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On the Discrete Criteria and Jørgensen Inequalities for $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$*

Jinghua YANG[1]

Abstract In this paper, the author gives the discrete criteria and Jørgensen inequalities of subgroups for the special linear group on $\overline{\mathrm{F}}((t))$ in two and higher dimensions.

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1 Introduction

Let $\mathrm{F}(t)$ be a local field with nontrivial non-archimedean value, where F is a finite field with p elements and p is a prime number. We denote the completion of algebraic closure of $\mathrm{F}(t)$ by $\overline{\mathrm{F}}((t))$. In this paper, we study the discrete criteria and Jørgensen's inequalities for a subgroup of the special linear group $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$.

It is an important topic to study Möbius maps in non-archimedean spaces (see [1–2, 4–5, 7–8]) especially the discrete criteria and Jørgensen inequalities for subgroups of Lie groups. In [2], Kato discussed the discrete criteria of groups of projective general linear group $\mathrm{PGL}(2, \mathbb{C}_p)$. In [1], Armitage and Parker studied the Jørgensen inequality for discrete subgroups of $\mathrm{SL}(2, \mathbb{Q}_p)$, and of $\mathrm{SL}(2, \mathrm{F}(t))$. In [6], Qiu, Yang and Yin gave the discrete criteria of $\mathrm{SL}(m, \mathbb{C}_p)$. Furthermore, the development of studying the moduli space of rational maps and the Kleinian group by arithmetic method arises our interest in studying the group $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$. Hence, we concern about the discrete criteria and Jørgensen inequalities for $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$.

The function field is different from the p-adic number field. The main difficulties that we should face are to estimate the distance between the primitive roots of unit and the unit, and show that there exists a finite number of extensions of degree $d \leq n$ for some given positive integer n. The character of the residue field of $\overline{\mathrm{F}}((t))$ is positive. We state our main theorem.

Theorem 1.1 *Let G be a subgroup of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ with no parabolic element. Then the group G is discrete if and only if any cyclic subgroup of G is discrete.*

Theorem 1.2 *Let $\mathbb{K}$ be a finite extension of $\mathrm{F}(t)$. If a discrete subgroup G of $\mathrm{SL}(2, \mathbb{K})$ contains elliptic elements of finite order only, then G is a finite group.*

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[1]Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail: davidyoung@amss.ac.cn
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Jørgensen's inequality is a necessary condition for the discreteness of subgroups of $\mathrm{SL}(2,\mathbb{C})$, which has been widely applied in many aspects such as the algebraic and geometric convergence of subgroups of $\mathrm{SL}(2,\mathbb{C})$ and the estimation of the volume of hyperbolic manifolds. It has been generalized by many authors in various cases, and also plays an important role in the p-adic analytic space. In [1], Armitage and Parker gave a version of Jørgensen's inequality of discrete subgroups for $\mathrm{SL}(2,\mathrm{F}(t))$. In this paper, we partially improve their results and generalize those to $\mathrm{SL}(m,\overline{\mathrm{F}}((t)))$.

Theorem 1.3 *Let $A\neq -I$ be an element of $\mathrm{SL}(2,\overline{\mathrm{F}}((t)))$. Let B be any element in $\mathrm{SL}(2,\overline{\mathrm{F}}((t)))$ such that B neither fixes nor interchanges the fixed points of A. If $G=\langle A,B\rangle$ is discrete with no parabolic elements, then* $\min\{|\mathrm{tr}^2(A)-4|,|\mathrm{tr}[A,B]-2|\}\geq 1$.

Theorem 1.4 *If a subgroup G of $\mathrm{SL}(m,\overline{\mathrm{F}}((t)))$ is discrete with no parabolic elements, then for each $g\in G\setminus\{I\}$, $\|g-I\|\geq 1$.*

2 Some Facts in $\overline{\mathrm{F}}((t))$

Let $p\geq 2$ be a prime number, and $\mathrm{F}(t)$ be the function field of the non-archimedean value, and $\overline{\mathrm{F}}((t))$ be the completion of the algebraic closure of $\mathrm{F}(t)$, namely $\overline{\mathrm{F}}((t))=\bigcup\limits_{n\geq 1}\overline{\mathrm{F}}((t^{\frac{1}{n}}))$. We denote the valuation group of $\overline{\mathrm{F}}((t))$ by $|\overline{\mathrm{F}}((t))^*|$, the integer ring by $\mathcal{O}_p=\{z\mid |z|\leq 1\}$, and the maximal ideal by $\mathcal{M}=\{z\mid |z|<1\}$. The absolute value satisfies the strong triangle inequality

$$|z-y|\leq \max\{|z|,|y|\}$$

for $x,y\in\overline{\mathrm{F}}((t))$. If x,y and z are points of $\overline{\mathrm{F}}((t))$ with $|x-y|<|x-z|$, then $|x-z|=|y-z|$.

