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Waring-Goldbach Problem: One Square and Nine Biquadrates*

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Abstract In this paper it is proved that every sufficiently large even integer N satisfying one of the congruence conditions $N \equiv 10, 58, 130, \text{or } 178 \pmod{240}$ may be represented as the sum of one square and nine fourth powers of prime numbers.

Keywords Waring-Goldbach problem, Hardy-Littlewood method
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1 Introduction

Let s and k be natural numbers and $k \geq 3$. The Diophantine equation

$$N = x^2 + y_1^k + y_2^k + \cdots + y_s^k \tag{1.1}$$

belongs to the small stock of variants of Waring's problem that have been studied by various writers since the early days of the Hardy-Littlewood method. A heuristical application of that method, based on a major arc analysis only, suggests that the number $R_{k,s}(N)$ of solutions to (1.1) in natural numbers $x, y_1, \cdots, y_s$ satisfies the asymptotic relation

$$R_{k,s}(N) = \frac{\Gamma\Big(\frac{3}{2}\Big)\Gamma\Big(1+\frac{1}{k}\Big)^s}{\Gamma\Big(\frac{1}{2}+\frac{s}{k}\Big)}\mathfrak{S}_{k,s}(N)N^{\frac{s}{k}-\frac{1}{2}}(1+o(1)), \tag{1.2}$$

provided that $s > \frac{1}{2}k$. Here the singular series is defined by

$$\mathfrak{S}_{k,s}(N) = \sum_{q=1}^{\infty} q^{-s-1} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{x=1}^{q} e\Big(\frac{ax^2}{q}\Big)\Big(\sum_{y=1}^{q} e\Big(\frac{ay^k}{q}\Big)\Big)^s e\Big(-\frac{aN}{q}\Big).$$

The first analysis of the problem was made by Stanley [9] in 1930. Following the pattern laid down by Hardy and Littlewood [3–4] in their classic series "Partitio Numerorum", she established the asymptotic formula (1.2) for $s \geq s_1(k)$ where

$$s_1(3) = 7, \quad s_1(4) = 14, \quad s_1(5) = 28, \quad s_1(k) = 2^{k-2}\Big(\frac{1}{2}k-1\Big) + O(k), \quad k > 5.$$

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Later, Sinnadurai [8] verified (1.2) for $R_{3,6}(N)$, and Hooley [6] gave a different proof for this result. Brüdern and Kawada [2] gave a proof of (1.2) for $R_{5,17}(N)$ and $R_{k,s}(N)$, when $k \geq 6$ for $s \geq 7 \cdot 2^{k-4} + 3$.

When $k = 4$, Brüdern [1] proved that every sufficiently large integer can be represented as the sum of one square and nine biquadrates, but he did not get the asymptotic formula for the number of representations.

In view of Brüdern's result, it is reasonable to expect that for every sufficiently large even integer N satisfying one of the congruence conditions $N \equiv 10, 58, 130,$ or $178 \pmod{240}$, the equation

$$N = p^2 + p_1^4 + p_2^4 + \cdots + p_9^4 \tag{1.3}$$

is solvable, where and below, the letter p, with or without subscript, always denotes a prime. The congruence conditions are necessary here, because for prime $p > 5$, we have

$$p^4 \equiv 1 \pmod{240}$$

and

$$p^2 \equiv 1 \text{ or } 49 \text{ or } 121 \text{ or } 169 \pmod{240}.$$

Motivated by [7], the Hardy-Littlewood method enables us to obtain the following result.

Theorem 1.1 *Every sufficiently large even integer N satisfying one of the congruence conditions $N \equiv 10, 58, 130,$ or $178 \pmod{240}$ may be represented as the sum of one square and nine fourth powers of prime numbers.*

