

Closed Strong Spacelike Curves, Fenchel Theorem and Plateau Problem in the 3-Dimensional Minkowski Space*

Nan YE¹ Xiang MA²

Abstract The authors generalize the Fenchel theorem for strong spacelike closed curves of index 1 in the 3-dimensional Minkowski space, showing that the total curvature must be less than or equal to 2π . Here the strong spacelike condition means that the tangent vector and the curvature vector span a spacelike 2-plane at each point of the curve γ under consideration. The assumption of index 1 is equivalent to saying that γ winds around some timelike axis with winding number 1. This reversed Fenchel-type inequality is proved by constructing a ruled spacelike surface with the given curve as boundary and applying the Gauss-Bonnet formula. As a by-product, this shows the existence of a maximal surface with γ as the boundary.

Keywords Fenchel theorem, Spacelike curves, Total curvature, Maximal surface

2010 MR Subject Classification 52A40, 53C42, 53C50

1 Introduction

To study the global properties of closed curves, an interesting idea is to associate a specific surface M with the given curve γ as boundary. Then we can control the geometry of γ by the information of M , and vice versa.

As an illustration, let us consider a closed smooth space curve γ in \mathbb{R}^n ($n \geq 3$) which is assumed to bound a minimal disk M . As an intrinsic result, the Gauss-Bonnet formula says

$$\int_M K dM + \int_{\partial M} \kappa_g ds = 2\pi, \quad (1.1)$$

where K is the Gauss curvature of M , dM is the area element with respect to the induced metric, κ_g is the geodesic curvature of the curve $\partial M = \gamma \subset M$, and s is the arc-length parameter. For such an Euclidean minimal surface, it is well-known that $K \leq 0$. There also holds

$$\kappa(p) \geq \kappa_g(p) = \kappa(p) \cdot \cos \theta_p$$

at any $p \in \gamma$, where θ_p is the angle between the tangent plane of M and the osculating plane of γ at p . Combining with these two facts, we immediately obtain the conclusion of the Fenchel

Manuscript received March 24, 2016. Revised May 10, 2016.

¹College of Mathematics and Econometrics, Hunan University, Changsha 410082, China.

E-mail: yenan@hnu.edu.cn

²Corresponding author. LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China. E-mail: maxiang@math.pku.edu.cn.

*This work was supported by the National Natural Science Foundation of China (No. 11471021) and the Fundamental Research Funds for the Central Universities of China (No. 531107050874).

theorem (see [4])

$$\int_{\gamma} \kappa ds \geq 2\pi.$$

The equality is attained exactly when M is flat and γ is a convex plane curve.

Notice that the total curvature gives a quantitative measure of the complexity of the space curve γ . A natural expectation is that when $\int_{\gamma} \kappa ds$ is small, γ should be simple, as confirmed by the Fary-Milnor theorem (see [7]) that when $n = 3$ and $\int_{\gamma} \kappa ds \leq 4\pi$, γ is always a trivial knot.

The next reasonable guess is that the solution M to the corresponding Plateau problem (see [2, 10]) should also be nice under similar conditions. A remarkable theorem due to Nitsche [8] states that a smooth Jordan curve with total curvature less or equal to 4π bounds a unique minimal disk. For a minimal surface $M \subset \mathbb{R}^n$ of arbitrary topological type with boundary $\gamma = \partial M$ and $\int_{\gamma} \kappa ds \leq 4\pi$, in 2002, Ekholm, White and Wienholtz [3] further proved that M must be smoothly embedded.

In this paper, we are motivated to consider a similar picture, namely a closed spacelike curve γ as the boundary of a spacelike (maximal) surface in the 3-dimensional Minkowski space \mathbb{R}_1^3 . We would like to find some appropriate assumptions on γ to guarantee that the Plateau problem has a solution. It is also desirable to give a prior estimation of the total curvature $\int_{\gamma} \kappa ds$ (i.e., a generalization of the Fenchel theorem and/or the Fary-Milnor theorem). We achieve these goals successfully. To state our result, let us introduce two definitions.

