Chin. Ann. Math. Ser. B
40(2), 2019, 187–198
DOI: 10.1007/s11401-019-0125-9

Chinese Annals of Mathematics, Series B

Joint Reducing Subspaces of Multiplication Operators and Weight of Multi-variable Bergman Spaces*

Hansong HUANG[1] Peng LING[2]

Abstract This paper mainly concerns a tuple of multiplication operators defined on the weighted and unweighted multi-variable Bergman spaces, their joint reducing subspaces and the von Neumann algebra generated by the orthogonal projections onto these subspaces. It is found that the weights play an important role in the structures of lattices of joint reducing subspaces and of associated von Neumann algebras. Also, a class of special weights is taken into account. Under a mild condition it is proved that if those multiplication operators are defined by the same symbols, then the corresponding von Neumann algebras are $*$-isomorphic to the one defined on the unweighted Bergman space.

Keywords Joint reducing subspaces, Von Neumann algebras, Weighted Bergman spaces
2000 MR Subject Classification 47A13, 47B35

1 Introduction

In this paper, denote by $\mathbb{D}$ the unit disk in the complex plane $\mathbb{C}$. Let Ω be a bounded domain in the complex space $\mathbb{C}^d$. Let $H^\infty(\Omega)$ denote the Banach space of all bounded holomorphic functions on Ω. For a nonnegative continuous function ω over Ω, the weighted Bergman space $L_a^2(\omega,\Omega)$ is the Hilbert space consisting of all holomorphic functions over $\{z\in\Omega:\omega(z)>0\}$ which are square integrable with respect to the measure $\omega(z)\mathrm{d}A(z)$, $\mathrm{d}A$ being the Lebesgue measure on Ω. In particular, if $\varphi\in H^\infty(\Omega)$ and $\varphi\not\equiv 0$, the weighted Bergman space $L_a^2(|\varphi|^2,\Omega)$ is called type-φ Bergman space. Specifically, letting $\varphi\equiv 1$ gives the ordinary Bergman space $L_a^2(\Omega)$.

Fix a bounded holomorphic function ϕ and a weight ω over Ω. Let M_ϕ denote the multiplication operator with the symbol ϕ on $L_a^2(\omega,\Omega)$, given by

$$M_\phi f=\phi f,\quad f\in L_a^2(\omega,\Omega).$$

In general, for a tuple $\Phi=\{\phi_j:1\le j\le n\}$, let $\{M_\Phi\}'$ denote the commutant of $\{M_{\phi_j}:1\le j\le n\}$, consisting of all bounded operators commuting with each operator M_{ϕ_j} $(1\le j\le n)$. It should be emphasized that M_Φ denotes a family of multiplication operators rather than a single vector-valued multiplication operator. Let $\mathcal{V}^*(\Phi,\omega,\Omega)$ denote the von Neumann algebra

$$\{M_{\phi_j},M_{\phi_j}^*:1\le j\le n\}',$$

Manuscript received May 15, 2016. Revised April 16, 2017.
[1]Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China. E-mail: hshuang@ecust.edu.cn
[2]School of Mathematics, Fudan University, Shanghai 200433, China. E-mail: pengling91@sina.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11471113, 11571064).

consisting of all bounded operators on $L_a^2(\omega,\Omega)$ which commutes with both M_{ϕ_j} and $M_{\phi_j}^*$ for all j. Write $\mathcal{W}^*(\Phi,\omega,\Omega)=\mathcal{V}^*(\Phi,\omega,\Omega)'$, which equals the von Neumann algebra generated by $\{M_{\phi_j}:1\leq j\leq n\}$. For $\omega\equiv 1$, we write $\mathcal{W}^*(\Phi,\Omega)$ for $\mathcal{W}^*(\Phi,\omega,\Omega)$, and $\mathcal{V}^*(\Phi,\Omega)$ for $\mathcal{V}^*(\Phi,\omega,\Omega)'$. There is a close connection between orthogonal projections in $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ and all joint reducing subspaces of $\{M_{\phi_j}:1\leq j\leq n\}$ (see [13–14, 17]). Precisely, the range of an orthogonal projection in $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ is exactly a joint reducing subspace of the tuple $\{M_{\phi_j}:1\leq j\leq n\}$, and vice versa. We say that $\mathcal{V}^*(\Phi,\omega,\Omega)$ is trivial if $\mathcal{V}^*(\Phi,\omega,\Omega)=\mathbb{C}I$; equivalently, $\{M_{\phi_j}:1\leq j\leq n\}$ has no nonzero joint reducing subspace other than the whole space $L_a^2(\omega,\Omega)$. In single-variable case, one can refer to [7–10, 12, 14–18, 20, 24–26] for work on commutants and reducing subspaces of multiplication operators and related von Neumann algebras. However, not much has been done for multi-variable cases. We call the reader's attention to [11, 19–21, 23, 27–28].

In this paper we mainly focus on the influence of the weight of the Bergman spaces on the structures of lattices for joint reducing subspaces for some M_Φ, and of related von Neumann algebras. To be precise, for $\alpha\in(-1,\infty)^d$, write $L_a^2(\alpha,\mathbb{D}^d)$ for the weighted Bergman space over $\mathbb{D}^d$ with weight

$$\prod_{j=1}^{d}(1-|z_j|^2)^{\alpha_j}\,\mathrm{d}A(z).$$

On one hand, we construct a concrete tuple of functions Φ and prove that for almost every $\alpha=(\alpha_1,\cdots,\alpha_d)\in(-1,\infty)^d$, M_Φ has no nontrivial joint reducing subspace and for some α, M_Φ has. This shows that weights play an important role on the structure of lattices for joint reducing subspaces. On another hand, we consider type-φ Bergman spaces. It is shown under a mild condition that the von Neumann algebra $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ is $*$-isomorphic to $\mathcal{V}^*(\Phi,\Omega)$. As an application, this gives an interesting corollary in single-variable case.

