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Stability of the Equilibrium to the Boltzmann Equation with Large Potential Force*

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Abstract The Boltzmann equation with external potential force exists a unique equilibrium—local Maxwellian. The author constructs the nonlinear stability of the equilibrium when the initial datum is a small perturbation of the local Maxwellian in the whole space $\mathbb{R}^3$. Compared with the previous result [Ukai, S., Yang, T. and Zhao, H.-J., Global solutions to the Boltzmann equation with external forces, *Anal. Appl.* (*Singap.*), **3**, 2005, 157–193], no smallness condition on the Sobolev norm H^1 of the potential is needed in our arguments. The proof is based on the entropy-energy inequality and the $L^2 - L^\infty$ estimates.

Keywords Boltzmann equation, Large potential force, Stability, Entropy-energy inequality, $L^2 - L^\infty$ method
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1 Introduction and Formulation

The time evolution of rarefied gas in an external field can be described by the classical Boltzmann equation with additional force term

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \Phi \cdot \nabla_v F = Q(F, F), \tag{1.1}$$

where $F = F(t, x, v)$ is a function describing the distribution of particles at the time $t \geq 0$, at the position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and with the velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The potential $\Phi = \Phi(x)$ is independent of the time t. The collision between particles is given by the standard Boltzmann collision operator $Q(F, G)$ with hard potential interactions:

$$Q(F, G) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |v - u|^\gamma [F(v')G(u') - F(v)G(u)] B(\theta) \mathrm{d}u \mathrm{d}\omega,$$

where $0 \leq \gamma \leq 1,\ \omega \in \mathbb{S}^2 = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$, the function $B(\theta)$ satisfies the Grad's angular cut-off assumption, i.e., $0 < B(\theta) \leq C|\cos\theta|$ with $\cos\theta = \frac{[(u-v)\cdot\omega]}{|u-v|}$, and u', v' related to u, v by the usual elastic collision relations

$$v' = v - [(v - u) \cdot \omega]\omega, \quad u' = u + [(v - u) \cdot \omega]\omega.$$

It is easy to check that the local Maxwellian

$$M(x, v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\Big\{ -\Phi(x) - \frac{|v|^2}{2} \Big\}$$

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is the unique stationary state to the Boltzmann equation (1.1). In [10–11], Ukai, Yang and Zhao studied the Cauchy problem of (1.1) with $\gamma = 1$ and obtained global existence and time decay rate of classical solutions near the equilibrium M. Later, in [2] they improved their previous results and obtained optimal decay rate of classical solutions. See also [9, 12] for the corresponding results for the hard potential case. Lei [8] studied the non-cutoff case of (1.1). Duan [1] studied the equation (1.1) in the torus of $\mathbb{R}^3$ and obtained the stability of the stationary state. Notice that the assumption that the Sobolev norm (e.g. H^4) of the external potential Φ is sufficiently small plays a crucial role in all the articles mentioned above and the methods developed there cannot be applied to the case when the Sobolev norm of Φ is large. Recently, Kim [7] studied the equation (1.1) with a large amplitude external potential in a periodic box of $\mathbb{R}^3$ and obtained the stability of the local Maxwellian M. It should be pointed that the periodic assumption plays a crucial role in the arguments of [7] and cannot be applied to the whole space case.

The goal of this paper is to construct the classical solutions for (1.1) near the equilibrium M with large amplitude Φ on H^1 in the whole space $\mathbb{R}^3$.

As usual, we introduce the standard perturbation $f(t,x,v)$ to M as

$$F(t,x,v) = M + \sqrt{M} f(t,x,v).$$

Denote $\mu(v) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\big\{-\frac{|v|^2}{2}\big\}$, then $M = \mu(v)\mathrm{e}^{-\Phi}$. A direct computation implies that $f(t,x,v)$ satisfies

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f + \mathrm{e}^{-\Phi} L f = \mathrm{e}^{-\frac{\Phi}{2}}\Gamma(f,f). \tag{1.2}$$

Here the nonlinear collision operator $\Gamma(g_1,g_2)$ takes the form

$$\begin{aligned}\Gamma(g_1,g_2) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g_1,\sqrt{\mu}g_2)\\ &= \iint_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\gamma\sqrt{\mu}(u) g_1(u')g_2(v')B(\theta)\mathrm{d}u\mathrm{d}\omega\\ &\quad - \Big[\iint_{\mathbb{R}^3\times\mathbb{S}^2} |(v-u)|^\gamma\sqrt{\mu}(u)g_1(u)B(\theta)\mathrm{d}u\mathrm{d}\omega\Big] g_2(v)\end{aligned}$$

and the linearized collision operator reads

$$Lf \equiv -\frac{1}{\sqrt{\mu}}[Q(\mu,\sqrt{\mu}f) + Q(\sqrt{\mu}f,\mu)] \equiv \nu(v)f - Kf.$$

It is well-known that K is a self-adjoint compact operator on $L^2(\mathbb{R}^3_v)$ and $\nu(v)$ is given by

$$\nu(v) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} |v-u|^\gamma \mu(u)B(\theta)\mathrm{d}u\mathrm{d}\omega = C\int_{\mathbb{R}^3} |v-u|^\gamma\mu(u)\mathrm{d}u$$

for some constant $C > 0$. Also, there exist positive constants ν_0, $\widetilde{C}_1$ and $\widetilde{C}_2$ such that

$$\nu_0 \le \widetilde{C}_1(1+|v|)^\gamma \le \nu(v) \le \widetilde{C}_2(1+|v|)^\gamma. \tag{1.3}$$

It is straightforward to verify that the number of particles and the sum of potential and kinetic energy are conserved under the evolution (1.1), thus we define the perturbation of the mass and the total energy as

$$\mathcal{M}(f(t)) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f(t,x,v) - M(x,v))\mathrm{d}v\mathrm{d}x, \tag{1.4}$$

$$\mathcal{E}(f(t)) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi(x)\Big)(F(t,x,v) - M(x,v))\mathrm{d}v\mathrm{d}x. \tag{1.5}$$

Moreover, by standard arguments it follows that the H-function of the perturbation f,

$$\mathcal{H}(f(t)) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} [F(t,x,v)\ln F(t,x,v) - M(x,v)\ln M(x,v)]\mathrm{d}v\mathrm{d}x, \tag{1.6}$$

does not increase in the evolution (1.1).

