Chin. Ann. Math. Ser. B
39(5), 2018, 791–804
DOI: 10.1007/s11401-018-0096-2

Chinese Annals of Mathematics, Series B

Strong Laws of Large Numbers for Sublinear Expectation under Controlled 1st Moment Condition*

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Abstract This paper deals with strong laws of large numbers for sublinear expectation under controlled 1st moment condition. For a sequence of independent random variables, the author obtains a strong law of large numbers under conditions that there is a control random variable whose 1st moment for sublinear expectation is finite. By discussing the relation between sublinear expectation and Choquet expectation, for a sequence of i.i.d random variables, the author illustrates that only the finiteness of uniform 1st moment for sublinear expectation cannot ensure the validity of the strong law of large numbers which in turn reveals that our result does make sense.

Keywords Sublinear expectation, Strong law of large numbers, Independence, Identical distribution, Choquet expectation
2000 MR Subject Classification 60F15

1 Introduction

In recent years, motivated by some problems in mathematical economics, statistics and financial mathematics, more and more researches on nonlinear expectation have appeared. People use nonlinear expectation to describe some phenomena in these fields which are difficult to be modeled exactly by classical probability theory. Choquet [5] first introduced the definition of capacity and it has been used in many fields of applied mathematics. To deal with the problems in risk measures, super-hedge pricing and modeling uncertainty in finance, Peng [12] initiated the definition of general sublinear expectation and the notion of independence and identical distribution for sublinear expectation. See more applications of nonlinear expectation for example, [3, 7, 11, 13, 15].

The major requirement for any probability theory is to give a frequentist justification to probability numbers via limit frequencies. The classical strong laws of large numbers (SLLNs for short) as fundamental limit theorems in probability theory play an important role in the development of probability and its applications. So the question arises naturally whether the SLLNs can be maintained in nonlinear expectation framework. There has been increasing interest in the investigation of SLLNs for nonlinear expectation. Marinacci [10] proved the SLLNs for a sequence of i.i.d random variables with respect to a totally monotone and continuous capacity under a multiplicative notion of independence. Maccheroni and Marinacci [9] introduced the

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*This work was supported by the National Natural Science Foundation of China (Nos. 11501325, 11231005).

definition of pairwise independence and proved a SLLN for a sequence of bounded i.i.d random variables with respect to a totally monotone capacity. They both indicate that any cluster point of empirical averages lies between the upper Choquet expectation $C_V[X_1]$ and the lower Choquet expectation $C_v[X_1]$ with probability one under capacity v. That is

$$v\Big(C_v[X_1] \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq C_V[X_1]\Big) = 1.$$

See more results regarding SLLNs for nonlinear expectations for example, [1–2, 4, 6, 8].

Moreover, the gap between the Choquet expectations $C_V[X_1]$ and $C_v[X_1]$ is bigger than that of the the upper expectation $\mathbb{E}[X_1]$ and lower expectation $\mathcal{E}[X_1]$. Chen et al. [4] obtained a more precise SLLN for a sequence of independent random variables under conditions of finite $(1+\alpha)$-th moment for upper expectation. That is

$$v\Big(\mathcal{E}[X_1] \leq \liminf_{n\to\infty} \frac{S_n}{n} \leq \limsup_{n\to\infty} \frac{S_n}{n} \leq \mathbb{E}[X_1]\Big) = 1.$$

Zhang [16] derived a SLLN of the above form for a sequence of negatively dependent identically distributed random variables under conditions of finite 1st moment for Choquet expectation. As we mentioned above, in sublinear situation, the Choquet expectation is larger than the upper expectation. And in classical probability theory, for a sequence of i.i.d random variables, the finiteness of 1st moment is the sufficient condition of the SLLNs. Our purpose in this paper is to study the SLLNs under some conditions of finite 1st moment for sublinear expectation.

First we obtain a SLLN under the condition that there is a random variable X such that for every $n \geq 1$, $|X_n| \leq |X|$ q.s. where X satisfies $\lim_{n\to\infty} \mathbb{E}[|X|I(|X|>n)] = 0$. This assumption looks like a control condition which is weaker than the uniform boundedness, but is stronger than $\sup_n \mathbb{E}[|X_n|] < \infty$. Furthermore, we discuss whether the SLLN can be maintained under conditions that the uniform 1st moment for sublinear expectation is finite. But by discussing the relation between Choquet expectation and sublinear expectation and putting forward a counterexample, we find out that the SLLN may not be valid when only the uniform 1st moment for sublinear expectation is bounded which in turn reveals the fundamental difference between classical probability and sublinear expectation.

The rest of this paper is organized as follows. In Section 2 we recall some basic concepts of sublinear expectation and some useful lemmas. In Section 3 we give our main result, the SLLN for sublinear expectation under controlled 1st moment condition. In Section 4 we give a counterexample to illustrate that the SLLN may not be true when only uniform 1st moment for sublinear expectation is finite.

2 Basic Concepts and Lemmas

We use the notations similar to that of Peng [14]. Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a subset of all random variables on $(\Omega, \mathcal{F})$ such that all $I_A \in \mathcal{H}$, where $A \in \mathcal{F}$ and if $X_1, X_2, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) function φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ. We consider $\mathcal{H}$ as the space of random variables.

Definition 2.1 *A sublinear expectation* $\mathbb{E}$ *on* $\mathcal{H}$ *is a functional* $\mathbb{E}$*:* $\mathcal{H} \to \overline{\mathbb{R}} := [-\infty, \infty]$ *satisfying the following properties: for all* $X, Y \in \mathcal{H}$*, we have*

(a) *Monotonicity: If* $X \geq Y$*, then* $\mathbb{E}[X] \geq \mathbb{E}[Y]$*.*

(b) *Constant preserving:* $\mathbb{E}[c] = c$*,* $\forall c \in \mathbb{R}$*.*

(c) *Positive homogeneity:* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$*,* $\forall \lambda \geq 0$*.*

(d) *Sub-additivity:* $\mathbb{E}[X+Y] \leq \mathbb{E}[X]+\mathbb{E}[Y]$ *whenever* $\mathbb{E}[X]+\mathbb{E}[Y]$ *is not of the form* $\infty - \infty$ *or* $-\infty + \infty$*.*

Remark 2.1 By combining (b) and (d) in Definition 2.1, we can easily obtain a basic property of sublinear expectation

(e) Translation invariance: $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, $\forall c \in \mathbb{R}$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. Given a sublinear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathcal{E}$ of $\mathbb{E}$ by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{H}.$$

It is evident that for all $X \in \mathcal{H}$, $\mathcal{E}[X] \leq \mathbb{E}[X]$.

