Chin. Ann. Math. Ser. B
39(5), 2018, 817–828
DOI: 10.1007/s11401-018-0098-0

Chinese Annals of Mathematics, Series B
© The Editorial Office of CAM and
Springer-Verlag Berlin Heidelberg 2018

Lie Triple Derivations on von Neumann Algebras*

Lei LIU[1]

Abstract Let $\mathcal{A}$ be a von Neumann algebra with no central abelian projections. It is proved that if an additive map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for any $a, b, c \in \mathcal{A}$ with $ab = 0$ (resp. $ab = P$, where P is a fixed nontrivial projection in $\mathcal{A}$), then there exist an additive derivation d from $\mathcal{A}$ into itself and an additive map $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a, b], c]$ with $ab = 0$ (resp. $ab = P$) such that $\delta(a) = d(a) + f(a)$ for any $a \in \mathcal{A}$.

Keywords Derivations, Lie triple derivations, von Neumann algebras
2000 MR Subject Classification 16W25, 47B47

1 Introduction

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$. Recall that an additive (a linear) map δ from $\mathcal{A}$ into itself is called an additive (a linear) derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. δ is called an additive (a linear) Lie derivation if $\delta([a, b]) = [\delta(a), b] + [a, \delta(b)]$ for all $a, b \in \mathcal{A}$, where $[a, b] = ab - ba$. More generally, δ is called an additive (a linear) Lie triple derivation if $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for all $a, b, c \in \mathcal{A}$. The structures of Lie triple derivations on some operator algebras were intensively studied (see [2, 7, 9] and references therein). Let M be a von Neumann algebra with no central abelian projections. Miers [9] proved that if $L : \mathcal{M} \to \mathcal{M}$ is a linear Lie triple derivation, then there exists an element $T \in \mathcal{M}$ and a linear map $f : \mathcal{M} \to \mathcal{Z}_{\mathcal{M}}$ which annihilates brackets such that $L(a) = aT - Ta + f(a)$ for any $a \in \mathcal{M}$.

In recent years, the local actions of derivations have been studied intensively. One direction is to study the conditions under which derivations of operator algebras can be completely determined by the actions on some elements concerning products. We say that an additive (a linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is derivable at a given point $G \in \mathcal{A}$, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$ with $ab = G$. This kind of maps were discussed by several authors (see [1, 3, 4–5, 11–12] and references therein). But, so far, there have been few papers on the study of the local actions of Lie triple derivations on operator algebras. We say that an additive (a linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is Lie triple derivable at a given point $G \in \mathcal{A}$, if $\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$ for all $a, b, c \in \mathcal{A}$ with $ab = G$. It is the aim of the present article to investigate the additive

Manuscript received October 30, 2015. Revised February 28, 2016.
[1]School of Mathematics and Statistics, Xidian University, Xi'an 710071, China; School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: leiliu@mail.xidian.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11401452) and the China Postdoctoral Science Foundation (No. 2015M581513).

(linear) Lie triple derivations on von Neumann algebras with no central abelian projections by the local actions. It is a generalization of the results in [9].

We need some notations and preliminaries about von Neumann algebras. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity I. $\mathcal{Z}_{\mathcal{A}} = \{z \in \mathcal{A} : za = az \text{ for all } a \in \mathcal{A}\}$ is called the center of $\mathcal{A}$. A projection P is called a central abelian projection if $P \in \mathcal{Z}_{\mathcal{A}}$ and $P\mathcal{A}P$ is abelian. We denote $\bar{a}$ be the central carrier of a, which is the smallest central projection satisfying $Pa = a$. It is well known that $\overline{a}$ is the projection whose range is the closed linear span of $\{\mathcal{A}a(h) : h \in H\}$. For each self-adjoint operator $r \in \mathcal{A}$, the core of r denoted by $\underline{r}$ is $\sup\{a \in \mathcal{Z}_{\mathcal{A}} : a = a^*,\ a \leq r\}$. If $P \in \mathcal{A}$ is a projection and $\underline{P} = 0$, we call P a core-free projection. It is easy to verify that $\underline{P} = 0$ if and only if $\overline{\mathrm{I} - P} = \mathrm{I}$. By [8, Lemma 4], we can say that $\mathcal{A}$ is a von Neumann algebra with no central abelian projections if and only if it has a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = \mathrm{I}$. We refer the reader to [6] for the theory of von Neumann algebras.

2 Characterizing Lie Triple Derivations by Acting on Zero-Product

In this section, we consider the question of characterizing Lie triple derivations by action at zero product on von Neumann algebras with no central abelian projections.

Theorem 2.1 *Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is an additive map satisfying*

$$\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$$

for all $a, b, c \in \mathcal{A}$ with $ab = 0$. Then there exists an additive derivation d from $\mathcal{A}$ into itself and an additive map $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a, b], c]$ when $ab = 0$ such that

$$\delta(a) = d(a) + f(a), \quad \forall a \in \mathcal{A}.$$

Note that every linear derivation of a von Neumann is inner (see [10]). By Theorem 2.1, the following corollary is immediate. It is a generalization of Theorem 1 in [9].

Corollary 2.1 Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a linear map satisfying

$$\delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]$$

for all $a, b, c \in \mathcal{A}$ with $ab = 0$. Then there exists an element $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a, b], c]$ when $ab = 0$ such that

$$\delta(a) = aT - Ta + f(a), \quad \forall a \in \mathcal{A}.$$

Proof of Theorem 2.1 By [8, Lemma 4], there is a projection $P \in \mathcal{A}$ such that $\underline{P} = 0$ and $\overline{P} = \mathrm{I}$. In what follows, we denote $P_1 = P$ and $P_2 = \mathrm{I} - P_1$. By the definitions of core and central carrier, P_2 is also a core-free projection and $\overline{P_2} = \mathrm{I}$. Set $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. For an operator $a_{ij} \in \mathcal{A}$, we always mean that $a_{ij} \in \mathcal{A}_{ij}$.

We shall organize the proof of Theorem 2.1 in a series of claims.

Claim 2.1 Let $a_{ii} \in \mathcal{A}_{ii},\ i = 1, 2$. If $a_{11}b_{12} = b_{12}a_{22}$ for all $b_{12} \in \mathcal{A}_{12}$, then $a_{11} + a_{22} \in \mathcal{Z}_{\mathcal{A}}$.

