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An Intrinsic Rigidity Theorem for Closed Minimal Hypersurfaces in $\mathbb{S}^5$ with Constant Nonnegative Scalar Curvature*

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Abstract Let M^4 be a closed minimal hypersurface in $\mathbb{S}^5$ with constant nonnegative scalar curvature. Denote by f_3 the sum of the cubes of all principal curvatures, by g the number of distinct principal curvatures. It is proved that if both f_3 and g are constant, then M^4 is isoparametric. Moreover, the authors give all possible values for squared length of the second fundamental form of M^4. This result provides another piece of supporting evidence to the Chern conjecture.

Keywords Chern conjecture, Isoparametric hypersurfaces, Scalar curvature, Minimal hypersurfaces in spheres
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1 Introduction

More than 40 years ago, Chern [6–7] proposed the following problem in several places.

Problem 1.1 *Let M^n be a closed minimal submanifold in $\mathbb{S}^{n+m}$ with the second fundamental form of constant length. Denote by $\mathcal{A}_n$ the set of all the possible values for the squared length of the second fundamental form of M^n. Is $\mathcal{A}_n$ a discrete set?*

The affirmative hand of this question is usually called the Chern conjecture.

Denote by B the second fundamental form of M^n and let $S := |B|^2$. Using the Gauss equations, one can easily deduce that

$$S = n(n-1) - R$$

with R denoting the scalar curvature of M^n. It means that S is in fact an intrinsic geometric quantity, and the Chern conjecture is equivalent to claiming that the scalar curvature R has gap phenomena for closed minimal submanifolds in Euclidean spheres.

Up to now, it is far from a complete solution of this problem, even in the case that M is a hypersurface (see [15, Problem 105]). Moreover, because all known examples of closed minimal hypersurfaces in $\mathbb{S}^{n+1}$ with constant scalar curvature are all isoparametric hypersurfaces (the definition of isoparametric hypersurfaces will be introduced in Section 2), mathematicians

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turned the hypersurface case of Chern conjecture into the following new formulation (see [12, 14]).

Conjecture 1.1 *Let M^n be a closed minimal hypersurface in $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is an isoparametric hypersurface.*

When $n = 2$, this conjecture is trivial. For the case that $n = 3$, Chang [4–5] gave a positive answer to the Chern conjecture. More precisely, it was shown that any closed minimal hypersurface M^3 in $\mathbb{S}^4$ with constant scalar curvature has to be isoparametric, and $\mathcal{A}_3 = \{0, 3, 6\}$.

For $n \geq 4$, the Chern conjecture remains open, although some partial results exist for low dimensions and with additional conditions for the curvature functions, such as the following theorem.

Theorem 1.1 (see [8]) *Let M^4 be a closed minimal Willmore hypersurface in $\mathbb{S}^5$ with constant nonnegative scalar curvature. Then M^4 is isoparametric.*

Theorem 1.2 (see [11]) *Let $M^6 \subset \mathbb{S}^7$ be a closed hypersurface with $H = f_3 = f_5 \equiv 0$, constant f_4 and $R \geq 0$. Then M^6 is isoparametric.*

Here and in the sequel

$$f_k := \sum_{i=1}^{n} \lambda_i^k$$

with $\lambda_1, \cdots, \lambda_n$ being the principal curvatures of M.

Note that in Theorem 1.1, the Willmore condition is equal to saying that $f_3 \equiv 0$. It is natural to ask whether this conclusion holds when $f_3 \equiv 0$ is replaced by a weaker condition that $f_3 \equiv \text{const}$. In this paper, we give a partial positive answer to the above question and obtain the main theorem as follows.

Theorem 1.3 *Let M^4 be a closed minimal hypersurface in $\mathbb{S}^5$ with constant nonnegative scalar curvature. If f_3 and the number g of distinct principal curvatures of M^4 are constant, then M^4 is isoparametric.*

Finally, in conjunction with the theory of isoparametric hypersurfaces in Euclidean spheres, we arrive at a classification result (see Theorem 3.1), which gave a piece of supporting evidence to the Chern conjecture.