Given $a\in\overline{\mathrm{F}}((t))$ and $r>0$, the open and closed disks of center a and radius r are defined by

$$D(a,r)^-=\{z\in\overline{\mathrm{F}}((t))\mid |z-a|<r\},$$
$$D(a,r)=\{z\in\overline{\mathrm{F}}((t))\mid |z-a|\leq r\}.$$

However, by the strong triangle inequality, topologically $D(a,r)^-$ and $D(a,r)$ are closed and open, and every point in disk $D(a,r)^-$ is the center. This denotes that if $x\in D(a,r)^-$, then $D(a,r)^-=D(x,r)^-$ (resp. $D(a,r)=D(x,r)$). If two disks D_1 and D_2 in $\overline{\mathrm{F}}((t))$ have non-empty intersection, then $D_1\subset D_2$, or $D_2\subset D_1$. For a set $E\subset\overline{\mathrm{F}}((t))$, denote $\mathrm{diam}(E)=\sup\limits_{z,w\in E}|z-w|$ the diameter of E in the non-archimedean metric. Especially, $\mathrm{diam}(D(a,r))=r$.

Let $\mathbb{P}^1(\overline{\mathrm{F}}((t)))$ be the projective space over $\overline{\mathrm{F}}((t))$ which can be viewed as $\mathbb{P}^1(\overline{\mathrm{F}}((t)))=\overline{\mathrm{F}}((t))\cup\{\infty\}$. The chordal distance on $\mathbb{P}^1(\overline{\mathrm{F}}((t)))$ can be defined by

$$\rho_v(z,w)=\frac{|z-w|}{\max\{1,|z|\}\max\{1,|w|\}}$$

for $z,w\in\overline{\mathrm{F}}((t))$,

$$\rho_v(z,w)=\frac{1}{\max\{1,|w|\}}$$

for $w \in \overline{\mathrm{F}}((t))$ and $z = \infty$, and

$$\rho_v(z, w) = 0$$

for $z = w = \infty$.

By the definition of the chordal distance and the strong triangle inequality, it is easy to show that if $|z| \le 1, |w| \le 1$, then $\rho_v(z, w) = |z - w|$, and if $|z| > 1, |w| \le 1$, then $\rho_v(z, w) = \frac{|z-w|}{|z|} = 1$, and if $|z| > 1, |w| > 1$, then $\rho_v(z, w) = \frac{|z-w|}{|z||w|} = \left|\frac{1}{z} - \frac{1}{w}\right|$.

Lemma 2.1 *The residue field* $\mathcal{O}_p/\mathcal{M} \cong \overline{\mathrm{F}}$.

Proof For any x in the finite extension of $\mathrm{F}(t)$, we can expand $x = \sum\limits_{i \ge k} a_i t^{\frac{i}{s}},\ k \in \mathbb{Z}$, and a_i in some finite extension of F. If $|x| = 1$, then $x = \sum\limits_{i \ge 0} a_i t^{\frac{i}{s}}$. Then $x \equiv a_0 \bmod \mathcal{M}$. If $x \in \overline{\mathrm{F}}((t))$, then there exists a sequence $\{x_n\}$ convergent to x, where each x_n is in some finite extension of $\mathrm{F}(t)$. Without loss of generality, we can assume that $|x_n - x_m| < 1$ for any $n, m \ge 1$. Let $x_n = \sum\limits_{i \ge k} a_{i,n} t^{\frac{i}{s}},\ k \in \mathbb{Z}$ and $x_m = \sum\limits_{i \ge k} a_{i,m} t^{\frac{i}{s}},\ k \in \mathbb{Z}$. Thus $a_{0,n} = a_{0,m}$. This implies that $|x - a_{0,n}| < 1$, namely $x \equiv a_{0,n} \bmod \mathcal{M}$.

Lemma 2.2 *Let d be an positive integer, and ζ be the primitive d-th root of unity, then* $|\zeta - 1| = 1$.

Proof By Lemma 2.1, $\zeta = a + u, a \in \overline{F}_p$, $|u| < 1$. Since

$$\zeta^d = a^d + \binom{d}{1} a^{d-1} u + \cdots + \binom{d}{d} u^d = 1,$$

we have $a^d \equiv 1 \bmod \mathcal{M}$. This implies that $|a - 1| = 1$.

