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A ϖ_n-Related Family of Homotopy Elements in the Stable Homotopy of Spheres*

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Abstract To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. Let p be a prime greater than 5. The authors make use of the May spectral sequence and the Adams spectral sequence to prove the existence of a ϖ_n-related family of homotopy elements, $\beta_1\omega_n\gamma_s$, in the stable homotopy groups of spheres, where $n > 3$, $3 \leq s < p-2$ and the ϖ_n-element was detected by X. Liu.

Keywords Stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence
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1 Introduction

Let S be the sphere spectrum localized at an odd prime number p. To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. One of the powerful tools to determine $\pi_*(S)$ is the classical Adams spectral sequence (see [1, 7]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}/p$ for the prime p,

$$\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p) \Longrightarrow \pi_{t-s}(S)$$

with differential $\mathrm{d}_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. Let $q = 2(p-1)$ as usual. From [6] we know that $\mathrm{Ext}_A^{1,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ is generated by $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}/p,\mathbb{Z}/p)$, $h_i \in \mathrm{Ext}_A^{1,qp^i}(\mathbb{Z}/p,\mathbb{Z}/p)$ $(i \geq 0)$. $\mathrm{Ext}_A^{2,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ is generated by a_1h_0, a_0^2, a_0h_i $(i > 0)$, g_i $(i \geq 0)$, k_i $(i \geq 0)$, b_i $(i \geq 0)$ and h_ih_j $(0 \leq i \leq j-2)$ which have degrees $2q+1$, 2, qp^i+1, $q(p^{i+1}+2p^i)$, $q(2p^{i+1}+p^i)$, qp^{i+1} and $q(p^i+p^j)$, respectively. In 1980, K. Aikawa [2] calculated $\mathrm{Ext}_A^{3,*}(\mathbb{Z}/p,\mathbb{Z}/p)$ by the λ-algebra.

Consider the Smith-Toda spectra $V(k)$ given in [8], and we have the following four cofiber

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sequences:

$$S \xrightarrow{p} S \xrightarrow{i_0} V(0) \xrightarrow{j_0} \Sigma S, \quad p \geq 2,$$
$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0), \quad p \geq 3,$$
$$\Sigma^{q(p+1)} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{q(p+1)+1} V(1), \quad p \geq 5,$$
$$\Sigma^{q(p^2+p+1)} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{q(p^2+p+1)+1} V(2), \quad p \geq 7.$$

Here α, β and γ are the v_1-, v_2- and v_3-mappings, respectively.

In 1998, Wang and Zheng [9] defined the third Greek letter family element $\widetilde{\gamma}_t$ in the ASS for $p \geq 7$ and $t \not\equiv 0, 1, 2 \bmod p$,

$$\widetilde{\gamma}_t \in \mathrm{Ext}_A^{t, q[tp^2+(t-1)p+(t-2)]+t-3}(\mathbb{Z}/p, \mathbb{Z}/p).$$

It is known that $\widetilde{\gamma}_t$ detects the stable homotopy γ-element $\gamma_t = j_0 j_1 j_2 \gamma^t i_2 i_1 i_0$.

In [5], Liu constructed a new nontrivial family of homotopy elements in the stable homotopy groups of spheres and proved the following theorems.

Theorem 1.1 (cf. [5]) *Let $p \geq 5$, $n \geq 3$. Then $k_0 h_n \neq 0 \in \mathrm{Ext}_A^{3, q(p^n+2p+1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a permanent cycle in the Adams spectral sequence and it converges to a non-trivial family of homotopy elements ϖ_n in the stable homotopy of spheres $\pi_{q(p^n+2p+1)-3}(S)$.*

In this paper, we make use of the above result to consider the composite map $\beta_1 \varpi_n \gamma_s$ and prove its non-triviality under some conditions. The main result can be stated as follows.

Theorem 1.2 *Let $p \geq 7$, $n > 3$, $3 \leq s < p - 2$. Then the family of homotopy elements, $\beta_1 \varpi_n \gamma_s$, is non-trivial in the stable homotopy of spheres.*

The paper is arranged as follows. After recalling some knowledge on the May spectral sequence in Section 2, we compute some May E_r-terms and Adams E_2-terms which are used in the proof of Theorem 1.2 in Section 3. Section 4 is devoted to showing Theorem 1.2.