2 Isoparametric Minimal Hypersurfaces in $\mathbb{S}^5$

Let M^n be an immersed hypersurface in $\mathbb{S}^{n+1}$. If M^n has constant principal curvatures, then M^n is said to be an isoparametric hypersurface. Each isoparametric hypersurface is an open subset of a level set of a so-called isoparametric function f. More precisely, there exists a smooth function $f : \mathbb{S}^{n+1} \to \mathbb{R}$ and $c \in \mathbb{R}$, such that $|\overline{\nabla} f|^2$ and $\overline{\Delta} f$ are both smooth functions of f ($\overline{\nabla}$ and $\overline{\Delta}$ are respectively the gradient operator and Laplace-Beltrami operator on $\mathbb{S}^{n+1}$), and $f(p) = c$ for each $p \in M$. Conversely, given an isoparametric function f, the level sets of f consist of a smooth family of isoparametric hypersurfaces and 2 minimal submanifolds of higher codimension (called focal submanifolds).

The following theorem reveals some important geometric properties of isoparametric minimal hypersurfaces in Euclidean spheres (see [1–2, 9–10]).

Theorem 2.1 *Let $f:\mathbb{S}^{n+1}\to\mathbb{R}$ be an isoparametric function. Then there exists a unique $c_0\in\mathbb{R}$, such that $M:=\{x\in\mathbb{S}^{n+1}: f(x)=c_0\}$ is an isoparametric minimal hypersurface. Let g be the number of distinct principal curvatures of M, $\lambda_1>\cdots>\lambda_g$ be the distinct principal curvatures, whose multiplicities are $m_1,\cdots,m_g$, respectively, and the denotation of S and R is the same as above. Then*

(1) $g=1,2,3,4$ *or* 6.

(2) *If $g=1$, then M has to be the totally geodesic great subsphere.*

(3) *If $g=2$, then M has to be a Clifford hypersurface, i.e.,*

$$M=M_{r,s}:=\mathbb{S}^r\Big(\sqrt{\frac{r}{n}}\Big)\times\mathbb{S}^s\Big(\sqrt{\frac{s}{n}}\Big),$$

where $1\le r<s\le n$ and $r+s=n$.

(4) *If $g=3$, then $m_1=m_2=m_3=2^r$ $(r=0,1,2$ or $3)$.*

(5) *There exists $\theta_0\in\big(0,\frac{\pi}{g}\big)$, such that*

$$\lambda_k=\cot\Big(\frac{(k-1)\pi}{g}+\theta_0\Big),\quad k=1,\cdots,g,$$
$$m_k=m_{k+2}\quad (k \mod g).$$

(6) *$R\ge 0$ and $S=(g-1)n$.*

Cartan [3] constructed an example of minimal hypersurface in $\mathbb{S}^5$ as follows.

Example 2.1 Denote

$$F:=\Big(\sum_i^3(x_i^2-x_{i+3}^2)\Big)^2+4\Big(\sum_i^3 x_ix_{i+3}\Big)^2.$$

For a number t with $0<t<\frac{\pi}{4}$, we denote by $M^4(t)$ a hypersurface in S^5 defined by the equation

$$F(x)=\cos^2(2t),\quad x=(x_1,\cdots,x_6)\in\mathbb{S}^5.$$

A straightforward calculation shows that $f:=F|_{\mathbb{S}^5}$ is an isoparametric function and $M^4(\frac{\pi}{8})$ is a minimal isoparametric hypersurface with 4 distinct principal curvatures, which is usually called the Cartan minimal hypersurface.

Takagi [13] proved that $M^4(\frac{\pi}{8})$, up to congruence, is the unique isoparmetric hypersurface in $\mathbb{S}^5$ with 4 distinct principal curvatures. In conjunction with Theorem 2.1, we obtain the following result.