Lemma 2.3 *Let $x \in \overline{\mathrm{F}}((t))$ with $|x| = 1$. Then the sequence $\{x^{p^n}\}$ has a convergent subsequence.*

Proof If $x^d = 1$ for some positive integer d, then we draw the conclusion. If $x^d \ne 1$ for any positive integer d, let $x = a + u$, where $a \in \overline{\mathrm{F}},\ |u| < 1$. By the structure of the finite field, there exists a positive integer N such that $a^{p^N} = a$. Since the character of $\overline{\mathrm{F}}$ is p, we have $x^{p^{Nk}} = a^{p^{Nk}} + u^{p^{Nk}} = a + u^{p^{Nk}}$. Thus $|x^{p^{Nk}} - a| = |u|^{p^{Nk}}$ tends to 0, as $k \to \infty$.

Lemma 2.4 *Let $g(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial in $\overline{\mathrm{F}}((t))[z]$. Given a fixed $r > 0$, if coefficients of $g(z)$ satisfy $|a_i| < r^{n-i}$, then all roots of the polynomial $g(z)$ are in the closed disk $D(0, r)$.*

Proof If $\alpha \notin D(0, r)$, then $|a_i \alpha^i| < r^{n-i} |\alpha^i| < |\alpha^n|$. By the ultrametric property, $|g(\alpha)| = |\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0| = |\alpha^n| > 0$. Then all roots of the polynomial are in the closed disk $D(0, r)$.

Lemma 2.5 *Let*

$$g_n(z) = z^m + \sum_{i=0}^{m-1} a_{in} z^i \tag{2.1}$$

be a sequence of polynomials in $\overline{\mathrm{F}}((t))[z]$. If all coefficients a_{in} tend to zero as $n \to \infty$, where $0 \le i \le m-1$, then all roots of $g_n(z)$ tend to zero as $n \to \infty$.

Proof For any $r > 0$, we can find a sufficiently large positive integer N such that for any $n > N$, $|a_{in}| < r^{n-i}$, since a_{in} tends to zero as $n \to \infty$, where $0 \leq i \leq m-1$. By Lemma 2.4 , all roots of $g_n(z)$ are in the closed disk $D(0,r)$. Since $r > 0$ is arbitrary, all roots of $g_n(z)$ tend to zero as $n \to \infty$.

3 Discrete Subgroups of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$

Since the product of all eigenvalues of $g \in \mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ is one, either the absolute value of each eigenvalue of g is one or there exists at least one eigenvalue whose absolute value is larger than 1. Thus each non-unit element $g \in \mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ falls into the following three classes:

(a) g is said to be parabolic if

(1) the absolute value of any eigenvalue of g is 1, and

(2) g can not be conjugated to a diagonal matrix.

(b) g is said to be elliptic if

(1) the absolute value of any eigenvalue of g is 1, and

(2) g can be conjugated to a diagonal matrix.

(c) g is said to be loxodromic if there exists at least one eigenvalue of g whose absolute value is larger than 1.

For $g = (a_{ij})$ in the matrix ring $\mathrm{M}(m, \overline{\mathrm{F}}((t)))$, the norm of g is defined by $\|g\| = \max\limits_{1\leq i\leq m, 1\leq j\leq m} \{|a_{ij}|\}$. Obviously, $\|g\| = 0$ implies that each $a_{ij} = 0$. It is easy to verify that $\|\alpha g\| = |\alpha|\|g\|$, $\|g + h\| \leq \max\{\|g\|, \|h\|\}$ and $\|gh\| \leq \|g\|\|h\|$.

We say that a subgroup G of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ is discrete if there exists $\delta = \delta(G) > 0$ such that each element $g \in G \setminus \{I\}$ satisfies $\|g - I\| > \delta$, where I denotes the identity.

Obviously, a subgroup G of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ is discrete if and only if any sequence consisting of distinct elements $g_n \in G$ is not a Cauchy sequence. Since $\|h^{-1}g_nh - h^{-1}gh\| \leq \|h^{-1}\|\|g_n - g\|\|h\|$, we have $\|h^{-1}g_nh - h^{-1}gh\| \to 0$, when $g_n \to g$, as $n \to \infty$. This means that conjugation does not change the discreteness.

We show that if G is a discrete subgroup of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$, then the elliptic element in G is of finite order.