Noticing that the H-function $\mathcal{H}(f)$ does not increase during the evolution (1.1) and the energy $\mathcal{E}(f)$ and the total masses $\mathcal{M}(f)$ are constants, we define the following non-increasing entropy-energy functional:

$$\mathcal{G}(f(t)) = \mathcal{H}(f(t)) + \mathcal{E}(f(t)) - \mathcal{M}(f(t))\Big(1 - \frac{3}{2}\ln(2\pi)\Big), \tag{1.7}$$

which plays a crucial role in the study of stability of the equilibrium.

Throughout this paper the letters C and C_i denote generic constants and may change from line to line. Denote by $\nabla_{x,v}$ the couple (∂_x, ∂_v). Our main result is the following theorem.

Theorem 1.1 *Let $w(v) = (\Lambda + |v|^2)^\beta$ for $\Lambda > 0$ and $\beta > \frac{3}{2}$. Assume that the potential $\Phi(x)$ satisfies $\partial_x^k\Phi \in L^\infty(\mathbb{R}^3)$ for $k = 1, 2$, $|\partial_x^2\Phi|_{L^\infty(\mathbb{R}^3)}$ is sufficiently small, and*

$$-\overline{\vartheta} \le \Phi(x) \le -\vartheta, \quad x \in \mathbb{R}^3 \tag{1.8}$$

for two constants ϑ and $\overline{\vartheta}$. Assume further that there exist constants $\delta > 0$ and $\Lambda > 0$ such that the initial datum $F(0) = M + \sqrt{M}f(0)$ satisfies

$$\|wf(0)\|_{L^\infty} + \sqrt{\mathcal{G}(f(0))} < \delta, \quad \|\nabla_{x,v}f(0)\|_{L^2} < +\infty.$$

Then the initial value problem for (1.2) *enjoys a unique global in time solution satisfying*

$$\sup_{0\le t\le\infty} \|wf(t)\|_{L^\infty} \le C_1[\|wf(0)\|_{L^\infty} + \sqrt{\mathcal{G}(f(0))}], \tag{1.9}$$

$$\|\nabla_{x,v}f(t)\|_{L^2} \le \mathrm{e}^{C_2 t}\|\nabla_{x,v}f(0)\|_{L^2}. \tag{1.10}$$

Remark 1.1 It turns out that $\mathcal{G}(f(0)) > 0$ if $\|wf(0)\|_{L^\infty}$ is sufficiently small, see Lemma 2.1 below.

Remark 1.2 The two constants ϑ and $\overline{\vartheta}$ in Theorem 1.1 are not necessary positive.

As pointed out in [3], due to the presence of a large amplitude potential Φ, we lose the control of the Sobolev estimate in higher order energy norms to the perturbation f. The proof of Theorem 1.1 is based on some ideas developed recently by Esposito, Guo and Marra [3] in studying the nonlinear stability of the phase state to the Vlasov-Boltzmann system of binary mixture. The strategy is to make a crucial use of the fundamental entropy-energy $\mathcal{G}(f)$ estimate to obtain a mixed $L^1 - L^2$ type of stability estimate and then bootstrap such a $L^1 - L^2$ stability to a L^∞ estimate to obtain pointwise stability estimate by following the curved trajectory induced by the force field.

The paper is organized as follows. In Section 2 we use the energy-entropy (1.7) to derive a mixed $L^1 - L^2$ estimate and state some results on the characteristics curves for the equation (1.1). In Section 3 we establish the nonlinear stability of the equilibrium in weighted L^∞ norm.

2 Entropy-Energy Estimate and Characteristics

In this section we first use the conservations of total energy and mass, and the entropy inequality to obtain a priori estimates on the deviation of the solution from the equilibrium.

Lemma 2.1 *There exist $\kappa > 0$ and $C_\kappa > 0$ such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big\{\frac{(F(t)-M)^2}{M}\mathbf{1}_{\{|F(t)-M|\le\kappa M\}} + |F(t)-M|\mathbf{1}_{\{|F(t)-M|\ge\kappa M\}}\Big\}\mathrm{d}v\mathrm{d}x \le \frac{1}{C_\kappa}\mathcal{G}(f(0)).$$

Proof The proof is similar to that of Lemma 2.2 in [3], so we present it here for completeness. First, we can construct solutions (see [4]) such that

$$\mathcal{G}(f(t)) \le \mathcal{G}(f(0)).$$

We expand $\mathcal{G}(f)$ at the equilibrium M and use (1.5) to cancel the linear part of the expansion, which takes the form

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3}\Big(1-\frac{3}{2}\ln(2\pi)\Big)(F(t)-M)\mathrm{d}v\mathrm{d}x\\ &+\iint_{\mathbb{R}^3\times\mathbb{R}^3}\Big\{\frac{|v|^2}{2}(F(t)-M)+(\ln M+1)(F(t)-M)+\Phi(F(t)-M)\Big\}\mathrm{d}v\mathrm{d}x.\end{aligned}$$

Indeed, since $\ln M = -\frac{3}{2}\ln(2\pi)-\Phi-\frac{|v|^2}{2}$, the above quantity is zero by construction. Therefore, we turn to the second order expansion of $\mathcal{G}(f)$. For some $\widetilde{F}$ between M and $F(t)$, we have

$$\mathcal{G}(f) = \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{2\widetilde{F}}\mathrm{d}v\mathrm{d}x.$$