This paper is arranged as follows. Section 2 presents some preliminaries, including the notion of local inverse. In Section 3, a concrete example is given to show that for different weights the joint reducing subspaces for some M_Φ can differ a lot. Section 4 shows that on distinct type-φ Bergman spaces the structure of the joint reducing subspaces for M_Φ is just the same under a mild condition.

2 Some Preliminaries

In this section, we will introduce some basic notations and give some preliminaries.

The notion of analytic continuation is important (see [22, Chapter 16]). A function element is an ordered pair (f,D), where D is a simply-connected domain and f is a holomorphic function on D. Two function elements (f_0,D_0) and (f_1,D_1) are called direct continuations if $D_0\cap D_1$ is not empty and $f_0=f_1$ holds on $D_0\cap D_1$. A curve is a continuous map from $[0,1]$ into $\mathbb{C}^d$. Given a function element (f_0,D_0) and a curve γ with $\gamma(0)\in D_0$, if there is a partition of $[0,1]$:

$$0=s_0<s_1<\cdots<s_n=1$$

and function elements (f_j,D_j) $(0\leq j\leq n)$ such that

(1) (f_j,D_j) and (f_{j+1},D_{j+1}) are direct continuations for all j with $0\leq j\leq n-1$;

(2) $\gamma[s_j,s_{j+1}]\subseteq D_j$ $(0\leq j\leq n-1)$ and $\gamma(1)\in D_n$,

then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along γ, and (f_0, D_0) is called to admit an analytic continuation along γ. In this case, we write $f_0 \sim f_n$. Clearly, $\sim$ defines an equivalence and we write $[f]$ for the equivalent class of f.

Let $\Omega \subseteq \mathbb{C}^d$. For a holomorphic mapping $\Phi = (\phi_1, \cdots, \phi_d) : \Omega \to \mathbb{C}^d$ and a subdomain Δ of Ω, a holomorphic function $\rho : \Delta \to \Omega$ is called a local inverse of Φ if $\Phi \circ \rho = \Phi$.

Recall that a subset E of Ω is called a zero variety of Ω if there is a nonconstant holomorphic function φ on Ω such that $E = \{z \in \Omega : \varphi(z) = 0\}$. A relatively closed subset $\mathcal{E}$ of Ω is called L_a^2-removable in Ω if each function in $L_a^2(\Omega - \mathcal{E})$ extends analytically to a function in $L_a^2(\Omega)$ (see [4, 6]). It is known that for a zero variety E of a domain Ω, E is L_a^2-removable in Ω (see [3, 5]).

A zero variety E of Ω is called good if for each point $\lambda \in \overline{\Omega}$, there is an open ball U centered at λ such that

$$U \cap \overline{E} = \{z \in U \cap \overline{\Omega} : \psi(z) = 0\},$$

where ψ is a holomorphic function on U (see [20]). Clearly, if φ is a nonconstant holomorphic function over $\overline{\Omega}$, the zero variety $Z(\varphi)$ is good. The following lemma comes from [20, Theorem 1.3].

Lemma 2.1 *Suppose that E is a good zero variety of a domain Ω in $\mathbb{C}^d$ and $F : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ such that the image of F contains an interior point. Then both $F(\overline{E})$ and $F^{-1}(F(\overline{E}))$ are L_a^2-removable.*

The following proposition is contained in [20, Proposition 3.5].

Proposition 2.1 *Suppose that E is a good zero variety of a domain Ω in $\mathbb{C}^d$. If $F : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ such that the image of F has an interior point. Then $\Omega \setminus F^{-1}(F(\overline{E}))$ is connected.*

We also need a result from operator theory in [14]. For the special case of Λ being a singlet, it is first proved in [2, Proposition 5.1] or [1, Proposition A.1].

Lemma 2.2 *Let $\mathcal{H}$ be a Hilbert space and let e_λ^k, f_μ^k $(1 \le k \le n$ and $\lambda, \mu \in \Lambda)$ be vectors in $\mathcal{H}$ satisfying*

$$\sum_{k=1}^n e_\lambda^k \otimes e_\mu^k = \sum_{k=1}^n f_\lambda^k \otimes f_\mu^k, \quad \lambda, \mu \in \Lambda.$$

Then there is an $n \times n$ numerical unitary matrix W such that

$$W \begin{pmatrix} e_\lambda^1 \\ \vdots \\ e_\lambda^n \end{pmatrix} = \begin{pmatrix} f_\lambda^1 \\ \vdots \\ f_\lambda^n \end{pmatrix}, \quad \lambda \in \Lambda.$$

3 An Example

Throughout this paper, we write $z = (z_1, \cdots, z_d)$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$. Let $L_a^2(\alpha, \mathbb{D}^d)$ denote the Bergman space with the weight $\prod_{j=1}^d (1 - |z_j|^2)^{\alpha_j} \, \mathrm{d}A(z)$.

In this section we mainly prove the following proposition.

Proposition 3.1 *Let* $(\alpha_1, \cdots, \alpha_d) \in (-1, \infty)^d$, *and write*

$$\Phi(z_1, \cdots, z_d) = \Big(z_1 + z_2 + \cdots + z_d, \sum_{1 \le i < j \le d} z_i z_j, \cdots, \prod_{1 \le j \le d} z_j\Big).$$

If there is no nonzero polynomial $P \in \mathbb{Z}[z]$ *satisfying* $P(\alpha_1, \cdots, \alpha_d) = 0$, *then*

$$\mathcal{V}^*(\Phi, \alpha, \mathbb{D}^d) = \mathbb{C}I.$$

Equivalently, M_Φ *has no nontrivial joint reducing subspace on* $L_a^2(\alpha, \mathbb{D}^d)$.