For any $x_{11} \in \mathcal{A}_{11},\ x_{12} \in \mathcal{A}_{12}$, we have $a_{11}x_{11}x_{12} = x_{11}x_{12}a_{22} = x_{11}a_{11}x_{12}$. Since $\overline{P_2} = \mathrm{I}$, which means that $\{\mathcal{A}P_2(h) : h \in H\}$ is dense in H, we get $a_{11}x_{11} = x_{11}a_{11}$, that is, $a_{11} \in \mathcal{Z}_{P_1\mathcal{A}P_1}$. By [6, Corollary 5.5.7], we know $\mathcal{Z}_{P_1\mathcal{A}P_1} = \mathcal{Z}_{\mathcal{A}}P_1$. So there exists $z_1 \in \mathcal{Z}_{\mathcal{A}}$ such that $a_{11} = z_1P_1$.

Similarly, we have $a_{22} = z_2P_2,\ z_2 \in \mathcal{Z}_{\mathcal{A}}$. It follows that $z_1b_{12} = a_{11}b_{12} = b_{12}a_{22} = z_2b_{12}$. Then $(z_1 - z_2)P_1 = 0$, which implies $(z_1 - z_2)\mathcal{A}P_1 = 0$. By $\overline{P_1} = \mathrm{I}$, we obtain $z_1 = z_2$. Hence $a_{11} + a_{22} \in \mathcal{Z}_{\mathcal{A}}$, the claim holds.

Moreover, for any $a_{12} \in \mathcal{A}_{12}$, since $a_{12}P_1 = 0$, we have

$$\begin{aligned}\delta(a_{12}) &= \delta([[a_{12}, P_1], P_1]) \\ &= [[\delta(a_{12}), P_1], P_1] + [[a_{12}, \delta(P_1)], P_1] + [[a_{12}, P_1], \delta(P_1)] \\ &= P_1\delta(a_{12})P_2 + P_2\delta(a_{12})P_1 + P_1\delta(P_1)a_{12} - a_{12}\delta(P_1)P_2 \\ &\quad + \delta(P_1)a_{12} - a_{12}\delta(P_1).\end{aligned}$$

Multiplying P_1 from the left side and P_2 from the right side of the above equation, we arrive at $P_1\delta(P_1)a_{12} = a_{12}\delta(P_1)P_2$. It follows from Claim 2.1 that $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}_{\mathcal{A}}$. Let $E = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$, and $\varphi = \delta - \delta_E$, where δ_E is the inner derivation given by $\delta_E(x) = xE - Ex$ for all $x \in \mathcal{A}$. It is not difficult to verify

$$\varphi(P_1) = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}_{\mathcal{A}}$$

and

$$\varphi([[a, b], c]) = [[\varphi(a), b], c] + [[a, \varphi(b)], c] + [[a, b], \varphi(c)]$$

for any $a, b, c \in \mathcal{A}$ with $ab = 0$.

Claim 2.2 $\varphi(P_2) \in \mathcal{Z}_{\mathcal{A}}$.

Since $P_2P_1 = 0$ and $\varphi(P_1) \in \mathcal{Z}_{\mathcal{A}}$, we have

$$0 = \varphi([[P_2, P_1], P_1]) = [[\varphi(P_2), P_1], P_1] = P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1.$$

For any $a_{12} \in \mathcal{A}_{12}$, since $P_2a_{12} = 0$, we get

$$\begin{aligned}-\varphi(a_{12}) &= \varphi([[P_2, a_{12}], P_2]) \\ &= [[\varphi(P_2), a_{12}], P_2] + [[P_2, \varphi(a_{12})], P_2] + [[P_2, a_{12}], \varphi(P_2)] \\ &= P_1\varphi(P_2)a_{12} - a_{12}\varphi(P_2)P_2 - P_1\varphi(a_{12})P_2 - P_2\varphi(a_{12})P_1 - a_{12}\varphi(P_2) + \varphi(P_2)a_{12}.\end{aligned}$$

Multiplying the above equation by P_1 from the left and by P_2 from the right, we obtain

$$P_1\varphi(P_2)a_{12} = a_{12}\varphi(P_2)P_2.$$

Then it follows that $P_1\varphi(P_2)P_1 + P_2\varphi(P_2)P_2 \in \mathcal{Z}_{\mathcal{A}}$ by Claim 2.1, and hence $\varphi(P_2) \in \mathcal{Z}_{\mathcal{A}}$.

Claim 2.3 $\varphi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij},\ 1 \le i \ne j \le 2$.

Since $a_{12}P_1 = 0$ and $\varphi(P_1) \in \mathcal{Z}_{\mathcal{A}}$, we get

$$\varphi(a_{12}) = \varphi([[a_{12}, P_1], P_1]) = P_1\varphi(a_{12})P_2 + P_2\varphi(a_{12})P_1.$$

Now it suffices to show that $P_2\varphi(a_{12})P_1 = 0$. Indeed, for any $b_{12} \in \mathcal{A}_{12},\ x \in \mathcal{A}$, it is easy to check that

$$0 = \varphi([[a_{12}, b_{12}], x]) = [[\varphi(a_{12}), b_{12}], x] + [[a_{12}, \varphi(b_{12})], x],$$

which leads to $[\varphi(a_{12}), b_{12}] + [a_{12}, \varphi(b_{12})] = z \in \mathcal{Z}_{\mathcal{A}}$. Then we have

$$\begin{aligned}[\varphi(a_{12}), b_{12}] &= z - [a_{12}, \varphi(b_{12})] \\ &= z + [[a_{12}, P_1], \varphi(b_{12})] \\ &= z + \varphi([[a_{12}, P_1], b_{12}]) - [[\varphi(a_{12}), P_1], b_{12}] \\ &= z - [[\varphi(a_{12}), P_1], b_{12}] \\ &= z - [P_2\varphi(a_{12})P_1, b_{12}].\end{aligned}$$

This together with $P_1\varphi(a_{12})P_1 = P_2\varphi(a_{12})P_2 = 0$ entails that $[P_2\varphi(a_{12})P_1, b_{12}] \in \mathcal{Z}_{\mathcal{A}}$. This leads to $[P_2\varphi(a_{12})P_1, b_{12}] = 0$. Then $P_2\varphi(a_{12})b_{12} = 0$. Since $\overline{P_2} = \mathrm{I}$, we have $P_2\varphi(a_{12})P_1 = 0$. Consequently, $\varphi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

Similarly, we can obtain $\varphi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Claim 2.4 There exist maps $f_i : \mathcal{A}_{ii} \to \mathcal{Z}_{\mathcal{A}}$ such that $\varphi(a_{ii}) - f_i(a_{ii}) \in \mathcal{A}_{ii}$ for any $a_{ii} \in \mathcal{A}_{ii},\ i = 1, 2$.