For some small number $0<\kappa<1$, we introduce the indicator functions $\chi^< = \mathbf{1}_{\{|F(t)-M|\le\kappa M\}}$ and $\chi^> = \mathbf{1}_{\{|F(t)-M|>\kappa M\}}$ and split the integral into

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{2\widetilde{F}}\chi^<\mathrm{d}v\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{2\widetilde{F}}\chi^>\mathrm{d}v\mathrm{d}x.$$

We first estimate $\frac{(F(t)-M)^2}{2\widetilde{F}}$ with $|F(t)-M|>\kappa M$. Notice that $|F(t)-M|>\kappa M$ implies either $F(t)\ge(1+\kappa)M$ or $F(t)\le(1-\kappa)M$. If $F(t)\ge(1+\kappa)M$, then $\widetilde{F}(t)\le F(t)$, thus we have

$$\frac{|F(t)-M|}{\widetilde{F}(t)} \ge \frac{|F(t)-M|}{F(t)} = 1-\frac{M}{F(t)} \ge 1-\frac{1}{1+\kappa} = \frac{\kappa}{1+\kappa}.$$

In the case of $F(t)\le(1-\kappa)M$, we have $\widetilde{F}(t)\le M$, thus

$$\frac{|F(t)(t)-M|}{\widetilde{F}(t)} \ge \frac{|F(t)-M|}{M} = 1-\frac{F(t)}{M} \ge 1-(1-\kappa) > \frac{\kappa}{1+\kappa}.$$

Combining these two cases and noticing $\widetilde{F}\le(1+\kappa)M$ for $|F(t)-M|\le\kappa M$, we conclude

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{2\widetilde{F}}\chi^<\mathrm{d}v\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{2\widetilde{F}}\chi^>\mathrm{d}v\mathrm{d}x\\ &\ge \frac{1}{2(1+\kappa)}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(F(t)-M)^2}{M}\chi^<\mathrm{d}v\mathrm{d}x + \frac{\kappa}{2(1+\kappa)}\int_{\mathbb{R}}\mathrm{d}x\int_{\mathbb{R}^3}|F(t)-M|\chi^>\mathrm{d}v\mathrm{d}x.\end{aligned}$$

In order to study the curved trajectory to the Boltzmann equation (1.1), we define the characteristics curves $[X(s;t,x,v),V(s;t,x,v)]$ for (1.1) passing through (t,x,v) at $s=t$, such that

$$\frac{\mathrm{d}X(s;t,x,v)}{\mathrm{d}s}=V(s;t,x,v),\quad X(t;t,x,v)=x; \tag{2.1}$$

$$\frac{\mathrm{d}V(s;t,x,v)}{\mathrm{d}s}=-\nabla_x\Phi\left(X(s;t,x,v)\right),\quad V(t;t,x,v)=v. \tag{2.2}$$

Lemma 2.2 *Fix $N>0$. Let $|v|\le N$. Then there exists $T_1>0$ and $0\le s<t\le T_1$ such that*

$$\frac{(t-s)^3}{2}\le\Big|\det\Big(\frac{\partial X(s;t,x,v)}{\partial v}\Big)\Big|\le 2(t-s)^3. \tag{2.3}$$

Proof Multiplying (2.2) by $V(s;t,x,v)$ and noticing (2.1), we obtain the conservation of particle energy

$$\frac{1}{2}|V(s;t,x,v)|^2+\Phi(X(s;t,x,v))=\frac{1}{2}|v|^2+\Phi(x). \tag{2.4}$$

For given $T_1>0$ and fixed $N>0$, noticing $-\overline{\vartheta}\le\Phi(x)\le-\vartheta$, we obtain from (2.4) and (2.1) that

$$|V(s;t,x,v)|\le|v|+2\sqrt{\|\Phi\|_{L^\infty}}\le N+\max\{|\vartheta|,|\overline{\vartheta}|\},$$
$$|X(s;t,x,v)-x|\le T_1(N+\max\{|\vartheta|,|\overline{\vartheta}|\}).$$

From (2.1)–(2.2), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\partial X(s;t,x,v)}{\partial v}=-\partial_x\nabla_x\Phi(X(s;t,x,v))\frac{\partial X(s;t,x,v)}{\partial v} \tag{2.5}$$

and we deduce that for $0\le s<t\le T_1$ and $|v|<N$,

$$\Big\|\frac{\partial X(s;t,x,v)}{\partial v}\Big\|_{L^\infty}\le C_{T_1}. \tag{2.6}$$

The Taylor expansion for $X(s;t,x,v)=(X_1(s;t,x,v),X_2(s;t,x,v),X_3(s;t,x,v))$ around t reads

$$\begin{aligned}\frac{\partial X_i(s;t,x,v)}{\partial v_j}&=\frac{\partial X_i(t;t,x,v)}{\partial v_j}+(s-t)\frac{\mathrm{d}}{\mathrm{d}s}\Big\{\frac{\partial X_i(s;t,x,v)}{\partial v_j}\Big\}\Big|_{s=t}\\&\quad+\frac{(s-t)^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{X_i(\overline{s}_{ij};t,x,v)}{\partial v_j}\\&=(s-t)+\frac{(s-t)^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\partial X_i(\overline{s}_{ij};t,x,v)}{\partial v_j}\end{aligned}$$

for some $\overline{s}_{ij}$ between s and t, $1\le i,j\le 3$. Hence the Jacobian matrix $\big(\frac{\partial X(s;t,x,v)}{\partial v}\big)$ is given by

$$\Big(\frac{\partial X(s;t,x,v)}{\partial v}\Big)=(s-t)\Big\{\mathbb{I}_{3\times3}+\frac{s-t}{2}\Big(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\partial X_i(\overline{s}_{ij};t,x,v)}{\partial v_j}\Big)\Big\},$$

where $\mathbb{I}_{3\times3}$ is the unit matrix. Noticing the fact that $|\partial_x\nabla_x\Phi(y)|_{L^\infty(\mathbb{R}^3)}$ is sufficiently small and using again (2.5) with $s=\overline{s}_{ij}$ and (2.6), we have

$$\Big|\frac{s-t}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{\partial X_i(\overline{s}_{ij};t,x,v)}{\partial v_j}\Big|\le|(t-s)C_{T_1}|\partial_x\nabla_x\Phi(y)|_{L^\infty}|\le\frac{1}{8}$$

for some suitable $T_1>0$. Therefore, the estimate (2.3) holds.