2 The May Spectral Sequence

From [7], there is a May spectral sequence $\{E_r^{s,t,*}, \mathrm{d}_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} \mid m > 0, i \geqslant 0) \otimes P[b_{m,i} \mid m > 0, i \geqslant 0] \otimes P[a_n \mid n \geqslant 0], \tag{2.1}$$

where $E(\ )$ denotes the exterior algebra, $P[\ ]$ denotes the polynomial algebra, and

$$h_{m,i} \in E_1^{1, 2(p^m-1)p^i, 2m-1}, \quad b_{m,i} \in E_1^{2, 2(p^m-1)p^{i+1}, p(2m-1)}, \quad a_n \in E_1^{1, 2p^n-1, 2n+1}.$$

One has

$$\mathrm{d}_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \tag{2.2}$$

and if $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$, then

$$\mathrm{d}_r(x \cdot y) = \mathrm{d}_r(x) \cdot y + (-1)^s x \cdot \mathrm{d}_r(y). \tag{2.3}$$

In particular, the first May differential d_1 is given by

$$\mathrm{d}_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}, \quad \mathrm{d}_1(a_i) = \sum_{0\leqslant k<i} h_{i-k,k}a_k, \quad \mathrm{d}_1(b_{i,j}) = 0. \tag{2.4}$$

There also exists a graded commutativity in the May spectral sequence as

$$x \cdot y = (-1)^{ss'+tt'} y \cdot x$$

for $x, y = h_{m,i}, b_{m,i}$ or a_n.

For each element $x \in E_1^{s,t,u}$, we define hdim $x = s$, intdim $x = t$, May$(x) = u$. Then we have

$$\begin{cases} \text{hdim } h_{i,j} = \text{hdim } a_i = 1, \\ \text{hdim } b_{i,j} = 2, \\ \text{intdim } h_{i,j} = q(p^{i+j-1} + \cdots + p^j), \\ \text{intdim } b_{i,j} = q(p^{i+j} + \cdots + p^{j+1}), \\ \text{intdim } a_i = q(p^{i-1} + \cdots + 1) + 1, \\ \text{intdim } a_0 = 1, \\ \text{May}(h_{i,j}) = \text{May}(a_{i-1}) = 2i - 1, \\ \text{May}(b_{i,j}) = (2i-1)p, \end{cases} \tag{2.5}$$

where $i \geqslant 1$, $j \geqslant 0$.

3 Some May E_r-Terms and Two Adams E_2-Terms

In this section, we first determine some May E_r-terms ($r \geq 1$). Then we give two important theorems about Adams E_2-term which will be used in the proof of Theorem 1.2.

Lemma 3.1 *Let $p \geq 7$, $n > 3$, $0 \leq s < p - 5$ and $r \geq 1$. Then the May E_1-term satisfies*

$$E_1^{s+8-r,t(s,n)+1-r,*} = \begin{cases} G_1, & r = 1 \text{ and } s = p-6, \\ G_2, & r = 1 \text{ and } s = p-7, \\ 0, & \text{otherwise}. \end{cases}$$

Here $t(s,n) = q[p^n + (s+3)p^n + (s+5)p + (s+2)] + s$, G_1 is the $\mathbb{Z}/p$-module generated by the unique element $a_3^{p-6}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}$ and G_2 is generated by the element $a_3^{p-7}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}$.

Proof When $r \geq s+2$, we can easily show that in the May spectral sequence

$$E_1^{s+8-r,t(s,n)+1-r,*} = 0. \tag{3.1}$$

Thus in the rest of the proof, we assume that $1 \leq r < s+2$.

