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Stability of Rarefaction Wave to the 1-D Piston Problem for the Pressure-Gradient Equations*

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Abstract The 1-D piston problem for the pressure gradient equations arising from the flux-splitting of the compressible Euler equations is considered. When the total variations of the initial data and the velocity of the piston are both sufficiently small, the author establishes the global existence of entropy solutions including a strong rarefaction wave without restriction on the strength by employing a modified wave front tracking method.

Keywords Piston problem, Pressure gradient equations, Rarefaction wave, Wave front tracking method, Interaction of waves
2000 MR Subject Classification 35A01, 35L50, 35Q35, 35R35, 76N10

1 Introduction

The piston problem is a special initial-boundary value problem in fluid dynamics which can be described as follows (see [8, 16]). In a thin long tube closed at one end by a piston and open at the other end, any motion of the piston causes the corresponding motion of the gas in the tube. More precisely, if the piston is pulled backward relatively to the gas, a rarefaction wave occurs and moves forward faster than the piston. Otherwise, a shock wave appears. In reality, there also exist many multidimensional piston models, for example, the surface of an inflatable balloon behaves as a spherically symmetric piston. The gas outside is compressed by the expansion of the balloon, then a shock appears. When the location of the piston initially degenerate into a single point, this phenomenon is related to explosive waves in physics. There are many literatures on the existence and stability of shock front solutions for the classical fluid. For one dimensional case, the global existence of strong shock front solutions to the 1-D piston problem for the compressible isentropic Euler equations was established by Wang [14] in BV space. For multidimensional case, the local existence of shock front solution to the axially symmetrical piston problem for the full Euler systems was considered in [13]. The authors in [6–7] considered the multidimensional axially symmetric piston problem and obtained the global existence and stability of weak and strong shock solutions, respectively, for the isentropic compressible Euler systems in BV space. When the function of piston boundary is smooth, Chen [5] studied the piecewise smooth solutions for the multidimensional piston problem and also established the global existence and stability of the shock front solutions for unsteady

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potential equations by partial hodograph transformation and nonlinear alternating iteration techniques. Under the framework of L^∞ space, the global entropy solution for the spherically symmetric piston problem to Euler equations was constructed by the shock capturing approach in [4].

Corresponding to the physical phenomenon when the piston is pulled back relatively to the gas, Ding-Kuang-Zhang [9] proved the stability of strong rarefaction wave to this problem for the full Euler equations from the mathematical point of view. In this paper, we are concerned with the 1-D piston problem for pressure gradient equations. Our motivation is to establish the stability of the strong rarefaction wave under the small perturbations of both the piston velocity and the initial data. Comparing with the results involving strong shock waves, one of the main difficulties is to capture and control the location of the strong rarefaction wave as well as its strength when weak waves interact with strong rarefaction waves by Glimm scheme. To overcome these obstacles, we employ the wave front tracking algorithm to construct the approximate solutions. Developing from [15], we impose some weights on weak waves interacting with strong rarefaction waves to measure the change of its strength after the interactions, and prove the monotonously decreasing of the Glimm functional. Different from [9], due to lack of the velocity in the eigenvalues of the pressure-gradient equations, we give some more restrictions on the velocity of the piston to make 1-waves interact with the piston boundary possibly.

The pressure gradient equations of the compressible Euler system can be described by (see [1, 11])

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t(\rho u) + \partial_x p = 0, \\ \partial_t\Big(\rho\Big(e + \dfrac{1}{2}u^2\Big)\Big) + \partial_x(\rho u) = 0, \end{cases} \tag{1.1}$$

where ρ, p and u represent the density, the pressure and the speed of the fluid, respectively, and e is the internal energy. For the polytropic gas, the constitutive relations are given by

$$p = \kappa \rho^\gamma \exp(S/c_v), \quad e = \frac{p}{(\gamma-1)\rho},$$

where S stands for the entropy.

System (1.1) comes from the flux-splitting method in numerical analysis on the Euler system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t\Big(\rho\Big(\dfrac{1}{2}u^2 + e\Big)\Big) + \partial_x\Big(u\Big(\dfrac{1}{2}\rho u^2 + \rho e + p\Big)\Big) = 0 \end{cases} \tag{1.2}$$

by separating the pressure from the inertia in the flux (see [1, 11]). The pressure gradient equations are still valid whenever the inertial effect is small compared to the pressure-gradient effect of the flow as to be negligible. Thus, the pressure-gradient equations (1.1) have their own physical meaning. Derived from the first equation of (1.1), ρ is independent of time. For simplicity, assume that $\rho \equiv 1$. Then system (1.1) can be written as

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \dfrac{1}{\gamma - 1}\partial_t p + p\partial_x u = 0. \end{cases} \tag{1.3}$$

Through the following transformation

$$\begin{cases} p = (\gamma - 1)\widetilde{p}, \\ t = \dfrac{1}{\gamma - 1}\widetilde{t}, \end{cases}$$

system (1.3) can be rewritten by

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t p + p\partial_x u = 0, \end{cases} \tag{1.4}$$

where $\widetilde{t}$ and $\widetilde{p}$ are still denoted as t and p provided no confusion occurs.

The eigenvalues of system (1.4) are

$$\lambda_1 = -\sqrt{p}, \quad \lambda_2 = \sqrt{p}. \tag{1.5}$$

The corresponding right eigenvectors are

$$r_1 = 2(1, -\sqrt{p})^{\mathrm{T}}, \quad r_2 = 2(1, \sqrt{p})^{\mathrm{T}}. \tag{1.6}$$

System (1.4) can be written in the general form of conservation law:

$$\partial_t W(U) + \partial_x H(U) = 0, \quad U = (u, p)^{\mathrm{T}}, \tag{1.7}$$

where

$$W(U) = \Big(u,\ p + \frac{1}{2}u^2\Big)^{\mathrm{T}}, \quad H(U) = (p,\ pu)^{\mathrm{T}}.$$

The entropy-entropy flux pair of system (1.7) is a pair of C^1 functions satisfying

$$\nabla_W \eta(W(U))\nabla_U H(U) = \nabla_U q(W(U)).$$

Suppose that the initial gas satisfies $u(x,0) = u_0(x)$, $p(x,0) = p_0(x)$, and the piston moves with a speed depending only on time t. Let the movement curve of the piston be $x = b(t)$ with the speed $b'(t)$. We study the state of the gas in domain $\Omega = \{(x,t) : x > b(t),\ t > 0\}$ with $\Gamma = \{(x,t) : x = b(t),\ t > 0\}$ (see Figure 1). Then the initial-boundary conditions for the piston problem can be described as

$$\begin{cases} (u,p)(x,0) = (u_0(x), p_0(x)), \\ u = b'(t) \quad \text{on } x = b(t). \end{cases} \tag{1.8}$$

When the initial data is constant, and the function of piston boundary is convex, denoted by $x = b_*(t)$ (see Figure 2), the corresponding initial boundary conditions can be reduced to the following:

$$\begin{cases} (u,p)(x,0) = (0, \overline{p})^{\mathrm{T}}, \\ u = b_*'(t), \quad \text{on } x = b_*(t), \end{cases} \tag{1.9}$$

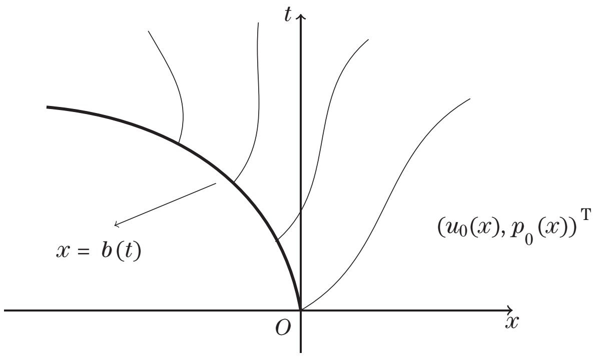

Figure 1 Definition domain.

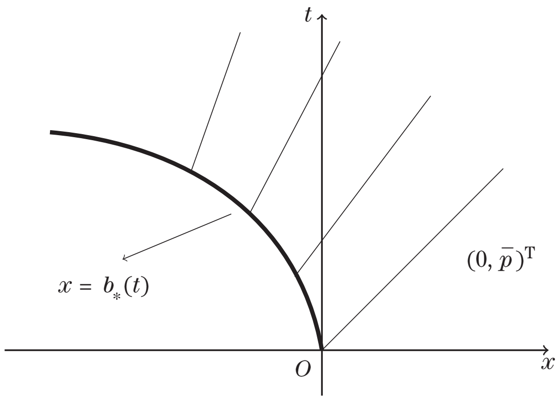

Figure 2 Background solution.

where $b'_*(t) \in (u_*,\ 0)$, u_* is the critical speed of the piston given by (2.10).