3 Weighted L^∞ Stability

In this section we shall use the entropy-energy inequality and the estimates on the characteristics to show the stability of the equilibrium. In fact, we obtain that the perturbation f is arbitrarily small at any positive time in a suitable weighted L^∞ norm provided that it is initially sufficiently small. We use the weight function $w(v) = (\Lambda + |v|^2)^\beta$ with $\beta > \frac{3}{2}$ and Λ a positive constant to be chosen later.

Lemma 3.1 *Let $h = wf$. Under the assumptions of Theorem* 1.1, *there exist* $0 < T_0 < T_1$, $\delta > 0$, $0 < \Pi < 1$, *and* $C_{T_0} > 0$ *such that, if* $\|h\|_{L^\infty} < \delta$, *then*

$$\|h(T_0)\|_{L^\infty} \le \Pi \|h(0)\|_{L^\infty} + C_{T_0}\sqrt{\mathcal{G}(f(0))}.$$

Proof We first write the equation for $h = wf$ from (1.2) as

$$\begin{aligned}&\partial_t h + v\cdot\nabla_x h - \nabla_x\Phi\cdot\nabla_v h + \mathrm{e}^{-\Phi}\nu(v)h\\ &= \mathrm{e}^{-\Phi}K_w(h) + \nabla_x\Phi\cdot\nabla_v w\frac{h}{w} + \mathrm{e}^{-\frac{\Phi}{2}}w\Gamma\Big(\frac{h}{w},\frac{h}{w}\Big),\end{aligned}\tag{3.1}$$

where $K_w(\cdot) = wK\big(\frac{\cdot}{w}\big)$. Denote $[X(s),V(s)] \equiv [X(s;t,x,v),V(s;t,x,v)]$. Noticing that

$$\frac{\mathrm{d}}{\mathrm{d}s}h(s,X(s;t,x,v),V(s;t,x,v)) = \partial_t h + \nabla_x h\cdot\frac{\mathrm{d}X}{\mathrm{d}s} + \nabla_v h\cdot\frac{\mathrm{d}V}{\mathrm{d}s},$$

the solution to the following transport equation

$$\partial_t h + v\cdot\nabla_x h + \nabla_x\Phi\cdot\nabla_v h + \mathrm{e}^{-\Phi}\nu(v)h = 0$$

can be written as

$$h(t,x,v) = \exp\Big\{-\int_0^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}h(0,X(0),V(0)),$$

where $\Phi(\tau)\equiv\Phi(X(\tau))$ and $\nu(\tau)\equiv\nu(V(\tau))$. Thus, for any (t,x,v), by integrating (3.1) along the backward trajectory (2.1)–(2.2) and applying the Duhamel principle, the solution $h(t,x,v)$ of the original nonlinear equation (3.1) can be written as

$$\begin{aligned}h(t,x,v) &= \exp\Big\{-\int_0^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}h(0,X(0),V(0))\\ &\quad+\int_0^t\exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}(\mathrm{e}^{-\Phi}K_w h)(s,X(s),V(s))\mathrm{d}s\\ &\quad+\int_0^t\exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\Big(\nabla_x\Phi\cdot\nabla_v w\frac{h}{w}\Big)(s,X(s),V(s))\mathrm{d}s\\ &\quad+\int_0^t\exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\Big[\mathrm{e}^{-\frac{\Phi}{2}}w\Gamma\Big(\frac{h}{w},\frac{h}{w}\Big)\Big](s,X(s),V(s))\mathrm{d}s.\end{aligned}\tag{3.2}$$

We note that, for $\Lambda \ge 1$,

$$\frac{w(v)}{w(v')} = \frac{[\Lambda+|v|^2]^\beta}{[\Lambda+|v'|^2]^\beta} \le C_\beta\frac{[\Lambda+|v'|^2]^\beta+|v'-v|^{2\beta}}{[\Lambda+|v'|^2]^\beta} \le C_\beta[1+|v'-v|^2]^\beta.\tag{3.3}$$

Fix a small constant $\epsilon > 0$. We can choose Λ sufficiently large such that

$$\Big|\frac{w'(v)}{w(v)}\Big| \le \epsilon.\tag{3.4}$$

Since $\nu(\tau) \geq \nu_0 > 0$ and $\Phi \leq -\vartheta$, the third term in (3.2) is bounded by

$$C\epsilon \exp\Big\{-\frac{e^{\vartheta}\nu_0 t}{2}\Big\} \sup_{0\leq s\leq t} \Big\{\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}.$$

For the last term in (3.2), by [5, Lemma 5], it follows

$$\Big\|w\Gamma\Big(\frac{h}{w},\frac{h}{w}\Big)(v)\Big\| \leq C\nu(v)\|h\|_{L^\infty}^2.$$

Noticing $\Phi \leq -\vartheta$, we get the bound for the last term by

$$\begin{aligned}
&C\int_0^t \exp\Big\{-\int_s^t e^{-\Phi(\tau)}\nu(\tau)d\tau\Big\}\nu(s)\|h(s)\|_{L^\infty}^2 ds \\
&\leq C \sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}^2 \\
&\quad \times \int_0^t \exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\}\nu(s)\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}ds.
\end{aligned}$$