Consider $g = w_1w_2\cdots w_l \in E_1^{s+8-r,t(s,n)-r+1,*}$ in the May spectral sequence, where w_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq r+j \leq n+1$, $0 \leq u+z \leq n$, $r > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that

$$\text{intdim } w_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,1}p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $w_i = a_{k_i}$, or $e_i = 0$. It follows that

$$\text{hdim } g = \sum_{i=1}^{l} \text{hdim } w_i = s + 8 - r \tag{3.2}$$

and

$$\begin{aligned}\operatorname{intdim} g &= \sum_{i=1}^{l} \operatorname{intdim} w_i \\ &= q\Big[\Big(\sum_{i=1}^{l} c_{i,n}\Big)p^n + \cdots + \Big(\sum_{i=1}^{l} c_{i,1}\Big)p + \Big(\sum_{i=1}^{l} c_{i,0}\Big)\Big] + \Big(\sum_{i=1}^{l} e_i\Big) \\ &= q[p^n + (s+3)p^2 + (s+5)p + (s+2)] + s + 1 - r. \end{aligned} \tag{3.3}$$

Note that hdim $h_{i,j}$ = hdim $a_i = 1$, hdim $b_{i,j} = 2$, $1 \le r < s+3$ and $0 \le s < p-5$. From hdim $g = \sum_{i=1}^{l}$ hdim $w_i = s + 8 - r$, we have $l \le s + 8 - r < p + 3 - r \le p + 2$.

We claim that $s + 1 - r \ge 0$. On the one hand, it is easy to get the following inequality

$$\sum_{i=1}^{l} e_i \le l \le p + 1$$

from the fact that $e_i = 0$ or 1. On the other hand, using $1 \le r < s+2$ and $p \ge 5$, we would also have the following inequality

$$\sum_{i=1}^{l} e_i = q + (s - r + 1) > 2p - 2 - 1 \ge p + 2,$$

which contradicts $\sum_{i=1}^{l} e_i \le l \le p + 1$. The claim is proved.

Using $0 \le s + 3,\ s + 1 - r < p$ and the knowledge on p-adic expression in number theory, we have

$$\begin{cases} \sum_{i=1}^{l} e_i = s + 1 - r, \\ \sum_{i=1}^{l} c_{i,0} = s + 2, \\ \sum_{i=1}^{l} c_{i,1} = s + 5, \\ \sum_{i=1}^{l} c_{i,2} = s + 3, \\ \sum_{i=1}^{l} c_{i,3} = 0 + \lambda_3 p, \quad \lambda_3 \ge 0, \\ \sum_{i=1}^{l} c_{i,4} + \lambda_3 = 0 + \lambda_4 p, \quad \lambda_4 \ge 0, \\ \vdots \\ \sum_{i=1}^{l} c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, \quad \lambda_{n-1} \ge 0, \\ \sum_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 1. \end{cases} \tag{3.4}$$

Consider the fifth equation of (3.4) $\sum_{i=1}^{l} c_{i,3} = 0 + \lambda_3 p$. By $c_{i,3} = 0$ or 1, and $l \le p + 1$, we get that $\lambda_3 = 0$ or 1.

Case 1 $\lambda_3 = 0$.

We claim that

$$\lambda_4 = 0.$$

If $\lambda_4 = 1$, we would have the following equations

$$\sum_{i=1}^{l} c_{i,2} = s+3, \quad \sum_{i=1}^{l} c_{i,3} = 0, \quad \sum_{i=1}^{l} c_{i,4} = p.$$

From $\sum\limits_{i=1}^{l} c_{i,2} = s+3$ and (2.5), there would be $s+3$ factors among g such that intdim $x_i = q$ (higher terms on $p+p^2$+lower terms on p) $+ \delta_i$, where δ_i may equal 0 or 1. Similarly, from $\sum\limits_{i=1}^{l} c_{i,4} = p$, there would be p factors among g such that intdim $w_i = q$ (high terms on $p+p^4$+lower terms on p) $+ \delta_i$. Thus, by $l \le p+1$ and (2.5), there would be at least $p+s+3-(p+1) = s+2$ factors in g such that intdim $w_i = q$ (higher terms on $p^4+p^3+p^2$+lower terms on p) $+ \delta_i$. Thus we would have

$$\sum_{i=1}^{l} c_{i,3} \ge s+2,$$

which contradicts $\sum\limits_{i=1}^{l} c_{i,3} = 0$. The claim is proved.

