus (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$$

on account of $(\nu)_+ = -(\nu)_-$ for $x \in [-\sqrt{2}, \sqrt{2}]$, and can be written as

$$\mathrm{i}N\Big(-\frac{\pi}{2} + F(z)\Big),$$

where

$$F(z) := \int_0^{\mathbf{z}} 2\zeta^2\sqrt{2-\zeta^2}\,\mathrm{d}\zeta \tag{5.43}$$

with branch cuts along $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$. It is clear that

$$F(0) = 0$$

and

$$F(z) \sim \frac{2}{3}\sqrt{2}z^3$$

as $z \to 0$. To make the mapping (5.42) one-to-one at least in a neighborhood of zero, it is natural to set

$$\zeta(z) = N^{\frac{1}{3}}\eta(z) = N^{\frac{1}{3}}\Big(\frac{3}{4}F(z)\Big)^{\frac{1}{3}}, \quad s = 0, \quad \theta = \frac{n\pi}{2}. \tag{5.44}$$

Since $s = 0 \in \mathbb{R}$, the RHP for $\widetilde{\Psi}(\zeta, s)$ is solvable (see the comment following $(\widetilde{\Psi}_{\mathrm{c}})$). Thus, $\overline{\Psi}(\zeta; s, \theta)$ is known. On account of $(\overline{\Psi}_{\mathbf{c}})$ and $(Q_{\mathbf{c}})$, we set

$$Q(z) = E_0(z)\overline{\Psi}\Big(N^{\frac{1}{3}}\eta(z); 0, \frac{n\pi}{2}\Big), \tag{5.45}$$

where

$$E_0 = \begin{cases} V_\infty(z), & \text{if } \operatorname{Im} z > 0, \\ V_\infty(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{5.46}$$

The jump condition $(V_{\infty,\mathrm{b}})$ for $V_\infty$ shows that $E_0(z)$ is actually analytic in $\mathbb{C} \setminus (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$. In fact, it follows from (4.21) and (5.13) that we have the explicit formula

$$E_0(z) = \begin{pmatrix} \dfrac{\beta(z) + \beta(z)^{-1}}{2} & \dfrac{\beta(z) - \beta(z)^{-1}}{2\mathrm{i}} \\ \dfrac{\beta(z) - \beta(z)^{-1}}{-2\mathrm{i}} & \dfrac{\beta(z) + \beta(z)^{-1}}{2} \end{pmatrix}, \tag{5.47}$$

where

$$\beta(z)=\Big(\frac{\sqrt{2}-z}{z+\sqrt{2}}\Big)^{\frac{1}{4}} \tag{5.48}$$

is analytic in $\mathbb{C}\setminus(-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)$.

In the literature, one may find the use of a "double-scaling" approach; that is, $c=-2+\rho N^{-\epsilon}$ for some constants $\rho$ and $\epsilon>0$. In this case, the parametrix in the neighborhood of the origin can still be constructed in terms of the function $\Psi_{\alpha}(\zeta,s)$ for the model RHP associated with Painlevé II equation. However, the variable $s$ depends on the parameter $c$; in our case (i.e., $c=-2$), we have $s=0$ (see [4, 6]).

### 5.2.3 Parametrices outside the origin

For $z$ outside the $\delta$-neighborhood $U_{\delta}$ of the origin, we will (as before) construct the parametrices by using Airy functions and elementary functions. To facilitate discussion, we divide the complex plane into four parts: $U_{\delta}$ and $\Omega_i$, $i=1,2,3$ (see Figure 3).

Let $\beta(z)$ be given as in (5.48), and let $Q(z)$ be given as in (5.45) with $\phi(z)$ and $\widetilde{\phi}(z)$ in (3.52) and (3.53), respectively. As in Subsection 5.1, we expect $U(z)$ to be asymptotically approximated by

$$U^*(z):=\begin{cases}E(z)M(z), & z\in\Omega_1,\\ \widetilde{E}(z)\widetilde{M}(z), & z\in\Omega_3,\\ e^{\frac{1}{2}Nl_N\sigma_3}V_\infty(z)e^{(N(g(z)-\frac{1}{2}l_N)-\alpha\log z)\sigma_3}, & z\in\Omega_2,\\ e^{\frac{1}{2}Nl_N\sigma_3}Q(z)e^{(N(g(z)-\frac{1}{2}l_N)-\alpha\log z)\sigma_3}, & z\in U_\delta,\end{cases} \tag{5.49}$$

where

$$\begin{aligned}E(z)&:=\sqrt{\pi}e^{\frac{1}{2}Nl_N\sigma_3}\begin{pmatrix}1&-1\\-i&-i\end{pmatrix}\Big(\frac{N^{\frac{1}{6}}\xi_N^{\frac{1}{4}}(z)}{\beta(z)}\Big)^{\sigma_3},\\ M(z)&:=Q(z)e^{(\frac{1}{2}(Nz^4+t\sqrt{N}z^2)-\alpha\log z)\sigma_3},\\ \xi_N&=f_N(z)=\Big(\frac{3}{2}\phi(z)\Big)^{\frac{2}{3}}\end{aligned} \tag{5.50}$$

for $z\in\Omega_1$, and

$$\begin{aligned}\widetilde{E}(z)&:=\sqrt{\pi}e^{\frac{1}{2}N\widetilde{l}_N\sigma_3}\begin{pmatrix}1&-1\\i&i\end{pmatrix}(N^{\frac{1}{6}}\widetilde{\xi}_N^{\frac{1}{4}}(z)\beta(z))^{\sigma_3},\\ \widetilde{M}(z)&:=Q(z)e^{(\frac{1}{2}(Nz^4+t\sqrt{N}z^2)-\alpha\log(-z))\sigma_3},\\ \widetilde{\xi}_N&=\widetilde{f}_N(z)=\Big(\frac{3}{2}\widetilde{\phi}(z)\Big)^{\frac{2}{3}}\end{aligned} \tag{5.51}$$

for $z\in\Omega_3$, and $l_N$ and $\widetilde{l}_N$ are shown in (3.51) and (3.67). $V_\infty$ is the matrix given in (4.21)–(4.22). Note that $U^*(z)$ also satisfies the jump condition in $(U_b)$ for $U(z)$ in Section 2.