Proposition 2.1 *Let M^4 be an isoparametric minimal hypersurface in $\mathbb{S}^5$. Then M^4, up to a congruence, is either an equator S^3, a Clifford hypersurface $\big(\mathbb{S}^1\big(\frac{1}{2}\big)\times\mathbb{S}^3\big(\frac{\sqrt{3}}{2}\big)$ or $\mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big)\times\mathbb{S}^3\big(\frac{\sqrt{2}}{2}\big)\big)$ or then Cartan minimal hypersurface $M^4(\frac{\pi}{8})$, and $S=0,4$ or 12.*

3 Proof of the Main Theorem

Let M^4 be an immersed hypersurface in $\mathbb{S}^5$. If ν is a local unit normal vector field along M, then there exists a pointwise symmetric bilinear form h on T_pM, such that

$$B = h\nu.$$

If $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a smooth orthonormal coframe field, then h can be written as

$$h = h_{ij}\omega_i \otimes \omega_j.$$

The covariant derivative ∇h with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = \mathrm{d}h_{ij} + \sum_k h_{kj}\omega_{ik} + \sum_k h_{ik}\omega_{jk}. \tag{3.1}$$

Here $\{\omega_{ij}\}$ is the connection forms of M^4 with respect to $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, which satisfy the following structure equations:

$$\begin{aligned} \mathrm{d}\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ \mathrm{d}\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l \end{aligned} \tag{3.2}$$

with R_{ijkl} denoting the coefficients of the Riemannian curvature tensor on M^4.

In this section, we shall give a proof of the main theorem in Section 1.

Proof of Theorem 1.3 We shall consider this problem case by case, according to the value of g, i.e., the number of distinct principal curvatures.

Case I $g = 1$.

In this case, all the principal curvatures are equal to 0 and hence M^4 is totally geodesic.

Case II $g = 2$.

Let λ and μ be distinct pricipal curvatures of M^4 with multiplicities $m_1 = k,\ m_2 = 4 - k$, respectively. We need to show that λ, μ are indeed constant functions.

Since $\lambda \neq \mu$, from

$$\begin{aligned} m_1\lambda + m_2\mu &= 0, \\ m_1\lambda^2 + m_2\mu^2 &= S, \end{aligned} \tag{3.3}$$

we can solve m_1, m_2 in terms of λ, μ and S, in other words, m_1, m_2 can be seen as continuous functions of λ, μ and S. In conjunction with the fact that m_1, m_2 take values in $\mathbb{Z}$, both m_1, m_2 are constant, so does k. Again from (3.3), we have

$$\lambda = \frac{\sqrt{k(4-k)S}}{2k}, \quad \mu = -\frac{\sqrt{kS}}{2\sqrt{4-k}}, \tag{3.4}$$

or

$$\lambda = -\frac{\sqrt{k(4-k)S}}{2k}, \quad \mu = \frac{\sqrt{kS}}{2\sqrt{4-k}}. \tag{3.5}$$

Thus λ and μ are both constant and M^4 is an isoparametric hypersurface.

Case III $g = 3$.

Let λ, μ, σ be distinct principal curvatures of M^4, with multiplicities p, q, r, respectively. Then

$$\begin{cases} p + q + r = 4, \\ p\lambda + q\mu + r\sigma = 0, \\ p\lambda^2 + q\mu^2 + r\sigma^2 = S, \\ p\lambda^3 + q\mu^3 + r\sigma^3 = f_3. \end{cases} \tag{3.6}$$

As in Case II, one can show that p, q, r are all constant integer-valued functions. Differentiating both sides of (3.6) gives

$$\begin{cases} p\mathrm{d}\lambda + q\mathrm{d}\mu + r\mathrm{d}\sigma = 0, \\ p\lambda\mathrm{d}\lambda + q\mu\mathrm{d}\mu + r\sigma\mathrm{d}\sigma = 0, \\ p\lambda^2\mathrm{d}\lambda + q\mu^2\mathrm{d}\mu + r\sigma^2\mathrm{d}\sigma = \dfrac{1}{3}\mathrm{d}f_3 = 0. \end{cases} \tag{3.7}$$