By induction on j, we have that

$$\lambda_j = 0, \quad 4 \le j \le n-1.$$

Then we have the following two cases.

Case 1.1 If there is a factor $h_{1,n}$ in g, we have that up to sign $g = h_{1,n}\widetilde{g}$ with $\widetilde{g} \in E_1^{s+7-r,q[(s+3)p^2+(s+5)p+(s+2)]+s+1-r,*}$.

By (2.5), $E_1^{s+6,q[(s+3)p^2+(s+5)p+(s+2)]+s+1-r,*} = \mathbb{Z}/p\{a_3^s h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}\}$. It follows that when $r=1$, the generator g exists and $g = a_3^s h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}$ up to sign.

When $r \ge 2$, we have $E_1^{s+7-r,q[(s+3)p^2+(s+5)p+(s+2)]+s+1-r,*} = 0$ by (2.5). Thus we have that in this case the generator g is impossible to exist.

Case 1.2 If there is a factor $b_{1,n-1}$ in g, then up to sign $g = b_{1,n-1}\widetilde{g}$ with $\widetilde{g} \in E_1^{s+6-r,*,*}$.

When $r=1$, from $E_1^{s+5,q[(s+3)p^2+(s+5)p+(s+2)]+s,*} = 0$, we know that the generator g is impossible to exist.

When $r \ge 2$, we can make use of (2.5) to get

$$E_1^{s+6-r,q[(s+3)p^2+(s+5)p+(s+2)]+s+1-r,*} = 0,$$

implying that the generator g is impossible to exist, either.

Case 2 $\lambda_3 = 1$.

If $r \ge 3$, we would have

$$l \le s+8-r < p+3-r \le p.$$

It is easy to see that λ_3 is impossible to equal 1. Thus in the rest of this case, we always assume

$$r \le 2.$$

From the sixth equation of (3.4) $\sum_{i=1}^{l} c_{i,4} + 1 = \lambda_4 p$ and $0 \le \sum_{i=1}^{l} c_{i,4} \le l \le p+1$, we can deduce

$$\lambda_4 = 1.$$

By induction on j,

$$\lambda_j = 1, \quad 4 \le j \le n-1.$$

Thus (3.4) can turn into

$$\begin{cases} \sum_{i=1}^{l} e_i = s + 1 - r, \\ \sum_{i=1}^{l} c_{i,0} = s + 2, \\ \sum_{i=1}^{l} c_{i,1} = s + 5, \\ \sum_{i=1}^{l} c_{i,2} = s + 3, \\ \sum_{i=1}^{l} c_{i,3} = p, \\ \sum_{i=1}^{l} c_{i,4} = p - 1, \\ \vdots \\ \sum_{i=1}^{l} c_{i,n-1} = p - 1, \\ \sum_{i=1}^{l} c_{i,n} = 0. \end{cases} \tag{3.5}$$

From the fifth equation of (3.5) $\sum_{i=1}^{l} c_{i,3} = p$, using $c_{i,3} = 0$ or 1, we can have that

$$l \ge p.$$

Note that $l \le s + 7$. Thus $s \ge p - 7$. By $0 \le s < p - 5$, s may equal $p - 6$ or $p - 7$.

Case 2.1 When $s = p - 6$, $g = w_1 w_2 \cdots w_l \in E_1^{p+2-r, t(p-6,n)+1-r,*}$. In this case, l may equal p or $p+1$.

Case 2.1.1 $l = p$. From the following two equations:

$$\sum_{i=1}^{l} e_i = p - 5 - r \quad \text{and} \quad \sum_{i=1}^{l} c_{i,n-1} = p - 1,$$

we have that up to sigh the generator g must be of the form

$$g = a_n^{p-6-r} x_{p-5-r} \cdots x_p.$$

In this case r must equal 1, then we have that up to sign

$$g = a_n^{p-7} x_{p-6} \cdots x_p,$$

where $x_{p-6} \cdots x_p \in E_1^{8, q[6p^{n-1} + \cdots + 6p^4 + 7p^3 + 4p^2 + 6p + 3] + 1, *} = 0$, which is trivial by (2.5). Thus, in this case g is impossible to exist.