## 5.3 Case for $c<-2$

Note that unlike cases (i) and (ii), here no special attention is needed for the singularity of the weight function at the origin. The main reason is that the interval $(-a,a)$, and hence the origin, lies outside the support of the equilibrium measure. Therefore, the orthogonal polynomials

$$\pi_{\mathbf{n}}(x) = S_{\mathbf{n}}(x;t)$$

in (1.2) have no zero in a small neighborhood of the origin.

In this subsection, we will construct an approximation $U^*(z)$ to the solution of the RHP for $U(z)$ as $n \to \infty$, by using only Airy functions and elementary functions. To facilitate the following discussions, we divide the complex plane into four parts: $\Omega_{\mathbf{i}}$, $i = 1, \cdots, 4$ (see Figure 4).

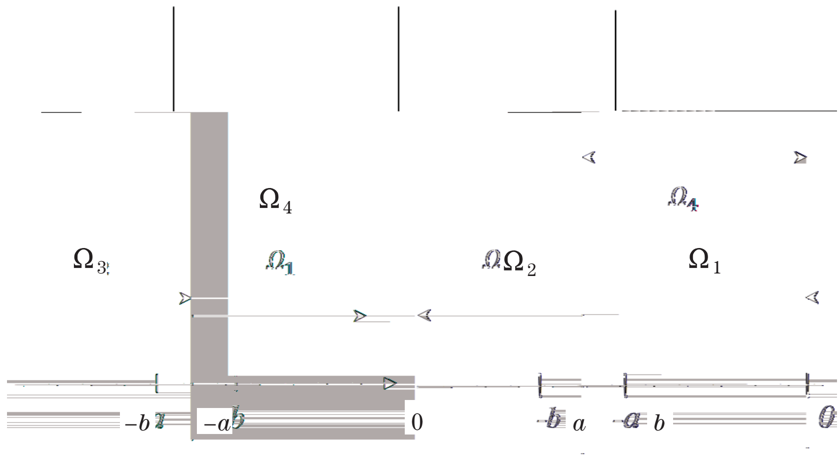


Figure 4 The domains $\Omega_1, \cdots, \Omega_4$.

In the view of (4.35), we can rewrite (4.37) as

$$U(z) \sim \frac{1}{2} e^{\frac{1}{2}\mathbf{N}\mathbf{l}_N \boldsymbol{\sigma}_3} D_\infty^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \widetilde{\gamma}(z)^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix} D^{-\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix} e^{[\mathbf{N}(\mathbf{g}(\mathbf{z})+\boldsymbol{\varphi}(\mathbf{z})-\frac{1}{2}\mathbf{l}_N)-\boldsymbol{\alpha}\log \mathbf{z}]\boldsymbol{\sigma}_3},$$

and from (3.111) we obtain

$$U(z) \sim \frac{1}{2} e^{\frac{1}{2}\mathbf{N}\mathbf{l}_N \boldsymbol{\sigma}_3} D_\infty^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & -1 \\ -\mathrm{i} & -\mathrm{i} \end{pmatrix} \widetilde{\gamma}(z)^{\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix} D^{-\boldsymbol{\sigma}_3} \begin{pmatrix} 1 & \mathrm{i} \\ -1 & \mathrm{i} \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix} e^{[\frac{1}{2}(\mathbf{N}\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)-\boldsymbol{\alpha}\log \mathbf{z}]\boldsymbol{\sigma}_3}. \tag{5.52}$$

To find an approximation to $U(z)$, we first look for a matrix which is asymptotic to

$$\begin{pmatrix} e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & \mathrm{i}e^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \end{pmatrix}. \tag{5.53}$$

From Proposition 3.5, it is clear that the function defined by

$$\xi_{\mathbf{N}} = f_{\mathbf{N}}(z) = \Big(\frac{3}{2}\phi(z)\Big)^{\frac{2}{3}} \tag{5.54}$$

is analytic in $\mathbb{C}\setminus(-\infty,a]$, where $\phi(z)$ is defined in (3.97) and depends on $N$. In particular, for $z\in\Omega_1\cap\mathbb{C}_+$, we have

$$0<\arg f_{\mathbf{N}}(z)<\pi. \tag{5.55}$$

Also, for $z\neq b$, $N^{\frac{2}{3}}\xi_{\mathbf{N}}=N^{\frac{2}{3}}f_{\mathbf{N}}(z)\to\infty$ as $n\to\infty$. Now, as in Subsection 5.1.2, we have from the asymptotic behavior of the Airy function (see [17, p. 392]),

$$\begin{pmatrix} e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & ie^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & ie^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})}\end{pmatrix}\sim 2\sqrt{\pi}(N^{\frac{2}{3}}f_{\mathbf{N}}(z))^{\frac{1}{4}\boldsymbol{\sigma}_3}\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}})\end{pmatrix} \tag{5.56}$$

for $z\in\Omega_1\cap\mathbb{C}_+$, where

$$w=e^{\frac{2\pi i}{3}}.$$

Also, we have

$$-\pi<\arg f_{\mathbf{N}}(z)<0$$

and

$$\begin{pmatrix} e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & ie^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} \\ -e^{-\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})} & ie^{\mathbf{N}\boldsymbol{\varphi}(\mathbf{z})}\end{pmatrix}\sim 2\sqrt{\pi}(N^{\frac{2}{3}}f_{\mathbf{N}}(z))^{\frac{1}{4}\boldsymbol{\sigma}_3}\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w\mathrm{Ai}(N^{\frac{2}{3}}w\xi_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\xi_{\mathbf{N}})\end{pmatrix} \tag{5.57}$$