Definition 2.2 *A set function* $V : \mathcal{F} \to [0, 1]$ *is called a capacity if it satisfies*

(a) $V(\emptyset) = 0$*,* $V(\Omega) = 1$*.*

(b) $V(A) \leq V(B)$*,* $A \subset B$*,* $A, B \in \mathcal{F}$*.*

A capacity V *is said to be sub-additive if it satisfies* $V(A \cup B) \leq V(A) + V(B)$*,* $A, B \in \mathcal{F}$*.*

In this paper we only consider the capacity generated by sublinear expectation. Given a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, we define a capacity: $V(A) := \mathbb{E}[I_A]$, $\forall A \in \mathcal{F}$ and also define the conjugate capacity: $v(A) := 1 - V(A^c)$, $\forall A \in \mathcal{F}$. It is clear that V is a sub-additive capacity and $v(A) = \mathcal{E}[I_A]$.

The corresponding Choquet expectation (Choquet integral) C_V is defined by

$$C_V[X] := \int_0^\infty V(X \geq t)\mathrm{d}t + \int_{-\infty}^0 [V(X \geq t) - 1]\mathrm{d}t.$$

Respectively if we change V to v, we can obtain the definition of C_v. Obviously, if V (or v) is the classical probability, then the Choquet expectation $C_V[X]$ (or $C_v[X]$) coincides with the classical expectation.

Definition 2.3 *A sublinear expectation* $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ *is said to be continuous if it satisfies*

(1) *Lower-continuity: If* $X_n \uparrow X$*, then* $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$*, where* $0 \leq X_n,\ X \in \mathcal{H}$*.*

(2) *Upper-continuity: If* $X_n \downarrow X$*, then* $\mathbb{E}[X_n] \downarrow \mathbb{E}[X]$*, where* $0 \leq X_n,\ X \in \mathcal{H}$*.*

A capacity $V : \mathcal{F} \to [0, 1]$ *is called a continuous capacity if it satisfies*

(1) *Lower-continuity: If* $A_n \uparrow A$*, then* $V(A_n) \uparrow V(A)$*, where* $A_n, A \in \mathcal{F}$*.*

(2) *Upper-continuity: If* $A_n \downarrow A$*, then* $V(A_n) \downarrow V(A)$*, where* $A_n, A \in \mathcal{F}$*.*

Now we give the following continuity properties of $\mathbb{E}$ and V and the proofs can be referred to Zhang [16].

Proposition 2.1 (1) *If $\mathbb{E}$ is lower-continuous, then it is countably sub-additive, i.e.,*

$$\mathbb{E}\Big[\sum_{n=1}^{\infty} X_n\Big] \leq \sum_{n=1}^{\infty} \mathbb{E}[X_n]$$

for any $0 \leq X_n$, $\sum\limits_{n=1}^{\infty} X_n \in \mathcal{H}$.

(2) *If V is lower-continuous, then it is countably sub-additive, i.e., $V\big(\bigcup\limits_{n=1}^{\infty} A_n\big) \leq \sum\limits_{n=1}^{\infty} V(A_n)$ for any $A_n \in \mathcal{F}$.*

(3) *If $\mathbb{E}$ is lower-continuous, then V induced by $\mathbb{E}$ is lower-continuous.*

Example 2.1 Let $\mathcal{P}$ be a family of probability measures defined on $(\Omega, \mathcal{F})$. For any random variable X, we define a upper expectation by

$$\widehat{\mathbb{E}}[X] = \sup_{Q \in \mathcal{P}} E_Q[X].$$

Then $\widehat{\mathbb{E}}[X]$ is a sublinear expectation. For any $X_n \uparrow X$, $0 \leq X_n$, $X \in \mathcal{H}$,

$$\begin{aligned}\widehat{\mathbb{E}}[X] &= \sup_{Q \in \mathcal{P}} E_Q[X] = \sup_{Q \in \mathcal{P}} \lim_n E_Q[X_n] = \sup_{Q \in \mathcal{P}} \sup_n E_Q[X_n] \\ &= \sup_n \sup_{Q \in \mathcal{P}} E_Q[X_n] = \sup_n \widehat{\mathbb{E}}[X_n] = \lim_n \widehat{\mathbb{E}}[X_n].\end{aligned}$$

So $\widehat{\mathbb{E}}$ is lower-continuous and then countably sub-additive. Moreover, we can also define the capacity $V(A) = \widehat{\mathbb{E}}[I_A] = \sup\limits_{Q \in \mathcal{P}} Q(A)$. By Proposition 2.1(3), we also have that V is lower-continuous and countably sub-additive.

Next we show the representation theorem of sublinear expectation introduced by Peng [14] and the proof can be found there.

Proposition 2.2 *Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space.*

(1) (see [14, Theorem 2.4]) *There exists a family of finitely additive probability measures $\{P_\theta : \theta \in \Theta\}$ defined on $(\Omega, \mathcal{F})$ such that for each $X \in \mathcal{H}$,*

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_{P_\theta}[X].$$

(2) (see [14, Lemma 3.4]) *For any fixed random variable $X \in \mathcal{H}$, there exists a family of probability measures $\{\mu_\theta\}_{\theta \in \Theta}$ defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R})$,*

$$\mathbb{E}[\varphi(X)] = \sup_{\theta \in \Theta} \int_{\mathbb{R}} \varphi(x) \mu_\theta(\mathrm{d}x).$$

Definition 2.4 *Given a capacity V, a set A is said to be a polar set if $V(A) = 0$. And we call a property holds "quasi-surely" (q.s.) if it holds outside a polar set.*

We adopt the following notion of independence and identical distribution for sublinear expectation which is initiated by Peng [14].