Thanks to

$$\frac{d}{ds}\Big[\exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\}\Big] = e^{\vartheta}\nu(s)\exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\},$$

we obtain, by integrating by parts, that

$$\begin{aligned}
&\int_0^t \exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\}\nu(s)\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}ds \\
&= e^{C_0}\Big(\exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\}\exp\{-e^{\vartheta}\nu_0 s\}\Big)\Big|_{s=0}^{s=t} \\
&\quad + \nu_0 e^{C_0}\int_0^t \exp\Big\{-\int_s^t e^{\vartheta}\nu(\tau)d\tau\Big\}\exp\{-e^{\vartheta}\nu_0 s\}ds \\
&\leq C(1+t)\exp\{-e^{\vartheta}\nu_0 t\}.
\end{aligned}$$

We shall mainly concentrate on the second term in (3.2). Let $\mathbf{k}(v,v')$ be the corresponding kernel associated with K in (3.1). Then the Grad's estimate implies that

$$|\mathbf{k}(v,v')| \leq C(|v-v'|+|v-v'|^{-1})\exp\Big\{-\frac{1}{8}|v-v'|^2 - \frac{1}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}\Big\}.$$

Denote $w(v) = (\Lambda+|v|^2)^{\beta}$ for the constants $\Lambda > 0$ and $\beta > \frac{3}{2}$. Then [5, Lemma 3], gives

$$\int_{\mathbb{R}^3}' \frac{w(v)}{w(v')}(|v-v'|+|v-v'|^{-1})\exp\Big\{-\frac{\sigma}{8}|v-v'|^2 - \frac{\sigma}{8}\frac{||v|^2-|v'|^2|^2}{|v-v'|^2}\Big\}dv \leq \frac{C}{1+|v|} \tag{3.5}$$

for some $0 < \sigma < 1$ and $C > 0$. Denote $K_w(\cdot) = wK\big(\frac{\cdot}{w}\big)$ and $\mathbf{k}(v,v')_w$ is the corresponding kernel associated with K_w. Then the estimate (3.5) implies that

$$\int_{\mathbb{R}^3} |\mathbf{k}_w(v,v')|dv' < \frac{\varpi}{1+|v|} \tag{3.6}$$

uniformly in Λ for some constant $\varpi > 0$.

Denote

$$[\widetilde{X}(s_1),\widetilde{V}(s_1)] \equiv [X(s_1;s,X(s;t,x,v),v'),V(s_1;s,X(s;t,x,v),v')].$$

We now use (3.2) for $h(s,X(s),v')$ again to evaluate

$$\{K_w h\}(s,X(s),V(s)) = \int_{\mathbb{R}^3} \mathbf{k}_w(V(s),v')h(s,X(s),v')\mathrm{d}v'.$$

In fact, we can bound the third term in (3.2) by

$$\begin{aligned}
&\int_0^t \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\mathrm{e}^{-\Phi(s)}\exp\Big\{-\int_0^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\
&\times \int_{\mathbb{R}^3} |\mathbf{k}_w(V(s),v')h(0,\widetilde{X}(0),\widetilde{V}(0))|\mathrm{d}v'\mathrm{d}s\\
&+\int_0^t\int_0^s \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\mathrm{e}^{-\Phi(s)}\exp\Big\{-\int_{s_1}^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\
&\times \int_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{k}_w(V(s),v')\mathbf{k}_w(\widetilde{V}(s_1),v'')h(s_1,\widetilde{X}(s_1),v'')|\mathrm{d}v'\mathrm{d}v''\mathrm{d}s_1\mathrm{d}s\\
&+\int_0^t\int_0^s \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\mathrm{e}^{-\Phi(s)}\exp\Big\{-\int_{s_1}^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\
&\times \int_{\mathbb{R}^3} \Big|\mathbf{k}_w(V(s),v')\Big\{\nabla_x\Phi\cdot\nabla_v w\frac{h}{w}\Big\}(s_1,\widetilde{X}(s_1),\widetilde{V}(s_1))\Big|\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s\\
&+\int_0^t\int_0^s \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\mathrm{e}^{-\Phi(s)}\exp\Big\{-\int_{s_1}^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\
&\times \int_{\mathbb{R}^3} \Big|\mathbf{k}_w(V(s),v')\Big\{\mathrm{e}^{-\Phi}w\Gamma\Big(\frac{h}{w},\frac{h}{w}\Big)\Big\}(s_1,\widetilde{X}(s_1),\widetilde{V}(s_1))\Big|\mathrm{d}v'\mathrm{d}s_1\mathrm{d}s. \qquad (3.7)
\end{aligned}$$

Since $\nu(\tau)\ge\nu_0$, by taking L^∞ norm for h and using the estimates (3.6) and (1.8), we bound the first term in (3.7) by $\varpi\mathrm{e}^{\overline{\vartheta}}t\exp\{-\nu_0\mathrm{e}^{\vartheta}t\}\|h_0\|_{L^\infty}$. Similarly, noticing the fact $\Phi\le-\vartheta$ and $\|\nabla_x\Phi\|_{L^\infty}\le C_0$, and using the estimate (3.4), the third term can be bounded by

$$C\epsilon\exp\Big\{-\frac{\mathrm{e}^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\le s\le t}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}$$

and the last nonlinear term can be bounded by

$$C(1+t)\exp\{-\mathrm{e}^{\vartheta}\nu_0 t\}\sup_{0\le s\le t}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}^2.$$

We now concentrate on the second term in (3.7) and we follow the same spirit of the proof of Theorem 6 in [5]. Since $\nabla_x\Phi\in L^\infty(\mathbb{R}^3)$, (2.2) implies that, for any $T>0$ and for fixed $N>0$ large enough, we have

$$\sup_{\substack{0\le t\le T\\0\le s\le T}}|V(s)-v|\le\frac{N}{2}.$$

Thanks to the estimate

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\mathbf{k}_w(V(s),v')\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{d}v'\mathrm{d}v''\le\frac{\varpi^2}{1+|V(s)|}, \qquad (3.8)$$

we divide the above integral into three cases according to the size of v,v',v'' and for each case, an upper bound of the second term in (3.7) will be obtained.