2 Notation and Some Lemmas

As usual, $\varphi(n)$ stands for the Euler function and $\tau(n)$ for divisor function. The letter ε denotes positive constant which is arbitrarily small. Suppose that x is a sufficiently large positive number. Consider an integer n with $n \equiv 7 \pmod{240}$, $x < n \leq 2x$, and write

$$P = \frac{1}{2}x^{\frac{1}{4}}.$$

We put

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \frac{13}{16}, \quad \lambda_4 = \left(\frac{13}{16}\right)^2,$$
$$\lambda_5 = \left(\frac{13}{16}\right)^2 \frac{91}{111}, \quad \lambda_6 = \lambda_7 = \left(\frac{13}{16}\right)^2 \frac{78}{111}$$

and

$$P_j = P^{\lambda_j}, \quad 1 \leq j \leq 7.$$

Let $r(n)$ denote the number of representations of n in the form

$$n = p_1^4 + p_2^4 + \cdots + p_7^4$$

with

$$P_j < p_j \le 2P_j, \quad 1 \le j \le 7.$$

We define the exponential sum

$$g_j(\alpha) = \sum_{P_j < p \le 2P_j} e(\alpha p^4)$$

and

$$S^*(q,a) = \sum_{\substack{n=1 \\ (n,a)=1}}^{q} e\Big(\frac{an^4}{q}\Big), \quad u_j(\beta) = \int_{P_j}^{2P_j} \frac{e(\beta t^4)}{\log t} \mathrm{d}t, \quad 1 \le j \le 7.$$

We have

$$r(n) = \int_0^1 \Big(\prod_{j=1}^{7} g_j(\alpha)\Big) e(-\alpha n) \mathrm{d}\alpha. \tag{2.1}$$

Let $L = (\log P)^B$, where B is a sufficiently large positive constant which will be determined later and

$$\mathfrak{M}(q,a) = \Big(\frac{a}{q} - \frac{P^{\frac{1}{4}}}{qP^4}, \frac{a}{q} + \frac{P^{\frac{1}{4}}}{qP^4}\Big], \quad \mathfrak{M} = \bigcup_{1 \le q \le P^{\frac{1}{4}}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q,a),$$
$$\mathfrak{m} = (0,1] \setminus \mathfrak{M}.$$

We define the multiplicative function $\omega(q)$ by taking

$$\omega(p^{4u+v}) = \begin{cases} 4p^{-u-\frac{1}{2}}, & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le 4. \end{cases}$$

Note that

$$q^{-\frac{1}{2}} \le \omega(q) \ll q^{-\frac{1}{4}}.$$

Lemma 2.1 *For $(q,a) = 1$, we have*

(i) $S^*(q,a) \ll q^{\frac{1}{2}+\varepsilon}$.

In particular, for $(p,a) = 1$ we have

(ii) $|S^*(p,a)| \le ((k, p-1) - 1)p^{\frac{1}{2}} + 1$,

(iii) $S^*(p^l, a) = 0$ *for* $l \ge \gamma(p)$,

where

$$\gamma(p) = \begin{cases} \theta + 2, & \text{if } p^\theta \| k,\ p \ne 2 \text{ or } p = 2,\ \theta = 0, \\ \theta + 3, & \text{if } p^\theta \| k,\ p = 2,\ \theta > 0. \end{cases}$$

Proof For (ii), see [11, Lemma 4.3]. For (i), see Chapter VI, Problem 14 in [12]. For (iii), see [5, Lemma 8.3].

Lemma 2.2 *We have*

$$\int_{\mathfrak{M}} \Psi(\alpha)^3 |g_6(\alpha)|^2 \mathrm{d}\alpha \ll Q^2 P^{-4},$$

where

$$\Psi(\alpha) = \frac{q^{4\varepsilon}\omega(q)}{1+P^4|\alpha - \frac{a}{q}|} \qquad \text{for } \alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}.$$

Proof See [7, Lemma 2.5].

Lemma 2.3 *For $\alpha \in \mathfrak{M}(q,a),\ 1 \le q \le \frac{P^2}{2},\ (a,q)=1,$ we have*

$$\sum_{P<p\le 2P} e(\alpha p^4) \ll P^{1-\frac{1}{32}+\varepsilon} + \frac{q^{\varepsilon}\omega(q)^{\frac12} P(\log P)^4}{\left(1+P^4|\alpha - \dfrac{a}{q}|\right)^{\frac12}}.$$

Proof See [7, Lemma 3.3].

Lemma 2.4 *We have*

$$\int_0^1 \Big|\prod_{j=2}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha \ll P^{\varepsilon} \prod_{j=2}^{7} P_j.$$

Proof See [7, Lemma 4.3].