Case 2.1.2 $l = p+1$. From the following two equations:

$$\sum_{i=1}^{l} e_i = p - 5 - r \quad \text{and} \quad \sum_{i=1}^{l} c_{i,n-1} = p - 1,$$

we have that up to sign the generator g must be of the following form:

$$g = a_n^{p-7-r} x_{p-6-r} \cdots x_{p+1}.$$

In this case r must equal 1, then we have that up to sign

$$g = a_n^{p-8} x_{p-7} \cdots x_{p+1},$$

where $x_{p-7} \cdots x_{p+1} \in E_1^{9,q[7p^{n-1}+\cdots+7p^4+8p^3+5p^2+7p+4]+2,*} = 0$, which is trivial by (2.5). Thus, in this case g is impossible to exist.

Case 2.2 When $s = p-7$, $g = w_1 w_2 \cdots w_l \in E_1^{p+1-r,t(p-7,n)+1-r,*}$.

Case 2.2.1 $l = p$. From the following two equations:

$$\sum_{i=1}^{l} e_i = p - 6 - r \quad \text{and} \quad \sum_{i=1}^{l} c_{i,n-1} = p - 1,$$

we have that up to sign the generator g must be of the form

$$g = a_n^{p-7-r} x_{p-7-r} \cdots x_p.$$

If $r = 1$, we have that up to sign

$$g = a_n^{p-8} x_{p-7} \cdots x_p,$$

where $x_{p-7} \cdots x_p \in E_1^{8,q[7p^{n-1}+\cdots+7p^4+8p^3+4p^2+6p+3]+1,*} = 0$. Thus, in this case g is impossible to exist.

If $r = 2$, we have that up to sign

$$g = a_n^{p-9} x_{p-8} \cdots x_p,$$

where $x_{p-8} \cdots x_p \in E_1^{9,q(8p^{n-1}+\cdots+8p^4+9p^3+5p^2+7p+4)+2,*} = 0$, implying that the generator g is impossible to exist.

Case 2.2.2 $l = p+1$. From the following two equations:

$$\sum_{i=1}^{l} e_i = p - 6 - r \quad \text{and} \quad \sum_{i=1}^{l} c_{i,n-1} = p - 1,$$

we have that up to sign the generator g must be of the form

$$g = a_n^{p-8-r} x_{p-7-r} \cdots x_{p+1},$$

where $x_{p-7-r} \cdots x_{p+1} \in E_1^{9,q(8p^{n-1}+\cdots+8p^4+9p^3+5p^2+7p+4)+2,*} = 0$. Then, in this case, g is impossible to exist.

From Cases 1 and 2, the lemma follows.

We need the following theorem about the γ-element.

Theorem 3.1 (cf. [4]) *Let $p \geq 7$, $0 \leq s < p-2$. Then the permanent cocycle*

$$a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_r^{s+3,t,*}$$

detects the second Greek letter element $\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the May spectral sequence, where $r \geq 1$, $t = (s+3)p^2q + (s+2)pq + (s+1)q + s$ and $\widetilde{\gamma}_{s+3}$ detects the γ-element

$$\gamma_{s+3} \in \pi_{(s+3)p^2q+(s+2)pq+(s+1)q-3}(S)$$

in the Adams spectral sequence.

Now we consider some results on the product $k_0 b_0 h_n \widetilde{\gamma}_{s+3}$.

Lemma 3.2 (1) *The product $k_0 b_0 h_n \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$ is represented by*

$$h_{2,0} h_{1,1} b_{1,0} h_{1,n} a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{s+8,t(s,n),*}$$

in the May spectral sequence, where $t(s,n) = q[p^n + (s+3)p^2 + (s+5)p + (s+2)] + s$.

