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The Strong Solution for the Viscous Polytropic Fluids with Non-Newtonian Potential

Qiu MENG[1] Hongjun YUAN[2]

Abstract The authors study an initial boundary value problem for the three-dimensional Navier-Stokes equations of viscous heat-conductive fluids with non-Newtonian potential in a bounded smooth domain. They prove the existence of unique local strong solutions for all initial data satisfying some compatibility conditions. The difficult of this type model is mainly that the equations are coupled with elliptic, parabolic and hyperbolic, and the vacuum of density causes also much trouble, that is, the initial density need not be positive and may vanish in an open set.

Keywords Compressible Navier-Stokes equations, Viscous polytropic fluids, Vacuum, Poincaré type inequality, Non-Newtonian potential
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1 Introduction

The motion of viscous polytropic fluids with nonnegative thermal conductivity under the self-gravitational force and outer power can be described by the model of the fluids dynamic, that is, the compressible full Navier-Stokes equations with non-Newtonian potential:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho e u) - \kappa\Delta e + P\operatorname{div} u = \frac{\mu}{2}(\nabla u + \nabla^{\mathrm{T}} u) : (\nabla u + \nabla^{\mathrm{T}} u) + \lambda(\operatorname{div} u)^2 + \rho h, \tag{1.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) + \nabla P + \rho\nabla\Phi = \rho f, \tag{1.3}$$

$$\operatorname{div}[(|\nabla\Phi|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla\Phi] = 4\pi g\Big(\rho - \frac{m_0}{|\Omega|}\Big), \quad p > 2 \tag{1.4}$$

in $(0,T)\times\Omega$ together with the boundary and the initial conditions

$$\nabla e\cdot n|_{\partial\Omega} = 0, \tag{1.5}$$

$$u|_{\partial\Omega} = 0, \tag{1.6}$$

$$\Phi|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\rho, \rho e, \rho u)|_{t=0} = (\rho_0, \rho_0 e_0, \rho_0 u_0). \tag{1.8}$$

Here the unknown functions ρ, e, u, P, Φ, f are the density, specific internal energy, velocity, pressure, non-Newtonian gravitational potential, outer power respectively, and h is a heat

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[1]Institute of Mathematics and Statistics, Beihua University, Jilin 132013, Jilin, China.
E-mail: mengqiu15@163.com
[2]Institute of Mathematics, Jilin University, Changchun 130012, China. E-mail: hjy@jlu.edu.cn

source. The physical constants μ, λ and κ are the shear viscosity, bulk viscosity and heat conductivity, respectively. μ and λ satisfy $\lambda + \frac{2}{3}\mu \geq 0$ and $\mu > 0$. $p > 2$ and $\varepsilon > 0$ are positive constants. In this paper, the internal energy e and the pressure P are assumed by $e = C_V\theta,\ P = R\rho\theta = (\gamma-1)\rho e$ with positive constants C_V, R and $\gamma = \frac{R}{C_V} > 1$. $\Omega \subset \mathcal{R}^3$ is a bounded domain with smooth boundary, n is the unit outward normal to $\partial\Omega$.

When $\Phi = 0$, the problem has received many studies. We refer the readers to the papers [6–7, 9–11, 13] for some local or global smooth solution in the absence of vacuum. But in the presence of vacuum, lots of results were obtained for viscous heat-conductivity compressible fluids with the uniqueness and existence results by [1, 3–4]. Especially, in [2], Cho and Kim proved the existence results for viscous polytropic fluids with vacuum. Very recently, the existence of local strong solutions to problem (1.1)–(1.4) under the Dirichlet boundary conditions was proved in [14].

The aim of this paper is to use the method in [2, 14] to prove the existence of unique local strong solutions to (1.1)–(1.8) with $\inf \rho_0 = 0$. Here it should be noted that, in [2, 14], the authors prescribed the Dirichlet boundary condition $e|_{\partial\Omega} = 0$ instead of (1.5). Here for technical reasons, their method could not deal with (1.5). We will use a Poincaré type inequality (2.17) due to Lions [8] and some careful estimates to circumvent this difficulty.

Moreover, using the method in [14], we can prove a similar existence result with $\kappa = 0$. Since the calculations are similar, we omit the details here.

In Section 2, we consider a linearized problem with $\kappa > 0$ and derive some local estimates for the solutions independent of the lower bound of the initial density, and in Section 3, we prove the existence theorem when $\kappa > 0$ by applying a classical iteration argument based on the uniform estimates.

2 A Priori Estimates for a Linearized Problem with $\kappa > 0$

In this section, we consider the following linearized problem with $\kappa > 0$:

$$\partial_t\rho + \operatorname{div}(\rho v) = 0, \tag{2.1}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho e v) - \kappa\Delta e + P\operatorname{div} v = \frac{\mu}{2}(\nabla v + \nabla^{\mathrm{T}} v) : (\nabla v + \nabla^{\mathrm{T}} v) + \lambda(\operatorname{div} v)^2 + \rho h, \tag{2.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho v \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) + \nabla P + \rho\nabla\Phi = \rho f, \tag{2.3}$$

$$\operatorname{div}[(|\nabla\Phi|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla\Phi] = 4\pi g\Big(\rho - \frac{m_0}{|\Omega|}\Big), \quad p > 2, \tag{2.4}$$

$$(\rho, \Phi, \rho e, \rho u)|_{t=0} = (\rho_0, \Phi_0, \rho_0 e_0, \rho_0 u_0) \qquad \text{in } \Omega, \tag{2.5}$$

$$(\nabla e \cdot n, u, \Phi) = (0,0,0) \qquad \text{on } (0,T)\times\partial\Omega, \tag{2.6}$$

where v is a known vector field on $(0,T)\times\Omega$ and Φ_0 is dependent on ρ_0 satisfying $\Delta\Phi_0 = 4\pi g\big(\rho_0 - \frac{m_0}{|\Omega|}\big)$ (m_0 is the initial mass). In fact, using the conservation of mass, we have $\int_\Omega \rho \mathrm{d}x = \int_\Omega \rho_0 \mathrm{d}x = m_0 > 0$.

Here we impose the following regularity conditions on the initial data, f and h:

$$\begin{aligned}
&\rho_0 \geq 0, \quad \rho_0 \in W^{1,q}(\Omega), \quad 3 < q \leq 6, \\
&e_0 \in H^2(\Omega), \quad \nabla e_0 \cdot n|_{\partial\Omega} = 0, \quad u_0 \in H^2(\Omega)\cap H_0^1(\Omega), \quad u_0|_{\partial\Omega} = 0, \\
&(h,f) \in C([0,T];L^2(\Omega)) \cap L^2(0,T;L^q(\Omega)), \quad (h_t, f_t) \in L^2(0,T;H^{-1}(\Omega)),
\end{aligned} \tag{2.7}$$

and the natural compatibility conditions:

$$\begin{aligned}
&-\kappa\Delta e_0 - \frac{\mu}{2}(\nabla u_0 + \nabla^{\mathrm{T}} u_0) : (\nabla u_0 + \nabla^{\mathrm{T}} u_0) - \lambda(\operatorname{div} u_0)^2 = \rho_0^{\frac{1}{2}} g_1,\\
&-\mu\Delta u_0 - (\lambda+\mu)\nabla(\operatorname{div} u_0) + \nabla P_0 = \rho_0^{\frac{1}{2}} g_2
\end{aligned} \tag{2.8}$$