Definition 2.5 (Independence) *Let $\boldsymbol{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ be two random variables on $(\Omega, \mathcal{H}, \mathbb{E})$. $\boldsymbol{Y}$ is said to be independent of $\boldsymbol{X}$, if for each test* *function* $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$, *we have* $\mathbb{E}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \mathbb{E}[\mathbb{E}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$ *whenever* $\overline{\varphi}(\boldsymbol{x}) := \mathbb{E}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ *for all* $\boldsymbol{x}$ *and* $\mathbb{E}[|\overline{\varphi}(\boldsymbol{X})|] < \infty$. $\{X_n\}_{n=1}^{\infty}$ *is said to be a sequence of independent random variables, if* X_{n+1} *is independent of* $(X_1, \cdots, X_n)$ *for each* $n \geq 1$.

Definition 2.6 (Identical Distribution) *Let* $\boldsymbol{X}_1, \boldsymbol{X}_2$ *be two* n*-dimensional random variables defined respectively on sublinear expectation spaces* $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. *They are called identically distributed if*

$$\mathbb{E}_1[\varphi(\boldsymbol{X}_1)] = \mathbb{E}_2[\varphi(\boldsymbol{X}_2)], \quad \forall \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n),$$

whenever the sublinear expectations are finite.

Definition 2.7 (IID Random Variables) *A sequence of random variables* $\{X_n\}_{n=1}^{\infty}$ *is said to be independent and identically distributed, if* X_{n+1} *is independent of* $(X_1, \cdots, X_n)$ *and* X_n *and* X_1 *are identically distributed for each* $n \geq 1$.

To prove our main results, we need the following basic lemmas for sublinear expectation. The proofs of Lemmas 2.1–2.2 can be found in [4].

Lemma 2.1 (Borel-Cantelli Lemma) *Let* $\{A_n\}_{n=1}^{\infty}$ *be a sequence of events in* $\mathcal{F}$ *and* V *be a capacity induced by lower-continuous sublinear expectation* $\mathbb{E}$. *If* $\sum\limits_{n=1}^{\infty} V(A_n) < \infty$, *then*

$$V\Big(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\Big) = 0.$$

Lemma 2.2 (Chebyshev's Inequality) *Let* $f(x) > 0$ *be a nondecreasing function on* $\mathbb{R}$. *Then for any* x,

$$V(X \geq x) \leq \frac{\mathbb{E}[f(X)]}{f(x)}, \quad v(X \geq x) \leq \frac{\mathcal{E}[f(X)]}{f(x)}.$$

As we all know, in classical probability theory, $\lim\limits_{n\to\infty} E[|X|I(|X| > n)] = 0$ and $E[|X|] < \infty$ are equivalent. But in sublinear situation, this property may not be true. The next lemma reveals the essential difference and the relation between sublinear expectation and Choquet expectation.

Lemma 2.3 *Given a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$, $\mathbb{E}$ *is lower-continuous and* V *is the induced capacity. Then*

(1) $C_V[|X|] < \infty$ *implies* $\lim\limits_{n\to\infty} \mathbb{E}[|X|I(|X| > n)] = 0$.

(2) $\lim\limits_{n\to\infty} \mathbb{E}[|X|I(|X| > n)] = 0$ *implies* $\mathbb{E}[|X|] < \infty$.

(3) $\lim\limits_{n\to\infty} \mathbb{E}[|X|I(|X| > n)] = 0$ *implies* $\lim\limits_{n\to\infty} \mathbb{E}[(|X| - n)^+] = 0$.

Proof (1) By the definition of C_V, we have

$$\begin{aligned} 2C_V[|X|] &= \int_0^{\infty} V(2|X| > t)\mathrm{d}t = \sum_{i=0}^{\infty} \int_i^{i+1} V(2|X| > t)\mathrm{d}t \\ &\geq \sum_{i=0}^{\infty} \int_i^{i+1} V(2|X| > i+1)\mathrm{d}t = \sum_{i=1}^{\infty} V(2|X| > i). \end{aligned}$$

It follows that $\sum\limits_{i=1}^{\infty} V(2|X| > i) < \infty$ and $\lim\limits_{n\to\infty} nV(|X| > n) = 0$.

By Proposition 2.2(2) and noticing that $\varphi_n(x) = (|x| - n)^+ \in C_{l,\mathrm{Lip}}(\mathbb{R})$, we have

$$
\begin{aligned}
&\mathbb{E}[|X|I(|X|>n)] \\
&\le \mathbb{E}[(|X|-n)^+] + \mathbb{E}[nI(|X|>n)] \\
&= \sup_{\theta\in\Theta}\int_{\mathbb{R}}(|x|-n)^+\mu_\theta(\mathrm{d}x) + nV(|X|>n) \\
&= \sup_{\theta\in\Theta}\Big(\int_{\mathbb{R}}|x|I(|x|>n)\mu_\theta(\mathrm{d}x) - \int_{\mathbb{R}} nI(|x|>n)\mu_\theta(\mathrm{d}x)\Big) + nV(|X|>n) \\
&= \sup_{\theta\in\Theta}\Big(\int_0^\infty \mu_\theta(|x|I(|x|>n)>t)\mathrm{d}t - n\mu_\theta(|x|>n)\Big) + nV(|X|>n) \\
&= \sup_{\theta\in\Theta}\Big(\int_0^n \mu_\theta(|x|>n)\mathrm{d}t + \int_n^\infty \mu_\theta(|x|>t)\mathrm{d}t - n\mu_\theta(|x|>n)\Big) + nV(|X|>n) \\
&= \sup_{\theta\in\Theta}\int_n^\infty \mu_\theta(|x|>t)\mathrm{d}t + nV(|X|>n) \\
&\le \int_n^\infty \sup_{\theta\in\Theta}\Big(\int_{\mathbb{R}} I(|x|>t)\mu_\theta(\mathrm{d}x)\Big)\mathrm{d}t + nV(|X|>n).
\end{aligned}
$$