In this paper, we mainly consider the initial data and the piston boundary satisfying the following assumptions:

(H_1) The initial data $U_0 = (u_0(x), p_0(x))^{\mathrm{T}}$ is a small perturbation of $\overline{U} = (0, \overline{p})^{\mathrm{T}}$, i.e.,

$$\|U_0 - \overline{U}\|_\infty + \mathrm{TV}.U_0(\cdot) \ll 1.$$

(H_2) The boundary of the piston $x = b(t)$ is Lipschitz continuous, and a small perturbation of the boundary $x = b_*(t)$. Moreover,

$$b(0) = 0, \quad b'_+(t) \in \mathrm{BV}(\mathbb{R}^+; \mathbb{R}),$$

where

$$b'_+(t) = \lim_{\tau \to t+} \frac{b(\tau) - b(t)}{\tau - t}.$$

In the following, we will give the definition of the entropy solutions of problem (1.7) and (1.8).

Definition 1.1 *A bounded measurable function $U(x,t)$ is an entropy solution of problem* (1.7) *and* (1.8), *if*

(i)

$$\int_\Omega (W(U)\varphi_t + H(U)\varphi_x)\mathrm{d}x\mathrm{d}t + \int_\Gamma (H(U) - W(U)b'(t))\varphi \mathrm{d}S + \int_0^{+\infty} W(U)(x,0)\varphi(x,0)\mathrm{d}x = 0$$

holds for any $\varphi(x,t) \in C_c^\infty(\overline{\Omega})$;

(ii) *The Lax entropy inequality holds in the sense of distributions:*

$$\partial_t \eta(U) + \partial_x q(U) \le 0 \quad \text{in } \mathcal{D}'(\Omega)$$

for any C^2 convex entropy-entropy flux pair $(\eta(U), q(U))$.

The main result of this paper is given in the following theorem.

Theorem 1.1 *Under the assumptions of* (H_1) *and* (H_2), *there exist positive constants ε, δ_0 and M_0 such that, if*

$$\|U_0 - \overline{U}\|_\infty + \mathrm{TV}.\{U_0(\cdot,[0,\infty))\} + |b'_+(0) - b'_{*+}(0)| + \mathrm{TV}.\{(b'_+ - b'_{*+})(\cdot)\} < \varepsilon, \tag{1.10}$$

then, there exists a global existence of entropy solution $U(x,t)$ to problem (1.7) *and* (1.8), *including a strong* 2*-rarefaction wave, which is a small perturbation of the corresponding solutions to problem* (1.7) *and* (1.9). *In addition, $\forall t > 0$, it satisfies*

$$U(x,t) \in \mathrm{BV}(\Omega) \cap D(\overline{U}, \delta_0), \tag{1.11}$$

and

$$\mathrm{TV}.\{U(\cdot,t) : [b(t), +\infty)\} \le M_0, \tag{1.12}$$

where $D(\overline{U}, \delta_0)$ is defined by (2.12) *in* §2, *and M_0 is a constant depending on the initial data, the background solution and* $\mathrm{TV}.\{b'(\cdot)\}$.

Remark 1.1 The background solution here means the solution with constant initial data and convex piston boundary, which will be further discussed in §2.

This paper is organized as follows. In §2, we first present some basic properties of elementary waves (shock and rarefaction waves). Then, we find the background solution and give the solvability of the Riemann piston problem for system (1.7). In §3, we construct approximate solutions by a modified wave front tracking method. In §4, we first consider the local interaction estimates of perturbation waves and their reflections on the piston and the strong rarefaction wave. Next, we construct the Glimm functional and prove its monotonicity. Then, the compactness and the convergence of the approximate solution follow.

2 Background Solution

In this section, we will establish the solvability to problem (1.7) and (1.9) when the initial data is constant, denoted by $(0, \overline{p})$, and the piston boundary is convex, denoted by $x = b_*(t)$. First of all, we consider the Riemann problem of (1.7) with initial data

$$U|_{t=t_0} := (u,p)|_{t=t_0} = \begin{cases} U_L, & x < x_0, \\ U_R, & x > x_0, \end{cases} \tag{2.1}$$

where $U_L = (u_L, p_L)$ and $U_R = (u_R, p_R)$ represent the left and right constant states, respectively. The solvability of the Riemann problem can be found in [10, 12] when $|U_L - U_R|$ is sufficiently small.

For any given left (right) state U_l (U_r), the shock (inverse shock) curve is the set of all possible states U which can be connected to U_l (U_r) on the right (left) by 1-or 2-shock wave, and denoted by S_1 or S_2 (S_1^{-1} or S_2^{-1}), respectively. Similarly, we denote by R_1 and R_2 (R_1^{-1} and R_2^{-1}) the 1- and 2-rarefaction (inverse rarefaction) wave curves. For our use, we express R_1^{-1} and R_2^{-1} explicitly as

$$R_1^{-1}(U_r) : u - u_r = -2(\sqrt{p} - \sqrt{p_r}), \quad p > p_r, \tag{2.2}$$

$$R_2^{-1}(U_r) : u - u_r = 2(\sqrt{p} - \sqrt{p_r}), \qquad p < p_r. \tag{2.3}$$

The Rankine-Hugoniot conditions of (1.7) satisfy that

$$\begin{cases} p - p_r = s_i(u - u_r), \\ pu - p_r u_r = s_i\Big(p + \dfrac{1}{2}u^2 - p_r - \dfrac{1}{2}u_r^2\Big), \end{cases} \tag{2.4}$$

where s_i denotes the velocity of the i-shock, $i = 1, 2$. We eliminate s_i to have

$$(p - p_r)^2 = \frac{1}{2}(p + p_r)(u - u_r)^2.$$

Therefore, we parameterize the inverse shock curves through U_r by

$$S_1^{-1}(U_r) : u - u_r = -\sqrt{\frac{2}{p + p_r}}(p - p_r), \quad u > u_r,\ p < p_r, \tag{2.5}$$

$$S_2^{-1}(U_r) : u - u_r = -\sqrt{\frac{2}{p + p_r}}(p - p_r), \quad u > u_r,\ p > p_r. \tag{2.6}$$

In addition, the Lax entropy conditions across the shock are

$$\lambda_1(U_r) < s_1 < \lambda_2(U_r), \quad s_1 < \lambda_1(U_l), \tag{2.7}$$

$$\lambda_1(U_l) < s_2 < \lambda_2(U_l), \quad s_2 > \lambda_2(U_r). \tag{2.8}$$

In the following, we give the background solution to problem (1.7) and (1.9). Let $\{t_k^*\}_{k=1}^{\infty}$ be a sequence of points with $t_0^* = 0 < t_k^* < t_{k+1}^*$ for any $k > 1$, and $\lim_{k\to+\infty} t_k^* = +\infty$. Suppose that

$$b_{*\Delta}(t) = b_*(t_k^*) + \tan\theta_k^*(t - t_k^*), \quad t_k^* < t < t_{k+1}^*, \tag{2.9}$$

where $0 > \theta_k^* > \theta_{k+1}^* > \arctan u_* + \delta$ for some $\delta > 0$, and

$$u_* = \inf\Big\{u : u = 2\sqrt{p} - 2\sqrt{\overline{p}},\ 0 > u > -\frac{2}{3}\sqrt{\overline{p}}\Big\}. \tag{2.10}$$

When $b(t) \equiv b_{*\Delta}(t)$, the solution of problem (1.7) and (1.9) is given by a constant state (u_{*k}, p_{*k}) which is connected to the initial state $(0, \overline{p})$ by 2-rarefaction wave fans issuing from the corner point $(b_*(t_k^*), t_k^*)$, $k \geq 0$ (see Figure 3). Meanwhile, the state $U_{*k} = (u_{*k}, p_{*k})$ satisfies

$$u_{*k} = 2\sqrt{p_{*k}} - 2\sqrt{\overline{p}}, \quad u_{*k} > -\sqrt{p_{*k}}. \tag{2.11}$$

Therefore, we can conclude this result as follows.

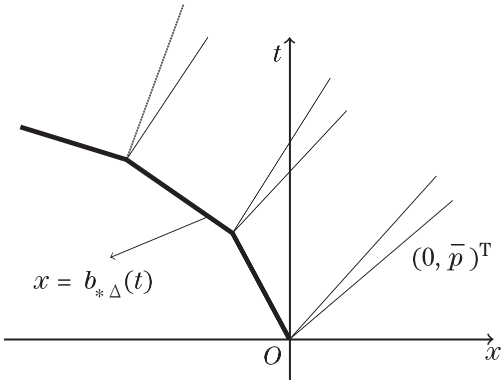

Figure 3 The piston boundary is piecewise linear.

Theorem 2.1 *Suppose that* $b'_{*\Delta}(t) \in (u_*, 0)$ *and* $b'_{*\Delta}(t)$ *is decreasing. If* TV. $b'_{*\Delta}(\cdot) < +\infty$, *then there exists a unique solution* $U_{*\Delta}(x,t)$ *of problem* (1.7) *and* (1.9), *which are connected to the initial state* $(0, \overline{p})$ *by the* 2*-rarefaction wave fans, satisfying* (2.11).

As a corollary, we can prove the similar result for more general boundary $x = b_*(t)$ as Theorem 2.1.