for $z\in\Omega_1\cap\mathbb{C}_-$. There is a result corresponding to (5.52) with $\phi(z)$ replaced by $\widetilde{\phi}(z)$ for $z\in\Omega_3$. Indeed, from (3.111) and (4.35), it follows that

$$\begin{aligned}U(z)\sim{}&\frac{1}{2}e^{\frac{1}{2}\mathbf{N}\widetilde{\mathbf{l}}_N\boldsymbol{\sigma}_3}e^{-\boldsymbol{\alpha}\boldsymbol{\pi}i\boldsymbol{\sigma}_3}D_\infty^{\boldsymbol{\sigma}_3}\begin{pmatrix}1 & -1\\ i & i\end{pmatrix}\widetilde{\gamma}(z)^{-\boldsymbol{\sigma}_3}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}D^{-\boldsymbol{\sigma}_3}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}^{-1}\\ &\times\begin{pmatrix} e^{-\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & -ie^{\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} \\ -e^{-\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & -ie^{\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})}\end{pmatrix}e^{[\frac{1}{2}(\mathbf{N}\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)-\boldsymbol{\alpha}\log(-\mathbf{z})]\boldsymbol{\sigma}_3}.\end{aligned} \tag{5.58}$$

Let

$$\widetilde{\xi}_{\mathbf{N}}=\widetilde{f}_{\mathbf{N}}(z)=\left(\frac{3}{2}\widetilde{\phi}(z)\right)^{\frac{2}{3}}, \tag{5.59}$$

which is analytic in $\mathbb{C}\setminus[-a,\infty)$. Also, note that

$$|\arg\widetilde{f}_{\mathbf{N}}(z)|<\pi \tag{5.60}$$

for $z\in\Omega_3$. Hence, as before, it can be shown that the matrix

$$\begin{pmatrix} e^{-\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & -ie^{\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} \\ -e^{-\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})} & -ie^{\mathbf{N}\widetilde{\boldsymbol{\varphi}}(\mathbf{z})}\end{pmatrix}$$

is the leading term in the asymptotic expansion of the matrices

$$2\sqrt{\pi}(N^{\frac{2}{3}}\widetilde{f}_{\mathbf{N}}(z))^{\frac{1}{4}\boldsymbol{\sigma}_3}\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w\mathrm{Ai}(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}}) \\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}})\end{pmatrix} \tag{5.61}$$

for $z \in \Omega_3 \cap \mathbb{C}_+$, and

$$2\sqrt{\pi}(N^{\frac{2}{3}}\widetilde{f}_{\mathbf{N}}(z))^{\frac{1}{4}\boldsymbol{\sigma}_3}\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}})\end{pmatrix} \tag{5.62}$$

for $z \in \Omega_3 \cap \mathbb{C}_-$. For $z \in \Omega_4$ and $z \in \Omega_2$, we replace $\phi(z)$ and $\widetilde{\phi}(z)$ by $\overline{\phi}(z)$, and define

$$\overline{\xi}_{\mathbf{N}} = \overline{f}_{\mathbf{N}}(z) = \left(\frac{3}{2}\overline{\phi}(z)\right)^{\frac{2}{3}}, \tag{5.63}$$

which is analytic in $\mathbb{C} \setminus (-\infty, -b] \cup [b, \infty)$ and

$$|\arg \overline{f}_{\mathbf{N}}(z)| < \pi$$

for $z \in \Omega_4$ and $z \in \Omega_2$. Thus, let us define the matrix function

$$Q(z) := \begin{cases}\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & -w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\xi_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_1 \cap \mathbb{C}_+,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w\mathrm{Ai}(N^{\frac{2}{3}}w\xi_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\xi_{\mathbf{N}}) & w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\xi_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_1 \cap \mathbb{C}_-,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w\mathrm{Ai}(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & -w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\widetilde{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_3 \cap \mathbb{C}_+,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{\xi}_{\mathbf{N}}) & w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\widetilde{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_3 \cap \mathbb{C}_-,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & -w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\overline{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & -w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\overline{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_4 \cap \mathbb{C}_+,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & w\mathrm{Ai}(N^{\frac{2}{3}}w\overline{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\overline{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_4 \cap \mathbb{C}_-,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & -w\mathrm{Ai}(N^{\frac{2}{3}}w\overline{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & -w^2\mathrm{Ai}'(N^{\frac{2}{3}}w\overline{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_2 \cap \mathbb{C}_+,\\
\begin{pmatrix}\mathrm{Ai}(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & w^2\mathrm{Ai}(N^{\frac{2}{3}}w^2\overline{\xi}_{\mathbf{N}})\\ \mathrm{Ai}'(N^{\frac{2}{3}}\overline{\xi}_{\mathbf{N}}) & w\mathrm{Ai}'(N^{\frac{2}{3}}w^2\overline{\xi}_{\mathbf{N}})\end{pmatrix}, & z \in \Omega_2 \cap \mathbb{C}_-.\end{cases} \tag{5.64}$$

The identities

$$\mathrm{Ai}(z) + w\mathrm{Ai}(wz) + w^2\mathrm{Ai}(w^2 z) = 0$$

and

$$\mathrm{Ai}'(z) + w^2\mathrm{Ai}'(wz) + w\mathrm{Ai}'(w^2 z) = 0$$

show that $Q(z)$ satisfies

$$Q_+(x) = Q_-(x)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad x \in \mathbb{R}. \tag{5.65}$$