For any $t \ge n$, let g_t be a function satisfying that its derivatives of each order are bounded, $g_t(x) = 1$ if $x \ge t$, $g_t(x) = 0$ if $x \le \frac{t}{2}$ and $0 \le g_t(x) \le 1$ for all x. Then we have

$$
g_t(\cdot) \in C_{l,\mathrm{Lip}}(\mathbb{R}) \quad \text{and} \quad I(x \ge t) \le g_t(x) \le I\Big(x > \frac{t}{2}\Big).
$$

Hence

$$
\begin{aligned}
\sup_{\theta\in\Theta}\int_{\mathbb{R}} I(|x|>t)\mu_\theta(\mathrm{d}x) &\le \sup_{\theta\in\Theta}\int_{\mathbb{R}} g_t(|x|)\mu_\theta(\mathrm{d}x) = \mathbb{E}[g_t(|X|)] \\
&\le \mathbb{E}\Big[I\Big(|X| > \frac{t}{2}\Big)\Big] = V(2|X|>t).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathbb{E}[|X|I(|X|>n)] &\le \int_n^\infty V(2|X|>t)\mathrm{d}t + nV(|X|>n) \\
&\le \sum_{i=n}^\infty V(2|X|>i) + nV(|X|>n) \\
&\to 0.
\end{aligned}
$$

(2) By the sub-additivity of $\mathbb{E}$, we have

$$
\begin{aligned}
\mathbb{E}[|X|] &= \mathbb{E}[|X|I(|X|\le n) + |X|I(|X|>n)] \\
&\le nV(|X|\le n) + \mathbb{E}[|X|I(|X|>n)].
\end{aligned}
$$

Taking n large sufficiently, there exists some K such that $\mathbb{E}[|X|I(|X|>n)] \le K$ and for this n, there holds $nV(|X|\le n) < \infty$. So we have $\mathbb{E}[|X|] < \infty$.

(3) Also by the sub-additivity of $\mathbb{E}$, we have

$$
\begin{aligned}
\mathbb{E}[(|X|-n)^+] &= \mathbb{E}[(|X|-n)I(|X|>n)] \\
&\le \mathbb{E}[|X|I(|X|>n)] + n\mathbb{E}[I(|X|>n)] \\
&\le 2\mathbb{E}[|X|I(|X|>n)] \\
&\to 0.
\end{aligned}
$$

Lemma 2.4 *If* $\mathbb{E}[|X|] < \infty$*, then* $|X| < \infty$ *q.s., i.e.,* $V(|X| = \infty) = 0$.

Proof

$$V(|X| = \infty) = V\Big(\bigcap_{i=1}^{\infty}\{|X| > i\}\Big) \le V(|X| > i) \le \frac{\mathbb{E}[|X|]}{i}.$$

Due to the finiteness of $\mathbb{E}[|X|]$, letting $i \to \infty$, we have $V(|X| = \infty) = 0$.

3 The Strong Law of Large Numbers under Controlled 1st Moment Condition

This section is devoted to state and prove the SLLN for sublinear expectation under controlled 1st moment condition. Before we state the main theorem, we need to prove some lemmas. The next lemma is initiated by Cozman [6].

Lemma 3.1 *If* X *satisfies* $a \le X \le b$ *and* $\mathbb{E}[X] \le 0$*, then for any* $s > 0$,

$$\mathbb{E}[\exp(sX)] \le \exp\Big(\frac{s^2(b-a)^2}{8}\Big). \tag{3.1}$$

Proof The result is trivial if $a = b$ or if $b < 0$. Now we consider the case $a \le 0 \le b$. By convexity of the exponential function, we have

$$\mathrm{e}^{sx} \le \frac{x-a}{b-a}\mathrm{e}^{sb} + \frac{b-x}{b-a}\mathrm{e}^{sa}, \quad \forall x \in [a, b].$$

Replacing x with X and taking integral on both sides of the above inequality, we have

$$\mathbb{E}[\mathrm{e}^{sX}] \le \mathbb{E}\Big[\frac{X-a}{b-a}\mathrm{e}^{sb} + \frac{b-X}{b-a}\mathrm{e}^{sa}\Big] = \mathbb{E}\Big[\frac{\mathrm{e}^{sb}-\mathrm{e}^{sa}}{b-a}X\Big] + \frac{b}{b-a}\mathrm{e}^{sa} - \frac{a}{b-a}\mathrm{e}^{sb}.$$

Since $(\mathrm{e}^{sb} - \mathrm{e}^{sa})(b-a) > 0$ and $\mathbb{E}[X] \le 0$, we have

$$\mathbb{E}[\mathrm{e}^{sX}] \le \frac{b}{b-a}\mathrm{e}^{sa} - \frac{a}{b-a}\mathrm{e}^{sb}.$$

Let $p = -\frac{a}{b-a}$ and $\varphi(z) = -pz + \log(1 - p + p\mathrm{e}^z)$. Then we can rewrite the above inequality as $\mathbb{E}[\mathrm{e}^{sX}] \le \mathrm{e}^{\varphi(s(b-a))}$.

By some ordinary calculations, we can obtain

$$\varphi'(z) = 1 - p + \frac{p-1}{1-p+p\mathrm{e}^z} \quad \text{and} \quad \varphi''(z) = \frac{(1-p)p\mathrm{e}^z}{(1-p+p\mathrm{e}^z)^2}.$$

So we have $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(z) \le \frac{1}{4}$.