It follows that

$$\frac{p\mathrm{d}\lambda}{\sigma - \mu} = \frac{q\mathrm{d}\mu}{\lambda - \sigma} = \frac{r\mathrm{d}\sigma}{\mu - \lambda} = \frac{\mathrm{d}f_3}{3D} = 0, \tag{3.8}$$

where $D := (\sigma - \mu)(\sigma - \lambda)(\mu - \lambda)$. Hence λ, μ and σ are all constant and M^4 is isoparametric. (In fact, Theorem 2.1 shows that there exists no isoparametric minimal hypersurface in $\mathbb{S}^5$ with $g = 3$, so this case cannot occur.)

Case IV $g = 4$.

Let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be distinct principal curvatures of M^4. We say that a coframe field (U, ω) is admissible (see [11]) if

(1) U is an open subset of M^4,

(2) $\omega := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a smooth orthonormal coframe field on U,

(3) $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$ is the volume form of M^4,

(4) $h = \sum\limits_i \lambda_i \omega_i \otimes \omega_i$.

Denote by $F := \{e_1, e_2, e_3, e_4\}$ the dual frame field of ω. Then it is easily-seen that, (U, ω) is admissible if and only if e_i is a unit principal vector associated to λ_i for each $1 \le i \le 4$, and $\{e_1, e_2, e_3, e_4\}$ is an oriented basis associated to the orientation of M^4. Therefore, for every $p \in M$, there exists an admissible coframe field (U, ω), such that $p \in U$.

Now we introduce a 3-form on M^4: For every admissible coframe field (U, ω), set

$$\psi := \sum_{1 \le i < j \le 4} (*(\omega_i \wedge \omega_j)) \wedge \omega_{ij}, \tag{3.9}$$

where $*$ is the Hodge star operator. If (U, ω) and $(\widetilde{U}, \widetilde{\omega})$ are both admissible coframe fields with $W := U \cap \widetilde{U} \neq \emptyset$, then on W, $\widetilde{\omega}_i = \alpha_i \omega_i$ for each $1 \le i \le 4$, where $\alpha_i = 1$ or -1 and $\prod\limits_{i=1}^{4} \alpha_i = 1$. Denote by $\{\widetilde{\omega}_{ij}\}$ the connection form with respect to $(\widetilde{U}, \widetilde{\omega})$. Then $\widetilde{\omega}_{ij} = \alpha_i \alpha_j \omega_{ij}$ and hence

$$(*(\widetilde{\omega}_i \wedge \widetilde{\omega}_j)) \wedge \widetilde{\omega}_{ij} = (*(\omega_i \wedge \omega_j)) \wedge \omega_{ij}$$

holds for any $i < j$. Therefore ψ is well-defined on M^4.

Now we compute the exterior differential of the form ψ. Due to the definition of the Hodge star operator, ψ can be written as

$$\begin{aligned}\psi &= \omega_1 \wedge \omega_2 \wedge \omega_{34} + \omega_2 \wedge \omega_3 \wedge \omega_{14} + \omega_3 \wedge \omega_1 \wedge \omega_{24} \\ &\quad + \omega_1 \wedge \omega_4 \wedge \omega_{23} + \omega_2 \wedge \omega_4 \wedge \omega_{31} + \omega_3 \wedge \omega_4 \wedge \omega_{12}. \end{aligned} \tag{3.10}$$

Substituting $h_{ij} = \lambda_i \delta_{ij}$ into (3.1), we have

$$\omega_{ij} = \frac{1}{\lambda_j - \lambda_i} \sum_k h_{ijk} \omega_k, \quad \forall i \neq j. \tag{3.11}$$