Proof Note that the d-th coordinate of Φ is $\prod_{1 \le j \le d} z_j$. In short, we write $M_z = \prod_{1 \le k \le d} M_{z_k}$. Then

$$M_z^* M_z(z^J) = \prod_{1 \le i \le d} \frac{1 + j_k}{\alpha_j + 2 + j_k} z^J, \quad J \in \mathbb{Z}_+^d.$$

Write

$$\lambda(J) = \prod_{1 \le i \le d} \frac{1 + j_k}{\alpha_j + 2 + j_k},$$

and we claim that $\lambda(J) = \lambda(J')$ if and only if $J = J'$.

In the case of $d = 1$, rewrite $J = j$ and $J' = j'$. Then $\lambda(J) = \lambda(J')$ can be written as

$$\frac{1 + j}{\alpha + 2 + j} = \frac{1 + j'}{\alpha + 2 + j'},$$

which holds if and only if $j = j'$. We will prove the claim by induction on d. Suppose that the claim is true for $d = m-1$ $(m \ge 2)$ and we proceed to check it for $d = m$. Write $J = (j_1, \cdots, j_m)$ and $J' = (j_1', \cdots, j_m')$. Assume $\lambda(J) = \lambda(J')$, and we have

$$\frac{1 + j_m}{\alpha_m + 2 + j_m} \prod_{1 \le k \le m-1} \frac{1 + j_k}{\alpha_k + 2 + j_k} = \frac{1 + j_m'}{\alpha_m + 2 + j_m'} \prod_{1 \le k \le m-1} \frac{1 + j_k'}{\alpha_k + 2 + j_k'}. \tag{3.1}$$

If $j_m = j_m'$, then

$$\prod_{1 \le i \le m-1} \frac{1 + j_k}{\alpha_k + 2 + j_k} = \prod_{1 \le i \le m-1} \frac{1 + j_k'}{\alpha_k + 2 + j_k'}.$$

By induction hypothesis we immediately get $J = J'$. Now assume $j_m \ne j_m'$ and we will derive a contradiction. In fact, by (3.1) α_m can be written as a rational function in $\alpha_1, \cdots, \alpha_{m-1}$, with integer coefficients. That is, there exists polynomials P_1 and P_2 in $\mathbb{Z}[z_1, \cdots, z_{m-1}]$ satisfying

$$\alpha_m = \frac{P_1(\alpha_1, \cdots, \alpha_{m-1})}{P_2(\alpha_1, \cdots, \alpha_{m-1})}.$$

Thus $P_1(\alpha_1, \cdots, \alpha_{m-1}) - \alpha_m P_2(\alpha_1, \cdots, \alpha_{m-1}) = 0$. Since $P_2 \not\equiv 0$, this derives a contradiction to finish the proof of the claim.

Let P_J denote the orthogonal projection from $L_a^2(\alpha, \mathbb{D}^d)$ onto $\mathbb{C}[z^J]$. Since $\lambda(J) = \lambda(J')$ if and only if $J = J'$, by spectrum theory we have

$$P_J \in \mathcal{W}^*(\Phi, \alpha, \mathbb{D}^d).$$

Based on this, one can show that there is no nontrivial joint reducing subspace for M_Φ. In fact, for each nonzero joint reducing subspace $\mathcal{M}$ for M_Φ one can pick a nonzero function h in $\mathcal{M}$. Write

$$h = \sum_{J \in \mathbb{Z}_+^d} c_J z^J,$$

and assume $c_{J_0} \neq 0$. Noting

$$P_{J_0}(h) = c_{J_0} z^{J_0} \in \mathcal{M},$$

we have $z^{J_0} \in M$. Then by a bit effort one will get $1 \in M$, and by induction one can prove that

$$z^J \in \mathcal{M}, \quad J \in \mathbb{Z}_+^d.$$

Hence $M = L_a^2(\alpha, \mathbb{D}^d)$. That is, M_Φ has only trivial joint reducing subspace. Equivalently,

$$\mathcal{V}^*(\Phi, \alpha, \mathbb{D}^d) \cong \mathbb{C}I.$$

The proof is complete.

Corollary 3.1 *Let $(\alpha_1, \cdots, \alpha_d) \in (-1, \infty)^d$, and write*

$$\Phi(z_1, \cdots, z_d) = \Big(z_1 + z_2 + \cdots + z_d, \sum_{1 \le i < j \le d} z_i z_j, \cdots, \prod_{1 \le j \le d} z_j\Big).$$

Then for almost everywhere $\alpha \in (-1, \infty)^d$, M_Φ has no nontrivial joint reducing subspace on $L_a^2(\alpha, \mathbb{D}^d)$.

Proof Let $\mathcal{E}$ denote the set of all $\alpha \in (-1, \infty)^d$, so that there is a nonzero polynomial $F \in \mathbb{Z}_+[z]$ satisfying $F(\alpha) = 0$. Let $\mathcal{Z}_F$ denote all $\alpha \in \mathbb{R}^d$ satisfying $F(\alpha) = 0$. If we can show that each set $\mathcal{Z}_F$ is of zero measure, then so is $\mathcal{E}$ since $\mathcal{E}$ is contained in a union of countably many $\mathcal{Z}_F$. Then by Proposition 3.1 we get the desired conclusion. Thus it suffices to show the following claim:

For each nonzero polynomial $F \in \mathbb{Z}_+[z]$, $\mathcal{Z}_F$ has zero measure.