Then by Taylor's theorem, we can obtain

$$\varphi(z) = \varphi(0) + z\varphi'(0) + \frac{z^2}{2}\varphi''(\xi) \le \frac{z^2}{8}, \quad \xi \in (0, z).$$

It follows that $\varphi(s(b-a)) \le \frac{s^2(b-a)^2}{8}$ and thus we obtain expression (3.1).

Lemma 3.2 *Given a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$*,* $\mathbb{E}$ *is lower-continuous and* V *is the induced capacity. Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of independent random variables satisfying* $\mathbb{E}[X_n] = \overline{\mu}$ *for each* $n \in \mathbb{N}^*$ *and* $|X_n - \overline{\mu}| \le 2n^{\frac{1}{2}-\alpha}$ *for some* $0 < \alpha < \frac{1}{2}$*. Let* $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$*. Then*

$$V\Big(\limsup_{n\to\infty} \frac{1}{n}S_n > \overline{\mu}\Big) = 0.$$

Proof By the lower-continuity of V, we only need to prove that for any $\varepsilon > 0$,

$$V\Big(\limsup_{n\to\infty}\Big\{\frac{1}{n}S_n \geq \overline{\mu}+\varepsilon\Big\}\Big) = 0.$$

By Lemma 2.2 and the independence of $\{X_n\}_{n=1}^{\infty}$, we have for any $\lambda > 0$,

$$\begin{aligned} V\Big(\frac{1}{n}S_n \geq \overline{\mu}+\varepsilon\Big) &= V\Big(\frac{S_n - n\overline{\mu}}{n} \geq \varepsilon\Big) = V(\lambda(S_n - n\overline{\mu}) \geq \lambda n\varepsilon) \\ &\leq \frac{1}{\mathrm{e}^{\lambda n\varepsilon}}\mathbb{E}[\mathrm{e}^{\lambda(S_n - n\overline{\mu})}] = \frac{1}{\mathrm{e}^{\lambda n\varepsilon}}\prod_{i=1}^{n}\mathbb{E}[\mathrm{e}^{\lambda(X_i-\overline{\mu})}]. \end{aligned}$$

By Lemma 3.1 and the fact that $-2i^{\frac{1}{2}-\alpha} \leq X_i - \overline{\mu} \leq 2i^{\frac{1}{2}-\alpha}$ and $\mathbb{E}[X_i - \overline{\mu}] = 0$ for each $i \geq 1$, we have

$$V\Big(\frac{1}{n}S_n \geq \overline{\mu}+\varepsilon\Big) \leq \frac{1}{\mathrm{e}^{\lambda n\varepsilon}}\prod_{i=1}^{n}\mathrm{e}^{\frac{\lambda^2(4i^{\frac{1}{2}-\alpha})^2}{8}} = \mathrm{e}^{2\lambda^2\sum\limits_{i=1}^{n} i^{1-2\alpha} - \lambda n\varepsilon} \leq \mathrm{e}^{2\lambda^2 n^{2-2\alpha} - \lambda n\varepsilon}.$$

Choosing $\lambda = \frac{\varepsilon}{4n^{1-2\alpha}}$, we have

$$V\Big(\frac{1}{n}S_n \geq \overline{\mu}+\varepsilon\Big) \leq \mathrm{e}^{-\frac{1}{8}\varepsilon^2 n^{2\alpha}}.$$

By noticing that $\sum\limits_{n=1}^{\infty}\mathrm{e}^{-\frac{1}{8}\varepsilon^2 n^{2\alpha}} < \infty$, we have

$$\sum_{n=1}^{\infty} V\Big(\frac{1}{n}S_n \geq \overline{\mu}+\varepsilon\Big) \leq \sum_{n=1}^{\infty}\mathrm{e}^{-\frac{1}{8}\varepsilon^2 n^{2\alpha}} < \infty.$$

By Lemma 2.1 we obtain the result.

Remark 3.1 By the method in Lemma 3.2, we can also obtain the SLLN for sublinear expectation for a sequence of uniformly bounded random variables.

The following theorem is the main result of this paper.

Theorem 3.1 *Given a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$, $\mathbb{E}$ *is lower-continuous and* V *is the induced capacity. Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of independent random variable with* $\mathbb{E}[X_n] = \overline{\mu}$ *and* $\mathcal{E}[X_n] = \underline{\mu}$ *for each* $n \in \mathbb{N}^*$. *Suppose that there is a random variable* X *satisfying* $|X_n| \leq |X|$ *q.s. for each* $n \in \mathbb{N}^*$ *and* $\lim\limits_{n\to\infty}\mathbb{E}[|X|I(|X| > n)] = 0$. *Let* $S_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$. *Then*

$$V\Big(\Big\{\liminf_{n\to\infty}\frac{1}{n}S_n < \underline{\mu}\Big\}\bigcup\Big\{\limsup_{n\to\infty}\frac{1}{n}S_n > \overline{\mu}\Big\}\Big) = 0$$

and

$$v\Big(\underline{\mu} \leq \liminf_{n\to\infty}\frac{1}{n}S_n \leq \limsup_{n\to\infty}\frac{1}{n}S_n \leq \overline{\mu}\Big) = 1.$$

Proof By the monotonicity and sub-additivity of V, we only need to prove

$$V\Big(\limsup_{n\to\infty}\frac{1}{n}S_n > \overline{\mu}\Big) = 0 \quad \text{and} \quad V\Big(\liminf_{n\to\infty}\frac{1}{n}S_n < \underline{\mu}\Big) = 0.$$

For some $0 < \alpha < \frac{1}{2}$, define $f_n(x) = (-n^{\frac{1}{2}-\alpha}) \vee (x \wedge n^{\frac{1}{2}-\alpha})$ and $\widehat{f}_n(x) = x - f_n(x)$. Then $f_n(\cdot), \widehat{f}_n(\cdot) \in C_{l,\mathrm{Lip}}(\mathbb{R})$. Let

$$Y_n = f_n(X_n - \overline{\mu}) - \mathbb{E}[f_n(X_n - \overline{\mu})] + \overline{\mu}$$

and $\overline{S_n} = \sum\limits_{i=1}^{n} Y_i$. Then Y_n, $n = 1, 2 \cdots$ are independent, $|Y_n - \overline{\mu}| \le 2n^{\frac{1}{2}-\alpha}$ and $\mathbb{E}[Y_n] = \overline{\mu}$. Then by Lemma 3.2, we have

$$V\Big(\limsup_{n\to\infty} \frac{1}{n}\overline{S_n} > \overline{\mu}\Big) = 0.$$