Definition 1.1 *A C^2 curve $\gamma \subset \mathbb{R}_1^3$ is called spacelike if at any point the tangent vector is spacelike, i.e., $\langle \gamma', \gamma' \rangle > 0$. It is called strong spacelike if its (unit) tangent vector $T = \gamma'(s)$ and the curvature vector $\kappa N = \gamma''(s)$ span a spacelike 2-plane at each point. In other words, the osculating plane at any point of γ is of rank-2 with a positive-definite inner product induced from \mathbb{R}_1^3 .*

Note that in the 3-dimensional Minkowski space, there exist neither closed timelike curves, nor closed spacelike curves with timelike normals. In contrast, the strong spacelike condition allows the length and curvature to be defined directly as before and admits closed examples. To avoid misunderstanding, we point out that a strong spacelike curve does not allow inflection points where the curvature vector is a zero vector. So we can assume $\kappa > 0$.

Definition 1.2 *The index of a closed spacelike curve in \mathbb{R}_1^3 is defined to be the winding number I of the tangent indicatrix (the image of the unit tangent vector T) around the Sitter sphere $\mathbb{S}_1^2 = \{X \in \mathbb{R}_1^3 \mid \langle X, X \rangle = 1\}$ (the usual one-sheet hyperboloid, which is homotopy equivalent to a circle). This index I is integer-valued and always assumed to be positive.*

Remark 1.1 Note that this definition is independent of the choice of the timelike direction, and hence it is well-defined. In contrast, for a closed curve in \mathbb{R}^3 , generally there is not a well-defined notion of index or winding number unless it is a plane curve. In the special case that $I = 1$, the closed curve winds around some timelike axis exactly for one cycle.

Now we can give the generalization of the Fenchel theorem in the 3-dimensional Minkowski space which is our first main result. This seems to be a new result to the best of our knowledge.

Theorem 1.1 (The Fenchel Theorem in \mathbb{R}_1^3) *Let γ be a closed strong spacelike curve in \mathbb{R}_1^3 with index 1. Then the total curvature satisfies $\int_\gamma k ds \leq 2\pi$. The equality holds if and only if it is a convex curve on a spacelike plane.*

Remark 1.2 The reversed inequality might look peculiar when compared with the Euclidean case. This can be explained as below. The total curvature of γ is equivalent to the length of the tangent indicatrix $T(\gamma)$. If we consider γ as a small perturbation of a closed convex plane curve, then the corresponding variation of $T(\gamma)$ in \mathbb{S}_1^2 is always along the time-like co-normal direction, and vibrates up and down along the equator, which makes the length $L(T(\gamma))$ less than the original length 2π of the equator. On the other hand, although a line of altitude Γ may have the length greater than 2π , it can not be realized as the tangent indicatrix $T(\gamma)$ of a closed strong spacelike curve γ , and hence it can not be a counterexample to our claim. This is because Γ always lies in a half space, and after integration, one gets a curve γ whose height function (with respect to a fixed timelike direction) increases monotonically, and thus it can not be closed.

To prove this reversed Fenchel inequality, here we adopt the same idea of constructing a surface M with $\partial M = \gamma$. Can we take M to be a spacelike surface with vanishing mean curvature (called maximal surface)? A known criterion for the existence of solutions to this Plateau problem is given as below.

Theorem 1.2 (see [1, 5]) *Given a compact, codimension two spacelike submanifold γ^{n-2} in \mathbb{R}_1^n without boundary, we suppose that there is a spacelike hypersurface M^{n-1} with $\partial M^{n-1} = \gamma^{n-2}$ and M^{n-1} is the graph of a C^2 function u defined over a compact domain Ω of spacelike subspace \mathbb{R}^{n-1} whose gradient is uniformly bounded, $|Du| < \delta < 1$. Then there exists a spacelike maximal hypersurface \overline{M}^{n-1} with $\partial \overline{M}^{n-1} = \gamma^{n-2}$ and this \overline{M}^{n-1} is also a graph over the same Ω .*

Thanks to this criterion, we need only to show the existence of a spacelike surface spanning γ . Fortunately, the assumption of γ being strong spacelike with index 1 ensures that γ has a simple shape: Its projection to a spacelike plane \mathbb{R}^2 must be a convex plane curve bounding a compact convex domain Ω (see Lemma 2.1); and any three points on γ span a spacelike plane (see Lemma 2.2). This enables us to show the existence of such a surface M by explicit construction.