Since $a_{11}P_2 = 0$ and from Claim 2.2, we have

$$0 = \varphi([[a_{11}, P_2], P_2]) = [[\varphi(a_{11}), P_2], P_2] = P_1\varphi(a_{11})P_2 + P_2\varphi(a_{11})P_1.$$

Moreover, for any $b_{22} \in \mathcal{A}_{22}$ and $x \in \mathcal{A}$, it is easy to check that

$$0 = \varphi([[a_{11}, b_{22}], x]) = [[\varphi(a_{11}), b_{22}], x] + [[a_{11}, \varphi(b_{22})], x],$$

which implies that $[\varphi(a_{11}), b_{22}] + [a_{11}, \varphi(b_{22})] \in \mathcal{Z}_{\mathcal{A}}$. Multiplying the above equation from both sides by P_2, we arrive at $[P_2\varphi(a_{11})P_2, b_{22}] \in \mathcal{Z}_{\mathcal{A}}P_2$, which leads to $[P_2\varphi(a_{11})P_2, b_{22}] = 0$. For every $a_{11} \in \mathcal{A}_{11}$, there is unique $Z \in \mathcal{Z}_{\mathcal{A}}$ such that $P_2\varphi(a_{11})P_2 = ZP_2$. So we can define $f_1 : \mathcal{A}_{11} \to \mathcal{Z}_{\mathcal{A}}$ by $f_1(a_{11}) = Z$ for all $a_{11} \in \mathcal{A}_{11}$.

Indeed, since $Z \in \mathcal{Z}_{\mathcal{A}}$, we have $P_1ZP_2 = P_2ZP_1 = 0$. Let $a_{11} = \widetilde{a}_{11}$ and $f_1(\widetilde{a}_{11}) = \widetilde{Z}$. It follows from the preceding argumentation that

$$Z = P_1ZP_1 + P_2\varphi(a_{11})P_2 \tag{2.1}$$

and

$$\widetilde{Z} = P_1\widetilde{Z}P_1 + P_2\varphi(\widetilde{a}_{11})P_2. \tag{2.2}$$

By (2.1)–(2.2), we have $Z - \widetilde{Z} = P_1ZP_1 - P_1\widetilde{Z}P_1$. Since [8, Lemma 14], that is, $P_1\mathcal{A}P_1 \bigcap \mathcal{Z}_{\mathcal{A}} = \{0\}$, we get $Z = \widetilde{Z}$.

Therefore, for any $a_{11} \in \mathcal{A}_{11}$, we have

$$\begin{aligned}\varphi(a_{11}) &= P_1\varphi(a_{11})P_1 + P_2\varphi(a_{11})P_2 \\ &= (P_1\varphi(a_{11})P_1 - P_1f_1(a_{11})P_1) + f_1(a_{11}) \\ &\in \mathcal{A}_{11} + \mathcal{Z}_{\mathcal{A}}.\end{aligned}$$

Similarly, we can define a map $f_2 : \mathcal{A}_{22} \to \mathcal{Z}_{\mathcal{A}}$ such that $\varphi(a_{22}) - f_2(a_{22}) \in \mathcal{A}_{22}$ for any $a_{22} \in \mathcal{A}_{22}$. So Claim 2.4 is true.

Now, we define two maps $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ and $d : \mathcal{A} \to \mathcal{A}$ respectively by

$$f(a) = f_1(P_1aP_1) + f_2(P_2aP_2), \quad \forall a \in \mathcal{A}$$

and

$$d(a) = \varphi(a) - f(a), \quad \forall a \in \mathcal{A}.$$

By the definition of d and Claim 2.4, we have $d(P_1) = d(P_2) = 0,\ d(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij},\ 1 \leq i, j \leq 2$ and $d(a_{ij}) = \varphi(a_{ij})$ for all $a_{ij} \in \mathcal{A}_{ij},\ 1 \leq i \neq j \leq 2$.

In the following we shall show that d is an additive derivation.

Claim 2.5 d is an additive map.

Since $d = \varphi - f$ and $f = f_1 + f_2$, we only need to show that f_1 and f_2 are additive maps.

For any $a_{11}, b_{11} \in \mathcal{A}_{11}$, it follows from (2.1) that

$$f_1(a_{11}) = P_1 f_1(a_{11}) P_1 + P_2 \varphi(a_{11}) P_2, \tag{2.3}$$

$$f_1(b_{11}) = P_1 f_1(b_{11}) P_1 + P_2 \varphi(b_{11}) P_2 \tag{2.4}$$

and

$$f_1(a_{11} + b_{11}) = P_1 f_1(a_{11} + b_{11}) P_1 + P_2 \varphi(a_{11} + b_{11}) P_2. \tag{2.5}$$

By (2.3)–(2.5), we get

$$\begin{aligned} &f_1(a_{11}) + f_1(b_{11}) - f_1(a_{11} + b_{11}) \\ &= P_1 f_1(a_{11}) P_1 + P_1 f_1(b_{11}) P_1 - P_1 f_1(a_{11} + b_{11}) P_1. \end{aligned}$$

Since $f_1(a_{11}) + f_1(b_{11}) - f_1(a_{11} + b_{11}) \in \mathcal{Z}_{\mathcal{A}}$ and $P_1 f_1(a_{11}) P_1 + P_1 f_1(b_{11}) P_1 - P_1 f_1(a_{11} + b_{11}) P_1 \in P_1 \mathcal{A} P_1$, we have $f_1(a_{11}) + f_1(b_{11}) - f_1(a_{11} + b_{11}) = 0$.

Similarly, f_2 is an additive map.

Claim 2.6 $d(a_{ii} b_{ij}) = a_{ii} d(b_{ij}) + d(a_{ii}) b_{ij}$ for any $a_{ii} \in \mathcal{A}_{ii},\ b_{ij} \in \mathcal{A}_{ij},\ 1 \leq i \neq j \leq 2$.