In addition we have

$$X_n = Y_n + \widehat{f}_n(X_n - \overline{\mu}) + \mathbb{E}[f_n(X_n - \overline{\mu})].$$

It follows

$$\frac{1}{n}S_n = \frac{1}{n}\overline{S_n} + \frac{1}{n}\sum_{i=1}^{n} \widehat{f}_i(X_i - \overline{\mu}) + \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[f_i(X_i - \overline{\mu})].$$

By the subadditivity and translation invariance of $\mathbb{E}$, we have

$$\begin{aligned}\mathbb{E}[f_i(X_i - \overline{\mu})] &= \mathbb{E}[X_i - \overline{\mu} - \widehat{f}_i(X_i - \overline{\mu})] \\ &\le \mathbb{E}[(X_i - \overline{\mu})] + \mathbb{E}[-\widehat{f}_i(X_i - \overline{\mu})] \\ &\le \mathbb{E}[|\widehat{f}_i(X_i - \overline{\mu})|].\end{aligned}$$

Therefore

$$\frac{1}{n}S_n \le \frac{1}{n}\overline{S_n} + \frac{1}{n}\sum_{i=1}^{n} |\widehat{f}_i(X_i - \overline{\mu})| + \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[|\widehat{f}_i(X_i - \overline{\mu})|]. \tag{3.2}$$

By $|X_n| \le |X|$ q.s., $\lim\limits_{n\to\infty} \mathbb{E}[|X|I(|X| > n)] = 0$ and Proposition 2.2(1), we have

$$\begin{aligned}\mathbb{E}[|\widehat{f}_n(X_n - \overline{\mu})|] &\le \mathbb{E}[|X_n - \overline{\mu}|I(|X_n - \overline{\mu}| > n^{\frac{1}{2}-\alpha})] \\ &\le \mathbb{E}[(|X_n| + |\overline{\mu}|)I(|X_n| + |\overline{\mu}| > n^{\frac{1}{2}-\alpha})] \\ &\le \mathbb{E}[(|X_n| + |\overline{\mu}|)I(|X_n| + |\overline{\mu}| > n^{\frac{1}{2}-\alpha})I(|X_n| \le |X|)] \\ &\quad + \mathbb{E}[(|X_n| + |\overline{\mu}|)I(|X_n| + |\overline{\mu}| > n^{\frac{1}{2}-\alpha})I(|X_n| > |X|)] \\ &\le \mathbb{E}[|X|I(|X| > n^{\frac{1}{2}-\alpha} - |\overline{\mu}|)] + |\overline{\mu}|\mathbb{E}[I(|X| > n^{\frac{1}{2}-\alpha} - |\overline{\mu}|)] \\ &\quad + \sup_{\theta\in\Theta} \int_{\{|X_n| > |X|\}} (|X_n| + |\overline{\mu}|)I(|X_n| + |\overline{\mu}| > n^{\frac{1}{2}-\alpha})\mathrm{d}P_\theta \\ &= \mathbb{E}[|X|I(|X| > n^{\frac{1}{2}-\alpha} - |\overline{\mu}|)] + |\overline{\mu}|V(|X| > n^{\frac{1}{2}-\alpha} - |\overline{\mu}|) \\ &\to 0.\end{aligned}$$

So we have

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[|\widehat{f}_i(X_i - \overline{\mu})|] \to \infty \quad \text{when } n \to \infty. \tag{3.3}$$

Furthermore by Proposition 2.1, Lemma 2.2 and Lemma 2.4, we have

$$
\begin{aligned}
& V\Big(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\widehat{f}_i(X_i-\overline{\mu})|>0\Big) \\
\leq\ & V\Big(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|X_i-\overline{\mu}|I(|X_i-\overline{\mu}|>i^{\frac{1}{2}-\alpha})>0\Big) \\
\leq\ & V\Big(\limsup_{n\to\infty}|X_n-\overline{\mu}|I(|X_n-\overline{\mu}|>n^{\frac{1}{2}-\alpha})>0\Big) \\
\leq\ & V\Big(\limsup_{n\to\infty}\frac{|X_n-\overline{\mu}|}{n^{\frac{1}{2}-\alpha}}\geq 1\Big) \\
\leq\ & V\Big(\Big\{\limsup_{n\to\infty}\frac{|X_n|}{n^{\frac{1}{2}-\alpha}}\geq 1\Big\}\bigcap\Big(\bigcap_{n=1}^{\infty}\{|X_n|\leq|X|\}\Big)\bigcap\{|X|<\infty\}\Big) \\
& +\sum_{n=1}^{\infty}V(|X_n|>|X|)+V(|X|=\infty) \\
=\ & V\Big(\Big\{\limsup_{n\to\infty}\frac{|X_n|}{n^{\frac{1}{2}-\alpha}}\geq 1\Big\}\bigcap\Big(\bigcap_{n=1}^{\infty}\{|X_n|\leq|X|\}\Big)\bigcap\{|X|<\infty\}\Big) \\
\leq\ & V\Big(\Big\{\limsup_{n\to\infty}\frac{|X|}{n^{\frac{1}{2}-\alpha}}\geq 1\Big\}\bigcap\{|X|<\infty\}\Big).
\end{aligned}
$$

So we have

$$
V\Big(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\widehat{f}_i(X_i-\overline{\mu})|>0\Big)=0. \tag{3.4}
$$

By (3.3), taking $\limsup_{n}$ on both sides of (3.2), we have

$$
\limsup_{n\to\infty}\frac{1}{n}S_n\leq\limsup_{n\to\infty}\frac{1}{n}\overline{S_n}+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\widehat{f}_i(X_i-\overline{\mu})|.
$$