Theorem 1.3 *Let γ be a closed strong spacelike C^2 (twice continuously differentiable) curve in \mathbb{R}_1^3 of index 1. Then there exists a surface M with the following properties and $\partial M = \gamma$:*

- (1) *M is a ruled surface;*
- (2) *M is the graph of a function defined on a convex domain Ω of a spacelike plane, and hence itself is a topological disk;*
- (3) *M is C^2 -smooth and spacelike (including the boundary points);*
- (4) *M has non-negative Gauss curvature.*

Compared with the Euclidean case, ruled surfaces in \mathbb{R}_1^3 are still saddle shaped, yet with non-negative Gauss curvature ($K \geq 0$) (see [9]). On the other hand, at a boundary point $p \in \gamma$,

we have $\kappa_g(p) = \kappa(p) \cosh(\theta_p) \geq \kappa(p)$, where $\theta_p \in \mathbb{R}$ is the so-called hyperbolic angle between the tangent plane of M and the osculating plane of γ at p . Applying the Gauss-Bonnet formula to M and using the similar argument as in \mathbb{R}^3 , we obtain the generalized Fenchel theorem (Theorem 1.1) immediately.

Moreover, based on Theorem 1.2, we can now confirm the existence of a solution to the Plateau problem. The second main result of this paper is as follows.

Theorem 1.4 *Let γ be a closed, strong spacelike curve in \mathbb{R}_1^3 with index $I = 1$. Then there exists a maximal surface \overline{M} with $\partial\overline{M} = \gamma$. This \overline{M} is a graph over a compact, convex domain $\Omega \subset \mathbb{R}^2$, thus itself is an embedded topological disk. Moreover, such a maximal surface \overline{M} is unique.*

The uniqueness of \overline{M} is a generalization of the aforementioned Nitsche's theorem and the classical Radó's theorem (see [10]) on minimal surfaces as below. Here we just remark that it is easier to obtain similar results in the Lorentz space, because requiring the surface to be spacelike is a strong restriction.

Theorem 1.5 (see [10]) *If $\gamma \subset \mathbb{R}^n$ has a one-to-one projection onto the boundary of a convex planar region Ω , then any minimal disk bounded by γ is the graph of a smooth function over Ω . In particular, it is smoothly embedded. If in addition $n = 3$, then there is only one disk and there are no minimal varieties of other topological types.*

We have found other proofs to the reversed Fenchel inequality (see Theorem 1.1). One of them uses a generalized Crofton formula (see [12]).

For higher dimensional spacelike submanifolds, one can also consider a suitable generalization of the strong spacelike condition and expect to find similar inequalities on various total curvature. This is an ongoing project of our research.

2 Basic Properties of Strong Spacelike Closed Curve of Index 1

The 3-dimensional Minkowski space \mathbb{R}_1^3 is endowed with a Lorentz inner product, expressed in a canonical coordinate system as

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 - x_3y_3, \quad X = (x_1, x_2, x_3), \quad Y = (y_1, y_2, y_3).$$

A vector X is called spacelike (lightlike, timelike, respectively) if $\langle X, X \rangle > 0$ ($= 0, < 0$, respectively). A timelike vector $X = (x_1, x_2, x_3)$ is called future-directed (past-directed) if $x_3 > 0$ ($x_3 < 0$).