Due to $b_{ij} a_{ii} = 0$, the following equations hold:

$$\begin{aligned} d(a_{ii} b_{ij}) &= \varphi(a_{ii} b_{ij}) \\ &= \varphi([[b_{ij}, a_{ii}], P_i]) \\ &= [[\varphi(b_{ij}), a_{ii}], P_i] + [[b_{ij}, \varphi(a_{ii})], P_i] \\ &= [[d(b_{ij}), a_{ii}], P_i] + [[b_{ij}, d(a_{ii})], P_i] \\ &= a_{ii} d(b_{ij}) + d(a_{ii}) b_{ij}. \end{aligned}$$

With the similar argument in Claim 2.6, we have the following claim.

Claim 2.7 $d(a_{ij} b_{jj}) = a_{ij} d(b_{jj}) + d(a_{ij}) b_{jj}$ for any $a_{ij} \in \mathcal{A}_{ij},\ b_{jj} \in \mathcal{A}_{jj},\ 1 \leq i \neq j \leq 2$.

Claim 2.8 $d(a_{ii} b_{ii}) = a_{ii} d(b_{ii}) + d(a_{ii}) b_{ii}$ for any $a_{ii}, b_{ii} \in \mathcal{A}_{ii},\ i = 1, 2$.

For any $b_{ij} \in \mathcal{A}_{ij}$, we have, from Claim 2.6, that

$$d(a_{ii} b_{ii} b_{ij}) = a_{ii} b_{ii} d(b_{ij}) + d(a_{ii} b_{ii}) b_{ij}.$$

At the same time,

$$\begin{aligned} d(a_{ii} b_{ii} b_{ij}) &= a_{ii} d(b_{ii} b_{ij}) + d(a_{ii}) b_{ii} b_{ij} \\ &= a_{ii} b_{ii} d(b_{ij}) + a_{ii} d(b_{ii}) b_{ij} + d(a_{ii}) b_{ii} b_{ij}. \end{aligned}$$

Comparing the above two equations, we get

$$d(a_{ii} b_{ii}) b_{ij} = a_{ii} d(b_{ii}) b_{ij} + d(a_{ii}) b_{ii} b_{ij}.$$

Note $\overline{P_j} = I$. It follows from the fact that $\{\mathcal{A}P_j(h) : h \in H\}$ is dense in H that $d(a_{ii}b_{ii}) = a_{ii}d(b_{ii}) + d(a_{ii})b_{ii}$.

Claim 2.9 $d(a_{ij}b_{ji}) = a_{ij}d(b_{ji}) + d(a_{ij})b_{ji}$ for any $a_{ij} \in \mathcal{A}_{ij},\ b_{ji} \in \mathcal{A}_{ji},\ 1 \leq i \neq j \leq 2$.

Since $P_2a_{12} = 0$ and $\varphi(P_2) \in \mathcal{Z}_{\mathcal{A}}$, we have

$$\begin{aligned}\varphi([[P_2, a_{12}], b_{21}]) &= [[P_2, \varphi(a_{12})], b_{21}] + [[P_2, a_{12}], \varphi(b_{21})]\\ &= [[P_2, d(a_{12})], b_{21}] + [[P_2, a_{12}], d(b_{21})]\\ &= b_{21}d(a_{12}) + d(b_{21})a_{12} - a_{12}d(b_{21}) - d(a_{12})b_{21}.\end{aligned}$$

Since $d(a) = \varphi(a) - f(a)$, $\forall a \in \mathcal{A}$,

$$d(b_{21}a_{12} - a_{12}b_{21}) - f(b_{21}a_{12} - a_{12}b_{21}) = b_{21}d(a_{12}) + d(b_{21})a_{12} - a_{12}d(b_{21}) - d(a_{12})b_{21}. \quad (2.6)$$

We shall show $f(b_{21}a_{12} - a_{12}b_{21}) = 0$. Multiplying the above equation by a_{12} to the left side and right side respectively, we obtain the following two equations:

$$a_{12}d(b_{21}a_{12}) - a_{12}f(b_{21}a_{12} - a_{12}b_{21}) = a_{12}b_{21}d(a_{12}) + a_{12}d(b_{21})a_{12} \quad (2.7)$$

and

$$d(a_{12})b_{21}a_{12} + a_{12}d(b_{21})a_{12} = d(a_{12}b_{21})a_{12} + a_{12}f(b_{21}a_{12} - a_{12}b_{21}). \quad (2.8)$$

Computing (2.7)–(2.8), we get

$$\begin{aligned}&a_{12}d(b_{21}a_{12}) + d(a_{12})b_{21}a_{12} - a_{12}f(b_{21}a_{12} - a_{12}b_{21})\\ &= a_{12}b_{21}d(a_{12}) + d(a_{12}b_{21})a_{12} + a_{12}f(b_{21}a_{12} - a_{12}b_{21}).\end{aligned}$$

It follows from Claims 2.6–2.7 that

$$a_{12}d(b_{21}a_{12}) + d(a_{12})b_{21}a_{12} = d(a_{12}b_{21}a_{12}) = a_{12}b_{21}d(a_{12}) + d(a_{12}b_{21})a_{12},$$

which combining with the above equation implies $a_{12}f(b_{21}a_{12} - a_{12}b_{21}) = 0$. Using polar decomposition of a_{12}, we have $V|a_{12}|f(b_{21}a_{12} - a_{12}b_{21}) = 0$, which yields $|a_{12}|f(b_{21}a_{12} - a_{12}b_{21}) = 0$. This leads to $f(b_{21}a_{12} - a_{12}b_{21})^*|a_{12}| = 0$, and so $a_{12}f(b_{21}a_{12} - a_{12}b_{21})^* = 0$.

Similarly, we have $b_{21}f(b_{21}a_{12} - a_{12}b_{21})^* = 0$.