Lemma 3.1 *Let I denote the unit matrix and J denote a nilpotent matrix in* $\mathrm{M}(m, \overline{\mathrm{F}}((t)))$. *Let $\lambda \in \overline{\mathrm{F}}((t))$ with $|\lambda| = 1$. If $f = \lambda I + J$, then the sequence $\langle f^{p^n} \rangle$ has a convergent subsequence. Especially, if $\lambda = 1$, then $\langle f^{p^n} \rangle$ is a periodic sequence.*

Proof Since J is a nilpotent matrix, there exists a positive integer N such that $J^N = 0$. Thus for any positive integer $k > N$, we have

$$f^k = (\lambda I + J)^k = \lambda^k I + \binom{k}{1}\lambda^{k-1}J + \cdots + \binom{k}{N}\lambda^{k-N}J^N.$$

Choose $k = p^{nm} > N$, then

$$f^{p^{nm}} = (\lambda I + J)^{p^{nm}} = \lambda^{p^{nm}} I.$$

By Lemma 2.3, $\{\lambda^{p^n}\}$ has a convergent subsequence. Therefore the sequence $\langle f^{p^n} \rangle$ has a convergent subsequence.

Theorem 3.1 *If the subgroup G of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ is discrete, then there is no elliptic element of infinite order in G.*

Proof Suppose that g is an elliptic element of infinite order in $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$. We can assume that

$$g = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix},$$

where $\lambda_i \in \overline{\mathrm{F}}((t))$ are eigenvalues of g with $|\lambda_i| = 1,\ 1 \le i \le m$.

Therefore $\lambda_i^s \neq \lambda_i^t$ for any positive integers s, t. By Lemma 2.3, the sequence $\{\lambda_i^{p^n}\}$ has the convergent subsequence. Thus $\{g^{p^n}\}$ is the sequence consisting of distinct elements and a convergent sequence. This contradicts the hypothesis. Thus there is no elliptic element of infinite order in G.

Lemma 3.2 *If $g_n \in \mathrm{SL}(m, \overline{\mathrm{F}}((t))) \to I$, as $n \to \infty$, then all eigenvalues of g_n tend to* 1*, as $n \to \infty$.*

Proof The eigenpolynomial $f_n(\lambda) = |\lambda I - g_n|$ tends to polynomial $(\lambda - 1)^m$, since g_n tends to I. By Lemma 2.5, all eigenvalues λ_n tend to 1.

Lemma 3.3 *If there exists a positive number $\delta = \delta(G)$ such that for any $g \in G$,* $\max\{|\lambda_1 - 1|, |\lambda_2 - 1|, \cdots, |\lambda_m - 1|\} \ge \delta$*, where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are eigenvalues of g, then G is discrete.*

Proof If G is not discrete, then there exists a sequence $\{g_n\}$ tending to I, as $n \to \infty$. By Lemma 3.2, we know that eigenvalues $\lambda_{1,n}, \lambda_{2,n}, \cdots, \lambda_{m,n}$ of g_n tend to 1 which implies that G is discrete.

Theorem 3.2 *Let G be a subgroup of $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ with no parabolic elements. Then G is discrete if and only if any cyclic subgroup of G is discrete.*

Proof $\Rightarrow$ It is obviously true.

$\Leftarrow$ By Theorem 3.1, we know that a subgroup G containing any elliptic element of infinite order is not discrete, which yields that there only exist loxodromic elements or elliptic elements of finite order.

If g is a loxodromic element, then let λ be the eigenvalue of g with $|\lambda| > 1$. By the ultrametric property, we have $|\lambda - 1| > 1$. If g is a elliptic element of the order n, namely $g^n = I$, where n is the smallest positive integer, then we can assume that g has the form

$$g = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix},$$

where λ_i is an eigenvalue of g, $1 \le i \le m$.

Since $g^n = I$, namely each eigenvalue λ_i of g satisfies $\lambda_i^n = 1$, by Lemma 2.2, we know that $|\lambda - 1| = 1$. By Lemma 3.3, we can see that G is discrete.

Example 3.1 Let G be generated by elements $f_n = z + t^n$, $n \ge 0$. Then G is not discrete but any cyclic group of G is discrete.

Proof Obviously, $f_n = z + t^n \to z$, as $n \to \infty$. We see that G is not discrete. Each cyclic group is generated by element $z + \sum\limits_{i\geq 0} a_i t^i,\ a_i \in \mathrm{F}_p$. Since each element f_n is parabolic, we know that f_n is of finite order which implies that the cyclic group generated by $\langle f_n \rangle$ is discrete.

Lemma 3.4 *The finite extension $\mathbb{K}$ of degree n of $\mathrm{F}(t)$ is $\mathrm{K}_n(t^{\frac{1}{m}})$, where K_n is a finite extension of F of degree $\leq n$, and $m \leq n$.*

Proof Since $\mathbb{K}$ is a finite extension of $\mathrm{F}(t)$, we see that $\mathbb{K}$ is also a local field. We denote the integer ring, maximal ideal and residue field of $\mathbb{K}$ by $\mathcal{O}_{\mathbb{K}} = \{z \mid |z| \leq 1\}$, $\mathcal{M}_{\mathbb{K}} = \{z \mid |z| < 1\}$ and $\mathrm{F}_{\mathbb{K}} = \mathcal{O}_{\mathbb{K}}/\mathcal{M}_{\mathbb{K}}$ respectively.