A 2-dimensional subspace V is called spacelike (lightlike, timelike, respectively) if the Lorentz inner product restricts to be a positive definite (degenerate, Lorentz, respectively) quadratic form on V . It is the orthogonal complement of a nonzero vector \vec{n} which is timelike (lightlike, spacelike, respectively). In particular, we can define the cross product of two spacelike vectors on a spacelike plane as below, which is always orthogonal to X and Y (when it is nonzero, we obtain a timelike normal vector),

$$X \times Y = (-x_2y_3 + x_3y_2, -x_3y_1 + x_1y_3, x_1y_2 - x_2y_1). \quad (2.1)$$

A curve or a surface in \mathbb{R}_1^3 is said to be spacelike (lightlike, timelike, respectively) if its tangent space at each point is spacelike (lightlike, timelike, respectively).

The strong spacelike condition for a curve γ is defined in the introduction (see Definition 1.1). A basic way to visualize the shape of γ is the parallel projection to a plane.

Definition 2.1 *When Σ is spacelike with unit timelike normal vector \vec{n} , the projection of any vector $\vec{v} \in \mathbb{R}_1^3$ to Σ is*

$$\sigma(\vec{v}) \triangleq \vec{v} + \langle \vec{v}, \vec{n} \rangle \vec{n}.$$

When Σ is a lightlike plane with lightlike normal \vec{n} , one should take care that $\vec{n} \in \Sigma$. Take an arbitrary lightlike vector \vec{n}^ so that $\langle \vec{n}, \vec{n}^* \rangle = 1$. Note that \vec{n}^* is transversal to Σ , and it is not unique. The projection of any vector $\vec{v} \in \mathbb{R}_1^3$ to Σ is defined as below which depends on both \vec{n} and \vec{n}^* :*

$$\sigma(\vec{v}) \triangleq \vec{v} - \langle \vec{v}, \vec{n} \rangle \vec{n}^*. \tag{2.2}$$

Lemma 2.1 (The Projection Lemma) *Let γ be a closed strong spacelike curve in \mathbb{R}_1^3 with $I = 1$. Let σ be the projection map to a spacelike or lightlike plane Σ in \mathbb{R}_1^3 . Then $\sigma(\gamma)$ is a strictly convex Jordan curve on Σ , and σ is a one-to-one correspondence.*

Proof Let s be the arclength parameter of γ . In the canonical coordinate system, the tangent vector of γ can be expressed in terms of the longitude and latitude parameters θ, ϕ ,

$$T(s) = (\cosh \phi(s) \cos \theta(s), \cosh \phi(s) \sin \theta(s), \sinh \phi(s)). \tag{2.3}$$

The strong spacelike assumption with $I = 1$ implies

$$\cosh^2 \phi \cdot \theta'(s)^2 - \phi'(s)^2 > 0, \tag{2.4}$$

and $\theta(s)$ ranges from 0 to 2π monotonically. Without loss of generality, we may assume that $\theta'(s) > 0$ everywhere.

When Σ is a spacelike plane, without loss of generality, we may take its normal vector $\vec{n} = (0, 0, 1)$. It follows from (2.3) that the projection $\sigma(\gamma)$ has tangent vector

$$\frac{d}{ds}(\sigma(\gamma)) = \sigma(T) = (\cosh \phi(s) \cos \theta(s), \cosh \phi(s) \sin \theta(s)). \tag{2.5}$$

So the tangent direction of $\sigma(\gamma)$ is the same as $(\cos \theta(s), \sin \theta(s))$, which rotates in a strictly monotonic manner with range $[0, 2\pi]$. Thus $\sigma(\gamma)$ must still be a closed and strictly convex curve on Σ . The 1-1 correspondence property is clear.

When Σ is a lightlike plane, without loss of generality, we suppose that it is orthogonal to $\vec{n} = (0, 1, 1)$ and transverse to $\vec{n}^* = (0, \frac{1}{2}, -\frac{1}{2})$. Let $\vec{e}_1 = (1, 0, 0)$. By Definition 2.1 and (2.2), the projection image $\sigma(\gamma)$ has tangent vector

$$\frac{d}{ds}(\sigma(\gamma)) = \sigma(T) = \cosh \phi(s) \cos \theta(s) \vec{e}_1 + \frac{\cosh \phi(s) \sin \theta(s) + \sinh \phi(s)}{2} \vec{n}. \tag{2.6}$$