(2) *For the generator of $E_1^{p+1,t(p-6,n),*}$, we have*

$$M(a_3^{p-6} h_{3,0} h_{2,0} h_{2,1} h_{1,1} h_{1,n} b_{2,0}) = 10p - 29.$$

For the generators of $E_1^{p,t(p-7,n),*}$, we have

$$M(a_3^{p-7} h_{3,0} h_{2,0} h_{2,1} h_{1,1} h_{1,n} b_{2,0}) = 10p - 36.$$

In particular,

$$M(h_{2,0} h_{1,1} b_{1,0} h_{1,n} a_3^s h_{3,0} h_{2,1} h_{1,2}) = p + 7s + 11.$$

Proof (1) Since it is known that $h_{1,i}$, $b_{1,i}$, $h_{2,0}h_{1,1}$ and $a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{*,*,*}$ are all permanent cocycles in the May spectral sequence and converge nontrivially to h_i, b_i, k_0, $\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ for $0 \leq s < p$ and $i \geq 0$ respectively (cf. Theorem 3.1), we have that

$$h_{2,0} h_{1,1} b_{1,0} h_{1,n} a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{s+8,t(s,n),*}$$

is a permanent cocycle in the May spectral sequence and converges to

$$k_0 b_0 h_n \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

(2) It is easy to get the desired results.

By Lemmas 3.1–3.2, we have the following corollary.

Corollary 3.1 *For the May E_1-module G_1 in Lemma 3.1, we have*

$$G_1 = E_1^{p+1,t(p-6,n),10p-29},$$

where

$$E_1^{p+1,t(p-6,n),10p-29} = \mathbb{Z}/p\{a_3^{p-6} h_{3,0} h_{2,0} h_{2,1} h_{1,1} h_{1,n} b_{2,0}\}.$$

For the May E_1-module G_2 in Lemma 3.1, we have

$$G_2 = E_1^{p,t(p-7,n),10p-36},$$

where

$$E_1^{p+1,t(p-7,n),10p-36} = \mathbb{Z}/p\{a_3^{p-7} h_{3,0} h_{2,0} h_{2,1} h_{1,1} h_{1,n} b_{2,0}\}.$$

To show the non-triviality of the product $k_0b_0h_0\widetilde{\gamma}_{s+3}$, we need to show the following two lemmas.

Lemma 3.3 *The May E_r-module $E_r^{p+1,t(p-6,n),10p-29}=0$ for $r\geq 2$.*

Proof From Corollary 3.1,

$$E_1^{p+1,t(p-6,n),10p-29}=\mathbb{Z}/p\{a_3^{p-6}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}\}.$$

By use of (2.2)–(2.3), we have that up to sign

$$\mathrm{d}_1(a_3^{p-6}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0})=\underline{a_3^{p-7}a_2h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,2}h_{1,n}b_{2,0}}+\cdots\neq 0,$$

showing

$$E_2^{p+1,t(p-6,n),10p-29}=0.$$

Then it follows that

$$E_r^{p+1,t(p-6,n),10p-29}=0$$

for $r\geq 2$. The proof of Lemma 3.3 is completed.

Lemma 3.4 *The May E_r-module $E_r^{p+1,t(p-7,n),10p-36}=0$ for $r\geq 2$.*

Proof From Corollary 3.1,

$$E_1^{p+1,t(p-7,n),10p-36}=\mathbb{Z}/p\{a_3^{p-7}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0}\}.$$

By use of (2.2)–(2.3), we have that up to sign

$$\mathrm{d}_1(a_3^{p-6}h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,n}b_{2,0})=\underline{a_3^{p-7}a_2h_{3,0}h_{2,0}h_{2,1}h_{1,1}h_{1,2}h_{1,n}b_{2,0}}+\cdots\neq 0,$$

showing

$$E_2^{p,t(p-7,n),10p-36}=0.$$

Then it follows that

$$E_r^{p,t(p-7,n),10p-36}=0$$

for $r\geq 2$. The proof of this lemma is completed.

By use of Lemmas 3.3–3.4, we can prove the non-triviality of the product $k_0b_0h_n\widetilde{\gamma}_{s+3}$ as follows.

Theorem 3.2 *Let $p\geq 7$, $n>3$, $0\leq s<p-5$. Then the product*

$$k_0b_0h_n\widetilde{\gamma}_{s+3}(\neq 0)\in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p,\mathbb{Z}/p),$$

where $t(s,n)=p^nq+(s+3)p^q+(s+5)pq+(s+2)q+s$.