for some $(g_1, g_2) \in L^2(\Omega)$, where $P_0 = (\gamma-1)\rho_0 e_0$. Roughly speaking, (2.8) is equivalent to the L^2-integrability of $\sqrt{\rho}e_t$ and $\sqrt{\rho}u_t$ at $t=0$, as can be shown formally by letting $t \to 0$ in (1.2) and (1.3). Hence the condition (2.8) plays a key role in deducing that $(e_t, u_t) \in L^2(0, T^*; H^1(\Omega))$ as well as $(\sqrt{\rho}e_t, \sqrt{\rho}u_t) \in L^\infty(0, T^*; L^2(\Omega))$ for some small time $T^* > 0$. This was observed and justified rigorously first by Salvi and Străskraba [12], and then by Cho, Choe and Kim [1, 4–5] for barotropic fluids, and by Cho and Kim [2] for the polytropic fluids. Naturally, the compatibility condition (2.8) is satisfied automatically for all initial data $\rho_0 \in W^{1,q}(\Omega)$, $e_0 \in H^2(\Omega)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, whenever ρ_0 is bounded away from zero.

For the known vector v, we assume that $v(0) = u_0$ and

$$\sup_{0\le t\le T^*} \|v(t)\|_{H_0^1(\Omega)} + \beta^{-1}\|v(t)\|_{H^2(\Omega)} + \int_0^{T^*} \|v_t(t)\|_{H_0^1(\Omega)}^2 + \|v(t)\|_{W^{2,q}(\Omega)}^2 \mathrm{d}t \le c_1 \tag{2.9}$$

for some fixed constants c_1, β and time T^* such that

$$1 < c_0 < c_1 < c_2 = \beta c_1, \quad 0 < T^* \le T$$

and

$$c_0 = 2 + \|\rho_0\|_{W^{1,q}(\Omega)} + \|(e_0, u_0)\|_{H^2(\Omega)} + \|(g_1, g_2)\|_{L^2(\Omega)}^2.$$

Here it should be emphasized that throughout the paper, C denotes a generic positive constant depending only on the fixed constants $\mu, \lambda, \kappa, C_V, R, p, q, \varepsilon, |\Omega|, m_0, T$ and the regularity of h, f, but independent of c_0, c_1, c_2 and β.

The following lemma is proved in [2, 14].

Lemma 2.1 *Assume that $\rho_0 \ge \delta > 0$ in Ω. Then there exists a unique solution ρ to the linear transport problem* (2.1) *and* (2.7) *such that*

$$\|\rho(t)\|_{W^{1,q}(\Omega)} \le Cc_0, \quad \|\rho_t(t)\|_{L^q(\Omega)} \le Cc_2^2 \tag{2.10}$$

for $0 \le t \le T^* \wedge T_1 = \min(T^*, T_1)$, where $T_1 = c_2^{-1} < 1$. Moreover,

$$C^{-1}\delta \le \rho(t, x) \le Cc_0 \tag{2.11}$$

for $0 \le t \le \min(T^*, T_1)$, $x \in \overline{\Omega}$.

The next lemma gives the estimates on the internal energy and hence on the pressure.

Lemma 2.2 *Assume further that $\rho_0 \ge \delta$ in Ω for some constant $\delta > 0$. Then there exists* *a unique strong solution e to the initial boundary value problem* (2.2) *and* (2.5)–(2.6) *such that*

$$\int_\Omega(\rho e_t^2+e^2+(\nabla e)^2)\mathrm{d}x+\int_0^t\int_\Omega(\nabla e_t)^2\mathrm{d}x\mathrm{d}s\le c_2\exp(Cc_0^4c_1), \tag{2.12}$$

$$\|e(t)\|_{H^2(\Omega)}\le c_2^2\exp(Cc_0^4c_1), \tag{2.13}$$

$$\int_0^t\|e(s)\|^2_{W^{2,q}(\Omega)}\mathrm{d}s\le c_2^6\exp(Cc_0^4c_1), \tag{2.14}$$

$$\|\nabla P(t)\|_{L^2(\Omega)}\le c_2^{\frac12}\exp(Cc_0^4c_1),\quad \|\nabla P(t)\|_{L^q(\Omega)}\le c_2^2\exp(Cc_0^4c_1),$$
$$\|P_t(t)\|_{L^2(\Omega)}\le c_2^{\frac52}\exp(Cc_0^4c_1) \tag{2.15}$$

for $0\le t\le T^*\wedge T_4$.

Proof We only need to prove the estimates. Applying $\frac{\partial}{\partial t}$ to (2.2) gives

$$\rho e_{tt}+\rho v\cdot\nabla e_t-\kappa\Delta e_t+(P\mathrm{div}\,v)_t=\mu(\nabla v+\nabla^{\mathrm{T}}v):(\nabla v_t+\nabla^{\mathrm{T}}v_t)+2\lambda\mathrm{div}\,v\mathrm{div}\,v_t$$
$$+(\rho h)_t-\rho_tv\nabla e-\rho v_t\nabla e-\rho_te_t.$$

Then multiplying this equation by e_t, integrating over Ω and using (2.1), we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho e_t^2\mathrm{d}x+\kappa\int_\Omega(\nabla e_t)^2\mathrm{d}x+\int_\Omega(P\mathrm{div}\,v)_t\cdot e_t\mathrm{d}x\\&=\int_\Omega[\mu(\nabla v+\nabla^{\mathrm{T}}v):(\nabla v_t+\nabla^{\mathrm{T}}v_t)+2\lambda\mathrm{div}\,v\mathrm{div}\,v_t\\&\quad+(\rho h)_t-\rho_tv\nabla e-\rho v_t\nabla e+\mathrm{div}(\rho v)e_t]\cdot e_t\mathrm{d}x,\end{aligned}$$

and hence

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho e_t^2\mathrm{d}x+\kappa\int_\Omega(\nabla e_t)^2\mathrm{d}x\\&\le C\int_\Omega(|\rho_t||v||\nabla e||e_t|+\rho|v_t||\nabla e||e_t|+\rho|v||\nabla e_t||e_t|+|P_t||\nabla v||e_t|\\&\quad+\rho|e||\nabla v_t||e_t|+|\nabla v||\nabla v_t||e_t|+|\rho_t||h||e_t|)\mathrm{d}x+\int_\Omega h_t\rho e_t\mathrm{d}x\\&=\sum_{i=1}^8\mathrm{I}_i.\end{aligned} \tag{2.16}$$

Making use of (2.9)–(2.11), we can estimate each term $\mathrm{I}_i,\ 1\le i\le 8$, as follows:

$$\begin{aligned}\mathrm{I}_1&=\int_\Omega|\rho_t||v||\nabla e||e_t|\mathrm{d}x\le\|\rho_t\|_{L^3(\Omega)}\|v\|_{L^\infty(\Omega)}\|\nabla e\|_{L^2(\Omega)}\|e_t\|_{L^6(\Omega)}\\&\le Cc_2^3\|\nabla e\|_{L^2(\Omega)}(\|e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)}).\end{aligned}$$