By the proof of Lemma 2.1, we see that $\mathrm{F}_{\mathbb{K}}$ is congruent to some finite extension of F. We claim that $\mathrm{F}_{\mathbb{K}}$ is some finite extension of F.

For any $x \in \mathrm{F}_{\mathbb{K}}$, we can write $x = a + u,\ a \in \overline{\mathrm{F}},\ |u| < 1$. There exists a positive integer N such that $a^{p^N} = a$ and $x^{p^N} = x$. Hence $x^{p^{Nk}} = a^{p^{Nk}} + u^{p^{Nk}} = a + u^{p^{Nk}} = a + u$ which implies that $u = 0$. Hence $\mathrm{F}_{\mathbb{K}} \subset \overline{\mathrm{F}}$, namely $\mathrm{F}_{\mathbb{K}}$ is some finite extension of F.

Since $\mathbb{K}$ is a finite extension of $\mathrm{F}(t)$ of degree n, we see that $\mathcal{O}_{\mathbb{K}} = \mathrm{F}_{\mathbb{K}}(\pi)$, where π is the uniformization element, and $|\pi| = |t|^{\frac{1}{m}}$. We claim that we can choose a uniformization element π as $t^{\frac{1}{n}}$.

Firstly, if we can expand $\pi = t^{\frac{1}{m}} + a_2 t^{\frac{2}{m}} + \cdots$, then $\pi - a_2\pi^2 = t^{\frac{1}{m}} + u$, where $|u| \leq |t|^{\frac{3}{m}}$. We write $\pi_1 = \pi - a_2\pi^2 = t^{\frac{1}{m}} + a_3 t^{\frac{3}{m}}$. Following this algorithm, let $\pi_2 = \pi_1 - a_3\pi^3, \cdots, \pi_{k+1} = \pi_k - a_{k+2}\pi^{k+2}, \cdots$. This implies that $|\pi_{k+1} - t^{\frac{1}{m}}| < |t|^{\frac{k+3}{nm}}$. Letting $k \to \infty$, we see that $\pi_k \to t^{\frac{1}{m}}$, namely $t^{\frac{1}{m}} \in \mathbb{K}$.

If $\pi = t^{\frac{1}{m}} + u$, where $|u| \leq |t|^{\frac{2}{m}}$, m is prime to p, then there exists a positive integer N such that $m \mid p^N - 1$. We consider $\pi^{p^{Nk}} = (t^{\frac{p^{Nk}}{m}} + u^{p^{Nk}})$. Since $\Big|\frac{u^{p^{Nk}}}{t^{\frac{p^{Nk}-1}{m}}}\Big| < |t|^{\frac{P^{Nk}-1}{m}}$, we see that $t^{\frac{1}{m}} + \frac{u^{p^{Nk}}}{t^{\frac{p^{Nk}-1}{m}}} = \frac{x^{p^{Nk}}}{t^{(p^{Nk}-1)}} \in \mathcal{O}_{\mathbb{K}}$. This implies that $t^{\frac{1}{m}} + \frac{u^{p^{Nk}}}{t^{\frac{p^{Nk}-1}{m}}} \to t^{\frac{1}{m}},\ k \to \infty,\ t^{\frac{1}{m}} \in \mathbb{K}$.

If $\pi = \big(\sum\limits_{i=1}^{p^r-1} t^{\frac{i}{p^r}}\big)u + at^{\frac{k}{p^s m}} + v$, where $u \in \mathcal{O}_{\mathbb{K}}, |v| < |t|^{\frac{k}{p^s m}}$, m is prime to p, we see that $x^{p^{r+s}} = \big(\sum\limits_{i=1}^{p^r-1} t^{ip^s}\big)u + a^{p^{r+s}} t^{k(\frac{p^r}{m})} + v^{p^{r+s}}$. This yields that $t^{k(\frac{p^r}{m})} + v^{(p^{s+r})} \in \mathcal{O}_{\mathbb{K}}$. Furthermore, by the proof above, we know that $t^{\frac{1}{m}} \in \mathcal{O}_{\mathbb{K}}$. This implies that $p \mid m$, since $\mathbb{K}$ is discrete valued field. However, this is a contradiction.