Then multiplying (2.6) by $f(b_{21}a_{12} - a_{12}b_{21})^*$ to the right side, we arrive at

$$f(b_{21}a_{12} - a_{12}b_{21})f(b_{21}a_{12} - a_{12}b_{21})^* = d(b_{21}a_{12} - a_{12}b_{21})f(b_{21}a_{12} - a_{12}b_{21})^*. \quad (2.9)$$

Due to Claim 2.8, the following equations hold:

$$\begin{aligned}&d(a_{12}b_{21})f(b_{21}a_{12} - a_{12}b_{21})^*\\ &= d(a_{12}b_{21}f(b_{21}a_{12} - a_{12}b_{21})^*) - a_{12}b_{21}d(P_1f(b_{21}a_{12} - a_{12}b_{21})^*P_1)\\ &= -a_{12}b_{21}d(P_1f(b_{21}a_{12} - a_{12}b_{21})^*P_1),\\ &d(b_{21}a_{12})f(b_{21}a_{12} - a_{12}b_{21})^*\\ &= d(b_{21}a_{12}f(b_{21}a_{12} - a_{12}b_{21})^*) - b_{21}a_{12}d(P_2f(b_{21}a_{12} - a_{12}b_{21})^*P_2)\\ &= -b_{21}a_{12}d(P_2f(b_{21}a_{12} - a_{12}b_{21})^*P_2).\end{aligned}$$

Putting the above two equations into (2.9), we have

$$\begin{aligned}&f(b_{21}a_{12}-a_{12}b_{21})f(b_{21}a_{12}-a_{12}b_{21})^*\\&=-b_{21}a_{12}d(P_2f(b_{21}a_{12}-a_{12}b_{21})^*P_2)+a_{12}b_{21}d(P_1f(b_{21}a_{12}-a_{12}b_{21})^*P_1).\end{aligned}$$

Multiplying the equation by $f(b_{21}a_{12}-a_{12}b_{21})^*$ to the left side, we get

$$f(b_{21}a_{12}-a_{12}b_{21})^*f(b_{21}a_{12}-a_{12}b_{21})f(b_{21}a_{12}-a_{12}b_{21})^*=0,$$

which implies $f(b_{21}a_{12}-a_{12}b_{21})=0$. So we arrive at

$$d(b_{21}a_{12}-a_{12}b_{21})=d(b_{21})a_{12}+b_{21}d(a_{12})-d(a_{12})b_{21}-a_{12}d(b_{21}).$$

This is equivalent to $d(a_{12}b_{21})=d(a_{12})b_{21}+a_{12}d(b_{21})$ and $d(b_{21}a_{12})=d(b_{21})a_{12}+b_{21}d(a_{12})$, as desired.

By Claims 2.5–2.9, we can conclude that d is an additive derivation. Hence we have $\delta(a)=\varphi(a)+\delta_E(a)=d(a)+f(a)+\delta_E(a)$, $\forall a\in\mathcal{A}$. Denote $\phi(a)=d(a)+\delta_E(a)$, then $\delta(a)=\phi(a)+f(a)$, $\forall a\in\mathcal{A}$. Clearly, ϕ is an additive derivation on $\mathcal{A}$ and f is an additive map from $\mathcal{A}$ to $\mathcal{Z}_{\mathcal{A}}$.

For $ab=0$, it follows that

$$\begin{aligned}f([[a,b],c])&=\delta([[a,b],c])-\phi([[a,b],c])\\&=[[\delta(a),b],c]+[[a,\delta(b)],c]+[[a,b],\delta(c)]-\phi([[a,b],c])\\&=[[\phi(a),b],c]+[[a,\phi(b)],c]+[[a,b],\phi(c)]-\phi([[a,b],c])\\&=0.\end{aligned}$$

3 Characterizing Lie Triple Derivations by Acting on Projection-Product

In this section, we consider the question of characterizing Lie triple derivations by acting at projection-product on von Neumann algebras without central abelian projections. The proof of the following theorem shares the similar outline as that of Theorem 2.1, but it needs different techniques.

Theorem 3.1 *Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections and P be a projection in $\mathcal{A}$ with $\underline{P}=0$ and $\overline{P}=\mathrm{I}$. Suppose that $\delta:\mathcal{A}\to\mathcal{A}$ is an additive map satisfying*

$$\delta([[a,b],c])=[[\delta(a),b],c]+[[a,\delta(b)],c]+[[a,b],\delta(c)]$$

for all $a,b,c\in\mathcal{A}$ with $ab=P$. Then there exists an additive derivation ϕ from $\mathcal{A}$ into itself and an additive map $f:\mathcal{A}\to\mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a,b],c]$ when $ab=P$ such that

$$\delta(a)=\phi(a)+f(a),\quad\forall a\in\mathcal{A}.$$

Note that all linear derivations of von Neumann algebras are inner (see [10]). We have the following corollary. It is a generalization of Theorem 1 in [9].

Corollary 3.1 Let $\mathcal{A}$ be a von Neumann algebra without central abelian projections and P be a projection in $\mathcal{A}$ with $\underline{P}=0$ and $\overline{P}=I$. Suppose that $\delta:\mathcal{A}\to\mathcal{A}$ is a linear map satisfying

$$\delta([[a,b],c])=[[\delta(a),b],c]+[[a,\delta(b)],c]+[[a,b],\delta(c)]$$

for all $a, b, c \in \mathcal{A}$ with $ab = P$. Then there exists an element $T \in \mathcal{A}$ and a linear map $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a, b], c]$ when $ab = P$ such that

$$\delta(a) = aT - Ta + f(a), \quad \forall a \in \mathcal{A}.$$

Proof of Theorem 3.1 We shall use the same symbols with that in Section 2.

For any $a_{12} \in \mathcal{A}_{12}$, since $(P_1 + a_{12})P_1 = P_1$, we obtain

$$\begin{aligned}\delta(a_{12}) &= \delta([[P_1 + a_{12}, P_1], P_1]) \\ &= [[\delta(P_1 + a_{12}), P_1], P_1] + [[P_1 + a_{12}, \delta(P_1)], P_1] + [[P_1 + a_{12}, P_1], \delta(P_1)] \\ &= [[\delta(a_{12}), P_1], P_1] + [[a_{12}, \delta(P_1)], P_1] + [[a_{12}, P_1], \delta(P_1)] \\ &= P_1\delta(a_{12})P_2 + P_2\delta(a_{12})P_1 + P_1\delta(P_1)a_{12} - a_{12}\delta(P_1)P_2 \\ &\quad + \delta(P_1)a_{12} - a_{12}\delta(P_1).\end{aligned}$$

Multiplying P_1 from the left side and P_2 from the right side of the above equation, we arrive at $P_1\delta(P_1)a_{12} = a_{12}\delta(P_1)P_2$. It follows from Claim 2.1 that $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}_{\mathcal{A}}$. Let $E = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$, and $\varphi = \delta - \delta_E$, where δ_E is the inner derivation. It is not difficult to verify that

$$\varphi(P_1) = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathcal{Z}_{\mathcal{A}}$$

and

$$\varphi([[a, b], c]) = [[\varphi(a), b], c] + [[a, \varphi(b)], c] + [[a, b], \varphi(c)]$$

for any $a, b \in \mathcal{A}$ with $ab = P_1$.