To estimate $\|e_t\|_{L^2(\Omega)}$, we use the following Poincaré type inequality (see [8]):

$$\|e_t\|_{L^2(\Omega)}\le C\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+Cc_0\|\nabla e_t\|_{L^2(\Omega)} \tag{2.17}$$

so that

$$
\begin{aligned}
\mathrm{I}_1 &\le Cc_0c_2^3\|\nabla e\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^2c_2^6\|\nabla e\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2+C\int_\Omega \rho e_t^2\mathrm{d}x,\\
\mathrm{I}_2 &= \int_\Omega \rho|v_t||\nabla e||e_t|\mathrm{d}x\\
&\le \|\rho\|_{L^\infty(\Omega)}\|v_t\|_{L^6(\Omega)}\|\nabla e\|_{L^2(\Omega)}\|e_t\|_{L^3(\Omega)}\\
&\le Cc_0\|\nabla v_t\|_{L^2(\Omega)}\|\nabla e\|_{L^2(\Omega)}(\|e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^2\|\nabla v_t\|_{L^2(\Omega)}\|\nabla e\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^4\|\nabla e\|_{L^2(\Omega)}^2\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\|\nabla v_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2\|\nabla e\|_{L^2(\Omega)}^2,\\
\mathrm{I}_3 &= \int_\Omega \rho|v||\nabla e_t||e_t|\mathrm{d}x\\
&\le \|\rho\|_{L^\infty(\Omega)}^{\frac12}\|v\|_{L^\infty(\Omega)}\|\nabla e_t\|_{L^2(\Omega)}\|\sqrt{\rho}e_t\|_{L^2(\Omega)}\\
&\le Cc_0^{\frac12}c_2\|\nabla e_t\|_{L^2(\Omega)}\|\sqrt{\rho}e_t\|_{L^2(\Omega)}\\
&\le Cc_0c_2^2\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2,\\
\mathrm{I}_4 &= \int_\Omega |P_t||\nabla v||e_t|\mathrm{d}x=\int_\Omega |\rho_t||e||\nabla v||e_t|+\rho|e_t||\nabla v||e_t|\mathrm{d}x\\
&\le C\|\rho_t\|_{L^3(\Omega)}\|\nabla v\|_{L^3(\Omega)}\|e\|_{L^6(\Omega)}\|e_t\|_{L^6(\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}\int_\Omega \rho e_t^2\mathrm{d}x\\
&\le Cc_2^3\|e\|_{L^6(\Omega)}(\|e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})+C\|v\|_{W^{2,q}(\Omega)}\int_\Omega \rho e_t^2\mathrm{d}x\\
&\le Cc_0c_2^3\|e\|_{H^1(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})+C\|v\|_{W^{2,q}(\Omega)}\int_\Omega \rho e_t^2\mathrm{d}x\\
&\le Cc_0^2c_2^6\|e\|_{H^1(\Omega)}^2+C\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2,\\
\mathrm{I}_5 &= \int_\Omega \rho|e||\nabla v_t||e_t|\mathrm{d}x\\
&\le C\|\rho\|_{L^\infty(\Omega)}\|\nabla v_t\|_{L^2(\Omega)}\|e\|_{L^3(\Omega)}\|e_t\|_{L^6(\Omega)}\\
&\le Cc_0^2\|e\|_{H^1(\Omega)}\|\nabla v_t\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^4\|e\|_{H^1(\Omega)}^2\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\|\nabla v_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2\|e\|_{H^1(\Omega)}^2,\\
\mathrm{I}_6 &= \int_\Omega |\nabla v||\nabla v_t||e_t|\mathrm{d}x\\
&\le C\|\nabla v\|_{L^3(\Omega)}\|\nabla v_t\|_{L^2(\Omega)}\|e_t\|_{L^6(\Omega)}\\
&\le Cc_0\|\nabla v\|_{L^2(\Omega)}^{\frac12}\|\nabla v\|_{H^1(\Omega)}^{\frac12}\|\nabla v_t\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0c_1^{\frac12}c_2^{\frac12}\|\nabla v_t\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^2c_1c_2\|\nabla v_t\|_{L^2(\Omega)}^2+C\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2,\\
\mathrm{I}_7 &= \int_\Omega |\rho_t||h||e_t|\mathrm{d}x\\
&\le \|\rho_t\|_{L^3(\Omega)}\|h\|_{L^2(\Omega)}\|e_t\|_{L^6(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\le Cc_0c_2^2\|h\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^2c_2^4\|h\|_{L^2(\Omega)}^2+C\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2,\\
\mathrm{I}_8&=\int_\Omega|\rho||h_t||e_t|\mathrm{d}x\\
&\le C\|h_t\|_{H^{-1}(\Omega)}\|\rho\|_{L^\infty(\Omega)}\|e_t\|_{H^1(\Omega)}\\
&\le Cc_0^2\|h_t\|_{H^{-1}(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})\\
&\le Cc_0^4\|h_t\|_{H^{-1}(\Omega)}^2+C\|\sqrt{\rho}e_t\|_{L^2(\Omega)}^2+\frac{\kappa}{16}\|\nabla e_t\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Substituting these estimates into (2.16), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho e_t^2\mathrm{d}x+\kappa\int_\Omega|\nabla e_t|^2\mathrm{d}x\\
&\le C(c_0c_2^2+\|v\|_{W^{2,q}(\Omega)}+c_0^4\|e\|_{H^1(\Omega)}^2)\int_\Omega\rho e_t^2\mathrm{d}x+Cc_0^2c_2^6\|e\|_{H^1(\Omega)}^2\\
&\quad+Cc_0^2c_1c_2\|\nabla v_t\|_{L^2(\Omega)}^2+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2\|e\|_{H^1(\Omega)}^2+Cc_0^2c_2^4\|h\|_{L^2(\Omega)}^2+Cc_0^4\|h_t\|_{H^{-1}(\Omega)}^2. \qquad (2.18)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^2+(\nabla e)^2\mathrm{d}x=2\int_\Omega ee_t+\nabla e\cdot\nabla e_t\mathrm{d}x\\
&\le C\|e\|_{L^2(\Omega)}\|e_t\|_{L^2(\Omega)}+C\|\nabla e\|_{L^2(\Omega)}\|\nabla e_t\|_{L^2(\Omega)}\\
&\le Cc_0\|e\|_{L^2(\Omega)}(\|\sqrt{\rho}e_t\|_{L^2(\Omega)}+\|\nabla e_t\|_{L^2(\Omega)})+C\|\nabla e\|_{L^2(\Omega)}\|\nabla e_t\|_{L^2(\Omega)}\\
&\le\frac{\kappa}{2}\|\nabla e_t\|_{L^2(\Omega)}^2+Cc_0^2\|e\|_{H^1(\Omega)}^2+C\int_\Omega\rho e_t^2\mathrm{d}x. \qquad (2.19)
\end{aligned}
$$

Combining this and (2.18), we deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho e_t^2+e^2+(\nabla e)^2\mathrm{d}x+\kappa\int_\Omega|\nabla e_t|^2\mathrm{d}x\\
&\le C(c_0c_2^2+\|v\|_{W^{2,q}(\Omega)})\int_\Omega\rho e_t^2\mathrm{d}x+c_0^4\Big(\int_\Omega\rho e_t^2\mathrm{d}x\Big)^2+(Cc_0^2c_2^6+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2)\|e\|_{H^1(\Omega)}^2\\
&\quad+c_0^4\|e\|_{H^1(\Omega)}^4+Cc_0^2c_1c_2\|\nabla v_t\|_{L^2(\Omega)}^2+Cc_0^2c_2^4\|h\|_{L^2(\Omega)}^2+Cc_0^4\|h_t\|_{H^{-1}(\Omega)}^2. \qquad (2.20)
\end{aligned}
$$