The above heuristic argument suggests that $U(z)$ is asymptotically approximated by

$$U^*(z) := \begin{cases} E(z)M(z), & z \in \Omega_1, \\ \widetilde{E}(z)\widetilde{M}(z), & z \in \Omega_3, \\ \overline{E}_1(z)\overline{M}_1(z), & z \in \Omega_4, \\ \overline{E}_2(z)\overline{M}_2(z), & z \in \Omega_2, \end{cases} \tag{5.66}$$

where

$$E(z) := \sqrt{\pi} e^{\frac{1}{2} N l_N \sigma_3} D_\infty^{\sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \widetilde{\gamma}(z)^{\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$
$$\times D(z)^{-\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} (N^{\frac{1}{6}} \xi_N^{\frac{1}{4}}(z))^{\sigma_3},$$
$$M(z) := Q(z) e^{[\frac{1}{2}(N z^4 + t\sqrt{N} z^2) - \alpha \log z]\sigma_3}$$

for $z \in \Omega_1$, and

$$\widetilde{E}(z) := \sqrt{\pi} e^{\frac{1}{2} N \widetilde{l}_N \sigma_3} e^{-\alpha \pi i \sigma_3} D_\infty^{\sigma_3} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \widetilde{\gamma}(z)^{-\sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$
$$\times D(z)^{-\sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}^{-1} (N^{\frac{1}{6}} \widetilde{\xi}_N^{\frac{1}{4}}(z))^{\sigma_3},$$
$$\widetilde{M}(z) := Q(z) e^{[\frac{1}{2}(N z^4 + t\sqrt{N} z^2) - \alpha \log(-z)]\sigma_3}$$

for $z \in \Omega_3$, where $\widetilde{l}_N = l_N + 2\pi i$, and

$$\overline{E}_1(z) := \sqrt{\pi} e^{\frac{1}{2} N \widetilde{l}_N \sigma_3} e^{-\alpha \pi i \sigma_3} D_\infty^{\sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \widetilde{\gamma}(z)^{\sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$
$$\times D(z)^{-\sigma_3} e^{\mp \frac{1}{2}\pi i N \sigma_3} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} (N^{\frac{1}{6}} \overline{\xi}_N^{\frac{1}{4}}(z))^{\sigma_3},$$
$$\overline{M}_1(z) := Q(z) e^{[\frac{1}{2}(N z^4 + t\sqrt{N} z^2) - \alpha \log(-z)]\sigma_3}$$

for $z \in \Omega_4 \cap \mathbb{C}_\pm$,

$$\overline{E}_2(z) := \sqrt{\pi} e^{\frac{1}{2} N l_N \sigma_3} D_\infty^{\sigma_3} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \widetilde{\gamma}(z)^{-\sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$
$$\times D(z)^{-\sigma_3} e^{\pm \frac{1}{2}\pi i N \sigma_3} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}^{-1} (N^{\frac{1}{6}} \overline{\xi}_N^{\frac{1}{4}}(z))^{\sigma_3},$$
$$\overline{M}_2(z) := Q(z) e^{[\frac{1}{2}(N z^4 + t\sqrt{N} z^2) - \alpha \log z]\sigma_3}$$

for $z \in \Omega_2 \cap \mathbb{C}_\pm$. From (5.66) it can be shown that

$$U_+^*(x) = U_-^*(x) \begin{pmatrix} 1 & |x|^{2\alpha} e^{-(N x^4 + t N^{\frac{1}{2}} x^2)} \\ 0 & 1 \end{pmatrix} \tag{5.67}$$

for all $x \in \mathbb{R} \setminus \{0\}$. Furthermore, $U^*(z)$ has the same large $z$ behavior as $U(z)$ shown in $(U_c)$.

## 6 Uniform Asymptotic Expansions

In this section, we will present uniform asymptotic expansions of the monic polynomials $\pi_n(x)$ in Theorem 2.1. As in the previous sections, there are three cases to be considered, but in all three cases, we will use the same notations with different meaning.

### 6.1 Case for $c > -2$

To derive such expansions, we define the matrix-valued function

$$S(z) := e^{-\frac{1}{2}\mathbf{N}\mathbf{l}_N\boldsymbol{\sigma}_3} U(z)U^*(z)^{-1} e^{\frac{1}{2}\mathbf{N}\mathbf{l}_N\boldsymbol{\sigma}_3}. \tag{6.1}$$

Since $U(z)$ and $U^*(z)$ have the same jump condition on $(-\delta,\delta)\setminus\{0\}$, $S(z)$ is analytic in $U_{\boldsymbol{\delta}}\setminus\{0\}$. In fact, 0 is a removable singularity of $S(z)$, (see [20, (6.2)–(6.9)]). Also, it is easily seen that $S(z)$ satisfies the following RHP:

($S_{\mathrm{a}}$) $S(z)$ is analytic for $z \in \mathbb{C}\setminus\Gamma$, where $\Gamma$ is the contour $\Gamma = \bigcup_{\mathbf{i}=1}^{8}\Gamma_{\mathbf{i}}$ shown in Figure 3.

($S_{\mathrm{b}}$) for $z \in \Gamma$,

$$S_+(z) = S_-(z)J(z),$$

where $J(z) := e^{-\frac{1}{2}\mathbf{N}\mathbf{l}_N\boldsymbol{\sigma}_3} U^*_-(z)(U^*_+(z))^{-1} e^{\frac{1}{2}\mathbf{N}\mathbf{l}_N\boldsymbol{\sigma}_3}$;

($S_{\mathrm{c}}$) for $z \in \mathbb{C}\setminus\Gamma$,

$$S(z) \to I$$

as $z \to \infty$.

To solve this problem, we first derive the asymptotic expansion of the jump matrix $J(z)$ as $n \to \infty$. This is done in exactly the same manner as in [20], and we have

$$J(z) \sim I + \sum_{\mathbf{k}=1}^{\infty} \frac{J_{\mathbf{k}}(z)}{N^{\mathbf{k}}}, \tag{6.2}$$

where the coefficients $J_{\mathbf{k}}(z)$ are explicitly given $2\times 2$ matrices; for details, see [20, p. 750].