Let $\pi = t^{\frac{1}{p^r m}} + u$, where m is prime to p with $|u| < |t|^{\frac{1}{p^r m}}$. Hence $\pi^{p^r} = t^{\frac{1}{m}} + u^{p^r}$. By the proof above, we know that $t^{\frac{1}{m}} \in \mathcal{M}_{\mathbb{K}}$. This implies that $p \mid m$, since $\mathbb{K}$ is discrete valued field. However, this is also a contradiction.

In the end, we see that $\mathbb{K} = \mathrm{F}_{\mathbb{K}}(t^{\frac{1}{m}})$. Since $[\mathbb{K} : \mathrm{F}_p(t)] = n$, we see that $n = [\mathbb{K} : \mathrm{F}_p(t)] = m[\mathrm{F}_{\mathbb{K}} : \mathrm{F}_p]$ which yields $m \leq n,\ [\mathrm{F}_{\mathbb{K}} : \mathrm{F}_p] \leq n$.

Lemma 3.5 *Given a positive integer n, there exists a finite number of extensions of degree $\leq n$.*

Proof Following Lemma 3.4, it is obvious.

Lemma 3.6 *There exist only finitely many primitive roots of unity in $\mathrm{K}_m(t^{\frac{1}{n}})$, where K_m is a finite extension of F of degree m.*

Proof Let $\lambda = a + u \in,\ |u| < 1,\ a \in \overline{\mathrm{F}}$ be a primitive root in $\mathrm{K}_m(t^{\frac{1}{n}})$. If $\lambda^s = 1$, where s is a positive integer which is prime to p, then there exists a positive integer N such that $s \mid p^N - 1$ and $a^{p^N} = a$.

Hence

$$\lambda^{p^N k} = (a+u)^{p^N k} = a^{p^{Nk}} + u^{p^{Nk}} = a + u^{p^{Nk}} = \lambda = a + u.$$

Let $k \to \infty$, and then $u = u^{p^{Nk}} = 0$, since $|u| < 1$.

Lemma 3.7 *If λ is the eigenvalue of the elliptic element g of finite order, then* $\leq |\mathrm{tr}(g) - 2| = 1$, *where* $\mathrm{tr}(g)$ *denotes the trace of* g.

Proof Let λ be the eigenvalue of the elliptic element g of finite order, namely λ is the primitive root of unity. By Lemma 2.2, $|\lambda - 1| = 1$. Since trace is invariant by conjugation, we have $\mathrm{tr}(g) = a + d = \lambda + \lambda^{-1}$ which implies that $|\mathrm{tr}(g) - 2| = |\lambda + \lambda^{-1} - 2| = \frac{|\lambda-1|^2}{|\lambda|} = 1$.

If an eigenvalue of g is -1, then the other eigenvalue is also -1, since the determinant is 1. Thus g can be conjugated to the diagonal matrix $-I$, and thus $g = h(-I)h^{-1} = -hh^{-1} = -I$.

Theorem 3.3 *Let $\mathbb{K}$ be a finite extension of* $\mathrm{SL}(2, \mathrm{F}(t))$. *If a discrete subgroup G of* $\mathrm{SL}(2, \mathbb{K})$ *contains elliptic elements of finite order only, then G is a finite group.*

Proof Let $\widetilde{\mathbb{K}}$ be a finite extension of $\mathbb{K}$ with $[\widetilde{\mathbb{K}} : \mathbb{K}] \leq 2$. Then $\widetilde{\mathbb{K}}_p$ is also a finite extension of $\mathrm{F}_p(t)$. Since $\mathrm{F}_p(t)$ is locally compact, we know that $\widetilde{\mathbb{K}}$ is locally compact. By Lemma 3.4, we see that $\widetilde{\mathbb{K}} \subset \mathrm{K}_p(t^{\frac{1}{n}})$.

For some fixed element $g \in G \setminus \{I\}$, we can assume that there exists an element $h \in G$ which can not commutate with $g \neq \pm I$, namely $\lambda^2 \neq 1$. Thus g, h can be respectively conjugated to

$$\overline{g} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \overline{h} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{K}(\lambda)).$$

Then the commutator

$$[\overline{g}, \overline{h}] = \overline{g}\overline{h}\overline{g}^{-1}\overline{h}^{-1} = \begin{pmatrix} ad - bc\lambda^{-2} & -ab\lambda^2 + ab \\ cd - cd\lambda^{-2} & -bc\lambda^2 + ad \end{pmatrix}.$$

Therefore $\mathrm{tr}[\overline{g}, \overline{h}] = 2ad - bc\big(\lambda^2 + \frac{1}{\lambda^2}\big) = 2 - bc\big[\big(\lambda + \frac{1}{\lambda}\big)^2 - 4\big] = 2 - bc\big(\lambda - \frac{1}{\lambda}\big)^2$. By Lemma 3.7, $\big|bc\big(\lambda - \frac{1}{\lambda}\big)^2\big| = 1$. Since λ^2 is also a primitive root of unity and $\lambda^2 \neq 1$, we have $\frac{|\lambda^2 - 1|}{|\lambda|} = 1$. Therefore $|bc| = 1$.