Corollary 2.1 *Suppose that* $b'_*(t) \in (u_*, 0)$ *and* $b'_*(t)$ *is monotonously decreasing. If* TV. $b'_*(\cdot) < +\infty$, *then there exists a unique solution* $U_*(x,t)$ *of problem* (1.7) *and* (1.9) *in the region* $\{U : u > u_*\}$ *that are connected to the initial state* $(0, \overline{p})$ *by strong* 2*-rarefaction wave fans for one dimensional piston problem with convex piston boundary* $x = b_*(t)$.

When the function $x = b(t)$ is a small perturbation of $b_*(t)$, some elementary waves will be produced. We introduce the perturbation domain $D(\overline{U}, \delta_0)$ defined as follows:

$$D(\overline{U}, \delta_0) = \{U : |u - 2\sqrt{p} + 2\sqrt{\overline{p}}| < \delta_0, \quad \delta_0 > u > u_* + \delta_0\} \tag{2.12}$$

for some $\delta_0 > 0$.

Hereinafter, we denote by $\alpha_i, \beta_i, \gamma_i$ the parameters of the corresponding i-waves, $i = 1, 2$, while by their absolute values the corresponding strengths of the waves. We also use the parameters to represent the i-waves provided no confusion occurs. We introduce the notation $\Psi(U_r; \alpha_1, \alpha_2)$ to represent the left state U_l and the right state U_r can be connected by 1-wave α_1 and 2-wave α_2 when $|U_l - U_r| \ll 1$. From [12], we have

$$\frac{\partial \Psi}{\partial \alpha_i}(U_r; \alpha_1, \alpha_2)\big|_{\alpha_1=\alpha_2=0} = -r_i(U_r), \quad i = 1, 2, \tag{2.13}$$

where r_i represents the right eigenvector of the system (1.1). In addition, $\alpha_i > 0$ along $R_i^{-1}(U_r)$, while $\alpha_i < 0$ along $S_i^{-1}(U_r)$. We can also use the notation

$$\Phi(U_l; \alpha_1, \alpha_2) = U_r,$$

and let (U_l, U_r) denote the nonlinear waves to solve the Riemann problem with the left state U_l and the right state U_r.

Remark 2.1 For any given state $U \in D(\overline{U}, \delta_0)$, we can parameterize the 2-inverse rarefaction wave curve $R_2^{-1}(U)$ by solving

$$\frac{\mathrm{d}\Psi(U;0,\sigma)}{\mathrm{d}\sigma} = -r_2(\Psi(U;0,\sigma)), \tag{2.14}$$

$$\Psi(U;0,0) = U. \tag{2.15}$$

3 Construction of the Approximate Solutions

In this section, we use the (piston) Riemann problem as building blocks to construct approximate solutions of problem (1.7) and (1.8) by a modified wave front tracking scheme.

3.1 Riemann problem

As mentioned in §2, the solution to the Riemann problem (2.1) is given by at most three constant states connected by shocks, or rarefaction waves. Exactly speaking, there exist C^2 curves: $\alpha_j \to \Phi_j(\alpha_j, U)$, $j = 1, 2$, such that

$$\Phi(U_L; \alpha_1, \alpha_2) := \Phi_2(\alpha_2, \Phi_1(\alpha_1, U_L)) = U_R, \tag{3.1}$$

where $|U_L - U_R| \ll 1$.

By the wave front tracking method, we adopt two types of the Riemann solver to Riemann problem (2.1).

Case 1 Accurate Riemann solver.

The accurate Riemann solver is as mentioned in §2, except that we replace every rarefaction wave R_i, $i = 1, 2$, by dividing into ν equal parts.

Suppose that the left state U_L and the middle state U_M are connected by 1-rarefaction wave α_1. If $\alpha_1 > 0$, then let $U_{0,0} = U_L$, $U_{0,\nu} = U_M$, for any $1 \le k \le \nu$,

$$U_{0,k} = \Phi_1\Big(\frac{1}{\nu}\alpha_1, U_{0,k-1}\Big), \quad x_{1,k} = x_0 + (t - t_0)\lambda_1(U_{0,k}).$$

Therefore, we can replace 1-rarefaction wave by

$$U_A^\nu = \begin{cases} U_L, & x < x_{1,1}, \\ U_{0,k}, & x_{1,k} < x < x_{1,k+1}, \\ U_M, & x_{1,\nu} < x < x_0 + (t - t_0)\lambda_1^*, \end{cases} \tag{3.2}$$

where $\lambda_1^* \in (\max \lambda_1, \min \lambda_2)$.

Similarly, we can approximate 2-rarefaction wave by ν 2-rarefaction fronts in the domain $\{(x,t) : x > x_0 + \lambda_*(t - t_0)\}$, $\lambda_* \in (\max \lambda_1, \min \lambda_2)$.

Case 2 Simplified Riemann solver.

Let $\widehat{\lambda}$ (strictly larger than all the characteristic speeds of system (1.7)) be the speed of non-physical waves, which are introduced so that the number of the wave fronts is finite for all $t \ge 0$. The strength of the non-physical waves is the error due to the simplified Riemann solver. It occurs in the following two cases:

Subcase 1 A j-wave β_j and an i-wave α_i interact at (x_0, t_0), $1 \le i \le j \le 2$. Suppose that U_L, U_M, and U_R are three constant states, satisfying

$$U_M = \Phi_j(\beta_j, U_L), \quad U_R = \Phi_i(\alpha_i, U_M). \tag{3.3}$$

We define an auxiliary right state

$$U_R' = \begin{cases} \Phi_j(\beta_j, \Phi_i(\alpha_i, U_L)), & j > i, \\ \Phi_j(\alpha_j + \beta_j, U_L), & j = i. \end{cases} \tag{3.4}$$

Then the simplified Riemann solver $U_S(U_L, U_R)$ at (x_0, t_0) of problem (1.7) and (2.1) can be given by

$$U_S(U_L, U_R) = \begin{cases} U_A^\nu(U_L, U_R'), & x - x_0 < \widehat{\lambda}(t - t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t - t_0), \end{cases} \tag{3.5}$$

where $U_A^\nu(U_L, U_R')$ is constructed by the accurate Riemann solver in Case 1. The non-physical wave is defined by

$$U_{np} = \begin{cases} U_R', & x - x_0 < \widehat{\lambda}(t - t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t - t_0) \end{cases} \tag{3.6}$$

and the strength of the non-physical wave is $|U_R - U_R'|$.

Subcase 2 A non-physical wave interacts with a weak i-wave front α_i $(i = 1, 2)$ from the right at (x_0, t_0). Suppose that the three states U_L, U_M, and U_R, satisfy

$$|U_M - U_L| = \epsilon, \quad U_R = \Phi_i(\alpha_i, U_M).$$

Then, the simplified Riemann solver $U_S(U_L, U_R)$ of problem (1.7) and (2.1) is

$$U_S(U_L, U_R) = \begin{cases} U_L, & x - x_0 < \lambda_i(U_L)(t - t_0), \\ \Phi_i(\alpha_i, U_L), & \lambda_i(U_L)(t - t_0) < x - x_0 < \widehat{\lambda}(t - t_0), \\ U_R, & x - x_0 > \widehat{\lambda}(t - t_0). \end{cases}$$

3.2 Piston Riemann problem

Let $h = \Delta t$ be a length in t. Choose a series of points $\{A_k\}|_{k=0}^{+\infty}$, where $A_k = (b(k\Delta t), k\Delta t)$, and connect A_k with A_{k+1}. In the sequel, the points A_k, $k \ge 0$, are called corner points. We introduce the notations $t_k = k\Delta t$, $b_k = b(k\Delta t)$. The movement curve of the piston is approximated by a piecewise linear function denoted as

$$x = b_\Delta(t) = b_k + \frac{b_{k+1} - b_k}{t_{k+1} - t_k}(t - t_k), \quad t_k \le t < t_{k+1}, \ \ k = 0, 1, \cdots.$$

In addition, denote by θ_k the angle between the straight line $x = b_k + \frac{b_{k+1} - b_k}{\Delta t}(t - t_k)$ and t-axis and by ω_k the angle between the straight lines $x = b_k + \tan\theta_k(t - t_k)$ and $x = b_{k-1} + \tan\theta_{k-1}(t - t_{k-1})$, where $\theta_k = \arctan\frac{b_{k+1} - b_k}{\Delta t}$. Then we have $\omega_k = \theta_k - \theta_{k-1}$. Let

$$\Omega_\Delta = \{(x, t) : x > b_\Delta(t), \ t > 0\}, \quad \Gamma_\Delta = \{(x, t) : x = b_\Delta(t), \ t > 0\},$$

and

$$\Omega_{\Delta,k} = \{(x,t) : x > b_{\Delta}(t), \quad kh \le t < (k+1)h\},$$

$$\Gamma_{\Delta,k} = \{(x,t) : x = b_{\Delta}(t), \quad kh \le t < (k+1)h\}.$$

We will define the approximate solutions near the piston boundary in $\{t_k \le t < t_{k+1}\} \cap \Omega_{\Delta}$ in two cases.