Case 1 $|v| \geq N$. In this case, since $|V(s)| \geq \frac{N}{2}$, the estimate (3.8) implies that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{k}_w(V(s),v')\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{d}v'\mathrm{d}v'' \leq \frac{\varpi}{1+|V(s)|} \leq \frac{2\varpi}{N}.$$

We can find an upper bound for the second term in (3.7) by

$$\begin{aligned}&\frac{C}{N}\int_0^t \exp\{-\mathrm{e}^{\vartheta}\nu_0(t-s)\Big\}\int_0^s \exp\{-\mathrm{e}^{\vartheta}\nu_0(s-s_1)\}\|h(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\ &\leq \frac{C}{N}\exp\Big\{-\frac{\mathrm{e}^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}.\end{aligned}$$

Case 2 $|v| \leq N$, $|v'| \geq 2N$, or $|v'| \leq 2N$, $|v''| \geq 3N$. Observe that $|V(s)-v'| \geq |v'-v| - |V(s)-v| \geq |v'|-|v|-|V(s)-v|$, $|\widetilde{V}(s_1)-v''| \geq |v''-v'|-|\widetilde{V}(s_1)-v'| \geq |v''|-|v'|-|\widetilde{V}(s_1)-v'|$, thus we have either $|V(s)-v'| \geq \frac{N}{2}$ or $|\widetilde{V}(s_1)-v''| \geq \frac{N}{2}$. Therefore, either one of the following is valid correspondingly for some $\sigma > 0$,

$$\begin{aligned}&|\mathbf{k}_w(V(s),v')| \leq C\mathrm{e}^{-\frac{\sigma}{8}N^2}|\mathbf{k}_w(V(s),v')|\mathrm{e}^{\frac{\sigma}{8}|V(s)-v'|^2},\\ &|\mathbf{k}_w(\widetilde{V}(s_1),v'')| \leq C\mathrm{e}^{-\frac{\sigma}{8}N^2}|\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{e}^{\frac{\sigma}{8}|\widetilde{V}(s_1)-v''|^2}.\end{aligned}$$

By (3.5), we have

$$\int_{\mathbb{R}^3} |\mathbf{k}_w(V(s),v')|\mathrm{e}^{\frac{\sigma}{8}|V(s)-v'|^2}\mathrm{d}v' + \int_{\mathbb{R}^3} |\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{e}^{\frac{\sigma}{8}|\widetilde{V}(s_1)-v''|^2}\mathrm{d}v'' < +\infty. \tag{3.9}$$

We use this bound to combine the cases of $|V(s)-v'| \geq \frac{N}{2}$ or $|\widetilde{V}(s_1)-v''| \geq \frac{N}{2}$ as

$$\int_0^t\int_0^s\Big\{\int_{|v|\leq N,|v'|\geq 2N,} + \int_{|v'|\leq 2N,|v''|\geq 3N}\Big\}.$$

We first integrate v' for the first integral and use (3.8) to integrate $\mathbf{k}_w$ over v''. We then integrate v'' for the second integral and use (3.8) to integrate $\mathbf{k}_w$ over v'. Noticing $\|\Phi\|_{L^\infty} \leq C_0$, we thus find an upper bound

$$\begin{aligned}&C\int_0^t\int_0^s \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau - \int_{s_1}^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\ &\times\Big\{\sup_v \int_{|v|\leq N,|v'|\geq 2N} |\mathbf{k}_w(V(s),v')|\mathrm{d}v'\Big\}\|h(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\ &+C\int_0^t\int_0^s \exp\Big\{-\int_s^t \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau - \int_{s_1}^s \mathrm{e}^{-\Phi(\tau)}\nu(\tau)\mathrm{d}\tau\Big\}\\ &\times\Big\{\sup_{v'} \int_{|v'|\leq 2N,|v''|\geq 3N} |\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{d}v''\Big\}\|h(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\ &\leq C\mathrm{e}^{-\frac{8N^2}{\sigma}}\int_0^t\int_0^s \exp\{-\mathrm{e}^{\vartheta}\nu_0(t-s_1)\}\|h(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s\\ &\leq C\mathrm{e}^{-\frac{8N^2}{\sigma}}\exp\Big\{-\frac{\mathrm{e}^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}.\end{aligned}$$

Case 3a $|v| \leq N$, $|v'| \leq 2N$, $|v''| \leq 3N$. This is the last remaining case because if $|v'| > 2N$, it is included in Case 2; while if $|v''| > 3N$, either $|v'| \leq 2N$ or $|v'| \geq 2N$ is also included in Case 2. We further assume that $t - s \leq \epsilon$. We can bound the second term in (3.7) by

$$C_N\int_{t-\epsilon}^t\int_0^s \exp\{-\mathrm{e}^{\vartheta}\nu_0(t-s_1)\}\|h(s_1)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s$$

$$\leq C_N\epsilon \exp\Big\{-\frac{e^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}. \tag{3.10}$$