Combining this with (3.4), we have

$$
\begin{aligned}
& V\Big(\limsup_{n\to\infty}\frac{1}{n}S_n>\overline{\mu}\Big) \\
\leq\ & V\Big(\limsup_{n\to\infty}\frac{1}{n}\overline{S_n}+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\widehat{f}_i(X_i-\overline{\mu})|>\overline{\mu}\Big) \\
\leq\ & V\Big(\limsup_{n\to\infty}\frac{1}{n}\overline{S_n}>\overline{\mu}\Big)+V\Big(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\widehat{f}_i(X_i-\overline{\mu})|>0\Big) \\
=\ & 0.
\end{aligned}
$$

Similarly, considering the sequence $\{-X_n\}_{n=1}^{\infty}$ with $\mathbb{E}[-X_n]=-\underline{\mu}$, we have the following e-quality

$$
V\Big(\limsup_{n\to\infty}\frac{1}{n}(-S_n)>-\underline{\mu}\Big)=0.
$$

This is equivalent to

$$
V\Big(\liminf_{n\to\infty}\frac{1}{n}S_n<\underline{\mu}\Big)=0.
$$

Remark 3.2 If $\mathbb{E}$ coincides with the classical expectation, i.e., $V = v = P$ and $\overline{\mu} = \underline{\mu} = E_P[X_1]$, where P is the classical probability, our SLLN reduces to the classical SLLN

$$P\Big(\lim_{n\to\infty}\frac{S_n}{n} = E_P[X_1]\Big) = 1.$$

Corollary 3.1 *Given a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$*,* $\mathbb{E}$ *is lower-continuous and* V *is the induced capacity. Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of independent random variables with* $\mathbb{E}[X_n] = \overline{\mu_n}$ *and* $\mathcal{E}[X_n] = \underline{\mu_n}$ *for each* $n \in \mathbb{N}^*$*. Suppose that there is a random variable* X *satisfying* $|X_n| \le |X|$ *q.s. for any* $n \in \mathbb{N}^*$ *and*

$$\lim_{n\to\infty}\mathbb{E}[|X|I(|X| > n)] = 0.$$

Then

$$V\Big(\Big\{\liminf_{n\to\infty}\frac{1}{n}S_n < \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\underline{\mu_i}\Big\}\bigcup\Big\{\limsup_{n\to\infty}\frac{1}{n}S_n > \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\overline{\mu_i}\Big\}\Big) = 0$$

and

$$v\Big(\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\underline{\mu_i} \le \liminf_{n\to\infty}\frac{1}{n}S_n \le \limsup_{n\to\infty}\frac{1}{n}S_n \le \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\overline{\mu_i}\Big) = 1.$$

Proof Take $Y_n = X_n - \overline{\mu_n}$. Then Y_n satisfies the conditions of Theorem 3.1 with $\mathbb{E}[Y_n] = 0$. Then we have

$$V\Big(\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}Y_i > 0\Big) = 0.$$

This implies

$$V\Big(\limsup_{n\to\infty}\frac{1}{n}S_n > \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\overline{\mu_i}\Big) = 0.$$

Similarly taking

$$Z_n = \underline{\mu_n} - X_n,$$

we can also obtain

$$V\Big(\liminf_{n\to\infty}\frac{1}{n}S_n < \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\underline{\mu_i}\Big) = 0.$$

4 The Strong Law of Large Numbers under Uniform 1st Moment Condition

In this section we discuss whether the SLLN is valid under conditions of the finiteness of uniform 1st moments for sublinear expectation.

Zhang [16] proved a SLLN for negatively dependent identically distributed random variables under conditions of finite 1st moment for Choquet expectation. By Lemma 2.3(1) and (3) and Theorem 3.1 in Zhang [16], we can obtain the following theorem.

Theorem 4.1 *Given a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$*,* $\mathbb{E}$ *is lower-continuous and* V *is the induced capacity. Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of i.i.d random variable with* $\mathbb{E}[X_1] = \overline{\mu}$ *and* $\mathcal{E}[X_1] = \underline{\mu}$*.*

(1) *Suppose that* $C_V[|X_1|]<\infty$. *Then*

$$V\Big(\Big\{\liminf_{n\to\infty}\frac{1}{n}S_n<\underline{\mu}\Big\}\bigcup\Big\{\limsup_{n\to\infty}\frac{1}{n}S_n>\overline{\mu}\Big\}\Big)=0.$$

(2) *Suppose that* V *is continuous. If*

$$V\Big(\limsup_{n\to\infty}\frac{|S_n|}{n}=\infty\Big)<1,$$

then $C_V[|X_1|]<\infty$.

By Lemma 2.3(1) and (2), there holds that $C_V[|X|]<\infty$ implies $\mathbb{E}[|X|]<\infty$. But the inverse result may not always be true. In some special cases, the finiteness of sublinear expectation can deduce the finiteness of Choquet expectation. For instance, if $\mathcal{P}$ only contains finite elements in it, we define a sublinear expectation

$$\mathbb{E}[X]=\sup_{P\in\mathcal{P}}E_P[X].$$

Then $\mathbb{E}[|X|]<\infty$ implies $C_V[|X|]<\infty$. But in some cases it is not true. We find out that if the the finiteness of sublinear expectation can deduce the finiteness of Choquet expectation, then by Theorem 4.1(1), one can obtain the SLLN. Otherwise if the sublinear expectation is finite but the Choquet expectation is infinite, then by Theorem 4.1(2), the SLLN is not valid.

Next we give an example to reveal that there does exist a sequence of random variables satisfying the conditions of Theorem 4.1 (2) for some certain constructed sublinear expectation such that the SLLN is not true.