Proof From Lemma 3.2(1), the product $k_0b_0h_n\widetilde{\gamma}_{s+3}\in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p,\mathbb{Z}/p)$ is represented by $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^sh_{3,0}h_{2,1}h_{1,2}\in E_1^{s+8,t(s,n),*}$ in the May spectral sequence. Now we show that nothing hits $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^sh_{3,0}h_{2,1}h_{1,2}$ under the May differential d_r for $r\geq 1$.

We divide the proof into the following three cases.

Case 1 When $0 \leq s < p-7$, from Lemma 3.1 we know that in the May spectral sequence

$$E_1^{s+7,t(s,n),*} = 0.$$

Then we have that in the May spectral sequence

$$E_r^{s+7,t(s,n),*} = 0 \quad (r \geq 1).$$

From (2.2), the permanent cocycle $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+8,t(s,n),*}$ does not bound and converges nontrivially to $k_0 b_0 h_n \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the May spectral sequence. It follows that $k_0 b_0 h_n \widetilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Case 2 When $s = p-7$, from Lemma 3.1 and Corollary 3.1, we have

$$G = E_1^{p,t(p-7,n),10p-36}.$$

By Corollary 3.2 [2], we have

$$M(h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2}) = 8p-38.$$

By direct computations, we have

$$M(E_1^{p,t(p-7,n),10p-36}) - (8p-38) = 2p+2 \geq 16.$$

Thus by the reason of May filtration, we have

$$h_{2,0}h_{1,1}h_{1,0}b_{1,n-1}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2} \notin \mathrm{d}_1(E_1^{p,t(p-7,n),10p-36}).$$

Moreover, by Lemma 3.4 one has

$$E_r^{p,t(p-7,n),10p-36} = 0 \quad (r \geq 2).$$

From the above discussion, the permanent cocycle $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2}$ cannot be hit by any differential in the May spectral sequence. Consequently, $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2}$ converges nontrivially to $k_0 b_0 h_n \widetilde{\gamma}_{p-4} \in \mathrm{Ext}_A^{p+1,t(p-7,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the May spectral sequence. It follows that

$$k_0 b_0 h_n \widetilde{\gamma}_{p-4} \neq 0 \in \mathrm{Ext}_A^{p+1,t(p-7,n)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Case 3 When $s = p-6$, from Lemma 3.1 and Corollary 3.1, we have

$$G = E_1^{p+1,t(p-6,n),10p-29}.$$

By Lemma 3.2, we have

$$M(h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-6}h_{3,0}h_{2,1}h_{1,2}) = 8p-31.$$

By direct computations, we have

$$M(E_1^{p+1,t(p-6,n),10p-29}) - (8p-31) = 2p-2 \geq 12.$$

Thus by the reason of May filtration, we have

$$h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-6}h_{3,0}h_{2,1}h_{1,2} \notin \mathrm{d}_1(E_1^{p+1,t(p-6,n),10p-29}).$$

Moreover, using Lemma 3.3, one has

$$E_r^{p+1,t(p-6,n),10p-29} = 0 \quad (r \geq 2).$$

From the above discussion, the permanent cocycle $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-6}h_{3,0}h_{2,1}h_{1,2}$ cannot be hit by any differential in the May spectral sequence. Thus, $h_{2,0}h_{1,1}b_{1,0}h_{1,n}a_3^{p-6}h_{3,0}h_{2,1}h_{1,2}$ converges nontrivially to $k_0b_0h_n\widetilde{\gamma}_{p-3}$ in the May spectral sequence. Consequently,

$$k_0b_0h_n\widetilde{\gamma}_{p-3} \neq 0 \in \mathrm{Ext}_A^{p+2,t(p-6,n)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

From Cases 1–3, the desired result follows.

Theorem 3.3 *Let $p \geq 7$, $n > 3$, $0 \leq s < p-5$, $2 \leq r \leq s+7$. Then*

$$\mathrm{Ext}_A^{s+8-r,t(s,n)+1-r}(\mathbb{Z}/p, \mathbb{Z}/p) = 0,$$

where $t(s,n) = q[p^n + (s+3)p^q + (s+5)pq + (s+2)] + s$.