3.2.1 Reflection on the approximate piston boundary

Assume that 1-wave α_1 hits the boundary at the non-corner point. Suppose that U_k and U_b denote the left and right states of α_1, respectively. Since the velocity of the flow close to the boundary has the same speed with that of the piston, the reflection wave is 2-wave, denoted by β_2. Let the left and right states of β_2 be U_{k+1} and U_b, respectively (see Figure 4). Then we have $(U_k, U_b) = (\alpha_1, 0), (U_{k+1}, U_b) = (0, \beta_2)$. Consider the piston problem as

$$\begin{cases} \partial_t W(U) + \partial_x H(U) = 0 & \text{in } \Omega_{\Delta,k}, \\ U|_{t=k\Delta t} = U_b, \\ u|_{x=b(t)} = \tan\theta_k & \text{on } \Gamma_{\Delta,k}. \end{cases} \tag{3.7}$$

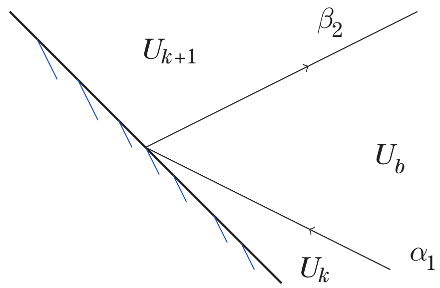

Figure 4 Reflection on the approximate piston boundary.

Lemma 3.1 *Assume that U_k and U_b are given as above, and $U_k, U_b \in O_\epsilon(U_{*k})$, where O_ϵ stands for a small neighborhood. Then problem* (3.7) *has a unique solution (β_2, U_{k+1}). Moreover,*

$$\beta_2 = K_{b_1}\alpha_1,$$

where $K_{b_1} = K_{b_1}(U_b, \alpha_1)$ is negative. Moreover, $K_{b_1}|_{\alpha_1=0} = -1$.

Proof It is easy to obtain that

$$\Psi_2(U_b; \alpha_1, 0) = \Psi_2(U_b; 0, \beta_2) = u_k. \tag{3.8}$$

Differentiate the equality (3.8) with respect to β_2, and let $\beta_2 = 0$. Then we have

$$\frac{\partial \Psi_2}{\partial \beta_2}\Big|_{\beta_2=0} = -r_{22}(U_b) = -2\sqrt{p_b} < 0.$$

By the implicit function theorem, near the point $\alpha_1 = 0$, there exists a function $f \in C^1$ such that

$$\beta_2 = f(\alpha_1). \tag{3.9}$$

Differentiating (3.8) with respect to α_1, we have

$$\frac{\partial \Psi_2}{\partial \beta_2}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\partial \Psi_2}{\partial \alpha_1} = 0. \tag{3.10}$$

Let $\alpha_1 = 0$, then

$$-r_{22}(U_b)\frac{\partial \beta_2}{\partial \alpha_1}\Big|_{\alpha_1=0} + r_{12}(U_b) = 0.$$

Hence

$$\frac{\partial \beta_2}{\partial \alpha_1}\Big|_{\alpha_1=0} = -1.$$

From the Taylor's expansion, we have

$$\beta_2 = f(\alpha_1) - f(0) + f(0) = K_{b_1}\alpha_1,$$

where $f(0) = 0$ and $K_{b_1}|_{\alpha_1=0} = -1$.

From (3.10), we can easily obtain that K_{b_1} is bounded. Therefore, we complete the proof of this lemma.

This lemma illustrates that 1-weak wave has changed into 2-weak wave while hitting the boundary by changing its type.

3.2.2 New waves issuing from the corner points

Suppose that U_k and U_{k+1} denote the states close to the piston boundary $x = b_{k-1} + \tan\theta_{k-1}(t - t_{k-1})$ and $x = b_k + \tan\theta_k(t - t_k)$ (see Figure 5), respectively, where $\omega_k = \theta_k - \theta_{k-1}$. Consider the following initial-boundary value problem

$$\begin{cases} \partial_t W(U) + \partial_x H(U) = 0 & \text{in } \Omega_{\Delta,k}, \\ U|_{t=k\Delta t} = U_k, & \\ u|_{x=b_\Delta(t)} = \tan\theta_k & \text{on } \Gamma_{\Delta,k}. \end{cases} \tag{3.11}$$

Lemma 3.2 *When $|\omega_k| \ll 1$ and $U_k \in O_\epsilon(U_{*k})$, problem (3.11) has a unique solution (β_2, U_{k+1}). Moreover,*

$$\beta_2 = K_{b_0}\omega_k,$$

where $K_{b_0} < 0$ is bounded, depending only on the background solution.

Proof From the definition of Ψ, we have

$$\Psi_2(U_k; 0, \beta_2) = u_{k+1}, \tag{3.12}$$

$$\arctan u_{k+1} - \arctan u_k = \omega_k. \tag{3.13}$$

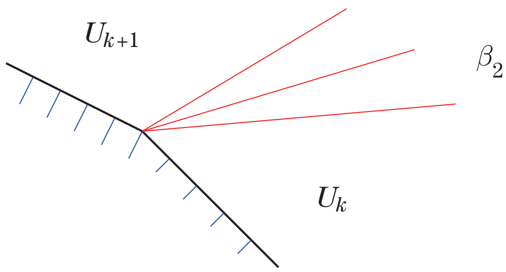

Figure 5 New wave issuing from the corner points.

It is easy to verify the condition of the implicit function theorem near the point $\omega_k = 0$. There exists a function $g \in C^1$ such that $\beta_2 = g(\omega_k)$. From Taylor's expansion,

$$\beta_2 = g(\omega_k) - g(0) + g(0) = K_{b_0}\omega_k,$$

where $g(0) = 0$.

Differentiating (3.13) with respect to ω_k, and letting $\omega_k = 0$, we have

$$\frac{1}{1+u_{k+1}^2}\frac{\partial \Psi_2}{\partial \beta_2}\frac{\partial \beta_2}{\partial \omega_k} - 1 = 0, \tag{3.14}$$

therefore

$$\frac{\partial \beta_2}{\partial \omega_k}\Big|_{\omega_k=0} = -\frac{1+u^2}{r_{22}(U)}\Big|_{U=U_k} = -\frac{1+u^2}{2\sqrt{p}}\Big|_{U=U_k} < 0.$$

Therefore, from (3.14), we can obtain that K_{b_0} is bounded.

Hence, we finish the proof of the lemma.

Lemma 3.3 *Suppose that* $\theta_{(k-1)} = \sum_{j=0}^{k-1} |\omega_j|$ *and assume that* $\theta_{(k-1)} < -\arctan u_*$, *and* $\theta_{(k-1)} + \arctan u_* < \omega_k < 0$. *Then problem* (3.11) *has a unique solution* U_{k+1}, *connected to the right state* U_k *by a rarefaction wave issuing from the corner point* $(b_k, k\Delta t)$. *Moreover,* $\beta_2 = O(1)\omega_k$.

Proof Differentiating (3.12) with respect to β_2, it yields that

$$\frac{\partial \Psi_2}{\partial \beta_2} = -r_{22}(U_{k+1}) = -2\sqrt{p_{k+1}} < 0.$$

We rewritten (3.13) as

$$\beta_2 \int_0^1 \Big(\frac{\partial}{\partial \alpha} \arctan \Psi_2(U_k; 0, \alpha)\Big|_{\alpha=\beta_2 t}\Big) \mathrm{d}t = \omega_k.$$

Hence, $0 < \beta_2 \le M|\omega_k|$. From the parametrization (2.14) of the rarefaction wave, we can easily obtain that $O(1)$ is bounded, depending only on the background solution and TV.$\{b'(\cdot)\}$.

3.3 Approximate solutions

For any $\nu \in \mathbb{N}$ such that

$$\frac{\sup\limits_{U} K_{b_1}}{\nu} \ll 1, \quad \frac{\sup\limits_{U,k}(|K_{b_0}||\omega_k|)}{\nu} \ll 1, \quad \frac{M \sup\limits_{k} |\omega_k|}{\nu} \ll 1,$$

we can construct a ν-approximate solution $U^\nu(x,t)$ by induction in the region Ω_Δ as follows.

Step 1 For $k = 1$, $U^{\nu,h}(x,t)$ on $\Omega_\Delta \cap \{0 \le t < \Delta t\}$ can be constructed by accurate Riemann solver to solve a series of Riemann (piston) problem, which can be carried out as shown in §3.1–§3.2. Approximate rarefaction wave or shock front is generated from the corner point A_0.

Step 2 By induction, we assume that ν, h-approximate solution $U^{\nu,h}$ has been constructed for $t < \tau$, for some $\tau > 0$, and assume that $U^{\nu,h}|_{t<\tau}$ consists of a finite number of wave fronts and some of them interact at $t = \tau$ at the first time. As shown in §3.1–§3.2, we solve the Riemann problem when two wave fronts interact, or the Riemann piston problem when a wave front hits the boundary or a new wave issues from a corner point. Assume that each front has been assigned a generation order in the following way.