For this, we use induction. For $d = 1$, $\mathcal{Z}_F$ is a finite set and obviously is of measure zero in $\mathbb{R}$. Suppose that the claim is true for $d = m - 1$ $(m \ge 2)$ and we proceed to check it for $d = m$. Without loss of generality, assume $\frac{\partial F}{\partial z_m} \not\equiv 0$. Write $z' = (z_1, \cdots, z_{m-1})$ and put

$$F(z) = a_n(z') z_m^n + \cdots + a_1(z') z_m + a_0(z'), \quad a_n \neq 0.$$

By induction hypothesis, $\mathcal{Z}_{a_n}$ is of measure zero in $\mathbb{R}^{m-1}$. Thus for almost everywhere $z' \in \mathbb{R}^{m-1}$, we have that $\{z_m \in \mathbb{R} : F(z) = 0\}$ is a finite set, and for such z',

$$\int_{\mathbb{R}} \chi_{\mathcal{Z}_F}(z', z_m) \mathrm{d}A(z_m) = 0.$$

Since $\mathcal{Z}_F$ is a closed set, by applying Fubini's theorem one can get that

$$\int_{\mathbb{R}^m} \chi_{\mathcal{Z}_F}(z) \mathrm{d}A(z) = 0,$$

namely, $\mathcal{Z}_F$ has zero measure. Thus we finish the proof of the claim as desired.

Let Φ be defined in Proposition 3.1 The following example shows that, for some $\alpha \in \mathbb{Z}^d$,

$$\mathcal{V}^*(\Phi, \alpha, \mathbb{D}^d) \neq \mathbb{C}I,$$

and especially, $\mathcal{V}^*(\Phi, \mathbb{D}^d) \neq \mathbb{C}I$.

Example 3.1 Write $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $d \geq 3$ and assume

$$\alpha_1 = \alpha_2.$$

As done above, set

$$\Phi(z_1, \cdots, z_d) = \Big(z_1 + z_2 + \cdots + z_d, \sum_{1 \leq i < j \leq d} z_i z_j, \cdots, \prod_{1 \leq j \leq d} z_j\Big).$$

Fix $t > 0$, and let $\mathcal{M}_t$ denote the closed subspace generated by

$$\{(tz_1 + z_2)z^J : J \in \mathbb{Z}_+^d\}.$$

Note that $\mathcal{M}_t$ is invariant under $\{M_{z_1+z_2}, M_{z_1 z_2}\}$ and $\{M^*_{z_1+z_2}, M^*_{z_1 z_2}\}$, as well as under $\{M_{z_j}, M^*_{z_j} : 2 < j \leq d\}$. Since Φ can be written as a function in z_1+z_2, $z_1 z_2$, and $z_3, \cdots, z_d$, $\mathcal{M}_t$ is invariant under the tuple of operators M_Φ and M^*_Φ. Hence $\mathcal{M}_t$ is a joint nontrivial reducing subspace of M_Φ. Equivalently, for such α with $\alpha_1 = \alpha_2$,

$$\mathcal{V}^*(\Phi, \alpha, \mathbb{D}^d) \neq \mathbb{C}I.$$

4 The Main Result

In this section, one will see that under a mild condition the multiplication operators M_Φ with the same symbols can induce $*$-isomorphic von Neumann algebras on distinct φ-type Bergman space. This is stated as below.

Theorem 4.1 *Suppose that the interior of $\overline{\Omega}$ equals Ω, $\Phi : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ and the image of Φ has an interior point. If φ is a nonconstant function holomorphic over $\overline{\Omega}$, then $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ is a finite dimensional von Neumann algebra. Furthermore, $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ is $*$-isomorphic to $\mathcal{V}^*(\Phi, \Omega)$.*

The von Neumann algebra $\mathcal{V}^*(\Phi, \Omega)$ is characterized in [20].

Proof Suppose that the interior of $\overline{\Omega}$ equals Ω, $\Phi : \Omega \to \mathbb{C}^d$ is holomorphic on $\overline{\Omega}$ and $\Phi(\Omega)$ contains an interior point. As follows, for a holomorphic map $F : \Omega \to \mathbb{C}^d$, denote by JF the determinant of the Jacobian of F.

First we prove that $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ is a finite dimensional von Neumann algebra. The main idea is borrowed from [14] and [20]. For this, we have $J\Phi \not\equiv 0$. Otherwise the complex dimension of $\Phi(\Omega)$ would be at most $d-1$, which leads to a contradiction. Write

$$\overline{Z} = \{z \in \overline{\Omega} : J\Phi(z) = 0\}.$$

For each point $\lambda \in \Omega \setminus \Phi^{-1}(\Phi(\overline{Z}))$, there exists a ball Δ ($\Delta \subseteq \Omega$) centered at λ satisfying

$$\Phi(\overline{\Delta}) \cap \Phi(\overline{Z}) = \emptyset,$$

and $\Phi|_\Delta$ is biholomorphic. Therefore, we have

$$J\Phi(w) \neq 0, \quad w \in \Phi^{-1}(\Phi(\overline{\Delta})) \cap \overline{\Omega},$$

and there is an open ball U_w of w such that

$$\Phi : U_w \to \Phi(U_w)$$

is biholomorphic. Note that $\Phi^{-1}(\Phi(\overline{\Delta})) \cap \overline{\Omega}$ is compact, and the union of all such balls U_w covers $\Phi^{-1}(\Phi(\overline{\Delta})) \cap \overline{\Omega}$. By a simple application of Henie-Borel's theorem, there exists finitely many of them: $U_1, \cdots, U_n$ satisfying