An appeal to Theorem 3 in [20] shows that the solution of the RHP for $S$ has the asymptotic expansion

$$S(z) \sim I + \sum_{\mathbf{k}=1}^{\infty} \frac{S_{\mathbf{k}}(z)}{N^{\mathbf{k}}} \tag{6.3}$$

as $n \to \infty$ uniformly for $z \in \mathbb{C}\setminus\Gamma$, where $N = n+\alpha$ and the coefficient functions $S_{\mathbf{k}}(z)$ can be determined recursively by

$$S_{\mathbf{k}}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \sum_{\mathbf{j}=1}^{\mathbf{k}} (S_{\mathbf{k}-\mathbf{j}})_-(\zeta) J_{\mathbf{j}}(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z}, \quad k = 1,2,\cdots \tag{6.4}$$

for $z \in \mathbb{C}\setminus\Gamma$.

The proofs of the following theorems can be carried out along the same lines as given in [19–20].

**Theorem 6.1** *Let* $\Omega_{\mathbf{i}}$, $i = 1,2,3$, *and* $U_{\boldsymbol{\delta}}$ *be the regions shown in Figure* 3. *With* $N = n+\alpha$, $t > -2\sqrt{N}$, $l_{\mathbf{N}}$ *and* $f_{\mathbf{N}}(z)$ *defined in* (3.32) *and* (5.17), *the asymptotic expansion of the polynomial* $\pi_{\mathbf{n}}(N^{\frac{1}{4}}z)$ *is given by*

$$\pi_{\mathbf{n}}(N^{\frac{1}{4}}z) = \sqrt{\pi} N^{\frac{n}{4}} e^{\frac{1}{2}(\mathbf{N}\mathbf{z}^4 + \mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2) + \frac{1}{2}\mathbf{N}\mathbf{l}_N} z^{-\boldsymbol{\alpha}}$$
$$\times [\mathrm{Ai}(N^{\frac{2}{3}} f_{\mathbf{N}}(z))\boldsymbol{A}(z,N) - \mathrm{Ai}'(N^{\frac{2}{3}} f_{\mathbf{N}}(z))\boldsymbol{B}(z,N)], \tag{6.5}$$

*where* $\boldsymbol{A}(z,N)$ *and* $\boldsymbol{B}(z,N)$ *are analytic in* $\Omega_1$ *and have asymptotic expansions*

$$\boldsymbol{A}(z,N)\sim\frac{N^{\frac{1}{6}}f_N^{\frac{1}{4}}(z)}{\beta(z)}\Big(1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)-\mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.6}$$

*and*

$$\boldsymbol{B}(z,N)\sim\frac{\beta(z)}{N^{\frac{1}{6}}f_N^{\frac{1}{4}}(z)}\Big(1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)+\mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.7}$$

*uniformly for* $z\in\Omega_1$. *In* (6.6)–(6.7), *the coefficient function* $S_{ij}^{(k)}, i,j=1,2$, *refers to the element in the* $i$*th row and* $j$*th column of the matrix* $S_k(z)$, *which is given in* (6.4). *The function* $\beta(z)$ *is given in* (4.15).

*Similarly, with* $\widetilde{f}_N(z)$ *given in* (5.22), *we have*

$$\begin{aligned}\pi_n(N^{\frac{1}{4}}z)&=(-1)^n\sqrt{\pi}N^{\frac{n}{4}}\mathrm{e}^{\frac{1}{2}(Nz^4+t\sqrt{N}z^2)+\frac{1}{2}Nl_N}(-z)^{-\alpha}\\&\quad\times[\mathrm{Ai}(N^{\frac{2}{3}}\widetilde{f}_N(z))\widetilde{\boldsymbol{A}}(z,N)-\mathrm{Ai}'(N^{\frac{2}{3}}\widetilde{f}_N(z))\widetilde{\boldsymbol{B}}(z,N)],\end{aligned} \tag{6.8}$$

*where* $\widetilde{\boldsymbol{A}}(z,N)$ *and* $\widetilde{\boldsymbol{B}}(z,N)$ *are analytic in* $\Omega_3$ *and have asymptotic expansions*

$$\widetilde{\boldsymbol{A}}(z,N)\sim N^{\frac{1}{6}}\widetilde{f}_N^{\frac{1}{4}}(z)\beta(z)\Big(1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)+\mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.9}$$

*and*

$$\widetilde{\boldsymbol{B}}(z,N)\sim\frac{1}{N^{\frac{1}{6}}\widetilde{f}_N^{\frac{1}{4}}(z)\beta(z)}\Big(1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)-\mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.10}$$

*uniformly for* $z\in\Omega_3$.

*Let* $\phi(z)$ *be defined as in* (3.33). *We have*

$$\begin{aligned}\pi_n(N^{\frac{1}{4}}z)&=\frac{1}{2}N^{\frac{n}{4}}\mathrm{e}^{\frac{1}{2}(Nz^4+t\sqrt{N}z^2)+\frac{1}{2}Nl_N-N\varphi(z)}z^{-\alpha}\\&\quad\times\Big[\Big(\Big(\frac{z-\alpha_t}{z+\alpha_t}\Big)^{\frac{1}{4}}+\Big(\frac{z-\alpha_t}{z+\alpha_t}\Big)^{-\frac{1}{4}}\Big)\boldsymbol{A}_1(z,N)\\&\quad+\Big(\Big(\frac{z-\alpha_t}{z+\alpha_t}\Big)^{\frac{1}{4}}-\Big(\frac{z-\alpha_t}{z+\alpha_t}\Big)^{-\frac{1}{4}}\Big)\boldsymbol{B}_1(z,N)\Big],\end{aligned} \tag{6.11}$$

*where* $\boldsymbol{A}_1(z,N)$ *and* $\boldsymbol{B}_1(z,N)$ *are analytic functions of* $z$ *in* $\Omega_2$. *In addition, they have asymptotic expansions*

$$\boldsymbol{A}_1(z,N)\sim1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)}{N^k} \tag{6.12}$$

*and*

$$\boldsymbol{B}_1(z,N)\sim\mathrm{i}\sum_{k=1}^{\infty}\frac{S_{12}^{(k)}(z)}{N^k}, \tag{6.13}$$

*uniformly in* $\Omega_2$.