Proof From Lemma 3.1, in this case

$$E_1^{s+8-r,t(s,n)+1-r,*} = 0.$$

By the May spectral sequence, the desired result follows.

4 Proof of the Main Result

We are now in a position to prove the main theorem in this paper. It is easy to see that to prove Theorem 1.2 is equivalent to proving the following theorem.

Theorem 4.1 *Let $p \geq 7$, $n > 3$, $0 \leq s < p-5$. Then the product*

$$k_0b_0h_n\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is a permanent cycle in the Adams spectral sequence, and converges nontrivially to the composite map

$$\beta_1\omega_n\gamma_{s+3} \in \pi_{t(s,n)-s-8}(S)$$

of order p, where $t(s,n) = q[p^n + (s+3)p^2 + (s+5)p + (s+2)] + s$.

Proof We know that β_1, ω_n and γ_{s+3} are represented in the Adams spectral sequence by b_0, k_0h_n and $\widetilde{\gamma}_{s+3}$, respectively. Thus, the composite map

$$\beta_1\omega_n\gamma_{s+3}$$

is represented by

$$k_0b_0h_n\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$$

up to nonzero scalar in the Adams spectral sequence.

By Theorem 3.1 and the knowledge of Yoneda products, we know that the composite

$$(j_0j_1j_2\gamma^{s+3}i_2i_1i_0)_* : \mathrm{Ext}_A^{0,*}(\mathbb{Z}/p, \mathbb{Z}/p) \xrightarrow{(i_2i_1i_0)_*} \mathrm{Ext}_A^{0,*}(V(2), \mathbb{Z}/p)$$
$$\xrightarrow{(j_0j_1j_2\gamma^{s+3})_*} \mathrm{Ext}_A^{s+3,*+(s+3)p^2q(s+2)pq+(s+1)q+s}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is a multiplication up to nonzero scalar by

$$\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,(s+3)p^2q(s+2)pq+(s+1)q+s}(\mathbb{Z}/p, \mathbb{Z}/p).$$

It follows that the composite map $\beta_1\omega_n\gamma_{s+3}$ is represented by

$$k_0b_0h_n\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$$

up to nonzero scalar in the Adams spectral sequence.

By Theorem 3.2, $k_0b_0h_n\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathbb{Z}/p, \mathbb{Z}/p)$ is non-trivial. Meanwhile, by Theorem 3.3 and (2.2), we see that $k_0b_0h_n\widetilde{\gamma}_{s+3}$ cannot be hit by any differential in the Adams spectral sequence. Consequently, the corresponding family of homotopy elements $\beta_1\omega_n\gamma_{s+3}$ in the stable homotopy groups of spheres is non-trivial and of order p. The proof of Theorem 4.1 is completed.

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References

[1] Adams, J. F., Stable homotopy and generalised homology, Univ. Chicago Press, Chicago, 1974.

[2] Aikawa, T., 3-dimensional cohomology of the mod p Steenrod algebra, *Math. Scand.,* **47**(1), 1980, 91–115.

[3] Cohen, R., Odd primary infinite families in stable homotopy theory, *Mem. Amer. Math. Soc.,* **30**(242), 1981, 92pp.

[4] Liu, X., A nontrivial product in the stable homotopy groups of spheres, *Sci. China Ser. A,* **47**(6), 2004, 831–841.

[5] Liu, X., Detection of some elements in the stable homotopy groups of spheres, *Chinese Ann. Math. Ser. B,* **29**(3), 2008, 291–316.

[6] Liulevicius, A., The factorizations of cyclic reduced powers by secondary cohomology operations, *Mem. Amer. Math. Soc.,* **42**, 1962, 112pp.

[7] Ravenel, D. C., Complex cobordism and stable homotopy groups of spheres, Academic Press, Orlando, 1986.

[8] Toda, H., On spectra realizing exterior parts of Steenord algebra, *Topology,* **10**, 1971, 55–65.

[9] Wang, X. and Zheng, Q., The convergence of $\widetilde{\alpha}_{s+4}^{(n)}h_0h_k$, *Sci. China Ser. A,* **41**(6), 1998, 622–628.