Set $\mathcal{Y}=\int_\Omega\rho e_t^2+e^2+(\nabla e)^2\mathrm{d}x$, then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y}&\le C(c_0^2c_2^6+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2+\|v\|_{W^{2,q}(\Omega)})\mathcal{Y}+c_0^4\mathcal{Y}^2\\
&\quad+Cc_0^2c_1c_2\|\nabla v_t\|_{L^2(\Omega)}^2+Cc_0^2c_2^4\|h\|_{L^2(\Omega)}^2+Cc_0^4\|h_t\|_{H^{-1}(\Omega)}^2.
\end{aligned}
$$

Multiplying the above inequality by $\exp\big(-C\int_0^t c_0^2c_2^6+Cc_0^4\|\nabla v_t\|_{L^2(\Omega)}^2+\|v\|_{W^{2,q}(\Omega)}\mathrm{d}\tau\big)$, and using $\lim\limits_{s\to0^+}\int_\Omega\rho e_t^2(s)\mathrm{d}x\le Cc_0^5$, integrating the resulting inequality, we deduce that

$$
\begin{aligned}
\mathcal{Y}(t)&\le\Big(Cc_0^5+c_0^4\int_0^t\mathcal{Y}^2\mathrm{d}s+Cc_0^2c_1^2c_2+Cc_0^4c_2^4\Big)\exp(Cc_1c_0^4)\\
&\le\Big(Cc_0^5c_1^2c_2^4+c_0^4\int_0^t\mathcal{Y}^2\mathrm{d}s\Big)\exp(Cc_1c_0^4)
\end{aligned}
$$

for $0 \le t \le T^* \wedge T_2 \wedge T_3$, where $c_0^{-2}c_2^{-6} = T_3 < T_2 = c_0^{-4}c_2^{-4} < T_1 = c_2^{-1}$. Now let $t \le T^* \wedge T_4$, where $T_4 \le \frac{1}{2Cc_0^8 c_1^2 c_2^4 \exp(Cc_1c_0^4)}$. Then it is easy to infer that

$$\mathcal{Y}(t) \le Cc_0^5 c_1^2 c_2^4 \exp(Cc_1c_0^4) \le c_2 \exp(Cc_1c_0^4) \tag{2.21}$$

for $0 \le t \le T^* \wedge T_4$.

This proves (2.12).

To obtain further estimates, we use the standard elliptic regularity theory to (2.2) and obtain

$$\begin{aligned}
\|\nabla e\|_{H^1(\Omega)} &\le C(\|\rho h\|_{L^2(\Omega)} + \|\rho e_t\|_{L^2(\Omega)} + \|\rho v\cdot\nabla e\|_{L^2(\Omega)} + \|P\operatorname{div} v\|_{L^2(\Omega)} \\
&\quad + \||\nabla v + \nabla^{\mathrm{T}} v|^2\|_{L^2(\Omega)} + \|(\nabla v)^2\|_{L^2(\Omega)} + \|\nabla e\|_{L^2(\Omega)}) \\
&\le C(\|\rho\|_{W^{1,q}(\Omega)}\|h\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)}^{\frac12}\|\sqrt{\rho}e_t\|_{L^2(\Omega)} + \|\rho\|_{W^{1,q}(\Omega)}\|v\|_{H^2(\Omega)}\|\nabla e\|_{L^2(\Omega)} \\
&\quad + \|\rho\|_{W^{1,q}(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)} + \|\nabla v\|_{L^4(\Omega)}^2 + \|\nabla e\|_{L^2(\Omega)}) \\
&\le Cc_0 + Cc_0^{\frac12}\|\sqrt{\rho}e_t\|_{L^2(\Omega)} + Cc_0c_2\|\nabla e\|_{L^2(\Omega)} + Cc_0c_1\|e\|_{L^2(\Omega)} + c_2^2 + \|\nabla e\|_{L^2(\Omega)} \\
&\le c_2^2\exp(Cc_1c_0^4).
\end{aligned} \tag{2.22}$$

This proves (2.13). Moreover

$$\begin{aligned}
\int_0^t |\nabla e|_{W^{1,q}(\Omega)}^2 \mathrm{d}s &\le C\int_0^t \Big(\|\rho h\|_{L^q(\Omega)}^2 + \|\rho e_t\|_{L^q(\Omega)}^2 + \|\rho v\cdot\nabla e\|_{L^q(\Omega)}^2 + \|P\operatorname{div} v\|_{L^q(\Omega)}^2 \\
&\quad + \Big\|\frac{\mu}{2}|\nabla v + \nabla^{\mathrm{T}} v|^2 + \lambda(\operatorname{div} v)^2\Big\|_{L^q(\Omega)}^2 + \|\nabla e\|_{L^q(\Omega)}^2\Big)\mathrm{d}s \\
&\le C\int_0^t (\|\rho\|_{W^{1,q}(\Omega)}^2\|h\|_{L^q(\Omega)}^2 + \|\rho\|_{W^{1,q}(\Omega)}^2\|e_t\|_{L^q(\Omega)}^2 + \|\rho\|_{W^{1,q}(\Omega)}^2\|v\|_{L^\infty(\Omega)}^2\|\nabla e\|_{L^q(\Omega)}^2 \\
&\quad + \|\rho\|_{W^{1,q}(\Omega)}^2\|\nabla v\|_{L^\infty(\Omega)}^2\|e\|_{L^q(\Omega)}^2 + \|\nabla v\|_{L^{2q}(\Omega)}^4 + \|\nabla e\|_{L^q(\Omega)}^2)\mathrm{d}s \\
&\le C\int_0^t c_0^2(\|h\|_{L^q(\Omega)}^2 + c_0^2\|e_t\|_{L^q(\Omega)}^2 + c_0^2c_2^2\|\nabla e\|_{L^q(\Omega)}^2 \\
&\quad + c_0^2\|v\|_{W^{2,q}(\Omega)}^2\|e\|_{L^q(\Omega)}^2 + \|\nabla e\|_{L^q(\Omega)}^2 + \|\nabla v\|_{W^{1,q}(\Omega)}^4)\mathrm{d}s \\
&\le c_2^6\exp(Cc_1c_0^4)
\end{aligned} \tag{2.23}$$

for $0 \le t \le T^* \wedge T_4$.

This proves (2.14).

Finally, recalling that $P = (\gamma-1)\rho e$, we also deduce from (2.22)–(2.23) that (2.15) holds.

The next lemma gives the estimate on the non-Newtonian gravitational potential.