Lemma 2.5 *For $1 \le i \le 7$, we have*

$$\int_0^1 \Big|g_i(\alpha)\prod_{j=2}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha \ll P_i^2 \Big(\prod_{j=2}^{7} P_j\Big)^2 P^{-4} (\log P)^2.$$

Proof See [7, Lemma 4.4].

3 Auxiliary Estimates

We introduce $v(x)$ to denote the set of integers $n \equiv 7 \pmod{240}$ with $x < n \le 2x$ such that

$$|r(n) - \mathfrak{S}(n)\mathfrak{J}(n)| > \Big(\prod_{j=1}^{7} P_j\Big) P^{-4} (\log P)^{-8}, \tag{3.1}$$

where

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{(S^*(q,a))^7}{\varphi^7(q)} e\Big(-\frac{a}{q}n\Big)$$

and

$$\mathfrak{J}(n) = \int_{-\infty}^{\infty} \Big(\prod_{j=1}^{7} u_j(\beta)\Big) e(-\beta n) \mathrm{d}\beta.$$

Lemma 3.1 (i) *The singular series* $\mathfrak{S}(n)$ *is convergent and* $\mathfrak{S}(n) > 0$.

(ii) *The singular integral*

$$\mathfrak{J}(n) \asymp \Big(\prod_{j=1}^{7} P_j\Big) P^{-4} (\log P)^{-7}.$$

Proof Let $L(q,n)$ denote the number of solutions of the congruence

$$u_1^4 + u_2^4 + \cdots + u_7^4 \equiv n \ (\operatorname{mod} q), \quad 1 \le u_j \le q,\ (u_j, q) = 1.$$

Then we have

$$\begin{aligned} pL(p,n) &= \sum_{a=1}^{p} S^*(p,a)^7 e\Big(-\frac{a}{p}n\Big) \\ &= (p-1)^7 + E_p, \end{aligned}$$

where

$$E_p = \sum_{a=1}^{p-1} S^*(p,a) e\Big(-\frac{a}{p}n\Big).$$

By Lemma 4.3 in [11] and Lemma 2.1 (ii), we have

$$|E_p| \le (p-1)(3\sqrt{p}+1)^7.$$

It is easy to see that

$$|E_p| < (p-1)^7 \quad \text{for } p \ge 29.$$

For $p = 2, 3, 5, \cdots, 23$, we can verify by hand that $L(p,n) > 0$. Hence we have $L(p,n) > 0$ for every prime.

Let

$$A(q,n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{(S^*(q,a))^7}{\varphi^7(q)} e\Big(-\frac{a}{q}n\Big).$$

Note the fact that $A(q,n)$ is multiplicative in q and by Lemma 2.1(iii), we have

$$\mathfrak{S}(n) = \Big(1 + \sum_{i=1}^{4} A(2^i, n)\Big) \prod_{p \ge 3} (1 + A(p,n)). \tag{3.2}$$

By Lemma 2.1(ii), for $p \ge 1000$, we have

$$|A(p,n)| \le \frac{(p-1)(3\sqrt{p}+1)^7}{(p-1)^7} \le \frac{1000}{p^2}$$

and

$$\prod_{p \ge 1000} (1 + A(p,n)) \ge \prod_{p \ge 1000} \Big(1 - \frac{1000}{p^2}\Big) > c > 0. \tag{3.3}$$

It is easy to verify that

$$1+\sum_{i=1}^{4}A(2^i,n)=\frac{L(16,n)}{\varphi^7(16)} \tag{3.4}$$

and

$$1+A(p,n)=\frac{L(p,n)}{(p-1)^7}\quad \text{for } p\neq 2. \tag{3.5}$$

Now by (3.2)–(3.5), we have $\mathfrak{S}(n)>0$. In view of Lemma 2.1(i), we obtain

$$\mathfrak{S}(n)\ll\sum_{q=1}^{\infty}q\frac{q^{\frac{7}{2}+\varepsilon}}{\varphi^7(q)}\ll 1.$$

The proof of (i) is completed.