The curvature being positive or not is an affine invariant property. So we need only to identify $\sigma(T)$ with the tuple $\tilde{T}(s) = (\cosh \phi(s) \cos \theta(s), \cosh \phi(s) \sin \theta(s) + \sinh \phi(s))$ and to show that

$$\det(\tilde{T}, \tilde{T}') = (\theta' \cosh^2 \phi + \phi' \cos \theta + \theta' \cosh \phi \sinh \phi \cdot \sin \theta)$$

has a fixed sign. Indeed, this can be shown by using the Cauch-Schwarz inequality and the strong spacelike property (2.4) as below:

$$|\phi' \cos \theta + \theta' \sinh \phi \cosh \phi \sin \theta| \leq \sqrt{\phi'^2 + \theta'^2 \sinh^2 \phi \cosh^2 \phi} < \theta' \cosh^2 \phi. \quad (2.7)$$

Thus the proof is completed.

Lemma 2.2 (The Section Lemma) *Under the above assumptions, let p_1, p_2, p_3 be arbitrarily chosen distinct points on γ . Then*

- (1) *the line segment $\overline{p_1 p_2}$ connecting p_1 and p_2 is spacelike.*
- (2) *p_1, p_2, p_3 span a spacelike plane.*
- (3) *the chord $p_1 p_2$ and the tangent line at p_1 span a spacelike plane.*

Proof We prove (2) by contradiction at first. Then (1) follows as a corollary. Suppose that $\{p_1, p_2, p_3\}$ are contained in a timelike plane (this includes the collinear case). We can always find a spacelike plane Σ , so that the orthogonal projection of the plane is a line on Σ . In particular, the projection of $\{p_1, p_2, p_3\}$ are collinear. On the other hand, the conclusion of Lemma 2.1 implies that these three points on a convex Jordan curve should always be distinct without lying on a line. This is a contradiction. The situation that $\{p_1, p_2, p_3\}$ span a lightlike plane can be ruled out in a similar way (as long as one can be careful about the definition of the projection map).

For conclusion (3), suppose otherwise that $\overline{p_1 p_2}$ and the tangent line at p_1 are contained in a timelike (or lightlike) plane. We can still choose a spacelike plane Σ orthogonal to the timelike plane (or transversal to the lightlike plane). Then the projection of $\overline{p_1 p_2}$ and the tangent line at p_1 are collinear, which contradicts Lemma 2.1 again.

3 The Ruled Spacelike Surface Spanning γ

As we pointed out in the introduction, the proof of the two main results can be reduced to showing the existence of a spacelike surface M in \mathbb{R}_1^3 with the given boundary and with non-negative Gauss curvature K . Generalizing the surface theory in \mathbb{R}^3 to the spacelike surfaces in \mathbb{R}_1^3 (see [9]), by the Gauss equation, it is easy to show $K = -\kappa_1 \kappa_2$, which means that the two principal curvatures κ_1, κ_2 should have opposite signs in general, i.e., the desired surface M should be saddle-shaped.

We tried several different saddle-shaped surfaces (like the graph of a harmonic function, or a minimal surface in \mathbb{R}^3 identified with this \mathbb{R}_1^3), with the same difficulty: How to show the constructed surface to be really spacelike in this ambient Minkowski space?

After many unsuccessful attempts, we noticed that Lemma 2.2 seems to be helpful. It guarantees that we can take finitely many points $\{p_i\}$ on γ and construct a polyhedron with spacelike triangular faces, which would serve as an approximation to the expected smooth spacelike surface. The obvious problem with this idea is that when we take more and more vertices $\{p_i\}$, we will not get a nice refinement of the triangulation in the usual sense; instead we would obtain something like a union of slim rulers. We were stuck here for some time until one day, the inspiration came that the limit shape might still exist as a smooth surface, which

must be a ruled surface. Realizing that ruled surface is automatically saddle-shaped, the rest thing to do was then clear.

