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Zeros of Monomial Brauer Characters*

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Abstract Let G be a finite group and p be a fixed prime. A p-Brauer character of G is said to be monomial if it is induced from a linear p-Brauer character of some subgroup (not necessarily proper) of G. Denote by $\mathrm{IBr}_m(G)$ the set of irreducible monomial p-Brauer characters of G. Let $H = G'\mathbf{O}^{p'}(G)$ be the smallest normal subgroup such that G/H is an abelian p'-group. Suppose that $g \in G$ is a p-regular element and the order of gH in the factor group G/H does not divide $|\mathrm{IBr}_m(G)|$. Then there exists $\varphi \in \mathrm{IBr}_m(G)$ such that $\varphi(g) = 0$.

Keywords Brauer character, Finite group, Vanishing regular element, Monomial Brauer character
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1 Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ denote the set of irreducible (complex) characters of G. For an element $g \in G$, when does there possibly exist a $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$? If this case occurs, g is called a vanishing element of G. A theorem of Burnside asserts that if $(\chi(1), |\mathrm{cl}(g)|) = 1$, where $\mathrm{cl}(g)$ denotes the conjugacy class of g in G, then either $g \in \mathbf{Z}(\chi)$ or $\chi(g) = 0$. If G is a p-solvable group and $\chi \in \mathrm{Irr}(G)$ is primitive of p-power degree, Navarro proved in [6] that $\chi(g) = 0$ for $g \in G$ if and only if $\chi(g_p) = 0$. The second author in [1] obtained a sufficient condition to decide when an element of a finite group can turn out to be a vanishing element. More precisely, let G' be the derived subgroup of G and $o(gG')$ be the order of gG' in G/G'; if $g \in G - G'$ and $(o(gG'), |\mathrm{Irr}(G)|) = 1$, then there exists a nonlinear $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$.

Let p be a fixed prime and denote by G^0 the set of p-regular elements, that is, $G^0 = \{g \in G \mid p \nmid o(g)\}$. Let R be the ring of algebraic integers in $\mathbb{C}$ and M be a maximal ideal containing pR of R. Then $F = R/M$ is a field with characteristic p. Under those circumstances, we consider p-Brauer characters. Now, under what conditions can a p-regular element be vanished by a nonlinear irreducible p-Brauer character? (Note that if g is a p-regular element of G, then g is said to be a vanishing regular element of G as long as $\varphi(g) = 0$, where φ is a nonlinear

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irreducible p-Brauer character of G.) Let $H = G'\mathbf{O}^{p'}(G)$ be the smallest normal subgroup such that G/H is an abelian p'-group. Suppose that $g \in G^0 - H^0$ and the order of gH in the factor group G/H is coprime to $|\mathrm{IBr}(G)|$, where $\mathrm{IBr}(G)$ is the set of irreducible p-Brauer characters of G. The first author, Wang and Zeng showed in [9] that there is a nonlinear irreducible Brauer character φ of G such that $\varphi(g) = 0$.

A monomial p-Brauer character of G is a p-Brauer character which is induced from a linear p-Brauer character of some subgroup (not necessarily proper) of G. This definition was first introduced by Okuyama [7] using module theory. If every irreducible p-Brauer character of G is monomial, then G is an M_p-group. Properties of M_p-groups were studied by Okuyama [7], Hanaki and Hida (see [2–3]). Let $\mathrm{IBr}_m(G)$ denote the set of irreducible monomial p-Brauer characters of G, and $\mathrm{LBr}(G)$ denote the set of linear p-Brauer characters of G. Obviously, for a finite group G and a fixed prime p,

$$\mathrm{LBr}(G) \subset \mathrm{IBr}_m(G) \subset \mathrm{IBr}(G).$$

Of course, G is an M_p-group if and only if $\mathrm{IBr}_m(G) = \mathrm{IBr}(G)$. Also, we have the following fact.

Proposition 1.1 *Let G be a solvable group, p be a fixed prime number and $P \in \mathrm{Syl}_p(G)$. If $\mathrm{IBr}_m(G) = \mathrm{LBr}(G)$, then $PG'' \lhd G$ and $G' \subset PG''$.*

Proof If G is abelian, then the result follows. Now we assume that G is not abelian. Thus $G'' < G'$ since G is solvable and we have that G/G'' is metabelian. And then G/G'' is an M_p-group by [7, Remark 3.5]. Since every $\varphi \in \mathrm{IBr}_m(G)$ is linear, it follows that $\mathrm{IBr}(G/G'') = \mathrm{LBr}(G/G'')$. Therefore, $PG'' \lhd G$ and $G' \subset PG''$ by [9, Lemma 2.1].

For notational convenience, we simply write Brauer characters for p-Brauer characters once a prime p is chosen. For other notations and terminologies, one can refer to [4–5]. Utilizing the action of linear Brauer characters on the set $\mathrm{IBr}_m(G)$, we have the following theorem.

Theorem 1.1 *Let G be a finite group, p be a fixed prime and let $H = G'\mathbf{O}^{p'}(G)$ be the smallest normal subgroup such that G/H is an abelian p'-group. Suppose that $g \in G^0$ and the order of gH in the factor group G/H does not divide $|\mathrm{IBr}_m(G)|$. Then there exists $\varphi \in \mathrm{IBr}_m(G)$ such that $\varphi(g) = 0$.*

If $p \nmid |G|$, then Brauer characters become the same as complex characters of G, and then our Theorem 1.1 agrees with [8, Theorem 1.4].