By change of variable, we get

$$u_j(\beta)=\int_{P_j^4}^{(2P_j)^4}\frac{x^{-\frac{3}{4}}e(\beta x)}{\log x}\mathrm{d}x.$$

From Fourier integral formula, we have

$$\mathfrak{J}(n)=\int_{\mathfrak{D}}\frac{(x_1x_2x_3\cdots x_7)^{-\frac{3}{4}}}{(\log n)(\log x_2)(\log x_3)\cdots(\log x_7)}\mathrm{d}x_2\mathrm{d}x_3\cdots\mathrm{d}x_7,$$

where $x_1=n-x_2-x_3-\cdots-x_7$, and where the region $\mathfrak{D}$ is the set of points $(x_2,x_3,\cdots,x_7)\in\mathbb{R}^n$ such that

$$P_j^4\le x_j\le(2P_j)^4,\quad 1\le j\le 7.$$

Let $\mathfrak{D}_0$ be the set of points $(x_2,x_3,\cdots,x_7)\in\mathbb{R}^n$ such that

$$P_j^4<x_j\le 2P_j^4,\quad 2\le j\le 7.$$

It is easy to see that $\mathfrak{D}_0\subset\mathfrak{D}$. Consequently, we have

$$\begin{aligned}\mathfrak{J}(n)&\gg(P_2^4P_3^4\cdots P_7^4)^{-\frac{3}{4}}(\log P)^{-7}\int_{\mathfrak{D}_0}\mathrm{d}x_2\mathrm{d}x_3\cdots\mathrm{d}x_7\\&\gg\Big(\prod_{j=1}^{7}P_j\Big)P^{-4}(\log P)^{-7}.\end{aligned} \tag{3.6}$$

By [10, Lemma 4.2], we have

$$u_j(\beta)=\int_{P_j}^{2P_j}\frac{e(\beta t^4)}{\log t}\mathrm{d}t\ll\frac{P_j}{(1+|\beta|P_j^4)\log P},\quad 1\le j\le 7. \tag{3.7}$$

Hence we obtain

$$\begin{aligned}\mathfrak{J}(n)&\ll\Big(\prod_{j=1}^{7}P_j\Big)(\log P)^{-7}\int_0^{\infty}\prod_{j=1}^{7}\frac{1}{1+P_j\beta}\mathrm{d}\beta\\&\ll\Big(\prod_{j=1}^{7}P_j\Big)P^{-4}(\log P)^{-7}.\end{aligned} \tag{3.8}$$

In view of (3.6) and (3.8), (ii) is proved.

Lemma 3.2 *Let* $V = \mathrm{card}(v(x))$. *For any* $A > 0$, *we have*

$$V \ll x(\log x)^{-A}.$$

Proof Define

$$\mathfrak{M}_0(q,a) = \Big(\frac{a}{q} - \frac{L}{P^4}, \frac{a}{q} + \frac{L}{P^4}\Big], \quad \mathfrak{M}_0 = \bigcup_{1 \le q \le L} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_0(q,a),$$

$$\mathfrak{m}_0 = (0,1] \setminus \mathfrak{M}_0.$$

By (2.1), we have

$$\begin{aligned} r(n) &= \int_{\mathfrak{M}_0} \Big(\prod_{j=1}^{7} g_j(\alpha)\Big) e(-\alpha n) \mathrm{d}\alpha + \int_{\mathfrak{m}_0} \Big(\prod_{j=1}^{7} g_j(\alpha)\Big) e(-\alpha n) \mathrm{d}\alpha \\ &:= M(n) + E(n). \end{aligned} \tag{3.9}$$

As an application of the Siegal-Walfisz theorem and summation by parts, for $\alpha \in \mathfrak{M}_0(q,a) \subseteq \mathfrak{M}_0$, we have

$$g_j(\alpha) = \frac{S^*(q,a)}{\varphi(q)} u_j\Big(\alpha - \frac{a}{q}\Big) + O(P_j L^{-5}), \quad 1 \le j \le 7. \tag{3.10}$$

By (3.10), we have

$$M(n) = \sum_{q \le L} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{(S^*(q,a))^7}{\varphi(q)^7} e\Big(-\frac{a}{q}n\Big) \int_{-\frac{L}{P^4}}^{\frac{L}{P^4}} \Big(\prod_{j=1}^{7} u_j(\beta)\Big) e(-\beta n) \mathrm{d}\beta + O\Big(\Big(\prod_{j=1}^{7} P_j\Big) P^{-4} L^{-1}\Big). \tag{3.11}$$