Suppose that there exist infinitely many distinct elements h_n which can not commutate with g, and let h_n have the following form

$$h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Since $a_n d_n - b_n c_n = 1$ and $|b_n c_n| = 1$, we see that $a_n d_n$ is also bounded. We also have a_n, d_n are bounded, since $a_n + d_n$ is bounded.

Assuming that b_n, c_n are bounded. Then a_n, d_n, b_n, c_n are all bounded. Since $\mathrm{F}_p(t)$ is locally compact, the sequence $\{a_n, d_n, b_n, c_n\}$ has convergent subsequences, Then h_n has the convergent subsequence, which contradicts the discreteness of G.

Suppose that $\{b_n\}$ or $\{c_n\}$ is unbounded. Without loss of generality, we suppose that $b_n \to \infty$, as $n \to \infty$. Since $|b_n c_n| = 1$, $c_n \to 0$, as $n \to \infty$. Consider the sequence $\{h_1 h_n\}$. Since

$$h_1 h_n = \begin{pmatrix} a_1 a_n + b_1 c_n & a_1 b_n + b_1 d_n \\ a_n c_1 + d_1 c_n & b_n c_1 + d_1 d_n \end{pmatrix},$$

we have $\mathrm{tr}[h_1 h_n] = a_1 a_n + b_1 c_n + b_n c_1 + d_1 d_n$. Since b_1, c_1 are nonzero and a_n, d_n are bounded, it follows $a_1 a_n + b_1 c_n + b_n c_1 + d_1 d_n \to \infty$, as $n \to \infty$. Therefore when n is sufficiently large, $h_1 h_n$ is a loxodromic element which contradicts the fact that G has elliptic elements only. Hence there do not exist infinitely many elements which can not commutate with g. Suppose that $h \in G$ can commutate with g. Then h and g can be conjugated to diagonal matrices simultaneously. Since eigenvalues of $h \in G$ are primitive roots of unity in $\widetilde{\mathbb{K}}$, by Lemma 3.4, there exist finitely many such h. Summing up, there are only finitely many elements in G.

But the result we proved above is not true for $\overline{\mathrm{F}}((t))$, even for $\overline{\mathrm{F}}$, since they are infinite extensions of $\mathrm{F}(t)$.

Example 3.2 The group $\mathrm{SL}(2, \overline{\mathrm{F}})$ is discrete.

Proof Since each $|x-1| = 1$, for any $1 \neq x \in \overline{\mathrm{F}}$, and the determinant of any element g is 1, we know that G is discrete.

4 Jørgensen's Inequality for $\mathrm{SL}(\mathrm{m}, \overline{\mathrm{F}}((t)))$

In [1], Armitage and Parker gave a version of Jørgensen's inequality in the non-archimedean metric space, especially for $\mathrm{SL}(2, \mathrm{F}(t))$.

Theorem 4.1 (see [1, Theorem 4.2]) *Let A be an element of $\mathrm{SL}(2, \mathrm{F}(t))$ conjugate to a diagonal matrix. Let B be any element of $\mathrm{SL}(2, \mathrm{F}(t))$ so that, when acting on $\mathrm{F}(t) \cup \{\infty\}$ via Möbius transformations, B neither fixes nor interchanges the fixed points of A. If $G = \langle A, B\rangle$ is discrete, then* $\max\{|\mathrm{tr}^2(A) - 4|, |\mathrm{tr}([A, B]) - 2|\} \geq 1$.

According to the results, the discrete subgroup does not contain any parabolic element which yields that a generator $A \in \mathrm{SL}(2, \overline{\mathrm{F}}((t)))$ can be conjugated to a diagonal matrix. If the subgroup G generated by $-I$ and $B \in \mathrm{SL}(2, \overline{\mathrm{F}}((t)))$, then the group $G = \{(-1)^i B^j\}$ is very trivial. Hence we do not consider $-I$ as the generator.

The Jørgensen's inequality is built for $\mathrm{SL}(2, \overline{\mathrm{F}}((t)))$.

Theorem 4.2 *Let $A \neq -I$ be an element of $\mathrm{SL}(2, \overline{\mathrm{F}}((t)))$. Let B be any element in $\mathrm{SL}(2, \overline{\mathrm{F}}((t)))$ such that B neither fixes nor interchanges the fixed points of A. If $G = \langle A, B\rangle$ is discrete with no parabolic elements, then* $\min\{|\mathrm{tr}^2(A) - 4|, |\mathrm{tr}[A, B] - 2|\} \geq 1$.