Case 3b $|v|\leq N$, $|v'|\leq 2N, |v''|\leq 3N$, and $t-s\geq\epsilon$. We can bound the second term in (3.7) by

$$C_N\int_0^{t-\epsilon}\int_{\mathbf{B}}\int_0^s \exp\{-e^{\vartheta}\nu_0(t-s_1)\}|\mathbf{k}_\omega(V(s),v')\mathbf{k}_\omega(\widetilde{V}(s_1),v'')|\|h(s)\|_{L^\infty}\mathrm{d}s_1\mathrm{d}s, \tag{3.11}$$

where $\mathbf{B}=\{|v'|\leq 2N,\ |v''|\leq 3N\}$. We notice that $\mathbf{k}_w(v,v')$ has a possible integrable singularity of the type $\frac{1}{|v-v'|}$. We can choose $\mathbf{k}_N(v,v')$ smooth with compact support such that

$$\sup_{|p|\leq 3N}\int_{|v'|\leq 3N}|\mathbf{k}_N(p,v')-\mathbf{k}_w(p,v')|\mathrm{d}v'\leq\frac{1}{N}. \tag{3.12}$$

Splitting

$$\begin{aligned}\mathbf{k}_w(V(s),v')\mathbf{k}_w(\widetilde{V}(s_1),v'') &= \{\mathbf{k}_w(V(s),v')-\mathbf{k}_N(V(s),v')\}\mathbf{k}_w(\widetilde{V}(s_1),v'')\\ &\quad+\{\mathbf{k}_w(\widetilde{V}(s_1),v'')-\mathbf{k}_N(\widetilde{V}(s_1),v'')\}\mathbf{k}_N(V(s),v')\\ &\quad+\mathbf{k}_N(V(s),v')\mathbf{k}_N(\widetilde{V}(s_1),v'').\end{aligned}$$

We then integrate the first term above in v'' and the second term above in v'. By (3.6), we can use such an approximation (3.12) to bound the s_1, s integration by

$$\begin{aligned}&\frac{C}{N}\exp\Big\{-\frac{e^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}\\ &\times\Big\{\sup_{|v'|\leq 2N}\int|\mathbf{k}_w(\widetilde{V}(s_1),v'')|\mathrm{d}v''+\sup_{|v|\leq 2N}\int|\mathbf{k}_w(V(s),v')|\mathrm{d}v'\Big\}\\ &+C_N\int_0^{t-\epsilon}\int_{\mathbf{B}}\int_0^s\exp\{-e^{\vartheta}\nu_0(t-s_1)\}|\mathbf{k}_N(V(s),v')\mathbf{k}_N(\widetilde{V}(s_1),v'')|\\ &\times|h(s_1,\widetilde{X}(s_1),v'')|\mathrm{d}s_1\mathrm{d}v'\mathrm{d}v''\mathrm{d}s.\end{aligned} \tag{3.13}$$

The first term in (3.13) is further bounded by

$$\frac{C}{N}\exp\Big\{-\frac{e^{\vartheta}\nu_0 t}{2}\Big\}\sup_{0\leq s\leq t}\Big\{\exp\Big\{\frac{e^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}.$$

Since $\mathbf{k}_N(V(s),v')$ and $\mathbf{k}_N(\widetilde{V}(s_1),v'')$ are bounded, the second term in (3.13) is controlled by

$$C\int_0^{t-\epsilon}\int_{\mathbf{B}}\int_0^s\exp\{-e^{\vartheta}\nu_0(t-s_1)\}|h(s_1,\widetilde{X}(s_1),v'')|\mathrm{d}s_1\mathrm{d}v'\mathrm{d}v''\mathrm{d}s. \tag{3.14}$$

To estimate this term, we introduce a new variable

$$y=\widetilde{X}(s_1)=X(s_1;s,X(s;t,x,v),v') \tag{3.15}$$

and apply Lemma 2.2 to $X(s_1;s,X(s;t,x,v),v')$ with $s=s_1,\ t=s,\ x=X(s;t,x,v)$, and $v=v'$. Noticing $0\leq s\leq t-\epsilon<t<T_1$, we have $|\frac{\mathrm{d}y}{\mathrm{d}v'}|\geq\frac{\epsilon^3}{8}$. Thanks to $\|\nabla_x\Phi\|_{L^\infty}\leq C$, we observe from (2.1)–(2.2) that

$$|v'-V(\tau)|\leq\int_\tau^s\|\nabla_x\Phi\|_{L^\infty}\mathrm{d}\tau\leq T_0\|\nabla_x\Phi\|_{L^\infty},$$

$$|y - X(s)| \leq \int_{s_1}^{s} |V(\tau)|\mathrm{d}\tau \leq T_0(|v'| + T_0\|\nabla_x\Phi\|_{L^\infty}) \leq C_{T_0,N}$$

for $|v'| \leq 2N$. By integrating over v' (bounded) and using the change of variable (3.15), we get

$$\begin{aligned}
&\int_0^{t-\epsilon}\int_{\mathbf{B}}\int_0^s \exp\{-\mathrm{e}^{\vartheta}\nu_0(t-s_1)\}|h(s_1,\widetilde{X}(s_1),v'')|\mathbf{1}_{\Omega\{X(s_1)\}}\mathrm{d}s_1\mathrm{d}v'\mathrm{d}v''\mathrm{d}s \\
&\leq \frac{C_N}{\epsilon^3}\int^{t-\epsilon}\int_{|y-X(s)|\leq C_{T_0,N}}\int_{|v''|\leq 3N}\int_0^s \exp\{-\mathrm{e}^{\vartheta}\nu_0(t-s_1)\}|h(s_1,y,v'')|\mathrm{d}s_1\mathrm{d}v''\mathrm{d}y\mathrm{d}s \\
&\leq \frac{C_{T_0,N}}{\epsilon^3}\sup_{0\leq s_1\leq T_0}\int_{|y-X(s)|\leq C_{T_0,N}}\int_{|v''|\leq 3N}|h(s_1,y,v'')|\mathrm{d}v''\mathrm{d}y \\
&\leq \int_{\{|F(t)-M|\geq\kappa M\}} + \int_{\{|F(t)-M|\geq\kappa M\}} \\
&\leq C_{T_0,N,\epsilon}[\mathcal{G}(f(0)) + \sqrt{\mathcal{G}(f(0))}\,],
\end{aligned}$$

where the fact $h = \frac{w(F-M)}{\sqrt{M}}$ (which is bounded by $F-M$ for $|v''| \leq 3N$) and Lemma 2.1 are used.