Lemma 2.3 *Assume that $\rho_0 \ge \delta > 0$ in Ω. Then there exists a unique strong solution Φ to the initial boundary value problem* (2.4)–(2.6) *such that*

$$\|\nabla\Phi\|_{W^{1,2}(\Omega)} \le Cc_0, \quad \|\nabla\Phi_t\|_{L^2(\Omega)} \le Cc_2^2. \tag{2.24}$$

Proof Multiplying (2.4) by Φ and integrating over Ω, we get

$$\begin{aligned}
\int_\Omega |\nabla\Phi|^p \mathrm{d}x &\le \int_\Omega (|\nabla\Phi|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla\Phi\cdot\nabla\Phi \mathrm{d}x \\
&= -\int_\Omega 4\pi g\rho - \frac{m_0}{|\Omega|}\cdot\Phi\mathrm{d}x \le C\int_\Omega \rho\cdot\Phi\mathrm{d}x + C\int_\Omega \Phi\mathrm{d}x.
\end{aligned}$$

If $p > 3$, we have

$$\|\nabla\Phi(t)\|_{L^p(\Omega)} \le C\|\rho(t)\|_{L^1(\Omega)};$$

if $2 < p \le 3$, we have

$$\|\nabla\Phi(t)\|_{L^p(\Omega)} \le C\|\rho(t)\|_{L^{\frac{3p}{4p-3}}(\Omega)} \le C\|\rho(t)\|_{L^{\frac{6}{5}}(\Omega)};$$

combining above inequalities, we obtain

$$\|\nabla\Phi(t)\|_{L^p(\Omega)} \le C\|\rho(t)\|_{L^{\frac{6}{5}}(\Omega)} \le Cc_0, \quad p > 2.$$

Next, differentiating (2.4) with respect to time, multiplying it by Φ_t and integrating over Ω, we get

$$\begin{aligned}\varepsilon^{\frac{p-2}{2}}\int_\Omega|\nabla\Phi_t|^2\mathrm{d}x &\le \int_\Omega(|\nabla\Phi|^2+\varepsilon)^{\frac{p-4}{2}}[(p-1)|\nabla\Phi|^2+\varepsilon]\nabla\Phi_t^2\mathrm{d}x\\ &= \int_\Omega[(|\nabla\Phi|^2+\varepsilon)^{\frac{p-2}{2}}\cdot\nabla\Phi]_t\cdot\nabla\Phi_t\mathrm{d}x\\ &= -\int_\Omega 4\pi g\rho_t\Phi_t\mathrm{d}x \le C\|\rho(t)\|_{L^2(\Omega)} + C_\varepsilon\|\nabla\Phi_t\|_{L^2(\Omega)}.\end{aligned}$$

Thus, we get

$$\int_\Omega|\nabla\Phi_t|^2\mathrm{d}x \le Cc_2^2.$$

Finally, let us estimate $\|\nabla\Phi(t)\|_{H^1(\Omega)}$. We consider (2.4)

$$\varepsilon^{\frac{p-2}{2}}|\nabla\Phi_x| \le \operatorname{div}[(|\nabla\Phi|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\Phi] = 4\pi g\Big(\rho - \frac{m_0}{|\Omega|}\Big),$$

so $\|\nabla\Phi(t)\|_{H^1(\Omega)} \le Cc_0$.

The next lemma gives the estimate on the velocity, it was proved in [2, 14].

Lemma 2.4 *Assume further that $\rho_0 \ge \delta$ in Ω for some constant $\delta > 0$. Then there exists a unique strong solution u to the initial boundary value problem* (2.3) *and* (2.5)–(2.6) *such that*

$$\|u(t)\|^2_{H_0^1(\Omega)} + \|\sqrt{\rho}u_t(t)\|^2_{L^2(\Omega)} + \int_0^t\|u_t(s)\|^2_{H^1(\Omega)}\mathrm{d}s \le Cc_0^5, \tag{2.25}$$

$$\|\nabla u(t)\|_{H^1(\Omega)} \le c_2^{\frac{1}{2}}\exp(Cc_0^4c_1), \tag{2.26}$$

$$\int_0^t\|u(s)\|^2_{W^{2,q}(\Omega)}\mathrm{d}s \le Cc_0^7 \tag{2.27}$$

for $0 \le t \le T^* \wedge T_5$, where $T_5 = c_2^{-5}\exp(-Cc_0^4c_1) \wedge Cc_2^{-6}$.

Let us define c_1, β and c_2 by $c_1 = Cc_0^7, c_2 = 2\exp(2C^2c_0^{11})$ and $\beta = \frac{c_2}{c_1} = \frac{2}{C}c_0^{-7}\exp(2C^2c_0^{11})$. Then we conclude from Lemmas 2.1–2.4 that

$$(\|\rho(t)\|_{W^{1,q}(\Omega)} + \|\Phi(t)\|_{W^{2,2}(\Omega)} + \|\rho_t(t)\|_{L^q(\Omega)} + \|\Phi_t(t)\|_{W^{1,2}(\Omega)} + \|e(t)\|_{H^2(\Omega)}) \le Cc_2^7, \tag{2.28}$$

$$\|(\sqrt{\rho}u_t, \sqrt{\rho}e_t)(t)\|_{L^2(\Omega)} + \int_0^t(\|e_t(s)\|^2_{H^1(\Omega)} + \|e(s)\|^2_{W^{2,q}(\Omega)})\mathrm{d}s \le Cc_2^7, \tag{2.29}$$

$$\|u(t)\|_{H_0^1(\Omega)} + \beta^{-1}\|u(t)\|_{H^2(\Omega)} + \int_0^t(\|u_t(s)\|^2_{H_0^1(\Omega)} + \|u(s)\|^2_{W^{2,q}(\Omega)})\mathrm{d}s \le c_1 \tag{2.30}$$

for $0 \le t \le T^* \wedge T_5$.

Now using the same proofs as that in [2, 14], we obtain the following lemma.

Lemma 2.5 *There exists a unique strong solution* (ρ, e, u, Φ) *to the linearized problem* (2.1)–(2.6) *in* $[0, T_*]$ *satisfying the estimates* (2.28)–(2.30) *as well as the regularity*

$$\begin{aligned}
&\rho \in C([0,T_*];W^{1,q}(\Omega)), \quad \rho_t \in C([0,T_*];L^q(\Omega)),\\
&\Phi \in C([0,T_*];W^{2,2}(\Omega)), \quad \Phi_t \in C([0,T_*];W^{1,2}(\Omega)),\\
&(e,u) \in C([0,T_*];H^2(\Omega)) \cap L^2(0,T_*;W^{2,q}(\Omega)),\\
&(e_t,u_t) \in L^2(0,T_*;H^1(\Omega)), \quad (\sqrt{\rho}e_t, \sqrt{\rho}u_t) \in L^\infty(0,T_*;L^2(\Omega)),
\end{aligned}$$

where $T_* = T^* \wedge T_5$.

3 An Existence Result for Polytropic Fluids with $\kappa > 0$

This section is devoted to proving the existence of a unique local solution with minimal regularity when $\kappa > 0$.

Theorem 3.1 *Let* $\kappa > 0$, *and assume that the initial data* $(\rho_0, e_0, u_0, \Phi_0)$ *satisfies* (2.7)–(2.8). *Then there exists a small time* $\widehat{T} > 0$ *and a unique strong solution* (ρ, e, u, Φ) *to the initial boundary value problem* (1.1)–(1.8) *such that*

$$\begin{aligned}
&\rho \in C([0,\widehat{T}];W^{1,q}(\Omega)), \quad \rho_t \in C([0,\widehat{T}];L^q(\Omega)),\\
&\Phi \in C([0,\widehat{T}];W^{2,2}(\Omega)), \quad \Phi_t \in C([0,\widehat{T}];W^{1,2}(\Omega)),\\
&(e,u) \in C([0,\widehat{T}];H^2(\Omega)) \cap L^2(0,\widehat{T};W^{2,q}(\Omega)),\\
&(e_t,u_t) \in L^2(0,\widehat{T};H^1(\Omega)), \quad (\sqrt{\rho}e_t, \sqrt{\rho}u_t) \in L^\infty(0,\widehat{T};L^2(\Omega)).
\end{aligned} \tag{3.1}$$

Proof Our proof will be based on the usual iteration argument and on the results (in particular, Lemma 2.5) in the last section.