$$\bigcup_{j=1}^{n} U_j \supseteq \Phi^{-1}(\Phi(\overline{\Delta})) \cap \Omega,$$

and $\Phi|_{U_j}$ is biholomorphic for each j. Suppose that Φ is holomorphic on a domain $\widetilde{\Omega}$ such that $\widetilde{\Omega} \supseteq \overline{\Omega}$. Writing

$$U_j' = U_j \cap \widetilde{\Omega} \cap \Phi^{-1}(\Phi(\Delta)), \quad j = 1, \cdots, K,$$

we still have

$$\bigcup_{j=1}^{n} U_j' \supseteq \Phi^{-1}(\Phi(\Delta)) \cap \Omega. \tag{4.1}$$

Since $\Phi|_{U_j'}$ are all biholomorphic, $\Phi(\lambda)$ is attained by Φ in U_j' at most once. Then by shrinking Δ, (4.1) shows that there are finitely many disjoint domains $\Delta_1, \cdots, \Delta_N$ and biholomorphic maps $\rho_1, \cdots, \rho_N$ such that

$$\bigsqcup_{j=1}^{N} \Delta_j = \Phi^{-1}(\Phi(\Delta)) \cap \Omega$$

and

$$\rho_j(\Delta_1) = \Delta_j, \quad \Phi \circ \rho_j = \Phi, \quad 1 \le j \le N,$$

where $\rho_1(z) \equiv z$ and $\Delta_1 = \Delta$. Again by shrinking Δ, we may require that all these U_j' have no intersection with $\partial\Omega$. In fact, recall that the interior of $\overline{\Omega}$ equals Ω. Then by analysis, one can prove that for each point $\zeta \in \partial\Omega$ and any open neighborhood $\mathcal{O}(\zeta)$ of ζ, $\mathcal{O}(\zeta) \setminus \Omega$ contains an open ball. Based on this, we may replace Δ with an open subset of Δ to ensure that U_1' have no intersection with $\partial\Omega$. Note that $U_2', \cdots, U_n'$ will shrink automatically. By shrinking Δ again, U_2' has no intersection with $\partial\Omega$. After finite steps, each U_j' has no intersection with $\partial\Omega$: Some of them become smaller, and some of them vanish. Besides, it is worthwhile to emphasize that if

$$\Phi^{-1}(\Phi(\lambda)) \cap \partial\Omega = \emptyset,$$

then Δ can be replaced with a smaller ball centered at λ.

Next we will use the method in [14] to give the representation of those operators in the von Neumann algebra $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ and the details are as follows. Let S be a unitary operator in $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$. Given any function g and h in $L_a^2(|\varphi|^2, \Omega)$, put

$$\widetilde{g} = Sg, \quad \widetilde{h} = Sh.$$

Then for any multivariate polynomials P and Q,

$$\langle P(\Phi)g, Q(\Phi)h\rangle = \langle P(\Phi)\widetilde{g}, Q(\Phi)\widetilde{h}\rangle.$$

That is

$$\int_\Omega ((P\overline{Q})\circ\Phi(w)g(w)\overline{h}(w) - (P\overline{Q})\circ\Phi(w)\widetilde{g(w)}\overline{\widetilde{h(w)}})|\varphi(w)|^2\mathrm{d}A(w) = 0. \tag{4.2}$$

Now set

$$\mathfrak{X} = \operatorname{span}\{p\overline{q} : \ p, q \text{ are polynomials in } d \text{ variables}\}.$$

By the Stone-Weierstrass theorem, a continuous function on $\overline{\Phi(\Omega)}$ can be uniformly approximated by a sequence of functions in $\mathfrak{X}$. Thus by (4.2),

$$\int_\Omega (u(\Phi(w))g(w)\overline{h}(w) - u(\Phi(w))\widetilde{g(w)}\overline{\widetilde{h(w)}})|\varphi(w)|^2\mathrm{d}A(w) = 0, \quad u \in C(\overline{\Phi(\Omega)}). \tag{4.3}$$

By Lebesgue's dominated convergence theorem, (4.3) holds for all u in $L^\infty(\Phi(\Omega))$. Let Δ be the ball in the above paragraph. For each u in $L^\infty(\Phi(\Delta))$, (4.3) gives that

$$\begin{aligned}&\int_{\Phi^{-1}(\Phi(\Delta))} u(\Phi(w))g(w)\overline{h}(w)|\varphi(w)|^2\mathrm{d}A(w)\\ =&\int_{\Phi^{-1}(\Phi(\Delta))} u(\Phi(w))\widetilde{g(w)}\overline{\widetilde{h(w)}}|\varphi(w)|^2\mathrm{d}A(w),\end{aligned}$$

and hence by requirements on Δ below (4.1),

$$\begin{aligned}&\int_\Delta u(\Phi(z))\sum_{j=1}^N (g\overline{h})\circ\rho_j(z)|(J\rho_j)(z)|^2|\varphi(\rho_j(z))|^2\mathrm{d}A(z)\\ =&\int_\Delta u(\Phi(z))\sum_{j=1}^N (\widetilde{g}\overline{\widetilde{h}})\circ\rho_j(z)|(J\rho_j)(z)|^2|\varphi(\rho_j(z))|^2\mathrm{d}A(z).\end{aligned}$$