Example 4.1 Let

$$\Omega_i=\{a_0,a_1,\cdots,a_n,\cdots\},\quad i=1,2,\cdots$$

be a family of full spaces, $\mathcal{F}_i$ be a family of sets each one of which contains all subsets of Ω_i and

$$\mathcal{P}_i=\{P_1,P_2,\cdots,P_n,\cdots\},\quad i=1,2,\cdots$$

be countable families of countable probability measures, where $P_1,P_2,\cdots,P_n,\cdots$ defined on each Ω_i by

$$P_i(a_j)=\begin{cases}1-\dfrac{1}{i}, & \text{if } j=0,\\ \dfrac{1}{i}, & \text{if } j=i,\\ 0, & \text{if } j\neq 0,i.\end{cases}$$

Define the sequence $\{X_n\}_{n=1}^\infty$ of random variables on each $(\Omega_i,\mathcal{F}_i)$, $i=1,2,\cdots$ by

$$X_n(a_j)=j+\frac{1}{2},\quad j=1,2,\cdots$$

for any $n\in\mathbb{N}^*$. Then $\{X_n\}_{n=1}^\infty$ satisfies that for any $i,j\geq 1$,

$$P_i(X_n>j)=\begin{cases}0, & \text{if } j>i,\\ \dfrac{1}{i}, & \text{if } j\leq i,\end{cases}\quad n=1,2,\cdots.$$

Define the full space

$$\Omega = \prod_{i=1}^{\infty} \Omega_i = \Omega_1 \times \Omega_2 \times \cdots .$$

Define the product σ-algebra on Ω

$$\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots .$$

Define the set $\mathcal{P}$ of probabilities on measure space $(\Omega, \mathcal{F})$ by

$$\mathcal{P} = \prod_{i=1}^{\infty} \mathcal{P}_i = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots = \{P_{i_1} \times P_{i_2} \times \cdots : i_j \in \{1, 2, \cdots\}\}.$$

We consider the sublinear expectation defined by upper expectation

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

Choose a sequence $\{Y_n\}_{n=1}^{\infty}$ of random variables on $(\Omega, \mathcal{F})$ defined by

$$Y_n(\omega) = Y_n(\omega_1, \omega_2, \cdots) = X_n(\omega_n).$$

It is easy to check that $\{Y_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables under sublinear expectation $\mathbb{E}$ and the induced capacity V is continuous.

Then we have

$$\begin{aligned}
\mathbb{E}[Y_1] &= \sup_{P \in \mathcal{P}} E_P[Y_1] = \sup_{P \in \mathcal{P}} \sum_{\omega} Y_1(\omega) P(\omega) \\
&= \sup_{P \in \mathcal{P}_1} \sum_{\omega_1} X_1(\omega_1) P(\omega_1) = \sup_{P \in \mathcal{P}_1} E_P[X_1] \\
&= \sup_{n \geq 1} \Big(\frac{1}{2} \Big(1 - \frac{1}{n} \Big) + \Big(n + \frac{1}{2} \Big) \frac{1}{n} \Big) \\
&= 1.5 < \infty,
\end{aligned}$$

But

$$\begin{aligned}
C_V[Y_1] &\geq \sum_{i=1}^{\infty} V(Y_1 > i) = \sum_{i=1}^{\infty} \sup_{P \in \mathcal{P}_1} P(X_1 > i) \\
&\geq \sum_{i=1}^{\infty} P_{i+1}(X_1 > i) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,
\end{aligned}$$

then by Theorem 4.1(2) the SLLN is not valid in this example.

References

[1] Agahi, H., Mohammadpour, A., Mesiar, R. and Ouyang, Y., On a strong law of large numbers for monotone measures, *Statis. Probab. Lett.*, **83**(4), 2013, 1213–1218.

[2] Chen, X., Strong law of large numbers under an upper probability, *Appl. Math.*, **3**, 2012, 2056–2062.

[3] Chen, Z. and Epstein, L., Ambiguity, risk, and asset returns in continuous time, *Econometrica*, **70**(4), 2002, 1403–1443.

[4] Chen, Z., Wu, P. and Li, B., A strong law of large numbers for non-additive probabilities, *Int. J. Approx. Reason.*, **54**(3), 2013, 365–377.

[5] Choquet, G., Theory of capacities, *Ann. Inst. Fourier*, **5**, 1953, 131–295.

[6] Cozman, F. G., Concentration inequalities and laws of large numbers under epistemic and regular irrelevance, *Int. J. Approx. Reason.*, **51**(9), 2010, 1069–1084.

[7] Huber, P. and Strassen, V., Minimax tests and the Neyman-Pearson lemma for capacities, *Ann. Stat.*, **1**(2), 1973, 251–263.

[8] Li, W. and Chen, Z., Laws of large numbers of negatively correlated random variables for capacities, *Acta Math. Appl. Sin. Engl. Ser.*, **27**(4), 2011, 749–760.

[9] Maccheroni, F. and Marinacci, M., A strong law of large number for capacities, *Ann. Probab.*, **33**(3), 2005, 1171–1178.

[10] Marinacci, M., Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econ. Theory*, **84**(2), 1999, 145–195.

[11] Peng, S., Backward SDE and related g-expectation, Backward Stochastic Differential Equations, *Pitman Research Notes in Math. Series*, **364**, 1997, 141–160.

[12] Peng, S., G-expectation, G-Brownian motion and related stochastic calculus of Ito type, Proceedings of the 2005 Abel Symposium, Springer-Verlag, Berlin, Heidelberg, 2007, 541–567.

[13] Peng, S., Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, *Stoch. Proc. Appl.*, **118**(12), 2008, 2223–2253.

[14] Peng, S., Nonlinear expectations and stochastic calculus under uncertainty, 2010, arXiv:1002.4546v1 [math.PR].

[15] Wasserman, L. and Kadane, J., Bayes's theorem for Choquet capacities, *Ann. Stat.*, **18**(3), 1990, 1328–1339.

[16] Zhang, L., Rosenthal's inequalities for independent and negatively dependent random variables under sub linear expectation with applications, *Sci. China Math.*, **59**(4), 2016, 751–768.