By collecting all the above terms, we conclude that, for $\mathcal{H}(g(0))$ small,

$$\begin{aligned}
&\sup_{0\leq s\leq T_0}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\} \\
&\leq (1+\varpi\mathrm{e}^{\overline{\vartheta}}T_0)\|h(0)\|_{L^\infty} + \Big(\frac{C_{T_0}}{N} + \epsilon C_{N,T_0} + \eta C_{N,T_0,\epsilon}\Big)\sup_{0\leq s\leq T_0}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\} \\
&\quad + C(1+T_0)\sup_{0\leq s\leq T_0}\Big\{\exp\Big\{\frac{\mathrm{e}^{\vartheta}\nu_0 s}{2}\Big\}\|h(s)\|_{L^\infty}\Big\}^2 + C_{T_0,N,\epsilon}\sqrt{\mathcal{H}(g(0))}.
\end{aligned}$$

Assume that $\sup\limits_{0\leq s\leq T_0}\|h(s)\|_{L^\infty}$ is sufficiently small. Since $|\partial_x^2\Phi|_{L^\infty(\mathbb{R}^3)}$ is sufficiently small, we can choose suitable large T_1 in Lemma 2.2 such that there exists a $T_0 < T_1$ satisfying

$$(1+\varpi\mathrm{e}^{\overline{\vartheta}}T_0)\exp\Big\{-\frac{\mathrm{e}^{\vartheta}\nu_0 T_0}{2}\Big\} \triangleq \lambda < 1,$$

then N sufficiently large, and finally ϵ sufficiently small to conclude our lemma.

Proof of Theorem 1.1 Assume that $\sup\limits_{0\leq t\leq\infty}\|h(t)\|_{L^\infty}$ is sufficiently small. We first establish (1.9). Choose any $n = 1,2,3,\cdots$ and use Lemma 3.1 repeatedly to get

$$\begin{aligned}
\|h(nT_0)\|_{L^\infty} &\leq \lambda\|h((n-1)T_0)\|_{L^\infty} + C_{T_0}\sqrt{\mathcal{G}(f(0))} \\
&\leq \lambda^2\|h((n-2)T_0)\|_{L^\infty} + \lambda C_{T_0}\sqrt{\mathcal{G}(f(0))} + C_{T_0}\sqrt{\mathcal{G}(f(0))} \\
&\leq \cdots \\
&\leq \lambda^n\|h(0)\|_{L^\infty} + C_{T_0}\sqrt{\mathcal{G}(f(0))}\{1+\lambda+\lambda^2+\cdots\} \\
&\leq \lambda^n\|h(0)\|_{L^\infty} + \frac{C_{T_0}\lambda}{1-\lambda}\sqrt{\mathcal{G}(f(0))}.
\end{aligned}$$

For any t, we can find n such that $nT_0 \leq t \leq (n+1)T_0$, and from L^∞ estimate from $[0,T_0]$, we conclude (1.9) by

$$\|h(t)\|_{L^\infty} \leq C_{T_0}\|h(nT_0)\| \leq C[\|h(0)\|_{L^\infty} + \sqrt{\mathcal{G}(f(0))}].$$

To prove (1.10), we take x and v derivatives to (1.2) to get

$$\begin{aligned}
&\{\partial_t + v\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_v + \mathrm{e}^{-\Phi}\nu(v)\}\partial_x f - \mathrm{e}^{-\Phi}K\partial_x f\\
&= \nabla_x\partial_x\Phi\cdot\nabla_v f + \mathrm{e}^{-\Phi}(\nu(v)-K)f - \frac{\partial_x\Phi}{2}\mathrm{e}^{-\Phi}\Gamma(f,f) + \mathrm{e}^{-\Phi}\partial_x\Gamma(f,f), \qquad (3.16)\\
&\{\partial_t + v\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_v + \mathrm{e}^{-\Phi}\nu(v)\}\partial_v f - \mathrm{e}^{-\Phi}K\partial_v f\\
&= -\partial_x f + \mathrm{e}^{-\Phi}\partial_v(\nu(v)))f + \mathrm{e}^{-\Phi}\partial_v\Gamma(f,f). \qquad (3.17)
\end{aligned}$$

By [6, Lemma 2.2], we have

$$\|\partial_v(Kf)\partial_v f\|_{L^1} \le \frac{1}{2}\int_{\mathbb{R}^3}\nu(v)|\partial_v f|^2\mathrm{d}v + C\|f\|_{L^2}^2.$$

Since $L=\nu-K\ge 0$, by multiplying (3.16) with $\partial_x f$ and (3.17) with $\partial_v f$ and then integrating them over $\mathbb{R}^3_{x,v}$ respectively, we can follow the procedures in [4] to get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x g\|_{L^2}^2 \le C\{\|\nabla_x\partial_x\Phi\|_{L^\infty} + \|h\|_{L^\infty}\}\|\nabla_{x,v}g\|_{L^2}^2 + C\|g\|_{L^2}^2,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_v g\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\nu(v)|\partial_v f|^2\mathrm{d}v \le C\|\partial_x g\|_{L^2}^2 + C\|g\|_{L^2}^2.
\end{aligned}$$

Hence (1.10) follows from the Gronwall Lemma since $\sup\limits_{0\le t\le\infty}\|h(t)\|_{L^\infty}$ is bounded by (1.9). With such an estimate, we easily obtain the uniqueness by taking L^2 estimate for the difference for (1.2). Therefore, we complete our proof of Theorem 1.1.

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