Noting that Φ is univalent on Δ and u can be an arbitrary function in $L^\infty(\Phi(\Delta))$, we immediately have that for $z \in \Delta$,

$$\sum_{j=1}^N (g\overline{h})\circ\rho_j(z)|(J\rho_j)(z)|^2|\varphi(\rho_j(z))|^2 = \sum_{j=1}^N (\widetilde{g}\overline{\widetilde{h}})\circ\rho_j(z)|(J\rho_j)(z)|^2|\varphi(\rho_j(z))|^2. \tag{4.4}$$

Let $\mathcal{H}$ be the Bergman space over Δ. For $1 \le j \le N$, $g \in L^2_a(\Omega)$, set

$$e_g^j = g(\rho_j(z))(J\rho_j)(z)\varphi(\rho_j(z)) \quad \text{and} \quad f_g^j = \widetilde{g}(\rho_j(z))(J\rho_j)(z)\varphi(\rho_j(z)),$$

all e_g^j and f_g^j lie in $\mathcal{H}$. By (4.4), the Berezin transforms of $\sum\limits_{j=1}^N e_g^k \otimes e_h^k$ and $\sum\limits_{j=1}^N f_g^k \otimes f_h^k$ are equal. By the property of Berezin transform,

$$\sum_{k=1}^N e_g^k \otimes e_h^k = \sum_{k=1}^N f_g^k \otimes f_h^k, \quad g, h \in L^2_a(\Omega).$$

Then applying Lemma 2.2 gives that there is an $N \times N$ unitary numerical matrix W satisfying

$$W \begin{pmatrix} g(\rho_1(w))J\rho_1(w)\varphi(\rho_1(z)) \\ \vdots \\ g(\rho_N(w))J\rho_N(w)\varphi(\rho_N(z)) \end{pmatrix} = \begin{pmatrix} \widetilde{g}(\rho_1(w))J\rho_1(w)\varphi(\rho_1(z)) \\ \vdots \\ \widetilde{g}(\rho_N(w))J\rho_N(w)\varphi(\rho_N(z)) \end{pmatrix}, \quad w \in \Delta.$$

By expanding the first raw of W, we get N constants $c_1, \cdots, c_N$ satisfying

$$\widetilde{g}(\rho_1(w))J\rho_1(w)\varphi(\rho_1(z)) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w)\varphi(\rho_j(z)).$$

Noting $\rho_1(w) \equiv w$,

$$\varphi(w)Sg(w) = \sum_{j=1}^{N} c_j g(\rho_j(w))(J\rho_j)(w)\varphi(\rho_j(w)), \quad w \in \Delta. \tag{4.5}$$

In the beginning we assume that S is a unitary operator in $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$. Since each operator in a von Neumann algebra is the linear span of finitely many unitary operators, an operator T in $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ has the same form as (4.5). Note that all such vectors $(c_1, \cdots, c_N)$ span a linear subspace of $\mathbb{C}^N$ with dimension not larger than N and T is uniquely determined by the formula (4.5) on Δ. Therefore,

$$\dim \mathcal{V}^*(\Phi, \Omega) \le N < \infty.$$

Some comments are in order. If in (4.5) $c_j \neq 0$ for some operator S, then ρ_j is called a representing local inverse for $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$. By the above proof, if there is a point $w \in \Delta$ such that $\rho_k(w) \notin \Omega$, then ρ_k does not appear in any representation (4.5) of S, that is, ρ_k is not representing. The same is true for each analytic continuation for ρ_k.

Next we will determine the generators of $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$. Write

$$\mathscr{Z} = \{z \in \overline{\Omega} : \varphi(z)J\Phi(z) = 0\},$$

and then $\Phi^{-1}(\Phi(\mathscr{Z}))$ is relatively closed in Ω. Let us call a local inverse ρ of $\Phi : \Omega \to \mathbb{C}^d$ φ-admissible if for each curve γ in $\Omega \setminus \overline{\Phi^{-1}(\Phi(\mathscr{Z}))}$, ρ admits analytic continuation with values in Ω. By Proposition 2.1, $\Omega \setminus \Phi^{-1}(\Phi(\mathscr{Z}))$ is connected. Then one can show that if ρ is a representing local inverse for $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$, then ρ is φ-admissible (see [26] or [17]). In the sequel, for a local inverse ρ of Φ, ρ^- always denotes the inverse of ρ. The proof of the theorem in [26, p. 526] shows that the class of all φ-admissible local inverses of Φ is closed under composition. If ρ is a φ-admissible local inverse, then its inverse ρ^- is also φ-admissible.

Rewrite

$$A_0 = \Phi^{-1}(\Phi(\mathscr{Z})).$$

Fix a representing local inverse ρ of Φ. As done in [13] or [16], define

$$\mathcal{E}_{[\rho]}h(w) = \sum_{\sigma \in [\rho]} \frac{\varphi(\sigma(w))}{\varphi(w)} h \circ \sigma(w)\, J\sigma(w), \quad w \in \Omega \setminus A_0, \tag{4.6}$$

where h is an arbitrary function over $\Omega \setminus A_0$ or Ω. The right-hand side of (4.6) is a finite sum as Φ is holomorphic over $\overline{\Omega}$. Also by the above paragraph, $\sigma(z) \in \Omega \setminus A_0$ if $z \in \Omega \setminus A_0$ and $\sigma \in [\rho]$.