(A) A wave front of order n hits the boundary at non-corner point $(b_\Delta(\tau), \tau)$. We solve the generated initial-boundary problem by the accurate Riemann solver as shown in Lemma 3.1, and the generation order of the outgoing 2-wave from the point $(b_\Delta(\tau), \tau)$ is $n+1$. If $n = \nu$, then the generation order of the outgoing wave is set to be $\nu + 1$.

(B) All the wave fronts issuing from the corner points can be constructed according to Lemmas 3.2–3.3, and have order 1.

(C) An i-wave front α_i of order n_1 interacts with a j-wave front β_j of order n_2 at the point (x_0, τ). We adopt the the following wave front algorithm:

(1) When $n_1, n_2 < \nu$, we adopt the accurate Riemann solver to construct the outgoing wave front, and assign the generation order of the l-wave by

$$\begin{cases} \max(n_1, n_2) + 1, & \text{if } l \ne i, j, \\ \min(n_1, n_2), & \text{if } l = i = j, \\ n_1, & \text{if } l = i \ne j, \\ n_2, & \text{if } l = j \ne i. \end{cases} \tag{3.15}$$

(2) When $\max(n_1, n_2) = \nu$, we adopt the simplified Riemann solver to construct the outgoing wave front at the interacting point (x_0, τ). We introduce the generation order of the i-wave front according to (3.15), $i = 1, 2$, and the generation order of the non-physical wave front is $\nu + 1$.

(3) When $n_1 = \nu + 1$ and $n_2 \le \nu$, α_i is non-physical wave front, we adopt the simplified Riemann solver to construct the outgoing wave front from (x_0, τ). The generation order of the outgoing non-physical wave front is $\nu + 1$, while the generation order of the physical wave front is the same as that of the incoming wave β_j.

The wave front tracking algorithm to construct the approximate solutions is given by the following:

Case 1 There are no more than two wave fronts interacting at the point (x_0, τ) by changing the speed of a single wave front by a quantity $O(1)2^{-\nu}$. Only one wave front hits Γ_Δ at the non-corner point and no front hits the corner point.

Case 2 If two wave fronts α and β interact at (x_0, τ), then the Riemann problem at the interacting point is solved in two ways:

(1) If $|\alpha\beta| > \mu_\nu$ and the two wave fronts are physical, where μ_ν is a fixed small parameter with $\mu_\nu \to 0$ as $\nu \to +\infty$, then we adopt the accurate Riemann solver.

(2) If $|\alpha\beta| < \mu_\nu$ and the two wave front are physical, or one of them is non-physical wave, then we adopt the simplified Riemann solver.

Case 3 The physical wave hits the boundary, or a new wave front issues from the corner point of the piston. We adopt the accurate Riemann solver.

4 Monotonicity of the Glimm Functional

In this section, we construct the Glimm functional and prove its monotonicity based on the local interaction estimates. The total variation of the approximate solutions is equivalent to the Glimm functional, which leads to the bounds on the total variation of the approximate solutions. First of all, we give the definition of the strong 2-rarefaction wave.

Definition 4.1 (Strong 2-Rarefaction Wave Front) *A wave front s is called a front of the strong rarefaction wave provided that s is a 2-rarefaction wave front with order 1. Otherwise, it is called a weak wave.*

For any weak wave α, denote its position and magnitude by $(x_\alpha(t), t)$ and α, respectively. For a front s of the strong rarefaction wave, denote the location and magnitude of s by $(x_s(t), t)$ and $s(t)$, respectively. Let

$$\Omega_{R_a}(t) = \{\omega(A_k) \mid \omega(A_k) \le 0,\ A_k = (b(k\Delta t), k\Delta t),\ t \le k\Delta t\},$$

where $\omega(A_k) = \theta_k - \theta_{k-1}$.

We first redefine the approaching waves as follows.

Definition 4.2 (Approaching Waves)

(1) $(\alpha_i, \beta_j) \in \mathcal{A}_1$: *Two weak waves α_i and β_j $(i, j \in \{1, 2\})$ located at points x_α and x_β respectively, with $x_\alpha < x_\beta$, satisfy the following condition: Either $i > j$ or $i = j$ and at least one of them is a shock.*

(2) $\alpha \in \mathcal{A}_s$: *A weak i-wave α is approaching a strong 2-rarefaction wave front if $(x_\alpha, t_\alpha) \in \Omega_-$, $i = 2$ and α_2 is a shock or $(x_\alpha, t_\alpha) \in \Omega_+$, $i = 1$, or $i = 2$ and α_2 is a shock, where*

$$\Omega_- = \{(x, t) : b(t) < x < x_s(t),\ t > 0\}, \quad \Omega_+ = \{(x, t) : x > x_s(t),\ t > 0\}.$$

Remark 4.1 The approaching waves in $\mathcal{A}_1$ are in fact the original approaching waves between weak waves.

Considering the interaction between the weak waves and the strong rarefaction wave fronts, we introduce some weights for the weak waves. For any weak wave α and any non-physical wave ϵ, at non-interacting point at time t, denote

$$R(t, \alpha, l) = \{s \mid s \text{ is a front of the strong rarefaction wave with } x_s(t) \le x_\alpha(t)\},$$
$$R(t, \epsilon, r) = \{s \mid s \text{ is a front of the strong rarefaction wave with } x_s(t) > x_\epsilon(t)\}$$

and

$$W(\alpha_i,t)=\exp(K_b\Sigma\{|\omega|:\omega\in\Omega_{R_a}(t)\}+K_\omega\Sigma\{|s(t)|:s\in R(t,\alpha_i,l)\}),\quad i=1,$$
$$W(\epsilon,t)=\exp(K_{np}\Sigma\{|s(t)|:s\in R(t,\epsilon,r)\}).$$

Now we introduce some notations as follows:

$$L_i(t)=\Sigma\{|\alpha_i|:\alpha_i\ \text{ is a weak } i\text{-physical wave}\},\quad 1\le i\le 2,$$

$$L_{np}(t)=\Sigma\{|\epsilon|:\epsilon\ \text{ is a non-physical wave}\}$$

and

$$Q_0(t)=\Sigma\{|\alpha_i||\beta_j|:(\alpha_i,\beta_j)\in\mathcal{A}_1\},$$

$$Q_{B_1}(t)=\Sigma\{|\alpha_i|W(\alpha_i,t):\ \alpha_i \text{ is a weak } i\text{-physical wave},\ i=1\},$$

$$Q_{B_2}(t)=\Sigma\{|\alpha_2|:\ \alpha_2 \text{ is a weak 2-wave}\},$$

$$Q_{np}(t)=\Sigma\{|\epsilon|W(\epsilon,t):\ \epsilon \text{ is a non-physical wave}\},$$

$$Q_c(t)=\Sigma\{|\omega(A_k)|:A_k=(b_\Delta(k\Delta t),k\Delta t)\text{ is a corner point and }\omega(A_k)>0,\ k\Delta t>t\}.$$

In order to assure that the Glimm functional is sufficiently small, we need to control the strength of the rarefaction wave before time t. For any $t\notin\{k\Delta t:k\in\mathbb{N}^+\}$, we define

$$F_1(t)=|\ \text{TV. }\{\arctan u^{\nu,h}(\cdot,t):[b_\Delta(t),+\infty)\}-\widehat{\theta}(t)|,$$

where

$$\widehat{\theta}(t)=\Sigma\{|\omega(A_k)|:A_k=(b_\Delta(k\Delta t),k\Delta t)\text{ is a corner point and }\omega(A_k)\le 0,\ k\Delta t<t\}.$$

In the sequel, we can redefine the Glimm functional as

$$L_w(t)=\sum_{i=1}^{2}L_i(t)+L_{np}(t),$$

$$Q(t)=K_0Q_0(t)+K_1Q_{B_1}(t)+Q_{B_2}(t)+Q_{np}(t)+K_cQ_c(t)$$

and

$$F_0(t)=L_w(t)+KQ(t),$$

$$F(t)=F_1(t)+C_*F_0(t),$$

where $C_*,\ K,\ K_0,\ K_1$ and K_c are positive constants, determined later.

In order to obtain the global interaction estimates and the bound of the total strength of the strong rarefaction waves, we give the following lemmas.

Lemma 4.1 *There exists a* $\delta'_*>0$ *such that for any state* $U_l\in D(\overline{U},\delta_0)$, *the function* $\arctan\Phi_2(U_l;0,\alpha_2)$ *is strictly increasing with respect to* α_2 *in* $\{\alpha_2\mid\alpha_2\ge-\delta'_*,\ \Phi(U_l;0,\alpha_2)\in D(\overline{U},\delta_0)\}$. *Moreover, there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1|\alpha_2|\le|\arctan\Phi_2(U_l;\ 0,\alpha_2)-\arctan u_l|\le C_2|\alpha_2|.\tag{4.1}$$

Proof When $\alpha_2 > 0$, by the properties of the rarefaction waves, we have

$$\begin{aligned}\frac{\partial}{\partial \alpha_2}\arctan u\Big|_{u=\Phi_2(U_l;\ 0,\ \alpha_2)} &= \frac{1}{1+u^2}(r_{22}(U))\Big|_{U=\Phi(U_l;\ 0,\ \alpha_2)}\\ &= \frac{2\sqrt{p}}{1+u^2}\Big|_{U=\Phi(U_l;\ 0,\ \alpha_2)} > 0\end{aligned}$$

for any $U_l, \Phi(U_l; 0, \alpha_2) \in D(\overline{U}, \delta_0)$.