Let $u^0 \in C([0,\infty);H_0^1(\Omega)\cap H^2(\Omega))\cap L^2(0,\infty;H^3(\Omega))$ be the solution to the linear parabolic problem

$$\begin{cases}
\omega_t - \Delta\omega = 0 & \text{in } (0,\infty)\times\Omega,\\
\omega(0) = u_0 & \text{in } \Omega,\\
\omega = 0 & \text{on } (0,\infty)\times\partial\Omega.
\end{cases}$$

Then we have

$$\sup_{0\le t\le T_*}(\|u^0(t)\|_{H_0^1(\Omega)} + \beta^{-1}\|u^0(t)\|_{H^2(\Omega)}) + \int_0^{T_*}(\|u_t^0(t)\|_{H_0^1(\Omega)}^2 + \|u^0(t)\|_{W^{2,q}(\Omega)}^2)\mathrm{d}t \le c_1.$$

Hence it follows from Lemma 2.5 that there exists a unique strong solution $(\rho^1, u^1, e^1, \Phi^1)$ to the linearized problem (2.1)–(2.6) with v replaced by u^0, which satisfies the regularity estimates (2.28)–(2.30). Similarly, we construct approximate solutions $(\rho^k, u^k, e^k, \Phi^k)$, inductively, as follows, assuming that u^{k-1} was defined for $k \ge 1$. Let $(\rho^k, u^k, e^k, \Phi^k)$ be the unique solution to the problem (2.1)–(2.6) with v replaced by u^{k-1}. Then it also follows from Lemma 2.5 that there exists a constant $\widetilde{C} > 1$ such that

$$\sup_{0\le t\le T_*}(\|\rho^k(t)\|_{W^{1,q}(\Omega)} + \|\Phi^k(t)\|_{W^{2,2}(\Omega)} + \|\rho^k_t(t)\|_{L^q(\Omega)} + \|\Phi^k_t(t)\|_{W^{1,2}(\Omega)})$$
$$+ \sup_{0\le t\le T_*}(\|e^k(t)\|_{H^2(\Omega)} + \|u^k(t)\|_{H^1_0(\Omega)\cap H^2(\Omega)}) + \sup_{0\le t\le T_*}\|(\sqrt{\rho^k}u^k_t, \sqrt{\rho^k}e^k_t)(t)\|_{L^2(\Omega)}$$
$$+ \int_0^{T_*}\|(e^k_t, u^k_t)(t)\|^2_{H^1(\Omega)} + \|(e^k, u^k)(t)\|^2_{W^{2,q}(\Omega)}\mathrm{d}t \le \widetilde{C} \tag{3.2}$$

for all $k \ge 1$. Throughout the proof, we denote by $\widetilde{C}$ a generic constant depending only on c_0 and the parameters of the constant C, but independent of k.

From now on, we show that the full sequence $(\rho^k, u^k, e^k, \Phi^k)$ converges to a solution to the original nonlinear problem (1.1)–(1.8) in a strong sense.

Let us define

$$\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \overline{e}^{k+1} = e^{k+1} - e^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k, \quad \overline{\Phi}^{k+1} = \Phi^{k+1} - \Phi^k.$$

Then from (2.1)–(2.4), we derive the equations for the differences

$$\overline{\rho}^{k+1}_t + \mathrm{div}(\overline{\rho}^{k+1}u^k) + \mathrm{div}(\rho^k\overline{u}^k) = 0, \tag{3.3}$$

$$\begin{aligned}&\rho^{k+1}\overline{e}^{k+1}_t + \rho^{k+1}u^k\cdot\nabla\overline{e}^{k+1} - \kappa\Delta\overline{e}^{k+1} = \frac{\mu}{2}(\nabla u^k + \nabla^{\mathrm{T}}u^k) : (\nabla u^k + \nabla^{\mathrm{T}}u^k)\\
&+ \lambda(\mathrm{div}\, u^k)^2 - \frac{\mu}{2}(\nabla u^{k-1} + \nabla^{\mathrm{T}}u^{k-1}) : (\nabla u^{k-1} + \nabla^{\mathrm{T}}u^{k-1}) - \lambda(\mathrm{div}\, u^{k-1})^2\\
&- \overline{\rho}^{k+1}e^k_t + \overline{\rho}^{k+1}(h - u^{k-1}\cdot\nabla e^k - (\gamma-1)e^k\mathrm{div}\, u^{k-1})\\
&- \rho^{k+1}(\overline{u}^k\cdot\nabla e^k + (\gamma-1)\overline{e}^{k+1}\mathrm{div}\, u^k + (\gamma-1)e^k\mathrm{div}\,\overline{u}^k),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\rho^{k+1}\overline{u}^{k+1}_t + \rho^{k+1}u^k\cdot\nabla\overline{u}^{k+1} - \mu\Delta\overline{u}^{k+1} - (\lambda+\mu)\nabla(\mathrm{div}\,\overline{u}^{k+1})\\
&= \overline{\rho}^{k+1}(f - \nabla\Phi^{k+1} - u^k_t - u^{k-1}\cdot\nabla u^k) - \rho^k\nabla\overline{\Phi}^{k+1} - \rho^{k+1}\overline{u}^k\cdot\nabla u^k\\
&- (\gamma-1)\nabla(\rho^{k+1}\overline{e}^{k+1} + \overline{\rho}^{k+1}e^k),\end{aligned} \tag{3.5}$$

$$\mathrm{div}[(|\nabla\Phi^{k+1}|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla\Phi^{k+1} - (|\nabla\Phi^k|^2 + \varepsilon)^{\frac{p-2}{2}}\nabla\Phi^k] = 4\pi g\overline{\rho}^{k+1}. \tag{3.6}$$

Multiplying (3.3) by $\overline{\rho}^{k+1}$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\overline{\rho}^{k+1}|^2\mathrm{d}x &= -\int_\Omega\mathrm{div}(\overline{\rho}^{k+1}u^k)\overline{\rho}^{k+1}\mathrm{d}x - \int_\Omega\mathrm{div}(\rho^k\overline{u}^k)\overline{\rho}^{k+1}\mathrm{d}x\\
&\le C\int_\Omega|\nabla u^k||\overline{\rho}^{k+1}|^2 + |\nabla\rho^k||\overline{u}^k||\overline{\rho}^{k+1}| + |\rho^k||\nabla\overline{u}^k||\overline{\rho}^{k+1}|\mathrm{d}x\\
&\le C(\|\nabla u^k\|_{W^{1,q}(\Omega)}\|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)}\\
&\quad + (\|\nabla\rho^k\|_{L^3(\Omega)} + \|\rho^k\|_{L^\infty(\Omega)})\|\nabla\overline{u}^k\|_{L^2(\Omega)}\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}).\end{aligned}$$

Hence, by virtue of Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)} \le A^k_\varepsilon(t)\|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\overline{u}^k\|^2_{L^2(\Omega)}, \tag{3.7}$$

where

$$A^k_\varepsilon(t) = \varepsilon\|\nabla u^k(t)\|_{W^{1,q}(\Omega)} + C\varepsilon^{-1}(\|\nabla\rho^k(t)\|^2_{L^3(\Omega)} + \|\rho^k(t)\|^2_{L^\infty(\Omega)}). \tag{3.8}$$