Chen [1] proved that if $|G/G'|$ is coprime to $|\mathrm{Irr}(G)|$ then $\mathbf{Z}(G) \leq G'$. In [9], the authors proved that if $|G/H|$ is coprime to $|\mathrm{IBr}(G)|$ then $\mathbf{Z}(G)^0 \leq H^0$, where $H = G'\mathbf{O}^{p'}(G)$. Replacing $\mathrm{IBr}(G)$ by $\mathrm{IBr}_m(G)$, we also have the following theorem.

Theorem 1.2 *Let G be a finite group and p be a fixed prime. If $|G/H|$ is coprime to $|\mathrm{IBr}_m(G)|$, where $H = G'\mathbf{O}^{p'}(G)$, then $\mathbf{Z}(G)^0 \subseteq H^0$.*

As an application of the action of linear Brauer characters on the set of irreducible monomial Brauer characters, we have the following theorem.

Theorem 1.3 *Let G be a finite group and p be a fixed prime. Denote by Ψ the sum of all the irreducible monomial Brauer characters of G. Then $\Psi(G^0 - H^0) = 0$, where $H = G'\mathbf{O}^{p'}(G)$.*

2 Proofs

Proof of Theorem 1.1 Let λ be a linear Brauer character of G and $\mathrm{IBr}_m(G) = \{\varphi_1, \varphi_2, \cdots, \varphi_l\}$. It follows from [9, Lemma 2.1] that $\{\lambda\varphi \mid \varphi \in \mathrm{IBr}_m(G)\} \subset \mathrm{IBr}(G)$. Since φ is a monomial Brauer character, we have that there exists a linear Brauer character μ of a subgroup K of G such that $\varphi = \mu^k$. Thus $\lambda\varphi = \lambda\mu^K = (\lambda_K\mu)^G \in \mathrm{IBr}_m(G)$ by the formula on [5, P. 175] and then $\{\lambda\varphi \mid \varphi \in \mathrm{IBr}_m(G)\} = \mathrm{IBr}_m(G)$.

Suppose that g is a p-regular element of G which satisfies the hypothesis of this theorem. Thus we have the following equality:

$$\begin{aligned}(\varphi_1 \cdots \varphi_l)(g) &= \varphi_1(g) \cdots \varphi_l(g) = ((\lambda\varphi_1)(g)) \cdots ((\lambda\varphi_l)(g)) \\ &= (\lambda(g))^l[(\varphi_1 \cdots \varphi_l)(g)].\end{aligned}$$

Assume that there does not exist any irreducible monomial Brauer character taking the value zero on g. Then $(\varphi_1 \cdots \varphi_l)(g) = \varphi_1(g) \cdots \varphi_l(g)$ should not be zero. It follows by the preceding equality that $(\lambda(g))^l = \lambda(g^l) = 1$ and then $g^l \in \ker\lambda$. Consequently, by the arbitrariness of λ and [9, Corollary 2.2], we see that

$$g^l \in \cap \ker\lambda = G'\mathbf{O}^{p'}(G) = H,$$

where λ runs over the set $\mathrm{LBr}(G)$. It follows that $o(gH)$ divides $l = |\mathrm{IBr}_m(G)|$, which violates the choice of gH.

Before proving Theorem 1.2, we first give a lemma.

Lemma 2.1 *Let G be a non-abelian finite group, $\mathbf{Z}(G)$ be the center of G and let p be a fixed prime. If $|\mathbf{Z}(G)/(\mathbf{Z}(G) \cap H)|$ is coprime to $|\mathrm{IBr}_m(G)|$, where $H = G'\mathbf{O}^{p'}(G)$, then $\mathbf{Z}(G)^0 \subseteq H^0$.*

Proof If $H = G$, it is clear that the conclusion is true. So we may suppose that H is a proper subgroup of G.

Notice that $\mathbf{Z}(G)$ is abelian. It follows from Clifford's theorem that every p-regular element of $\mathbf{Z}(G)$ is not a vanishing regular element of G. Assume that $\mathbf{Z}(G)^0$ is not contained in H^0. Thus there exists an element $g \in \mathbf{Z}(G)^0$, but $g \notin H^0$. Note that $o(gH)$ is coprime to $|\mathrm{IBr}_m(G)|$ and $o(gH)$ does not divide $|\mathrm{IBr}_m(G)|$. Therefore, it follows from Theorem 1.1 that there exists an irreducible monomial Brauer character of G which vanishes on g, a contradiction.

Proof of Theorem 1.2 By Lemma 2.1, we conclude Theorem 1.2 immediately.

Now we prove Theorem 1.3.

Proof of Theorem 1.3 Observe that it is understood that if $G^0 - H^0 = \emptyset$, then $\Psi(G^0 - H^0) = 0$.

Suppose that there exists $x \in G^0 - H^0$ such that $\Psi(x) \neq 0$. Let $\lambda \in \mathrm{LBr}(G)$. Then $\lambda\Psi = \Psi$ and we have

$$\lambda(g)\Psi(g) = \Psi(g)$$

for every $g \in G^0$. In particular, since $\Psi(x) \neq 0$ and $\lambda(x)\Psi(x) = \Psi(x)$, we see that $\lambda(x) = 1$. Note that λ is arbitrary. Then by [9, Corollary 2.2] again it follows that

$$x \in \cap \ker\lambda = G'\mathbf{O}^{p'}(G) = H,$$

where λ runs over $\mathrm{LBr}(G)$, and we arrive at a contradiction.

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