*Finally, when* $z \in U_{\delta}$*, we have*

$$\pi_n(N^{\frac{1}{4}}z) = \sqrt{\frac{\pi}{2}} N^{\frac{n+2}{4}} \mathrm{e}^{\frac{1}{2}(Nz^4 + t\sqrt{N}z^2) + \frac{1}{2}Nl_N - \frac{1}{4}\pi \mathrm{i}} \eta_N(z)^{\frac{1}{2}} z^{-\alpha} \times [J_{\alpha+\frac{1}{2}}(N\eta_N(z)) \boldsymbol{A}_0(z,N) + J_{\alpha-\frac{1}{2}}(N\eta_N(z)) \boldsymbol{B}_0(z,N)], \tag{6.14}$$

*where* $\eta_N(z)$ *is defined in* (5.8)*,* $\boldsymbol{A}_0(z,N)$ *and* $\boldsymbol{B}_0(z,N)$ *are analytic functions in* $U_{\delta}$ *with asymptotic expansions*

$$\boldsymbol{A}_0(z,N) \sim \sum_{k=0}^{\infty} \frac{A_{0,k}(z)}{N^k}, \quad \boldsymbol{B}_0(z,N) \sim \sum_{k=0}^{\infty} \frac{B_{0,k}(z)}{N^k}$$

*as* $n \to \infty$*. The leading coefficients are given by*

$$A_{0,0}(z) = (E_0)_{11}(z) + \mathrm{i}(E_0)_{12}(z) \quad \text{and} \quad B_{0,0}(z) = \mathrm{i}(E_0)_{11}(z) + (E_0)_{12}(z),$$

*where* $E_0(z)$ *is given in* (5.13)*.*

## 6.2 Cases for $c \le -2$

Similarly, we have the following two theorems on global asymptotic expansions of the polynomials $\pi_n(x)$ in Theorem 2.1.

**Theorem 6.2** *Let* $\Omega_i$*,* $i = 1,2,3$*, and* $U_{\delta}$ *be the regions shown in Figure* 3*, with* $N = n+\alpha$*,* $t = -2\sqrt{N}$*,* $l_N$ *and* $f_N(z)$ *defined in* (3.51) *and* (5.50)*. The asymptotic expansion of the polynomial* $\pi_n(N^{\frac{1}{4}}z)$ *is given*

$$\pi_n(N^{\frac{1}{4}}z) = \sqrt{\pi} N^{\frac{n}{4}} \mathrm{e}^{\frac{1}{2}N(z^4-2z^2)+\frac{1}{2}Nl_N} z^{-\alpha} \times [\mathrm{Ai}(N^{\frac{2}{3}} f_N(z)) \boldsymbol{A}(z,N) - \mathrm{Ai}'(N^{\frac{2}{3}} f_N(z)) \boldsymbol{B}(z,N)], \tag{6.15}$$

*where* $\boldsymbol{A}(z,N)$ *and* $\boldsymbol{B}(z,N)$ *are analytic in* $\Omega_1$ *and have asymptotic expansions*

$$\boldsymbol{A}(z,N) \sim \frac{N^{\frac{1}{6}} f_N^{\frac{1}{4}}(z)}{\beta(z)} \Big(1 + \sum_{k=1}^{\infty} \frac{S_{11}^{(k)}(z) - \mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.16}$$

*and*

$$\boldsymbol{B}(z,N) \sim \frac{\beta(z)}{N^{\frac{1}{6}} f_N^{\frac{1}{4}}(z)} \Big(1 + \sum_{k=1}^{\infty} \frac{S_{11}^{(k)}(z) + \mathrm{i}S_{12}^{(k)}(z)}{N^k}\Big) \tag{6.17}$$

*uniformly for* $z \in \Omega_1$*. In* (6.16)*–*(6.17)*, the function* $\beta(z)$ *is given in* (4.22)*. The coefficient function* $S_{ij}^{(k)}, i,j = 1,2$*, refers to the element in the ith row and jth column of the matrix* $S_k(z)$*, which is given in* (6.4)*. In* (6.4)*, the coefficients* $J_k(z)$ *are* $2\times 2$ *matrices explicitly given in* [19, p. 148]*.*

*Let* $\phi(z)$ *be defined as in* (3.52)*. We have*

$$\pi_n(N^{\frac{1}{4}}z) = \frac{1}{2} N^{\frac{n}{4}} \mathrm{e}^{\frac{1}{2}N(z^4-2z^2)+\frac{1}{2}Nl_N - N\varphi(z)} z^{-\alpha} \times \Big[\Big(\Big(\frac{z-\sqrt{2}}{z+\sqrt{2}}\Big)^{\frac{1}{4}} + \Big(\frac{z-\sqrt{2}}{z+\sqrt{2}}\Big)^{-\frac{1}{4}}\Big) \boldsymbol{A}_1(z,N)$$

$$+\Big(\Big(\frac{z-\sqrt{2}}{z+\sqrt{2}}\Big)^{\frac{1}{4}}-\Big(\frac{z-\sqrt{2}}{z+\sqrt{2}}\Big)^{-\frac{1}{4}}\Big)\boldsymbol{B}_1(z,N)\Big], \tag{6.18}$$

*where* $\boldsymbol{A}_1(z,N)$ *and* $\boldsymbol{B}_1(z,N)$ *are analytic functions of* $z$ *in* $\Omega_2$*. In addition, they have asymptotic expansions*

$$\boldsymbol{A}_1(z,N)\sim 1+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)}{N^k} \tag{6.19}$$

*and*

$$\boldsymbol{B}_1(z,N)\sim \mathrm{i}\sum_{k=1}^{\infty}\frac{S_{12}^{(k)}(z)}{N^k}, \tag{6.20}$$

*uniformly in* $\Omega_2$.