Multiplying (3.4) by $\overline{e}^{k+1}$, integrating over Ω and recalling that

$$\partial_t\rho^{k+1}+\operatorname{div}(\rho^{k+1}u^k)=0, \tag{3.9}$$

and using the Poincaré type inequality (2.17), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{e}^{k+1}|^2\mathrm{d}x+\kappa\int_\Omega|\nabla\overline{e}^{k+1}|^2\mathrm{d}x\\
&\le C\int_\Omega |\overline{\rho}^{k+1}||e_t^k||\overline{e}^{k+1}|+|\overline{\rho}^{k+1}|(|u^{k-1}||\nabla e^k|+|e^k||\operatorname{div}u^{k-1}|)|\overline{e}^{k+1}|\\
&\quad+\rho^{k+1}(|\overline{u}^k||\nabla e^k|+|\overline{e}^{k+1}||\operatorname{div}u^k|+|e^k||\operatorname{div}\overline{u}^k|)|\overline{e}^{k+1}|\\
&\quad+(|\nabla u^k|+|\nabla u^{k-1}|)|\nabla\overline{u}^k||\overline{e}^{k+1}|+|\overline{\rho}^{k+1}||h||\overline{e}^{k+1}|\mathrm{d}x\\
&\le C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|e_t^k\|_{H^1(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})\\
&\quad+C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|u^{k-1}\|_{L^6(\Omega)}\|\nabla e^k\|_{L^6(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})\\
&\quad+C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|e^k\|_{L^\infty(\Omega)}\|\nabla u^{k-1}\|_{L^3(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})\\
&\quad+C\|\rho^{k+1}\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}\|\overline{u}^k\|_{L^6(\Omega)}\|\nabla e^k\|_{L^3(\Omega)}+C\|\nabla u^k\|_{L^\infty(\Omega)}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}^2\\
&\quad+\|\rho^{k+1}\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}\|e^k\|_{L^\infty(\Omega)}\|\nabla\overline{u}^k\|_{L^2(\Omega)}\\
&\quad+C(\|\nabla u^k\|_{L^3(\Omega)}+\|\nabla u^{k-1}\|_{L^3(\Omega)})\|\nabla\overline{u}^k\|_{L^2(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})\\
&\quad+\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|h\|_{L^3(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}).
\end{aligned}$$

Hence it follows from (3.2) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}^2+\kappa\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}^2\\
&\le C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|e_t^k\|_{H^1(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})\\
&\quad+C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}(\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}+\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)})+\|\nabla\overline{u}^k\|_{L^2(\Omega)}\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}\\
&\quad+C\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}\|\nabla\overline{u}^k\|_{L^2(\Omega)}+C\|\nabla u^k\|_{L^\infty(\Omega)}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

By virtue of Young's inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}^2+\kappa\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}^2\\
&\le B^k(t)(\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}^2+\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}^2)+C\|\nabla\overline{u}^k\|_{L^2(\Omega)}^2,
\end{aligned}\tag{3.10}$$

where

$$B^k(t)=C(1+\|\nabla u^k\|_{L^\infty(\Omega)}+\|e_t^k\|_{H^1(\Omega)}^2). \tag{3.11}$$

Furthermore, multiplying (3.6) by $\overline{\Phi}^{k+1}$ and integrating over Ω, and letting $\Theta(s)=(s^2+\varepsilon)^{\frac{p-2}{2}}s$, we get

$$\begin{aligned}
&\int_\Omega \operatorname{div}[(|\nabla\Phi^{k+1}|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\Phi^{k+1}-(|\nabla\Phi^k|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\Phi^k]\overline{\Phi}^{k+1}\mathrm{d}x\\
&=\int_\Omega\Big\{\int_\Omega\Theta'[\theta\nabla\Phi^{k+1}+(1-\theta)\nabla\Phi^k]\mathrm{d}\theta\Big\}(\nabla\overline{\Phi}^{k+1})^2\mathrm{d}x
\end{aligned}$$

by virtue of

$$\Theta'(s) = [(s^2+\varepsilon)^{\frac{p-2}{2}}s]' = (s^2+\varepsilon)^{\frac{p-4}{2}}[(p-1)s^2+\varepsilon] \ge \varepsilon^{\frac{p-2}{2}}.$$

Thus, we have

$$\int_\Omega |\nabla\overline{\Phi}^{k+1}|^2 \mathrm{d}x \le C\|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)}. \tag{3.12}$$

Furthermore, differentiating (2.4) in which one increases the index k and $k+1$ with respect to time, respectively, multiplying them by Φ^k and Φ^{k+1}, then integrating over Ω, we can easily deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\overline{\Phi}^{k+1}|^p \mathrm{d}x \le C\|\overline{\rho}_t^{k+1}\|^2_{L^2(\Omega)} + C\|\nabla\overline{\Phi}^{k+1}\|^2_{L^2(\Omega)}. \tag{3.13}$$

Finally, multiplying (3.5) by $\overline{u}^{k+1}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \mu\int_\Omega |\nabla\overline{u}^{k+1}|^2\mathrm{d}x + (\lambda+\mu)\int_\Omega |\mathrm{div}\,\overline{u}^{k+1}|^2\mathrm{d}x \\
&\le C\int_\Omega |\overline{\rho}^{k+1}|(|f| + |\nabla\Phi^{k+1}| + |u_t^k| + |u^{k-1}\cdot\nabla u^k|)|\overline{u}^{k+1}| + |\rho^k||\nabla\overline{\Phi}^{k+1}||\overline{u}^{k+1}| \\
&\quad + |\rho^{k+1}||\overline{u}^k||\nabla u^k||\overline{u}^{k+1}| + (|\rho^{k+1}||\overline{e}^{k+1}| + |\overline{\rho}^{k+1}||e^k|)|\nabla\overline{u}^{k+1}|.
\end{aligned}$$

Then it follows from (3.2) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|^2_{L^2(\Omega)} + \mu\|\nabla\overline{u}^{k+1}\|^2_{L^2(\Omega)} \\
&\le C\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}(\|f\|_{L^3(\Omega)} + \|\nabla\Phi^{k+1}\|_{L^3(\Omega)} + \|u_t^k\|_{L^3(\Omega)} \\
&\quad + \|u^{k-1}\|_{L^\infty(\Omega)}\|\nabla u^k\|_{L^3(\Omega)})\|\overline{u}^{k+1}\|_{L^6(\Omega)} + \|\rho^k\|_{L^3(\Omega)}\|\nabla\overline{\Phi}^{k+1}\|_{L^2(\Omega)}\|\overline{u}^{k+1}\|_{L^6(\Omega)} \\
&\quad + \|\rho^{k+1}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\overline{u}^k\|_{L^6(\Omega)}\|\nabla u^k\|_{L^3(\Omega)}\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2(\Omega)} \\
&\quad + \|\rho^{k+1}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)} + \|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|e^k\|_{L^\infty(\Omega)}\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)} \\
&\le C(1+\|u_t^k\|_{L^3(\Omega)})\|\overline{\rho}^{k+1}\|_{L^2(\Omega)}\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)} + C\|\nabla\overline{\Phi}^{k+1}\|_{L^2(\Omega)}\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)} \\
&\quad + C\|\nabla\overline{u}^k\|_{L^2(\Omega)}\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2(\Omega)} + \|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|_{L^2(\Omega)}.
\end{aligned}$$