Proof If $[A, B] = I$, then $ABA^{-1}B^{-1} = I$. This implies that $AB = BA$, which means that B can fix or interchange the fixed point of A. This contradicts the hypothesis.

We assume that $[A, B] \neq I$. Let λ and $\frac{1}{\lambda}$ be eigenvalues of A. If A is a loxodromic element, we can assume that $|\lambda| > 1$. Hence $\left|\lambda - \frac{1}{\lambda}\right| = |\lambda| > 1$, and then $|\mathrm{tr}^2(A) - 4| = \left|\left(\lambda + \frac{1}{\lambda}\right)^2 - 4\right| = \left|\lambda - \frac{1}{\lambda}\right|^2 = |\lambda| > 1$. If A is an elliptic element of finite order, then $|\lambda - 1| = 1$.

Let ζ and ζ^{-1} be the eigenvalues of the $[A, B]$, and then $\zeta \neq 1$. If $[A, B]$ is a loxodromic element, we can assume that $|\zeta| > 1$. Hence $\left|\zeta - \frac{1}{\zeta}\right| = |\zeta| > 1$ which implies that $|\mathrm{tr}[A, B] - 2| = \left|\left(\zeta + \frac{1}{\zeta}\right) - 2\right| = \frac{|\zeta - 1|^2}{|\zeta|} = |\zeta| > 1$. If $[A, B]$ is an elliptic element of finite order, then $|\zeta - 1| = 1$.

In [3], Martin discussed the group generated by finitely many elements, and estimated the maximum distance between the generator and the identity, and gave a version of Jørgensen's inequality for the real Möbius transform in higher dimensions.

Theorem 4.3 (see [3, Theorem 4.5]) *Let f and g be Möbius transformations of S^n. If f and g together generate a discrete non-elementary group, then* $\max\{\|g^i f g^{-i} - I\| : i = 0, 1, 2, \cdots, n\} > 2 - \sqrt{3}$.

Lemma 4.1 *If $g \in \mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ and $\|g - I\| < 1$, then all eigenvalues of g are in $D(1, 1)^-$.*

Proof Let $g = (b_{ij}) \in \mathrm{SL}(m, \overline{\mathrm{F}}((t)))$. Since $\|g - I\| < 1$, we have $|b_{ij} - \delta_{ij}| < 1$, where $\delta_{ij} = 1$, if $i = j$; otherwise $\delta_{ij} = 0$, if $i \neq j$.

Then eigenpolynomial

$$
\begin{aligned}
|\lambda I - g| &= \begin{vmatrix} \lambda - b_{11} & -b_{12} & \cdots & -b_{1m} \\ -b_{21} & \lambda - b_{22} & \cdots & -b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & -b_{m(m-1)} & \lambda - b_{mm} \end{vmatrix} \\
&= \begin{vmatrix} (\lambda - 1) + 1 - b_{11} & -b_{12} & \cdots & -b_{1m} \\ -b_{21} & (\lambda - 1) + 1 - b_{22} & \cdots & -b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & -b_{m(m-1)} & (\lambda - 1) + 1 - b_{mm} \end{vmatrix}.
\end{aligned}
$$

The eigenpolynomial can be expressed as $G(\lambda - 1) = (\lambda - 1)^m + a_{m-1}(\lambda - 1)^{m-1} + \cdots + a_0$, where the coefficient a_i of eigenpolynomial $G(\lambda - 1)$ is a combination of the $c_{ij} = \delta_{ij} - b_{ij}$ by product or addition. By the ultrametric property, we have $|a_i| \leq \max\{|c_{ij}|\} < 1$. Since $|a_i|^{\frac{1}{m-i}} < 1$, there exists a positive number r satisfying $0 < r < 1$ such that $|a_i| < r^{m-i}$. By Lemma 2.4, each eigenvalue of g is in $D(1, 1)^-$.

Theorem 4.4 *If a subgroup G of* $\mathrm{SL}(m, \overline{\mathrm{F}}((t)))$ *is discrete with no parabolic elements, then for each $g \in G \setminus \{I\}$, $\|g - I\| \geq 1$.*

Proof By Theorem 3.1, we know that each element in G is either a loxodromic element or an elliptic element of finite order. If g in G is a loxodromic element, then at least one eigenvalue λ whose absolute value is larger than 1. Hence $|\lambda - 1| = |\lambda| > 1$. If g is an elliptic element of finite order, then each eigenvalue λ satisfies $|\lambda - 1| = 1$.

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