Combining (3.11) and (3.2) yields

$$\begin{aligned} \mathrm{d}\omega_1 &= -(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3 + \omega_{14} \wedge \omega_4) \\ &= (\cdots) \wedge \omega_2 - \frac{1}{\lambda_3 - \lambda_1}(h_{131}\omega_1 + h_{134}\omega_4) \wedge \omega_3 \\ &\quad - \frac{1}{\lambda_4 - \lambda_1}(h_{141}\omega_1 + h_{143}\omega_3) \wedge \omega_4. \end{aligned}$$

Hence

$$\begin{aligned} &\mathrm{d}\omega_1 \wedge \omega_2 \wedge \omega_{34} \\ &= -\Big[\frac{h_{113}h_{443}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_4)} + \frac{h_{114}h_{334}}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3)} + \frac{h_{134}^2}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)}\Big] * 1, \end{aligned} \tag{3.12}$$

where we have used Codazzi equations. A similar calculation shows

$$\begin{aligned} &\omega_1 \wedge \mathrm{d}\omega_2 \wedge \omega_{34} \\ &= \Big[\frac{h_{223}h_{443}}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + \frac{h_{224}h_{334}}{(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} + \frac{h_{234}^2}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)}\Big] * 1. \end{aligned} \tag{3.13}$$

By the structure equations,

$$\begin{aligned} \mathrm{d}\omega_{34} &= -\omega_{31} \wedge \omega_{32} \wedge_{24} + \frac{1}{2} \sum_{k,l} R_{34kl} \omega_k \wedge \omega_l \\ &= \Big[\frac{h_{331}h_{441}}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{h_{332}h_{442}}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} - \frac{h_{134}^2}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \\ &\quad - \frac{h_{234}^2}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} + R_{3434}\Big] \omega_3 \wedge \omega_4 + (\cdots) \wedge \omega_1 + (\cdots) \wedge \omega_2. \end{aligned} \tag{3.14}$$

Combining (3.12)–(3.14) gives

$$\begin{aligned} &\mathrm{d}(\omega_1 \wedge \omega_2 \wedge \omega_{34}) = \mathrm{d}\omega_1 \wedge \omega_2 \wedge \omega_{34} - \omega_1 \wedge \mathrm{d}\omega_2 \wedge \omega_{34} + \omega_1 \wedge \omega_2 \wedge \mathrm{d}\omega_{34} \\ &= \Big[\frac{h_{331}h_{441}}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{h_{332}h_{442}}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} - \frac{h_{113}h_{443}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_4)} \\ &\quad - \frac{h_{114}h_{334}}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3)} - \frac{h_{223}h_{443}}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + \frac{h_{224}h_{334}}{(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} \\ &\quad - \frac{2h_{134}^2}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} - \frac{2h_{234}^2}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} + R_{3434}\Big] * 1. \end{aligned} \tag{3.15}$$

Similarly, one can compute the exterior differential of each term of (3.10); taking the sum of these equations, we arrive at

$$\mathrm{d}\psi = \Big(\frac{1}{2}R - \sum_{l=1}^{4} \mathrm{I}_l\Big) * 1, \tag{3.16}$$

where

$$\mathrm{I}_l = \sum_{l\neq i<j\neq l} \frac{h_{iil}h_{jjl}}{(\lambda_l-\lambda_i)(\lambda_l-\lambda_j)}, \quad \forall l = 1,2,3,4. \tag{3.17}$$

Taking the exterior differential of

$$\begin{cases} \sum_i h_{ii} = 0, \\ \sum_{i,j} h_{ij}^2 = S = \text{const.}, \\ \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = f_3 = \text{const.} \end{cases} \tag{3.18}$$

implies that

$$\begin{cases} \sum_i h_{iik} = 0, \\ \sum_i \lambda_i h_{iik} = 0, \\ \sum_i \lambda_i^2 h_{iik} = 0 \end{cases} \tag{3.19}$$

holds for each $1 \le k \le 4$. Especially, letting $k := 1$ gives

$$\begin{cases} h_{111} + h_{221} + h_{331} + h_{441} = 0, \\ \lambda_1 h_{111} + \lambda_2 h_{221} + \lambda_3 h_{331} + \lambda_4 h_{441} = 0, \\ \lambda_1^2 h_{111} + \lambda_2^2 h_{221} + \lambda_3^2 h_{331} + \lambda_4^2 h_{441} = 0. \end{cases} \tag{3.20}$$