Hence by virtue of Young's inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|^2_{L^2(\Omega)} + \mu\|\nabla\overline{u}^{k+1}\|^2_{L^2(\Omega)} \\
&\le C_\varepsilon^k(t)(\|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)} + \|\nabla\overline{\Phi}^{k+1}\|^2_{L^2(\Omega)} + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|^2_{L^2(\Omega)}) \\
&\quad + C\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\overline{u}^k\|^2_{L^2(\Omega)},
\end{aligned} \tag{3.14}$$

where

$$C_\varepsilon^k(t) = C\Big(1 + \frac{1}{\varepsilon} + \|u_t^k\|^2_{L^3(\Omega)}\Big). \tag{3.15}$$

Therefore, combining (3.7), (3.10), (3.13)–(3.14) and defining

$$\Psi^{k+1} = \|\overline{\rho}^{k+1}\|^2_{L^2(\Omega)} + \|\nabla\overline{\Phi}^{k+1}\|^p_{L^p(\Omega)} + \frac{\varepsilon}{\kappa}\|\sqrt{\rho^{k+1}}\overline{e}^{k+1}\|^2_{L^2(\Omega)} + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|^2_{L^2(\Omega)},$$

we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{k+1}+\varepsilon\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}^2+\frac{\mu}{2}\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)}^2+\|\nabla\overline{\Phi}^{k+1}\|_{L^2(\Omega)}^2\\&\le E_\varepsilon^k(t)\Psi^{k+1}+3\varepsilon\widetilde{C}\|\nabla\overline{u}^k\|_{L^2(\Omega)}^2,\end{aligned}\tag{3.16}$$

where

$$E_\varepsilon^k(t)=\widetilde{C}+A_\varepsilon^k(t)+B^k(t)+C_\varepsilon^k(t).\tag{3.17}$$

From (3.2), (3.8), (3.11) and (3.15), we see

$$\int_0^t E_\varepsilon^k(s)\mathrm{d}s\le\widetilde{C}+\widetilde{C}t+\widetilde{C}\varepsilon+\widetilde{C}\varepsilon t.$$

Hence choose $T_{**}\le\varepsilon<1$ so that we easily deduce

$$\begin{aligned}&\Psi^{k+1}+\int_0^t\varepsilon\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}^2+\mu\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)}^2+\|\nabla\overline{\Phi}^{k+1}\|_{L^2(\Omega)}^2\mathrm{d}s\\&\le\varepsilon\exp(\widetilde{C})\int_0^t\|\nabla\overline{u}^k\|_{L^2(\Omega)}^2\mathrm{d}s\end{aligned}\tag{3.18}$$

for $0\le t\le\widehat{T}=T_*\wedge T_{**}$.

Now taking ε so that $\varepsilon\exp(\widetilde{C})\le\frac{\mu}{2}$ and hence

$$\begin{aligned}&\sum_{k=1}^\infty\sup_{0\le t\le\widehat{T}}\Psi^{k+1}(t)+\sum_{k=1}^\infty\int_0^t\varepsilon\|\nabla\overline{e}^{k+1}\|_{L^2(\Omega)}^2\\&+\mu\|\nabla\overline{u}^{k+1}\|_{L^2(\Omega)}^2+\|\nabla\overline{\Phi}^{k+1}\|_{L^2(\Omega)}^2\mathrm{d}t\le\widetilde{C}<\infty.\end{aligned}\tag{3.19}$$

Therefore, we conclude that the full sequence (ρ^k,e^k,u^k,Φ^k) converges to a limit (ρ,e,u,Φ) in the following strong sense

$$\begin{aligned}&\rho^k\to\rho\quad\text{in }L^\infty(0,\widehat{T};L^2(\Omega)),\qquad\Phi^k\to\Phi\quad\text{in }L^2(0,\widehat{T};H^1(\Omega)),\\&\Phi^k\to\Phi\quad\text{in }L^\infty(0,\widehat{T};W^{1,p}(\Omega)),\quad(e^k,u^k)\to(e,u)\quad\text{in }L^2(0,\widehat{T};H^1(\Omega)).\end{aligned}\tag{3.20}$$

It is easy to prove that the limit (ρ,e,u,Φ) is a weak solution to the original nonlinear problem (1.1)–(1.8). Furthermore, it follows from (3.2) that (ρ,e,u,Φ) satisfies the following regularity estimate:

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\le t\le\widehat{T}}\|(\sqrt{\rho}u_t,\sqrt{\rho}e_t)(t)\|_{L^2(\Omega)}+\int_0^{\widehat{T}}\|(e_t,u_t)(t)\|_{H^1(\Omega)}^2+\|(e,u)(t)\|_{W^{2,q}(\Omega)}^2\mathrm{d}t\\&+\sup_{0\le t\le\widehat{T}}\big(\|\rho(t)\|_{W^{1,q}(\Omega)}+\|\Phi(t)\|_{W^{2,2}(\Omega)}+\|\rho_t(t)\|_{L^q(\Omega)}+\|\Phi_t(t)\|_{W^{1,2}(\Omega)}\\&+\|e(t)\|_{H^2(\Omega)}+\|u(t)\|_{H_0^1(\Omega)\cap H^2(\Omega)}\big)\le\widetilde{C}.\end{aligned}$$

This proves the existence of a strong solution. Then adapting the argument in [2, 14], we can easily prove the time-continuity of the solution (ρ,e,u,Φ). To prove the uniqueness, let (ρ_1,e_1,u_1,Φ_1) and (ρ_2,e_2,u_2,Φ_2) be two strong solutions to the problem (1.1)–(1.8) with the regularity (3.1) and we denote by $(\overline{\rho}, \overline{e}, \overline{u}, \overline{\Phi}) = (\rho_1 - \rho_2, e_1 - e_2, u_1 - u_2, \Phi_1 - \Phi_2)$. Then following the same arguments as in the derivations of (3.7), (3.10) and (3.13)–(3.14), we can show

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{\rho}\|^2_{L^2(\Omega)} + \|\nabla\overline{\Phi}\|^p_{L^p(\Omega)} + \varepsilon\|\sqrt{\rho_1}\overline{e}\|^2_{L^2(\Omega)} + \|\sqrt{\rho_1}\overline{u}\|^2_{L^2(\Omega)})\\&+\varepsilon\|\nabla\overline{e}\|^2_{L^2(\Omega)} + \mu\|\nabla\overline{u}\|^2_{L^2(\Omega)} + \|\nabla\overline{\Phi}\|^2_{L^2(\Omega)}\\&\leq F(t)(\|\overline{\rho}\|^2_{L^2(\Omega)} + \|\sqrt{\rho_1}\overline{e}\|^2_{L^2(\Omega)} + \|\sqrt{\rho_1}\overline{u}\|^2_{L^2(\Omega)})\end{aligned}$$

for some $F(t) \in L^1(0, \widehat{T})$. Therefore, in view of Gronwall's inequality, we conclude that $\overline{\rho} = \overline{e} = \overline{u} = \overline{\Phi} = 0$ in $(0, \widehat{T}) \times \Omega$, which implies the uniqueness of strong solution. This completes the proof of the Theorem 3.1.

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