Step 1 Construct a ruled surface M directly with $\partial M = \gamma$.

Step 2 Show that M is differentiable (using a suitable parametrization).

Step 3 Show that M is immersed and spacelike (with the help of Lemma 2.2).

From now on, following these three steps, we give a proof of Theorem 1.3. As a preparation, fix the canonical coordinate system (x_1, x_2, x_3) in \mathbb{R}_1^3 . Choose the coordinate plane Ox_1x_2 as the target spacelike plane Σ to make projection.

Step 1 First, we give an explicit construction of the ruled surface M together with its parametrization. Take two points p and q on γ which divide γ into two arcs with equal length L . Denote these two arcs as $\gamma_0(s), \gamma_1(s)$ with arc-length parameter $s \in [0, L]$ and tangent vectors $\gamma'_0(s) = T_0(s), \gamma'_1(s) = T_1(s)$, respectively. The ruled surface $X(s, t)$ is given as

$$X(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s) = X\left(s, \frac{1}{2}\right) + \left(t - \frac{1}{2}\right)\vec{v}(s), \tag{3.1}$$

where $X\left(s, \frac{1}{2}\right) = \frac{1}{2}[\gamma_0(s) + \gamma_1(s)]$, $\vec{v}(s) \triangleq \gamma_1(s) - \gamma_0(s)$, $s \in [0, L]$, $t \in [0, 1]$.

Step 2 $X\left(s, \frac{1}{2}\right)$ and $\vec{v}(s)$ are obviously C^2 curves, because γ is so. We have

$$\frac{\partial X}{\partial t} = \gamma_1(s) - \gamma_0(s) = \vec{v}(s), \tag{3.2}$$

$$\frac{\partial X}{\partial s} = (1 - t)T_0(s) + tT_1(s). \tag{3.3}$$

In particular, $X(s, t)$ is a C^2 ruled surface even on a larger domain $s \in [0, L], t \in (-\epsilon, 1 + \epsilon)$ for some $\epsilon > 0$.

Step 3 Finally, we will verify that $X(s, t)$ is a spacelike surface including the boundary points.

At the end point p with $s = 0$, we use the expansion

$$\gamma_0(s) = p + T_0(0)s + \kappa(0)N_0(0)\frac{s^2}{2} + o(s^2), \quad \gamma_1(s) = p + T_1(0)s + \kappa(0)N_1(0)\frac{s^2}{2} + o(s^2),$$

where $T_0(0) = -T_1(0)$ is the tangent vector of γ at p , $N_0(0) = N_1(0)$ is the normal vector, and $\kappa(0) > 0$ is the curvature at the same point. This implies that $u = s^2$ is a regular parameter for the curve

$$X\left(s, \frac{1}{2}\right) = p + \kappa(0)N_0(0)\frac{s^2}{2} + o(s^2) = p + \kappa(0)N_0(0)\frac{u}{2} + o(u)$$

with tangent vector $\frac{\kappa(0)N_0(0)}{2} \neq \vec{0}$. Thus the tangent plane of $X(s, t)$ at $s = 0$ is clearly spanned by $N_0(0), T_0(0)$, hence it is the spacelike osculating plane of the strong spacelike γ at p .

At a generic boundary point, said to be $s \in (0, L)$ and $t = 0$, the tangent plane spanned by $T_0(s), \vec{v}(s)$ is spacelike according to the conclusion Lemma 2.2(3). The argument is the same when $s \in (0, L)$, $t = 1$.

Moreover, we can choose the orientation suitably so that $T_0(s) \times \vec{v}(s)$ and $T_1(s) \times \vec{v}(s)$ are both future-directed timelike normal vector, because Lemma 2.1 tells us that after projection to Ox_1x_2 -plane, $T_0(s), T_1(s)$ will point to the same side of $\vec{v}(s)$. Since all future-directed timelike

vectors form the positive lightcone which is convex, we know that $(1-t)T_0(s) \times \vec{v}(s) + tT_1(s) \times \vec{v}(s)$ is also a future-directed timelike vector. This is nothing else but exactly the normal vector of $X(s, t)$ at an interior point with $s \in (0, L)$ and $t \in (0, 1)$, because by (3.2)–(3.3), one finds

$$\frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t} = (1-t)[T_0(s) \times \vec{v}(s)] + t[T_1(s) \times \vec{v}(s)].$$

Thus $X(s, t)$ is spacelike and immersed at any interior point.