When $-\delta'_* \le \alpha_2 < 0$, we can similarly obtain (4.1). Therefore, the proof of this lemma is completed.

Lemma 4.2 *Suppose that the three constant states $U_1, U_2, U_3 \in D(\overline{U}, \delta_0)$, satisfying $U_2 = \Phi(U_1; \alpha_1, \alpha_2)$, and that U_2 and U_3 are connected by a non-physical wave front with the strength ϵ. Then*

$$|u_3 - 2\sqrt{p_3} - u_1 + 2\sqrt{p_1}| = O(1)(|\alpha_1| + |\alpha_2^-| + |\epsilon|),$$

where $\alpha_2^- = \min\{\alpha_2, 0\}$ and $O(1)$ is bounded.

Proof From the mean value theorem and the expression of rarefaction wave curves, we have

$$(u - 2\sqrt{p})(\Phi(U; 0, \alpha_2)) = (u - 2\sqrt{p})(U)$$

for any $\alpha_2 > 0$.

Therefore, we complete the proof of this lemma.

Suppose that $(\alpha_i, \beta_j) \in \mathcal{A}_1$ $(1 \le i, j \le 2)$. We use ϵ and s to represent a non-physical wave and a 2-strong rarefaction wave front, respectively. Let

$$E_{\nu,h}(\tau) = \begin{cases} |\alpha_i||\beta_j|, & \text{Case 1},\\ |\alpha_1|, & \text{Case 2},\\ |\omega_k|, & \text{Case 3},\\ |\alpha_2| \text{ or } |s|, & \text{Case 4},\\ |\alpha_1||s|, & \text{Case 5},\\ |\epsilon||s|, & \text{Case 6}.\end{cases} \tag{4.2}$$

Suppose that two weak wave fronts interact, or a weak wave front hits at the non-corner point on the piston boundary, or a new wave front issues from the corner point due to the turning of the piston boundary at the interaction time τ.

In the following, we prove that the Glimm functional is decreasing based on the local interaction estimates. Before the interaction time τ, we give the inductive hypotheses.

$A_1(\tau-)$: Before τ, the strength of every wave front is less than δ_*.

$A_2(\tau-)$: $U^{\nu,h}|_{t<\tau} \in D(\overline{U}, \delta_*)$.

$A_3(\tau-)$: $\sum\limits_k |\omega(A_k)| \le -\arctan u_* - \delta_*$,

where u_* is the critical speed of the piston.

Theorem 4.1 *Suppose that* $\mathrm{A}_1(\tau-)$–$\mathrm{A}_3(\tau-)$ *hold for any* $k\Delta t \le \tau < (k+1)\Delta t$. *Then there exist positive constants* δ_1, C_*, K_0, K_1 *and* K_c *such that if*

$$F(\tau-) < \delta_1, \tag{4.3}$$

then

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}E_{\nu,h}(\tau) \tag{4.4}$$

and

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}E_{\nu,h}(\tau). \tag{4.5}$$

Moreover, it holds that:

(i) *Before* τ_1, *the strength of the wave front is less than* δ_*.

(ii) *There exists a positive constant* $C > 0$, *independent of* τ_1 *and* τ *such that for any* 2-*strong rarefaction wave front* $s_0(t)$,

$$|s_0(t)| \le \frac{C}{\nu}. \tag{4.6}$$

(iii) *For any fixed* $\nu \in \mathbb{N}^+$, *the number of the wave fronts of the approximate solution* U^ν *is finite and the total strength of the non-physical wave is* $O(1)2^{-\nu}$, *i.e., there exists a positive constant* C_{np}, *independent of* ν *and* τ_1, *such that*

$$\sum_{\epsilon_0} |\epsilon_0(t)| \le \frac{C_{np}}{2^\nu} \tag{4.7}$$

for any $t < \tau_1$, *where* τ_1 *is next to* τ *when the wave interaction occurs.*

Proof Based on the local interaction estimates, the proof can be divided into the following six cases.

Case 1 Interaction between the weak waves.

Let the two weak waves be α_i and β_j interacting on the line $t = \tau$ $(i, j = 1, 2)$, where $\tau \notin \{k\Delta t : k \text{ is a positive integer}\}$, let γ_l be the generated waves, $l = 1, 2$, and let ϵ be the outgoing non-physical wave. By a standard procedure (see [3]), we obtain the following uniform estimates for the interactions between weak waves.

Lemma 4.3 *It holds that*

$$\gamma_i = \alpha_i + O(1)|\alpha_i||\beta_j|, \quad \gamma_j = \beta_j + O(1)|\alpha_i||\beta_j| \quad \text{for } i \ne j \tag{4.8}$$

and for $i = j$, *it satisfies that*

$$\gamma_i = \alpha_i + \beta_j + O(1)|\alpha_i||\beta_j|, \tag{4.9}$$

$$\gamma_l = O(1)|\alpha_i||\beta_j|, \quad l \ne i, \ \epsilon = O(1)|\alpha_i||\beta_j|, \tag{4.10}$$

where $O(1)$ *is bounded.*

Based on the estimates (4.8)–(4.10), we have

$$L_i(\tau+) - L_i(\tau-) = O(1)|\alpha_i||\beta_j|, \quad i = 1, 2,$$
$$F(\tau+) - F(\tau-) = K_0(O(1)|\alpha_i||\beta_j|L_\omega(\tau-) - |\alpha_i||\beta_j|) + O(1)|\alpha_i||\beta_j|.$$

When $L_w(\tau-)$ is sufficiently small, we can choose K_0 large enough such that

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\alpha_i||\beta_j|.$$

Case 2 Reflection on the piston boundary.

Assume that a weak 1-wave front α_1 hits the boundary at the point $(b_\Delta(\tau), \tau)$, where $\tau \notin \{k\Delta t, k \text{ is a positive integer}\}$. Denote the outgoing wave by β_2. From Lemma 3.1, we have

$$L_1(\tau+) - L_1(\tau-) = -|\alpha_1|,$$
$$L_2(\tau+) - L_2(\tau-) = K_{b_1}|\alpha_1|,$$
$$Q(\tau+) - Q(\tau-) = K_0(|\beta_2| - |\alpha_1|)L_w(\tau-) - K_1 W(\alpha_1, \tau-)|\alpha_1| + |\beta_2|.$$

Therefore, when $L_w(\tau-)$ is sufficiently small, we can choose K_1 suitably large, i.e., $K_1 > K_1^*$, such that

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|\alpha_1|. \tag{4.11}$$

Then, we can choose K and C_* large enough such that

$$F(\tau+) - F(\tau-) < -\frac{1}{4}|\alpha_1|. \tag{4.12}$$

Case 3 New waves issuing from the boundary corner.

Suppose that the flow moves past the corner point $A_k(b(k\Delta t), k\Delta t)$ for some $k > 0$. Let the new outgoing wave be β_2, and denote the turning angle ω_k between the two approximate piston boundary $x = b_k + \tan\theta_k(t - t_k)$ and $x = b_{k-1} + \tan\theta_{k-1}(t - t_{k-1})$. From Lemma 3.2, we have the following cases:

(1) When $\omega_k > 0$, we have

$$L_1(\tau+) - L_1(\tau-) = 0,$$
$$L_2(\tau+) - L_2(\tau-) = |\beta_2|,$$
$$Q(\tau+) - Q(\tau-) \le K_0|\beta_2|L_w(\tau-) + |\beta_2| - K_c|\omega_k|.$$

Thus, when $L_w(\tau-)$ is sufficiently small, we can choose K_c large enough such that

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|\omega_k|. \tag{4.13}$$

On the other hand, we can obtain that

$$F_1(\tau+) - F_1(\tau-) = O(1)|\omega_k|. \tag{4.14}$$

Therefore, according to (4.13)–(4.14), we can choose K and C_* large enough such that

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\omega_k|.$$

(2) When $\omega_k < 0$, from Lemma 3.3 and the wave front tracking algorithm, β_2 is a front of 2-strong rarefaction wave. We have

$$\begin{aligned} &L_i(\tau+) - L_i(\tau-) = 0, \quad i = 1, 2, \\ &Q(\tau+) - Q(\tau-) = K_1 \sum_{i=1}(W(\beta_i, \tau+)|\beta_i| - W(\beta_i, \tau-)|\beta_i|), \end{aligned}$$

where

$$\begin{aligned} W(\beta_i, \tau+) - W(\beta_i, \tau-) &= \exp(O^+(1))(\exp(K_\omega|\beta_2|) - \exp(K_b|\omega_k|)) \\ &= \exp(O^+(1))(K_\omega|\beta_2| - K_b|\omega_k|), \end{aligned}$$

hence, when $L_w(\tau-)$ is sufficiently small, we can choose K_b large enough such that

$$Q(\tau+) - Q(\tau-) \le -\frac{1}{2}|\omega_k|. \tag{4.15}$$

In addition, it holds

$$F_1(\tau+) - F_1(\tau-) = O(1)|\omega_k|. \tag{4.16}$$

By combining (4.15) with (4.16), when K and C_* are large enough, it yields that

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\omega_k|.$$

Case 4 Interaction between the 2-strong rarefaction wave front and 2-weak shocks from the left (the right).