Since $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are distinct at every point, we can express $h_{ii1},\ i = 2,3,4$, in terms of h_{111}:

$$h_{ii1} = -\frac{\prod_{j\neq i,1}(\lambda_j - \lambda_1)}{\prod_{j\neq i,1}(\lambda_j - \lambda_i)} h_{111}, \quad \forall i = 2,3,4. \tag{3.21}$$

Let $K := \det h$ be the Gauss-Kronecker curvature of M^4 and denote

$$\mathrm{d}K = \sum_i K_i \omega_i.$$

Then

$$K_1 = \sum_{i=1}^{4}\Big(h_{ii1}\prod_{j\neq i}\lambda_j\Big) = -(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)h_{111} \tag{3.22}$$

and hence

$$h_{ii1} = \frac{K_1}{\prod\limits_{j\neq i}(\lambda_j - \lambda_i)}. \tag{3.23}$$

In a similar way, we have

$$h_{iil} = \frac{K_l}{\prod\limits_{j\neq i}(\lambda_j - \lambda_i)}, \quad \forall i, l = 1, 2, 3, 4. \tag{3.24}$$

Substituting (3.24) into (3.17), we deduce that

$$\mathrm{I}_l = K_l^2 \sum_{l\neq i<j\neq l} \frac{1}{(\lambda_l - \lambda_i)(\lambda_l - \lambda_j)\prod\limits_{m\neq i}(\lambda_m - \lambda_i)\prod\limits_{m\neq j}(\lambda_m - \lambda_j)}. \tag{3.25}$$

More precisely,

$$\begin{aligned}
\mathrm{I}_1 &= K_1^2 \sum_{1\neq i<j\neq 1} \frac{1}{(\lambda_1 - \lambda_i)(\lambda_1 - \lambda_j)\prod\limits_{m\neq i}(\lambda_m - \lambda_i)\prod\limits_{l\neq j}(\lambda_l - \lambda_j)} \\
&= K_1^2\Big[\frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\prod\limits_{m\neq 2}(\lambda_m - \lambda_2)\prod\limits_{l\neq 3}(\lambda_l - \lambda_3)} \\
&\quad + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_4)\prod\limits_{m\neq 2}(\lambda_m - \lambda_2)\prod\limits_{l\neq 3}(\lambda_l - \lambda_4)} \\
&\quad + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)\prod\limits_{m\neq 3}(\lambda_m - \lambda_3)\prod\limits_{l\neq 4}(\lambda_l - \lambda_4)}\Big] \\
&= -\frac{K_1^2}{D^2}[(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)^2 + (\lambda_3 - \lambda_4)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)^2 \\
&\quad + (\lambda_2 - \lambda_4)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)^2],
\end{aligned} \tag{3.26}$$

where $D := \prod\limits_{1\leq i<j\leq 4}(\lambda_j - \lambda_i)$. Similarly, one computes

$$\begin{aligned}
\mathrm{I}_2 &= -\frac{K_2^2}{D^2}[(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_1) + (\lambda_3 - \lambda_4)(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_1) \\
&\quad + (\lambda_1 - \lambda_4)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)^2)],
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\mathrm{I}_3 &= -\frac{K_3^2}{D^2}[(\lambda_4 - \lambda_3)^2(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1) + (\lambda_2 - \lambda_4)(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_1) \\
&\quad + (\lambda_1 - \lambda_4)(\lambda_1 - \lambda_3)^2(\lambda_1 - \lambda_2)]
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\mathrm{I}_4 &= -\frac{K_4^2}{D^2}[(\lambda_3 - \lambda_4)^2(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1) + (\lambda_2 - \lambda_4)^2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1) \\
&\quad + (\lambda_1 - \lambda_4)^2(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)].
\end{aligned} \tag{3.29}$$