Now we organize the proof in a series of claims.

Claim 3.1 $\varphi(P_2) \in \mathcal{Z}_{\mathcal{A}}$.

Since $(P_1 + P_2)P_1 = P_1$ and $\varphi(P_1) \in \mathcal{Z}_{\mathcal{A}}$, we have

$$0 = \varphi([[P_1 + P_2, P_1], P_1]) = [[\varphi(P_1 + P_2), P_1], P_1] = P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1.$$

For any $a_{12} \in \mathcal{A}_{12}$, since $(P_1 + a_{12})(P_1 + P_2 - a_{12}) = P_1$, we get

$$\begin{aligned}\varphi(a_{12}) &= \varphi([[P_1 + a_{12}, P_1 + P_2 - a_{12}], P_1]) \\ &= [[\varphi(a_{12}), -a_{12}], P_1] + [[P_1 + a_{12}, \varphi(P_2) - \varphi(a_{12})], P_1] \\ &= P_2\varphi(a_{12})P_1 + P_1\varphi(a_{12})P_2 + P_1\varphi(P_2)a_{12} - a_{12}\varphi(P_2)P_2.\end{aligned}$$

Multiplying the above equation by P_1 from the left and by P_2 from the right, we obtain

$$P_1\varphi(P_2)a_{12} = a_{12}\varphi(P_2)P_2.$$

It follows from Claim 2.1 that $P_1\varphi(P_2)P_1 + P_2\varphi(P_2)P_2 \in \mathcal{Z}_{\mathcal{A}}$. Hence $\varphi(P_2) \in \mathcal{Z}_{\mathcal{A}}$.

Claim 3.2 $\varphi(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij},\ 1 \le i \ne j \le 2$.

Since $(P_1 + a_{12})P_1 = P_1$ and $\varphi(P_1) \in \mathcal{Z}_{\mathcal{A}}$, we get

$$\varphi(a_{12}) = \varphi([[P_1 + a_{12}, P_1], P_1]) = P_1\varphi(a_{12})P_2 + P_2\varphi(a_{12})P_1.$$

Now, for any $b_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} 0 &= \varphi([[P_1 + b_{12}, P_1], b_{12}]) \\ &= [[\varphi(b_{12}), P_1], b_{12}] + [[b_{12}, P_1], \varphi(b_{12})] \\ &= P_2\varphi(b_{12})b_{12} - b_{12}\varphi(b_{12})P_1 + \varphi(b_{12})b_{12} - b_{12}\varphi(b_{12}). \end{aligned}$$

Multiplying the above equation from both side by P_2, we arrive at $P_2\varphi(b_{12})b_{12} = 0$. Moreover, it follows that

$$\begin{aligned} 0 &= \varphi([[P_1 + a_{12}, P_1], b_{12}]) \\ &= [[\varphi(a_{12}), P_1], b_{12}] + [[a_{12}, P_1], \varphi(b_{12})] \\ &= P_2\varphi(a_{12})b_{12} - b_{12}\varphi(a_{12})P_1 + \varphi(b_{12})a_{12} - a_{12}\varphi(b_{12}). \end{aligned}$$

Multiplying the equation by b_{12} from the right and for the fact $P_2\varphi(b_{12})b_{12} = 0$, we obtain $b_{12}\varphi(a_{12})b_{12} = 0$. By linearizing, we get $b_{12}\varphi(a_{12})d_{12} + d_{12}\varphi(a_{12})b_{12} = 0$ for any $b_{12},\ d_{12} \in \mathcal{A}_{12}$. It is not difficult to check

$$P_2\varphi(a_{12})b_{12}\varphi(a_{12})[b_{12}\varphi(a_{12})d_{12}]\varphi(a_{12})P_1 + P_2\varphi(a_{12})b_{12}\varphi(a_{12})[d_{12}\varphi(a_{12})b_{12}]\varphi(a_{12})P_1 = 0,$$

that is,

$$P_2\varphi(a_{12})b_{12}\varphi(a_{12})d_{12}\varphi(a_{12})b_{12}\varphi(a_{12})P_1 = 0.$$

As von Neumann algebras are semiprime, we see $P_2\varphi(a_{12})b_{12}\varphi(a_{12})P_1 = 0$. Then $P_2\varphi(a_{12})P_1 = 0$. Consequently, $\varphi(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$.

Similarly, we can obtain $\varphi(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$.

Claim 3.3 There exists a map f_1 on $\mathcal{A}_{11}$ such that $\varphi(a_{11}) - f_1(a_{11}) \in \mathcal{A}_{11}$ for all $a_{11} \in \mathcal{A}_{11}$.

First suppose that a_{11} is invertible in $\mathcal{A}_{11}$, i.e., there exists $a_{11}^{-1} \in \mathcal{A}_{11}$ such that $a_{11}^{-1}a_{11} = a_{11}a_{11}^{-1} = P_1$. Since $a_{11}^{-1}a_{11} = P_1$, we have

$$0 = \varphi([[a_{11}^{-1}, a_{11}], P_1]) = [[\varphi(a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}, \varphi(a_{11})], P_1].$$

It follows from $(P_2 + a_{11}^{-1})a_{11} = P_1$ and Claim 3.1 that

$$\begin{aligned} 0 &= \varphi([[P_2 + a_{11}^{-1}, a_{11}], P_1]) \\ &= [[\varphi(a_{11}^{-1}), a_{11}], P_1] + [[P_2 + a_{11}^{-1}, \varphi(a_{11})], P_1] \\ &= P_1\varphi(a_{11})P_2 + P_2\varphi(a_{11})P_1. \end{aligned}$$

Moreover, for any $b_{22} \in \mathcal{A}_{22}$ and $x \in \mathcal{A}$, since $(a_{11}^{-1} + b_{22})a_{11} = P_1$, it is easy to check that

$$\begin{aligned} 0 &= \varphi([[a_{11}^{-1} + b_{22}, a_{11}], x]) \\ &= [[\varphi(a_{11}^{-1} + b_{22}), a_{11}], x] + [[a_{11}^{-1} + b_{22}, \varphi(a_{11})], x] \\ &= [[\varphi(b_{22}), a_{11}], x] + [b_{22}, \varphi(a_{11})], x], \end{aligned}$$

which implies that $[\varphi(b_{22}), a_{11}] + [b_{22}, \varphi(a_{11})] \in \mathcal{Z}_{\mathcal{A}}$. Multiplying the above equation from both sides by P_2, we arrive at $[b_{22}, P_2\varphi(a_{11})P_2] \in \mathcal{Z}_{\mathcal{A}}P_2$. So we get $[b_{22}, P_2\varphi(a_{11})P_2] = 0$. Then there exists $Z \in \mathcal{Z}_{\mathcal{A}}$ such that $P_2\varphi(a_{11})P_2 = ZP_2$.