Suppose that a front of the strong rarefaction wave denoted by s and a 2-weak shock α_2 from the left interact at the time $t = \tau$, where $\tau \notin \{k\Delta t : k \text{ is a positive integer}\}$. Let γ_1, γ_2 and ϵ be the outgoing 1-wave, 2-wave and non-physical wave, respectively. In a similar way to the argument of Lemma 4.3, we have the following result.

Lemma 4.4 *It holds that*

$$\begin{aligned} &\gamma_1 = O(1)|\alpha_2||s|, \\ &\gamma_2 = \alpha_2 + s + O(1)|\alpha_2||s|, \quad \epsilon = O(1)|\alpha_2||s|. \end{aligned}$$

Based on this lemma, we have the following cases.

Subcase 4.1 $\gamma_2 \ge 0$. It holds that

$$|\gamma_2| = |s| - |\alpha_2| + O(1)|\alpha_2||s|$$

and

$$\begin{aligned}
L_1(\tau+) - L_1(\tau-) &= O(1)|\alpha_2||s|, \\
L_2(\tau+) - L_2(\tau-) &= -|\alpha_2|, \\
Q(\tau+) - Q(\tau-) &= K_0(|\gamma_1| - |\alpha_2|)L_w(\tau-) \\
&\quad + K_1\Big(\sum_{i=1}\sum_{\beta_i\neq\gamma_i}(W(\beta_i,\tau+)|\beta_i| - W(\beta_i,\tau-)|\beta_i|) \\
&\quad + W(\gamma_1,\tau+)|\gamma_1|\Big) - |\alpha_2| + W(\epsilon,\tau+)|\epsilon| \\
&< 0,
\end{aligned}$$

where $W(\beta_i,\tau+)|\beta_i| - W(\beta_i,\tau-)|\beta_i| < 0$.

Therefore, we have

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|\alpha_2|. \tag{4.17}$$

So, when K and C_* are suitably large, we have

$$F(\tau+) - F(\tau-) < -\frac{1}{4}|\alpha_2|. \tag{4.18}$$

Subcase 4.2 $\gamma_2 < 0$. We have

$$|\gamma_2| = |\alpha_2| - |s| + O(1)|\alpha_2||s|$$

and

$$\begin{aligned}
L_1(\tau+) - L_1(\tau-) &= O(1)|\alpha_2||s|, \\
L_2(\tau+) - L_2(\tau-) &= -|s| + O(1)|\alpha_2||s|, \\
Q(\tau+) - Q(\tau-) &= K_0(|\gamma_1| + |\gamma_2| - |\alpha_2|)L_w(\tau-) \\
&\quad + K_1\Big(\sum_{i=1}\sum_{\beta_i\neq\gamma_i}(W(\beta_i,\tau+)|\beta_i| - W(\beta_i,\tau-)|\beta_i|) \\
&\quad + W(\gamma_1,\tau+)|\gamma_1|\Big) + |\gamma_2| - |\alpha_2| + W(\epsilon,\tau+)|\epsilon| \\
&\leq 0,
\end{aligned}$$

where

$$W(\beta_i,\tau+) = \exp(-K_\omega|s|)W(\beta_i,\tau-).$$

Therefore, we have

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|s|.$$

When K and C_* are suitably large, we have

$$F(\tau+) - F(\tau-) < -\frac{1}{4}|s|. \tag{4.19}$$

The interaction between 2-strong rarefaction wave front and 2-weak waves from the right can be proved similarly.

Case 5 Interaction between the 2-strong rarefaction wave front and 1-weak waves or 2-shock waves from the right.

Suppose that the left and right states of the strong 2-rarefaction wave front denoted by s are U_l and U_m, respectively. The incoming 1-weak wave α_1 from the right connects the states U_m and U_r. Let the outgoing wave fronts be γ_1, s' and the non-physical wave be ϵ, respectively, satisfying

$$\gamma_1 = (U_l, U'_m), \quad s' = (U'_m, U'_r), \quad \epsilon = (U'_r, U_r).$$

In a similar way to the proof of Lemma 4.3, we have following result.

Lemma 4.5 *It satisfies that*

$$\begin{aligned} &\gamma_1 = \alpha_1 + O(1)|\alpha_1||s|, \\ &s' = s + O(1)|\alpha_1||s|, \quad \epsilon = O(1)|\alpha_1||s|, \end{aligned}$$

where $O(1)$ is bounded, depending only on the system.

Based on this lemma, we have

$$\begin{aligned} &L_w(\tau+) - L_w(\tau-) = O(1)|\alpha_1||s|, \\ &K_0Q_0(\tau+) + Q_{B_2}(\tau+) + Q_{np}(\tau+) + K_cQ_c(\tau+) \\ &- K_0Q_0(\tau-) - Q_{B_2}(\tau-) - Q_{np}(\tau-) - K_cQ_c(\tau-) \\ = {} & O(1)|\alpha_1||s|L_w(\tau-). \end{aligned}$$

When $L_w(\tau-)$ is sufficiently small, we choose K_w large enough such that

$$\begin{aligned} &\sum_{i=1} Q_{B_1}(\tau+) - \sum_{i=1} Q_{B_1}(\tau-) \\ = {} & |\alpha_1|W(\alpha_1, \tau-)((1 + O(1)|s|)\mathrm{e}^{-K_w|s|} - 1) \\ & + \sum_{i=1} \sum_{\beta_i \neq \gamma_i} |\beta_i|W(\beta_i, \tau-)(\mathrm{e}^{O(1)K_w|\alpha_1||s|} - 1) \\ \leq {} & -\frac{K_w}{2}|\alpha_1||s|. \end{aligned}$$

Therefore

$$Q(\tau+) - Q(\tau-) \leq -\frac{1}{2}|\alpha_1||s|. \tag{4.20}$$

Next we need to make estimate for $F_1(\tau)$. Denote

$$\arctan u(\Phi(U; \alpha_1, \alpha_2)) = h(U; \alpha_1, \alpha_2).$$

Hence we have

$$\begin{aligned} &\arctan u'_r - \arctan u'_m - \arctan u_m + \arctan u_l \\ = {} & h(U_l; \gamma_1, s') - h(U_l; \gamma_1, 0) - h(U_l; 0, s) + h(U_l; 0, 0) \\ = {} & h(U_l; \alpha_1, s) - h(U_l; \alpha_1, 0) - h(U_l; 0, s) + h(U_l; 0, 0) + O(1)|\alpha_1||s| \\ = {} & O(1)|\alpha_1||s|. \end{aligned}$$

Similarly, it holds that

$$\arctan u'_m - \arctan u_l - \arctan u_r + \arctan u_m = O(1)|\alpha_1||s|,$$
$$\arctan u_r - \arctan u'_r = O(1)|\alpha_1||s|.$$

So we have

$$\begin{aligned} &\text{TV. } \{\arctan u^{\nu,h}(\cdot,\tau+) : [b_\Delta(\tau+), +\infty)\} - \text{TV. } \{\arctan u^{\nu,h}(\cdot,\tau-) : [b_\Delta(\tau-), +\infty)\} \\ &= |\arctan u_r - \arctan u'_r| + |\arctan u'_r - \arctan u'_m| + |\arctan u'_m - \arctan u_l| \\ &\quad - |\arctan u_r - \arctan u_m| - |\arctan u_m - \arctan u_l| \\ &= O(1)|\alpha_1||s|. \end{aligned}$$

Therefore

$$F_1(\tau+) - F_1(\tau-) = O(1)|\alpha_1||s|. \tag{4.21}$$

Finally, from (4.20)–(4.21), we choose K and C_* large enough such that

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\alpha_1||s|.$$

Case 6 Interaction between the 2-strong rarefaction front and a non-physical wave.

Suppose that a front of the strong rarefaction wave denoted by s and a non-physical wave ϵ interact at the time $t = \tau$, where $\tau \notin \{k\Delta t : k \text{ is a positive integer}\}$. Let s_0 and ϵ_0 be the outgoing rarefaction wave and non-physical wave, respectively. Then, we have the following lemma.