Observing that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$, we can derive estimates as follows:

$$\mathrm{I}_1 = -\frac{K_1^2}{D^2}[(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)^2 + (\lambda_3 - \lambda_4)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)^2$$

$$
\begin{aligned}
&\quad + (\lambda_2-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)^2] \\
&\le -\frac{K_1^2}{D^2}[(\lambda_4-\lambda_3)(\lambda_3-\lambda_2)(\lambda_4-\lambda_1)^2 + (\lambda_3-\lambda_4)(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)^2 \\
&\quad + (\lambda_2-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)^2] \\
&= -\frac{K_1^2}{D^2}[(\lambda_4-\lambda_3)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)(\lambda_4+\lambda_3-2\lambda_1) \\
&\quad + (\lambda_2-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)^2] \\
&\le 0,
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
\mathrm{I}_2 &= -\frac{K_2^2}{D^2}[(\lambda_4-\lambda_3)(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_1) + (\lambda_3-\lambda_4)(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_1) \\
&\quad + (\lambda_1-\lambda_4)(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)^2)] \\
&\le -\frac{K_2^2}{D^2}[(\lambda_4-\lambda_3)(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_1) + (\lambda_3-\lambda_4)(\lambda_3-\lambda_2)^2(\lambda_4-\lambda_1) \\
&\quad + (\lambda_1-\lambda_4)(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)^2)] \\
&= -\frac{K_2^2}{D^2}[(\lambda_4-\lambda_3)(\lambda_4-\lambda_1)(\lambda_4-\lambda_3)(\lambda_4+\lambda_3-2\lambda_2) \\
&\quad + (\lambda_1-\lambda_4)(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)^2)] \\
&\le 0.
\end{aligned} \tag{3.31}
$$

In the same way, $\mathrm{I}_3 \le 0,\ \mathrm{I}_4 \le 0$.

Note that M^4 is closed. Integrating both sides of (3.16) on M^4 and then using Stokes's theorem gives

$$
0 = \int_{M^4} \mathrm{d}\psi = \frac{1}{2}\int_{M^4} R \ *1\mathrm{d}\psi - \int_{M^4} \sum_k \mathrm{I}_k \ *1\mathrm{d}\psi. \tag{3.32}
$$

Since $R \ge 0$ and $\mathrm{I}_k \le 0$ for $k = 1,2,3,4$, it follows that $R = 0$ and $\mathrm{I}_k = 0,\ k = 1,2,3,4$. From (3.25), $\mathrm{d}K = 0$, so $\prod\limits_{i=1}^{4} \lambda_i = K = \text{const}$. In conjunction with $\sum\limits_i \lambda_i = 0$, $\sum\limits_i \lambda_i^2 = S = \text{const}.$ and $\sum\limits_i \lambda_i^3 = f_3 = \text{const}.$, one can easily deduce that λ_i $(1 \le i \le 4)$ are all constant on M. Thus M^4 is an isoparametric hypersurface.

Combining Theorem 1.3 and Proposition 2.1 yields a classification theorem as follows.

Theorem 3.1 *Let M^4 be a closed minimal hypersurface in $\mathbb{S}^5$ with constant nonnegative scalar curvature. If f_3 and the number g of distinct principal curvatures of M^4 are constant, then M^4, up to a congruence, is either an equator S^3, a Clifford hypersurface $\big(\mathbb{S}^1\big(\frac{1}{2}\big) \times \mathbb{S}^3\big(\frac{\sqrt{3}}{2}\big)$ or $\mathbb{S}^2\big(\frac{\sqrt{2}}{2}\big) \times \mathbb{S}^3\big(\frac{\sqrt{2}}{2}\big)\big)$ or then Cartan minimal hypersurface $M^4(\frac{\pi}{8})$. Let S denote the squared length of the second fundamental form of M^4. Then $S = 0, 4$ or 12.*

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