Finally, it is easy to see that a spacelike ruled surface M in the Minkowski space must have non-negative Gauss curvature. Because the normal curvature along a straight line on M is always zero, the principal curvatures κ_1, κ_2 at every point of M must have opposite signs, or one of them vanishes. Then the Gauss equation in \mathbb{R}_1^3 ($K = -\kappa_1\kappa_2$) verifies our claim. (Intuitively, a non-planar ruled surface in the 3-dimensional real linear space is always saddle-shaped, no matter whether the ambient metric is positive definite or not.) This finishes the proof of Theorem 1.3, and establishes Theorem 1.1.

Remark 3.1 Another candidate for saddle-shaped surfaces is the graph of any harmonic function on the complex plane. Let $\Omega \subset \mathbb{R}^2$ be the compact convex domain bounded by the projection image of γ . Precisely speaking, we need to solve the Dirichlet problem for a harmonic function u defined in Ω , with the boundary value assigned by the height of γ (i.e., $\gamma = \{(z, u(z)) \mid z \in \partial\Omega\}$). The existence of a solution u is no doubt. Yet the non-trivial part is to guarantee that the graph $\widetilde{M} = \{(z, u(z)) \mid z \in \Omega\}$ is spacelike, i.e., to prove a gradient estimate $|Du| < 1$. This depends on the boundary value determined by γ and the assumption that γ is strong spacelike. The Poisson integral formula combined with some numerical evidence can be used to obtain the desired estimation when Ω is a circular disk. But this method is not successful in the general case.

4 The Maximal Spacelike Surface Spanning γ

In the final part, we prove Theorem 1.4. Given a strong spacelike closed curve γ in \mathbb{R}_1^3 with index $I = 1$, Lemma 2.1 guarantees that on any spacelike plane Σ , there is a compact convex domain Ω bounded by the projection image of γ . Then Theorem 1.3 and Theorem 1.2 imply the existence of a maximal surface \overline{M} as a graph over this domain.

We want to show that any maximal surface M' with $\partial M' = \gamma$ must be a graph over the same $\Omega \subset \Sigma$ and agrees with \overline{M} . The proof is divided into three steps.

First, the shadow of M' , i.e., the projection image of M' on Σ , must be a subset of Ω . Otherwise, there would exist an interior point $p \in M'$ which is projected to the boundary of the shadow. Then the normal vector of M' at p must be a spacelike vector parallel to Σ . This is impossible for a spacelike surface.

Second, the projection from M' to Ω must be onto. As a subset of Ω , the shadow of M' is closed, because it is the image of a compact set under a continuous map, so it is still compact, namely, bounded and closed. On the other hand, the shadow of M' is open, because at any interior point of M' , the spacelike immersion condition guarantees that the projection map is a local diffeomorphism from M' to Ω . Combining together, the shadow of M' is exactly Ω . Moreover, M' is a local graph over Ω .

Third, notice that the projection from M' to Ω is a local homomorphism from a compact topological space to a connected topological space, by a well-known proposition, such a map must be a covering map. Because the disk Ω is simply connected, this covering map is always one-to-one, hence a global homomorphism.

Finally, we will prove that the two maximal graphs M' and \overline{M} over Ω must be the same. This follows from [1, Proposition 1.1] by taking the prescribed mean curvature H to be identically 0. This establishes the uniqueness and the whole Theorem 1.4.