If a_{11} is not invertible in $\mathcal{A}_{11}$, we may find a sufficiently big number n such that $nP_1 - a_{11}$ is invertible in $\mathcal{A}_{11}$. It follows from the preceding case that $P_1\varphi(nP_1-a_{11})P_2+P_2\varphi(nP_1-a_{11})P_1 = 0$, and $P_2\varphi(nP_1-a_{11})P_2 = ZP_2$. Since $\varphi(P_1) \in \mathcal{Z}_{\mathcal{A}}$, we also have $P_1\varphi(a_{11})P_2+P_2\varphi(a_{11})P_1 = 0$ and $P_2\varphi(a_{11})P_2 \in \mathcal{Z}_{\mathcal{A}}P_2$. Without loss of generality, we still denote $P_2\varphi(a_{11})P_2 = ZP_2$.

We define $f_1 : \mathcal{M}_{11} \to \mathcal{Z}_{\mathcal{M}}$ by $f_1(a_{11}) = Z$ for all $a_{11} \in \mathcal{A}_{11}$. With the similarly argument as in Claim 2.4, we know f_1 is well defined. Hence

$$\varphi(a_{11}) = (P_1\varphi(a_{11})P_1 - P_1f_1(a_{11})P_1) + f_1(a_{11}) \in \mathcal{A}_{11} + \mathcal{Z}_{\mathcal{A}}.$$

Claim 3.4 There exists a map f_2 on $\mathcal{A}_{22}$ such that $\varphi(a_{22})-f_2(a_{22}) \in \mathcal{A}_{22}$ for any $a_{22} \in \mathcal{A}_{22}$.

For any $a_{22} \in \mathcal{A}_{22}$, since $(P_1 + a_{22})P_1 = P_1$, we have

$$0 = \varphi([[P_1 + a_{22}, P_1], P_1]) = P_1\varphi(a_{22})P_2 + P_2\varphi(a_{22})P_1.$$

The rest step is similar to the proof of Claim 3.3.

Now, we define two maps $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ and $d : \mathcal{A} \to \mathcal{A}$ respectively by

$$f(a) = f_1(P_1aP_1) + f_2(P_2aP_2), \quad \forall a \in \mathcal{A}$$

and

$$d(a) = \varphi(a) - f(a), \quad \forall a \in \mathcal{A}.$$

By the definition of d and Claim 3.4, we have $d(P_1) = d(P_2) = 0,\ d(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij},\ 1 \le i, j \le 2$ and $d(a_{ij}) = \varphi(a_{ij})$ for all $a_{ij} \in \mathcal{A}_{ij},\ 1 \le i \ne j \le 2$.

In the following we shall show that d is an additive derivation.

Claim 3.5 d is an additive map.

The proof is similar to that of Claim 2.5.

Claim 3.6 $d(a_{11}b_{12}) = a_{11}d(b_{12}) + d(a_{11})b_{12}$ for any $a_{11} \in \mathcal{A}_{11},\ b_{12} \in \mathcal{A}_{12}$.

If a_{11} is invertible in $\mathcal{A}_{11}$, then for any $x_{12} \in \mathcal{A}_{12}$, we have $(a_{11}^{-1}x_{12} + a_{11}^{-1})a_{11} = P_1$. It follows that

$$\begin{aligned} d(a_{12}) &= d([[a_{11}^{-1}x_{12} + a_{11}^{-1}, a_{11}], P_1]) \\ &= \varphi([[a_{11}^{-1}x_{12} + a_{11}^{-1}, a_{11}], P_1]) \\ &= [[\varphi(a_{11}^{-1}x_{12} + a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}x_{12} + a_{11}^{-1}, \varphi(a_{11})], P_1] \\ &= [[d(a_{11}^{-1}x_{12} + a_{11}^{-1}), a_{11}], P_1] + [[a_{11}^{-1}x_{12} + a_{11}^{-1}, d(a_{11})], P_1] \\ &= [[d(a_{11}^{-1}x_{12}), a_{11}], P_1] + [[a_{11}^{-1}x_{12}, d(a_{11})], P_1] \\ &= a_{11}d(a_{11}^{-1}x_{12}) + d(a_{11})a_{11}^{-1}x_{12}. \end{aligned}$$

Replacing b_{12} with $a_{11}^{-1}x_{12}$, we have $d(a_{11}b_{12}) = a_{11}d(b_{12}) + d(a_{11})b_{12}$.

If a_{11} is not invertible in $\mathcal{A}_{11}$, we may find a sufficiently big number n such that $nP_1 - a_{11}$ is invertible in $\mathcal{A}_{11}$. Then $d((nP_1 - a_{11})a_{12}) = (nP_1 - a_{11})d(a_{12}) + d(nP_1 - a_{11})a_{12}$. Clearly, P_1 is invertible in $\mathcal{A}_{11}$, so we get $d(a_{11}b_{12}) = a_{11}d(b_{12}) + d(a_{11})b_{12}$ from the above equation.

Claim 3.7 $d(a_{21}b_{11}) = a_{21}d(b_{11}) + d(a_{21})b_{11}$ for any $a_{21} \in \mathcal{A}_{21},\ b_{11} \in \mathcal{A}_{11}$.

Considering $a_{11}(x_{21}a_{11}^{-1} + a_{11}^{-1}) = P_1$ and using the same approach in Claim 3.6, we know that Claim 3.7 is true.

Claim 3.8 $d(a_{22}b_{21}) = a_{22}d(b_{21}) + d(a_{22})b_{21}$ for any $a_{22} \in \mathcal{A}_{22},\ b_{21} \in \mathcal{A}_{21}$.