*Let* $(\psi_1(\zeta,s),\psi_2(\zeta,s))$ *be the unique solution of equation* (5.35) *characterized by the asymptotic behavior*

$$\mathrm{e}^{\mathrm{i}(\frac{4}{3}\zeta^3+s\zeta)}\begin{pmatrix}\psi_1(\zeta,s)\\ \psi_2(\zeta,s)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+O(\zeta^{-1})$$

*as* $\zeta\to\infty$ *for* $0<\arg\zeta<\pi$*, and let* $\zeta(z)$ *be as shown in* (5.44)*. We have*

$$\begin{aligned}\pi_n(N^{\frac{1}{4}}z)=&\sqrt{\pi}N^{\frac{n}{4}}\mathrm{e}^{\frac{1}{2}N(z^4-2z^2)+\frac{1}{2}Nl_N}z^{-\alpha}\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}\alpha}\\ &\times\psi_1(N^{\frac{1}{3}}\eta(z),0)\boldsymbol{A}_1(z,N)+\psi_2(N^{\frac{1}{3}}\eta(z),0)\boldsymbol{B}_1(z,N))\end{aligned} \tag{6.21}$$

*as* $n\to\infty$ *uniformly for* $z\in U_\delta$*, where*

$$\boldsymbol{A}_1(z,N)\sim\sum_{k=0}^{\infty}\frac{A_k(z)}{N^{\frac{k}{3}}},\quad \boldsymbol{B}_1(z,N)\sim\sum_{k=0}^{\infty}\frac{B_k(z)}{N^{\frac{k}{3}}}$$

*with*

$$A_0(z)=\mathrm{e}^{\frac{\pi\mathrm{i}n}{2}}\Big(\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\Big(\frac{\sqrt{2}-z}{z+\sqrt{2}}\Big)^{\frac{1}{4}}+\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\Big(\frac{\sqrt{2}-z}{z+\sqrt{2}}\Big)^{-\frac{1}{4}}\Big)$$

*and*

$$B_0(z)=\mathrm{e}^{-\frac{\pi\mathrm{i}n}{2}}\Big(\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\Big(\frac{\sqrt{2}-z}{z+\sqrt{2}}\Big)^{\frac{1}{4}}+\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\Big(\frac{\sqrt{2}-z}{z+\sqrt{2}}\Big)^{-\frac{1}{4}}\Big).$$

*When* $z\in\Omega_3$*, a corresponding asymptotic expansion can be obtained by using the reflection formula* $\pi_n^{(n)}(z)=(-1)^n\pi_n^{(n)}(-z)$.

**Theorem 6.3** *Let* $\Omega_i$*,* $i=1,\cdots,4$ *be the regions shown in Figure 4, with* $N=n+\alpha$*,* $t<-2\sqrt{N}$*,* $l_N$ *and* $f_N(z)$ *defined in* (3.96) *and* (5.54)*, respectively. The asymptotic expansion of the polynomial* $\pi_n(N^{\frac{1}{4}}z)$ *is given by*

$$\begin{aligned}\pi_n(N^{\frac{1}{4}}z)=&\sqrt{\pi}N^{\frac{n}{4}}\mathrm{e}^{\frac{1}{2}(Nz^4+t\sqrt{N}z^2)+\frac{1}{2}Nl_N}z^{-\alpha}\\ &\times[\mathrm{Ai}(N^{\frac{2}{3}}f_N(z))\boldsymbol{A}(z,N)-\mathrm{Ai}'(N^{\frac{2}{3}}f_N(z))\boldsymbol{B}(z,N)],\end{aligned} \tag{6.22}$$

*where* $\boldsymbol{A}(z,N)$ *and* $\boldsymbol{B}(z,N)$ *are analytic in* $\Omega_1$ *and have asymptotic expansions*

$$\boldsymbol{A}(z,N)\sim N^{\frac{1}{6}}f_N^{\frac{1}{4}}(z)\Big(\boldsymbol{W}_{11}(z)+\sum_{k=1}^{\infty}\frac{S_{11}^{(k)}(z)\,\boldsymbol{W}_{11}(z)-\mathrm{i}S_{12}^{(k)}(z)\,\boldsymbol{W}_{21}(z)}{N^k}\Big) \tag{6.23}$$

*and*

$$\boldsymbol{B}(z,N)\sim\frac{1}{N^{\frac{1}{6}}f_{\mathbf{N}}^{\frac{1}{4}}(z)}\Big(\boldsymbol{W}_{12}(z)+\sum_{\mathbf{k}=1}^{\infty}\frac{S_{11}^{(\mathbf{k})}(z)\,\boldsymbol{W}_{12}(z)-\mathrm{i}S_{12}^{(\mathbf{k})}(z)\,\boldsymbol{W}_{22}(z)}{N^{\mathbf{k}}}\Big) \tag{6.24}$$