Lemma 4.6 *It holds that*

$$s_0 = s,$$
$$\epsilon_0 = \epsilon + O(1)|\epsilon||s|.$$

In this case,

$$\begin{aligned} &Q(\tau+) - Q(\tau-) \\ &= W(\epsilon_0, \tau+)|\epsilon_0| - W(\epsilon, \tau-)|\epsilon| \\ &= W(\epsilon_0, \tau+)|\epsilon_0| - W(\epsilon_0, \tau+)\exp(K_{np}|s|)|\epsilon| \\ &= W(\epsilon_0, \tau+)|\epsilon_0|(1 + O(1)|s| - \exp(K_{np}|s|)). \end{aligned}$$

Therefore, if K_{np} is large enough, then we have

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}|\epsilon||s|.$$

Hence, when K and C_* are large enough, we have

$$F(\tau+) - F(\tau-) \le -\frac{1}{4}|\epsilon||s|.$$

Combining the estimates of Case 1-Case 6, we can prove (4.4)–(4.5). Therefore, there exist positive constants δ_1, K_0, K_c and C_*, independent of τ, ν, such that if $F(\tau-) < \delta_1$, then the strength of every wave front is less than δ_*.

In order to prove (ii), we introduce a new functional for any 2-strong rarefaction wave front s as

$$F_2(t) = |s(t)|\mathrm{e}^{L_s(t)+K_2Q(t)},$$

where

$$L_s(t) = \sum_{\alpha}\{|\alpha(t)| : \alpha \in \mathcal{A}_s\}$$

and K_2 is a positive constant to be determined later.

Similar to the proof of (4.5), there exists a positive constant K_2 independent of τ and ν such that

$$F_2(\tau+) < F_2(\tau-). \tag{4.22}$$

Hence we have

$$F_2(t) \leq F_2(k_0\Delta t) \leq O(1)|s(k_0\Delta t)|,$$

where $(b(k_0\Delta t), k_0\Delta t)$ is the point from which the rarefaction wave front s issues. On the other hand, by the construction of the approximate solution, we have

$$|s(k_0\Delta t)| \leq O(1)\frac{1}{\nu},$$

which along with (4.22) yields (ii).

Finally, we need to show (iii). From (4.4), we know that $Q(t)$ is decreasing and bounded. By the wave front tracking algorithm, when $|\alpha||\beta| > \frac{1}{2^\nu}$, we adopt the accurate Riemann solver. From (4.4), we have

$$Q(\tau+) - Q(\tau-) < -\frac{1}{2}E_{\nu,h}(\tau).$$

Therefore, the number of the interaction is finite. Hence, the number of the new physical waves is finite. On the other hand, when we adopt the simplified Riemann solver, the number of the wave fronts keeps the same after the interaction.

Hence, for any $k\Delta t \leq t < (k+1)\Delta t$, the number of the wave fronts is finite. On the other hand, we know that the non-physical wave is introduced by the simplified Riemann solver. Then, the number of the non-physical waves is also finite. By a standard procedure (see [3]), we can also prove that the total strength of the non-physical waves at any $t < \tau_1$ is small, satisfying (4.7).

In conclusion, we complete the proof of this theorem.

In the following, we will show that the inductive hypotheses $A_2(\tau-)$ is reasonable, and the total strength of the 2-strong rarefaction fronts is finite.

Lemma 4.7 *For any $\tau < t < \tau_1$, the following holds:*

(i) *There exist positive constants* η_0 *and* δ_2 *independent of* ν, h *such that*

$$|U^{\nu,h}(x,t) - \overline{U}|_{x>\eta_0} \leq \delta_2. \tag{4.23}$$

(ii) $U^{\nu,h}(\cdot,t)|_{t<\tau_1} \in D(\overline{U}, \delta_{**})$.

(iii) *For any* 2*-strong rarefaction wave front* $s_0(t)$*, it satisfies that*

$$\sum_{s_0} |s_0(t)| \leq O(1)(|\arctan u_*| + 2\delta_0 + L_w(t)), \tag{4.24}$$

where τ *and* τ_1 *are given in Theorem* 4.1.

Proof Based on Theorem 4.1, there exists a positive constant η_0 such that when $x > \eta_0$, there exists a positive constant ϵ_* such that

$$\text{TV. } \{U^{\nu,h}(\cdot,t) : [\eta_0, +\infty)\} < O(1)\varepsilon_*.$$

Due to the hypothesis $A_2(\tau-)$, there exist positive constants $0 < \delta_{**} < \frac{\delta_*}{2}$ and δ_2 such that when $x > \eta_0$, we have

$$\begin{aligned} |U^{\nu,h}(x,t) - \overline{U}| &\leq \text{TV. } \{U^{\nu,h}(x,t) : [\eta_0, +\infty)\} + |U^{\nu,h}(+\infty,t) - \overline{U}| \\ &\leq O(1)\epsilon_* + \delta_{**} \\ &< \delta_2. \end{aligned}$$

From Lemma 4.2, we have

$$\begin{aligned} |(u - 2\sqrt{p})(U^{\nu,h}) - (u - 2\sqrt{p})(\overline{U})| &\leq \sum_i |(u - 2\sqrt{p})(U_i^{\nu,h}) - (u - 2\sqrt{p})(U_{i-1}^{\nu,h})| \\ &= O(1)(|\alpha_1| + |\alpha_2^-| + |\epsilon|) \\ &= O(1)F(t), \end{aligned}$$

where $U_i^{\nu,h}$ and $U_{i-1}^{\nu,h}$ are the constant states connected by the physical waves α_i, $i = 1, 2$, or non-physical wave ϵ.

Next, we need to make estimate for $\arctan u$. Due to (i), there exists a positive constant C such that

$$|\arctan u^{\nu,h}(x,t) - \arctan \overline{u}| \leq C\delta_2. \tag{4.25}$$

Suppose that $x_0 < x_1 < \cdots < x_{N-1} < x_N$ are the discontinuities of the approximate solution $U^{\nu,h}(x,t)$. From Lemma 4.1, we have

$$\begin{aligned} &\arctan u^{\nu,h}(x,t) \\ &= \arctan u^{\nu,h}(x,t) - \arctan u^{\nu,h}(2\eta_0,t) + \arctan u^{\nu,h}(2\eta_0,t) - \arctan \overline{u}, \\ &= \sum_{1\leq k\leq N} (\arctan u^{\nu,h}(x_k,t) - \arctan u^{\nu,h}(x_{k-1},t)) + \arctan u^{\nu,h}(2\eta_0,t) - \arctan \overline{u} \\ &\leq O(1)F(t) + C\delta_2. \end{aligned}$$

On the other hand, it holds

$$\begin{aligned}
&\arctan u^{\nu,h}(x,t)\\
=&\arctan u^{\nu,h}(x,t)-\arctan u^{\nu,h}(2\eta_0,t)+\arctan u^{\nu,h}(2\eta_0,t)-\arctan \overline{u}\\
\geq& -\mathrm{TV.}\arctan u^{\nu,h}(\cdot,t)-C\delta_2\\
\geq& -F_0(t)-\widehat{\theta}-C\delta_2\\
\geq& \theta_*+\delta_{**}.
\end{aligned}$$

Finally, from Lemma 4.1 and (i), we have

$$\begin{aligned}
\sum_{s_0}|s_0(t)|\leq& \frac{1}{C_1}|\arctan u^{\nu,h}(\Phi(U^{\nu,h}(x,t);0,s_0))-\arctan u^{\nu,h}(x,t)|\\
\leq& \frac{1}{C_1}|\arctan u^{\nu,h}(b_\Delta(t),t)-\arctan \overline{u}|+\frac{1}{C_1}|\arctan u^{\nu,h}(x,t)-\arctan\overline{u}|\\
&+\frac{C_2}{C_1}L_w(t)\\
\leq& O(1)(|\arctan u_*|+\delta_0+L_w(t)).
\end{aligned}$$

Therefore, we complete the proof of this lemma.

Finally, combining Lemmas 4.3–4.7 altogether, we can obtain the following theorem.

Theorem 4.2 *There exist positive constants ε and M_0 independent of ν, h such that, if*

$$\|U_0-\overline{U}\|_\infty+|b'_+(0)-b'_{*+}(0)|+\mathrm{TV.}\{U_0(\cdot):[0,\infty)\}+\mathrm{TV.}\{(b'_+-b'_{*+})(\cdot)\}<\varepsilon, \tag{4.26}$$

then

$$\mathrm{TV.}\{U^{\nu,h}(\cdot,t):[b(t),+\infty)\}<M_0. \tag{4.27}$$

Proof By Theorem 4.1 and induction hypothesis, if $F(0+)$ is sufficiently small, then for any $0<\tau<t$, we deduce that

$$F(t+)<F(0+)-\frac{1}{4}\sum_{\tau>0}E_{\nu,h}(\tau).$$

Since

$$F(0+)=O(1)\Big(\|U_0-\overline{U}\|_\infty+\mathrm{TV.}\ \{U_0(\cdot):[0,\infty)\}+\sum_{k\geq 0}|\omega_k^+|\Big)$$

and

$$\sum_{k\geq 0}|\omega_k^+|=O(1)(\mathrm{TV.}\ \{(\mathrm{b}'_+-\mathrm{b}'_{*+})(\cdot)\}+|\mathrm{b}'_+(0)-\mathrm{b}'_{*+}(0)|),$$

we combine the above estimates to get the desired result. The proof is complete.

By Theorem 4.2, the proof of the convergence of the approximate solution $U^{\nu,h}(x,t)$ is a standard procedure, also see [2–3]. Therefore, we complete the proof of Theorem 1.1.

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