Then the formula (4.6) makes sense. By the proof of [16, Lemma 6.3], we naturally get that $\mathcal{E}_{[\rho]}$ maps each function in $L_a^2(|\varphi|^2,\Omega)$ to a function in $L_a^2(|\varphi|^2,\Omega\setminus A_0)$, and $\mathcal{E}_{[\rho]}$ is bounded. Since $\mathcal{Z}$ is a good zero variety, by Lemma 2.1, A_0 is L_a^2-removable. Then

$$\mathcal{E}_{[\rho]}: L_a^2(|\varphi|^2,\Omega)\to L_a^2(|\varphi|^2,\Omega)$$

is a well-defined linear bounded operator. Furthermore, following the proof of [16, Lemma 6.3] yields that $\mathcal{E}_{[\rho]}^*=\mathcal{E}_{[\rho^-]}$. Since both $\mathcal{E}_{[\rho]}$ and $\mathcal{E}_{[\rho^-]}$ commute with the tuple of operators M_Φ, they both lie in $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$. Furthermore, it is not difficult to see that $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ is generated by $\mathcal{E}_{[\rho]}$ where ρ runs over φ-admissible local inverses of Φ. Recall that 1-admissible local inverses are ordinary admissible ones.

It remains to prove that $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ is $*$-isomorphic to $\mathcal{V}^*(\Phi,\Omega)$. Let us compare $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ with $\mathcal{V}^*(\Phi,\Omega)$. Note that by Lemma 2.1, a φ-admissible local inverse ρ also defines an operator $\mathcal{E}_{[\rho]}$ in $\mathcal{V}^*(\Phi,\Omega)$. Thus φ-admissible local inverses are representing for $\mathcal{V}^*(\Phi,\Omega)$, and hence they are exactly 1-admissible local inverses; they also have the same equivalent classes. We now write $\mathcal{E}^\varphi_{[\rho]}$ and $\mathcal{E}_{[\rho]}$ to distinguish them:

$$\mathcal{E}^\varphi_{[\rho]}\in\mathcal{V}^*(\Phi,|\varphi|^2,\Omega),\quad \mathcal{E}_{[\rho]}\in\mathcal{V}^*(\Phi,\Omega).$$

This gives a one-to-one correspondence Θ from $\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ to $\mathcal{V}^*(\Phi,\Omega)$:

$$\begin{aligned}\Theta:\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)&\to\mathcal{V}^*(\Phi,\Omega),\\ \sum_j c_j\mathcal{E}^\varphi_{[\rho_j]}&\mapsto\sum_j c_j\mathcal{E}_{[\rho_j]},\end{aligned}$$

where two sums are finite. It is straightforward to check that Θ is a linear bijection and for each $S\in\mathcal{V}^*(\Phi,|\varphi|^2,\Omega)$ we have $\Theta(S^*)=\Theta(S)^*$.

For two admissible local inverses ρ and σ, for $z\notin\Phi^{-1}(\Phi(\mathcal{Z}))$ and f in $L_a^2(|\varphi|^2,\Omega)$,

$$\begin{aligned}\varphi(z)(\mathcal{E}^\varphi_{[\rho]}\mathcal{E}^\varphi_{[\sigma]}f)(z)&=\sum_{\rho_j\in[\rho]}(\mathcal{E}^\varphi_{[\sigma]}f)\circ\rho_j(z)\varphi(\rho_j(z))J\rho_j(z)\\ &=\sum_{\sigma_k\in[\sigma]}\Big[\frac{\varphi(\sigma_k(w))}{\varphi(w)}f\circ\sigma_k(w)J\sigma_k(w)\Big]\circ\rho_j(z)\,\varphi(\rho_j(z))J\rho_j(z)\\ &=\sum_{j,k}\varphi(\sigma_k\circ\rho_j(z))f(\sigma_k\circ\rho_j(z))(J\sigma_k)(\rho_j(z))J\rho_j(z)\\ &=\sum_{j,k}\varphi(\sigma_k\circ\rho_k(z))f(\sigma_k\circ\rho_k(z))J(\sigma_k\circ\rho_j)(z).\end{aligned}$$

By (4.5) this can be written as

$$\mathcal{E}^\varphi_{[\rho]}\mathcal{E}^\varphi_{[\sigma]}=\mathcal{E}^\varphi_{[\sigma]\circ[\rho]},$$

and similarly,

$$\mathcal{E}_{[\rho]}\mathcal{E}_{[\sigma]}=\mathcal{E}_{[\sigma]\circ[\rho]}.$$

Then we have

$$\Theta(\mathcal{E}^\varphi_{[\rho]}\mathcal{E}^\varphi_{[\sigma]})=\Theta(\mathcal{E}^\varphi_{[\rho]})\Theta(\mathcal{E}^\varphi_{[\sigma]}).$$

By linearity of Θ,

$$\Theta(ST)=\Theta(S)\Theta(T),\quad S,T\in\mathcal{V}^*(\Phi,|\varphi|^2,\Omega).$$

Hence Θ defines an $*$-isomorphism between $\mathcal{V}^*(\Phi, |\varphi|^2, \Omega)$ and $\mathcal{V}^*(\Phi, \Omega)$. The proof is finished.

Example 4.1 Fix $k = 1, 2, \cdots$. The Bergman space $L_a^2(|z|^{2k}, \mathbb{D})$ has a orthogonal basis

$$z^{-k}, z^{-k+1}, \cdots .$$

Let ϕ be a nonconstant function and ϕ is holomorphic over $\overline{\mathbb{D}}$. Then by Theorem 4.1,

$$\mathcal{V}^*(\phi, |z|^{2k}, \mathbb{D}) \cong \mathcal{V}^*(\phi, \mathbb{D}), \quad k = 1, 2, \cdots .$$

But it is known that there exists a finite Blaschke product B such that ϕ can be written as a function of B and $\{M_\phi\}' = \{M_B\}'$ (see [26]). Hence

$$\mathcal{V}^*(\phi, \mathbb{D}) = \mathcal{V}^*(B, \mathbb{D}).$$

In particular, for a finite Blaschke product B_0,

$$\mathcal{V}^*(B_0, |z|^{2k}, \mathbb{D}) \cong \mathcal{V}^*(B_0, \mathbb{D}), \quad k = 1, 2, \cdots .$$

The idea of the above example gives the following corollary of Theorem 4.1.