Due to $(P_1 + a_{22} - a_{22}b_{21})(P_1 + b_{21}) = P_1$, we compute

$$\begin{aligned}
-d(b_{21}) &= d([[P_1 + a_{22} - a_{22}b_{21}, P_1 + b_{21}], P_1]) \\
&= \varphi([[P_1 + a_{22} - a_{22}b_{21}, P_1 + b_{21}], P_1]) \\
&= [[d(P_1 + a_{22} - a_{22}b_{21}), P_1 + b_{21}], P_1] \\
&\quad + [[P_1 + a_{22} - a_{22}b_{21}, d(P_1 + b_{21})], P_1] \\
&= a_{22}d(b_{21}) - d(a_{22}b_{21}) + d(a_{22})b_{21} - d(b_{21}),
\end{aligned}$$

that is, $d(a_{22}b_{21}) = a_{22}d(b_{21}) + d(a_{22})b_{21}$.

Considering $(P_1 + a_{12})(P_1 - b_{22} + a_{12}b_{22}) = P_1$, we arrive at the following claim.

Claim 3.9 $d(a_{12}b_{22}) = a_{12}d(b_{22}) + d(a_{12})b_{22}$ for any $a_{12} \in \mathcal{A}_{12},\ b_{22} \in \mathcal{A}_{22}$.

Claim 3.10 $d(a_{ii}b_{ii}) = a_{ii}d(b_{ii}) + d(a_{ii})b_{ii}$, $i = 1, 2$.

It is similar to Claim 2.8.

Claim 3.11 $d(a_{ij}b_{ji}) = a_{ij}d(b_{ji}) + d(a_{ij})b_{ji}$ for any $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$, $1 \le i \ne j \le 2$.

Since $(a_{12} + P_1)P_1 = P_1$, we have

$$\begin{aligned}
\varphi(b_{21}a_{12} - a_{12}b_{21}) &= \varphi([[a_{12} + P_1, P_1], b_{21}]) \\
&= [[d(a_{12} + P_1), P_1], b_{21}] + [[a_{12} + P_1, P_1], d(b_{21})] \\
&= b_{21}d(a_{12}) + d(b_{21})a_{12} - a_{12}d(b_{21}) - d(a_{12})b_{21}.
\end{aligned}$$

Since $d(a) = \varphi(a) - f(a),\ \forall a \in \mathcal{A}$,

$$\begin{aligned}
&d(b_{21}a_{12} - a_{12}b_{21}) - f(b_{21}a_{12} - a_{12}b_{21}) \\
&= b_{21}d(a_{12}) + d(b_{21})a_{12} - a_{12}d(b_{21}) - d(a_{12})b_{21}.
\end{aligned}$$

With the same approach as in Claim 2.9, we can get $f(b_{21}a_{12} - a_{12}b_{21}) = 0$. So we arrive at

$$d(b_{21}a_{12} - a_{12}b_{21}) = d(b_{21})a_{12} + b_{21}d(a_{12}) - d(a_{12})b_{21} - a_{12}d(b_{21}).$$

This is equivalent to $d(b_{21}a_{12}) = d(b_{21})a_{12} + b_{21}d(a_{12})$ and $d(a_{12}b_{21}) = d(a_{12})b_{21} + a_{12}(b_{21})$. Consequently, Claim 3.11 is true.

So we can conclude that d is an additive derivation by Claims 3.5–3.11. Hence we have $\delta(a) = \varphi(a) + \delta_E(a) = d(a) + f(a) + \delta_E(a)$, $\forall a \in \mathcal{A}$. Denote $\phi(a) = d(a) + \delta_E(a)$, then $\delta(a) = \phi(a) + f(a)$, $\forall a \in \mathcal{A}$. Clearly, ϕ is an additive derivation on $\mathcal{A}$ and f is an additive map from $\mathcal{A}$ to $\mathcal{Z}_{\mathcal{A}}$.

With the similar argument as in the proof of Theorem 2.1, we can verify the additive map $f : \mathcal{A} \to \mathcal{Z}_{\mathcal{A}}$ vanishing at every second commutator $[[a, b], c]$ when $ab = P$.

Acknowledgement The author wish to give his thanks to the referees and the editor for their helpful comments and suggestions.

References

[1] Alaminos, J., Extremera, J., Villena, A. R. and Bresar, M., Characterizing homomorphisms and derivations on C*-algebras, *Proc. Roy. Soc. Edinburgh Sect. A*, **137**, 2007, 1–7.

[2] Benkovic, D., Lie triple derivations of unital algebras with idempotents, *Linear Multilinear Algebra*, **63**(1), 2015, 141–165.

[3] Bresar, M., Characterizing homomorphisms, derivations and multipliers in rings with idempotents, *Proc. Roy. Soc. Edinburgh Sect. A*, **137**, 2007, 9–21.

[4] Chebotar, M. A., Ke, W.-F. and Lee, P.-H., Maps characterized by action on zero products, *Pacific J. Math.*, **216**, 2004, 217–228.

[5] Jing, W., Lu, S. and Li, P., Characterizations of derivations on some operator algebras, *Bull. Austral. Math. Soc.*, **66**, 2002, 227–232.

[6] Kadison, R. V. and Ringrose, J. R., Fundamentals of the Theory of Operator Algebras, Vol. I, Academic Press, New York, 1983, Vol. II, Academic Press, New York, 1986.

[7] Li, J. and Shen, Q., Characterizations of Lie higher and Lie triple derivations on triangular algebras, *J. Korean Math. Soc.*, **49**(2), 2012, 419–433.

[8] Miers, C. R., Lie isomorphisms of operator algebras, *Pacific J. Math.*, **38**, 1971, 717–735.

[9] Miers, C. R., Lie triple derivations of von Neumann algebras, *Proc. Amer. Math. Soc.*, **71**, 1978, 57–61.

[10] Sakai, S., Derivations of W*-algebras, *Ann. Math.*, **83**, 1966, 273–279.

[11] Zhang, Y., Hou, J. and Qi, X., Characterizing derivations for any nest algebras on Banach spaces by their behaviors at an injective operator, *Linear Algebra Appl.*, **449**, 2014, 312–333.

[12] Zhu, J. and Zhao, S., Characterizations of all-derivable points in nest algebras, *Proc. Amer. Math. Soc.*, **141**, 2013, 2343–2350.