Remark 4.1 A famous result in minimal surface and calibrated geometry is that a minimal graph $M = (x, y, u(x, y))$ over a 2-dimensional domain Ω minimizes the area among all surfaces in $\Omega \times \mathbb{R}$ with the same boundary (see [6]). The standard proof is extending the normal vector field N to $\Omega \times \mathbb{R}$ by parallel translation in the vertical direction, and applying Stokes' theorem to the integral of the divergence $\operatorname{div} N$ in the volume enclosed by M and any other surface M' in $\Omega \times \mathbb{R}$ with the same boundary. Following $\operatorname{Area}(M') \geq \operatorname{Area}(M)$, the equality holds if and only if $M = M'$. This method can be used here in almost the same form to obtain the uniqueness result in the proof above.

Remark 4.2 The regularity assumption on γ might be weakened. For example, consider piecewise smooth (C^2) closed curve with vertices $\{\gamma(s_i) : i = 1, 2, \dots\}$ and index $I = 1$. Assume that the Lorentz cross product of two tangent vectors $\gamma'(s_i + 0)$ and $\gamma'(s_i - 0)$ at the same vertex $\gamma(s_i)$ is always a future-directed timelike vector. This is a natural discrete version of the strong spacelike condition. The Euclidean angle between $\gamma'(s_i + 0)$ and $\gamma'(s_i - 0)$ is the contribution at this vertex to the total curvature. Then one can prove similar results. Indeed, we can establish a Fenchel-type theorem for a strong spacelike polygon (formed by straight line segment) of index $I = 1$ and taking limit to obtain Theorem 1.1 in [11].

On the other hand, notice that one important feature of Theorem 1.2 and other results in [1] is that they made no assumption about the smoothness of the boundary or boundary data. If we can define rectifiable spacelike curves and find appropriate generalization of the strong spacelike condition, it is hopeful to obtain the same results using similar methods.

Acknowledgements The authors are grateful for discussing with Donghao Wang on various constructions of the desired saddle shaped surface with given boundary γ and on the gradient estimation problem. The second author would like to thank Keomkyo Seo for introducing the paper [3] and motivating his interest in this topic. The authors are also thankful for the suggestions of the referee helping them to clarify several points and improve the presentation of this paper. The authors are grateful to Dinghong Liu for pointing out a gap in the original argument of Section 4.

References

- [1] Bartnik, R. and Simon, L., Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Commun. Math. Phys.*, **87**, 1982, 131–152.
- [2] Douglas, J., Solution of the problem of Plateau, *Trans. Amer. Math. Soc.*, **33**(1), 1931, 263–321.
- [3] Ekholm, T., White, B. and Wienholtz, D., Embeddedness of minimal surfaces with total boundary curvature at most 4π , *Ann. of Math. (2)*, **155**(1), 2002, 209–234.

- [4] Fenchel, W., The differential geometry of closed space curves, *Bull. Amer. Math. Soc.*, **57**, 1951, 44–54.
- [5] Flaherty, F. J., The boundary value problem for maximal hypersurfaces, *Proc. Natl. Acad. Sci. USA.*, **76**(10), 1979, 4765–4767.
- [6] Lawson, H. B., Stokes' Theorem and Minimal Surfaces, *Contemp. Math.*, **395**, Amer. Math. Soc., Providence, RI, 2006, 67–82.
- [7] Milnor, J. W., On the total curvature of knots, *Ann. of Math.*, **52**(2), 1950, 248–257.
- [8] Nitsche, J. C. C., A new uniqueness theorem for minimal surfaces, *Arch. Rational Mech. Anal.*, **52**, 1973, 319–329.
- [9] O'Neill, B., *Semi-Riemannian Geometry, with Applications to Relativity*, Pure and Applied Mathematics, **130**, Academic Press, New York, 1983.
- [10] Rado, T., *On the Problem of Plateau*, Springer-Verlag, New York, 1971.
- [11] Ye, N., Fenchel-type inequality and curve shortening flow in 3-dimensional Lorentz space, Ph.D. Thesis, Peking University, Beijing, 2016.
- [12] Ye, N., Ma, X. and Wang D. H., The Fenchel-type inequality in 3-dimensional Lorentz space and a Crofton formula, *Annals of Global Analysis and Geometry*, DOI: 10.1007/s10455-016-9510-8.