Corollary 4.1 *Suppose that both ϕ and φ are holomorphic functions over $\overline{\mathbb{D}}$ and they are not constant. Then there is a finite Blaschke product B and a function $\widetilde{\phi} \in H^\infty(\mathbb{D})$ such that $\phi = \widetilde{\phi}(B)$ and*

$$\mathcal{V}^*(\phi, |\varphi|^2, \mathbb{D}) = \mathcal{V}^*(B, |\varphi|^2, \mathbb{D}).$$

Furthermore, $\mathcal{V}^*(B, |\varphi|^2, \mathbb{D})$ is $*$-isomorphic to $\mathcal{V}^*(B, \mathbb{D})$.

References

[1] Arveson, W., An Invitation to C*-Algebras, GTM **39**, Springer-Verlag, New York, 1998.

[2] Arveson, W., A Short Course on Spectral Theory, GTM **209**, Springer-Verlag, New York, 2001.

[3] Bell, S., The Bergman kernel function and proper holomorphic mappings, *Trans. Amer. Math. Soc.,* **270**, 1982, 685–691.

[4] Bjorn, A., Removable singularities for weighted Bergman spaces, *Czechoslovak Mathematical Journal*, **56**, 2006, 179–227.

[5] Bochner, S., Weak solutions of linear partial differential equations, *J. Math. Pure Appl.,* **35**, 1956, 193–202.

[6] Carleson, L., Selected Problems on Exceptional Sets, Van Nostrand, Princeton, NJ, 1967.

[7] Cowen, C., The commutant of an analytic Toeplitz operator, *Trans. Amer. Math. Soc.,* **239**, 1978, 1–31.

[8] Cowen, C., The commutant of an analytic Toeplitz operator, II, *Indiana Univ. Math. J.,* **29**, 1980, 1–12.

[9] Cowen, C., An analytic Toeplitz operator that commutes with a compact operator and a related class of Toeplitz operators, *J. Funct. Anal.,* **36**, 1980, 169–184.

[10] Cowen, C. and Wahl, R., Commutants of finite Blaschke product multiplication operators, preprint.

[11] Dan, H. and Huang, H., Multiplication operators defined by a class of polynomials on $L_a^2(\mathbb{D}^2)$, *Integr. Equ. Oper. Theory,* **80**, 2014, 581–601.

[12] Douglas, R., Putinar, M. and Wang, K., Reducing subspaces for analytic multipliers of the Bergman space, *J. Funct. Anal.,* **263**, 2012, 1744–1765.

[13] Douglas, R., Sun, S. and Zheng, D., Multiplication operators on the Bergman space via analytic continuation, *Adv. Math.,* **226**, 2011, 541–583.

[14] Guo, K. and Huang, H., On multiplication operators of the Bergman space: Similarity, unitary equivalence and reducing subspaces, *J. Operator Theory,* **65**, 2011, 355-378.

[15] Guo, K. and Huang, H., Multiplication operators defined by covering maps on the Bergman space: The connection between operator theory and von Neumann algebras, *J. Funct. Anal.,* **260**, 2011, 1219–1255.

[16] Guo, K. and Huang, H., Geometric constructions of thin Blaschke products and reducing subspace problem, *Proc. London Math. Soc.,* **109**, 2014, 1050–1091.

[17] Guo, K. and Huang, H., Multiplication Operators on the Bergman Space, Lecture Notes in Mathematics, **2145**, Springer-Verlag, Heidelberg, 2015.

[18] Guo, K., Sun, S., Zheng, D. and Zhong, C., Multiplication operators on the Bergman space via the Hardy space of the bidisk, *J. Reine Angew. Math.,* **629**, 2009, 129–168.

[19] Guo, K. and Wang, X., Reducing subspaces of tensor products of weighted shifts, *Sci. China Ser. A,* **59**, 2016, 715–730.

[20] Huang, H. and Zheng, D., Multiplication operators on the Bergman space of bounded domains in $\mathbb{C}^d$, 2015, arXiv: math.OA//1511.01678v1.

[21] Lu, Y. and Zhou, X., Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk, *J. Math. Soc. Japan,* **62**(3), 2010, 745–765.

[22] Rudin, W., Real and Complex Analysis, 3rd ed., McGraw-Hill Book Co., New York, 1987.

[23] Shi, Y. and Lu, Y., Reducing subspaces for Toeplitz operators on the polydisk, *Bull. Korean Math. Soc.,* **50**, 2013, 687–696.

[24] Sun, S., Zheng, D. and Zhong, C., Classification of reducing subspaces of a class of multiplication operators via the Hardy space of the bidisk, *Canad. J. Math.,* **62**, 2010, 415–438.

[25] Thomson, J., The commutant of a class of analytic Toeplitz operators, II, *Indiana Univ. Math. J.,* **25**, 1976, 793–800.

[26] Thomson, J., The commutant of a class of analytic Toeplitz operators, *Amer. J. Math.,* **99**, 1977, 522–529.

[27] Tikaradze, A., Multiplication operators on the Bergman spaces of pseudoconvex domains, *New York J. Math.,* **21**, 2015, 1327–1345.

[28] Wang, X., Dan, H. and Huang, H., Reducing subspaces of multiplication operators with the symbol $\alpha z^k + \beta w^l$ on $L_a^2(\mathbb{D}^2)$, *Sci. China Ser. A,* **58**, 2015, 1–14.