*uniformly for* $z\in\Omega_1$. *Furthermore,*

$$\boldsymbol{W}=\frac{1}{2}\begin{pmatrix}D_\infty D(z)^{-1} & D_\infty D(z)\\ D_\infty^{-1}D(z) & D_\infty^{-1}D(z)^{-1}\end{pmatrix}\begin{pmatrix}\widetilde{\gamma}+\widetilde{\gamma}^{-1} & \widetilde{\gamma}+\widetilde{\gamma}^{-1}\\ \widetilde{\gamma}-\widetilde{\gamma}^{-1} & -\widetilde{\gamma}+\widetilde{\gamma}^{-1}\end{pmatrix};$$

*the coefficient functions* $S_{\mathbf{ij}}^{(\mathbf{k})}$ *and* $\boldsymbol{W}_{\mathbf{ij}}$ *in* (6.23)–(6.24), $i,j=1,2$, *refer to the elements in the* $i$*th row and* $j$*th column of the matrices* $S_{\mathbf{k}}(z)$ *and* $\boldsymbol{W}(z)$, *respectively. The function* $\widetilde{\gamma}(z)$ *is given in* (4.34). *Moreover,* $D(z)$ *and* $D_\infty$ *are given in* (4.31)–(4.32).

*Similarly, with* $\overline{f}_{\mathbf{N}}(z)$ *given in* (5.63), *we have*

$$\begin{aligned}\pi_{\mathbf{n}}(N^{\frac{1}{4}}z)&=\sqrt{\pi}N^{\frac{n}{4}}\mathrm{e}^{\frac{1}{2}(\mathbf{N}\mathbf{z}^4+\mathbf{t}\sqrt{\mathbf{N}}\mathbf{z}^2)+\frac{1}{2}\mathbf{N}\mathbf{l}_N}z^{-\boldsymbol{\alpha}}\\&\quad\times[\mathrm{Ai}(N^{\frac{2}{3}}\overline{f}_{\mathbf{N}}(z))\overline{\boldsymbol{A}}(z,N)-\mathrm{Ai}'(N^{\frac{2}{3}}\overline{f}_{\mathbf{N}}(z))\overline{\boldsymbol{B}}(z,N)],\end{aligned} \tag{6.25}$$

*where* $\overline{\boldsymbol{A}}(z,N)$ *and* $\overline{\boldsymbol{B}}(z,N)$ *are analytic in* $\Omega_2$ *and have asymptotic expansions*

$$\overline{\boldsymbol{A}}(z,N)\sim N^{\frac{1}{6}}f_{\mathbf{N}}^{\frac{1}{4}}(z)\Big(\overline{\boldsymbol{W}}_{11}(z)+\sum_{\mathbf{k}=1}^{\infty}\frac{S_{11}^{(\mathbf{k})}(z)\overline{\boldsymbol{W}}_{11}(z)-\mathrm{i}S_{12}^{(\mathbf{k})}(z)\overline{\boldsymbol{W}}_{21}(z)}{N^{\mathbf{k}}}\Big) \tag{6.26}$$

*and*

$$\overline{\boldsymbol{B}}(z,N)\sim\frac{1}{N^{\frac{1}{6}}f_{\mathbf{N}}^{\frac{1}{4}}(z)}\Big(\overline{\boldsymbol{W}}_{12}(z)+\sum_{\mathbf{k}=1}^{\infty}\frac{S_{11}^{(\mathbf{k})}(z)\overline{\boldsymbol{W}}_{12}(z)-\mathrm{i}S_{12}^{(\mathbf{k})}(z)\overline{\boldsymbol{W}}_{22}(z)}{N^{\mathbf{k}}}\Big) \tag{6.27}$$

*uniformly for* $z\in\Omega_2$. *In* (6.26)–(6.27),

$$\overline{\boldsymbol{W}}=\frac{1}{2}\begin{pmatrix}D_\infty D(z)^{-1}\mathrm{e}^{\pm\frac{1}{2}\mathbf{N}\boldsymbol{\pi}\mathrm{i}} & D(z)D_\infty\mathrm{e}^{\mp\frac{1}{2}\mathbf{N}\boldsymbol{\pi}\mathrm{i}}\\ D_\infty^{-1}D(z)^{-1}\mathrm{e}^{\pm\frac{1}{2}\mathbf{N}\boldsymbol{\pi}\mathrm{i}} & D_\infty^{-1}D(z)\mathrm{e}^{\mp\frac{1}{2}\mathbf{N}\boldsymbol{\pi}\mathrm{i}}\end{pmatrix}\begin{pmatrix}\widetilde{\gamma}+\widetilde{\gamma}^{-1} & \widetilde{\gamma}+\widetilde{\gamma}^{-1}\\ \widetilde{\gamma}^{-1}-\widetilde{\gamma} & \widetilde{\gamma}-\widetilde{\gamma}^{-1}\end{pmatrix}.$$

*When* $z\in\Omega_3$ *and* $z\in\Omega_4$, *a corresponding asymptotic expansion can be obtained by using the reflection formula* $\pi_{\mathbf{n}}^{(\mathbf{n})}(z)=(-1)^{\mathbf{n}}\pi_{\mathbf{n}}^{(\mathbf{n})}(-z)$.

By Theorem 2.1, there exists a $2\times 2$ matrix $Y_1$ such that

$$Y(z)=\Big[I+\frac{Y_1}{z}+O\Big(\frac{1}{z^2}\Big)\Big]z^{\mathbf{n}\boldsymbol{\sigma}_3}\quad\text{as } z\to\infty \tag{6.28}$$

(see condition $(Y_{\rm c})$). The recurrence coefficient $\beta_{\mathbf{n}}(t;\alpha)$ in (1.2) is related to $Y_1$ by $\beta_{\mathbf{n}}(t;\alpha)=(Y_1)_{12}(Y_1)_{21}$ (see, e.g., [6, (5.2)]). Thus, the asymptotics of the recurrence coefficient $\beta_{\mathbf{n}}(t;\alpha)$ can also be derived by using the asymptotics of $Y$ as $n\to\infty$.

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