Write

$$\mathfrak{S}(Q,n) = \sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{(S^*(q,a))^7}{\varphi^7(q)} e\Big(-\frac{a}{q}n\Big)$$

and

$$\mathfrak{J}(Q,n) = \int_{-\frac{Q}{P^4}}^{\frac{Q}{P^4}} \Big(\prod_{j=1}^{7} u(\beta)\Big) e(-\beta n) \mathrm{d}\beta.$$

From Lemma 2.1(i), we get

$$|\mathfrak{S}(n) - \mathfrak{S}(L,n)| \ll \sum_{q > L} q \frac{q^{\frac{7}{2}+\varepsilon}}{\varphi^7(q)} \ll \frac{1}{L}. \tag{3.12}$$

In view of (3.7), we have

$$\begin{aligned} |\mathfrak{J}(n) - \mathfrak{J}(L,n)| &\ll \int_{|\beta| > \frac{L}{P^4}} \prod_{j=1}^{7} \frac{P_j}{(1 + |\beta| P_j^4) \log P} \mathrm{d}\beta \\ &\ll \Big(\prod_{j=1}^{7} P_j\Big) P^{-4} L^{-1}. \end{aligned} \tag{3.13}$$

From (3.11)–(3.13) and Lemma 3.1, we have

$$M(n) = \mathfrak{S}(n)\mathfrak{J}(n) + O\Big(\Big(\prod_{j=1}^{7} P_j\Big)P^{-4}L^{-1}\Big). \tag{3.14}$$

By (3.1), (3.9) and (3.14), for $n \in v(x)$ we have

$$|E(n)| > \Big(\prod_{j=1}^{7} P_j\Big)P^{-4}(\log x)^{-8}$$

and

$$\sum_{n\in v(x)} |E(n)| > V \cdot \Big(\prod_{j=1}^{7} P_j\Big)P^{-4}(\log x)^{-8}. \tag{3.15}$$

On the other hand, by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sum_{n\in v(x)} |E(n)| &= \sum_{n\in v(x)} \eta_n E(n) = \int_{\mathfrak{m}_0} \Big(\prod_{j=1}^{7} g_j(\alpha)\Big) \sum_{n\in v(x)} \eta_n e(-\alpha n)\mathrm{d}\alpha \\ &\ll \Big(\int_{\mathfrak{m}_0} \Big|\prod_{j=1}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{2}} \Big(\int_0^1 \Big|\sum_{n\in v(x)} \eta_n e(-\alpha n)\Big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{2}} \\ &\ll V^{\frac{1}{2}}\Big(\int_{\mathfrak{m}_0} \Big|\prod_{j=1}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{2}}, \end{aligned} \tag{3.16}$$

where $|\eta(n)| = 1$.

From (3.15)–(3.16), we conclude that

$$V \ll P^8(\log x)^2\Big(\prod_{j=1}^{7} P_j\Big)^{-2} \int_{\mathfrak{m}_0} \Big|\prod_{j=1}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha. \tag{3.17}$$

Write

$$T = \int_{\mathfrak{m}_0} \Big|\prod_{j=1}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha.$$

By Lemma 2.3, for $\alpha = \frac{a}{q} + \beta \in \mathfrak{m}$, we have

$$\begin{aligned} g_1(\alpha) &\ll P^{1-\frac{1}{32}+\varepsilon} + \frac{q^{\varepsilon}\omega(q)^{\frac{1}{2}}P(\log P)^4}{(1+P^4|\alpha - \frac{a}{q}|)^{\frac{1}{2}}} \\ &\ll P^{1-\frac{1}{32}+\varepsilon} + \frac{q^{-\frac{1}{8}+\varepsilon}P^{1+\varepsilon}}{(1+P^4|\alpha - \frac{a}{q}|)^{\frac{1}{8}}} \\ &\ll P^{1-\frac{1}{32}+\varepsilon}, \end{aligned} \tag{3.18}$$

and for $\alpha \in \mathfrak{M} \cap \mathfrak{m}_0$, we have

$$\begin{aligned} g_1(\alpha) &\ll P^{1-\frac{1}{32}+\varepsilon} + \frac{q^{\varepsilon}\omega(q)^{\frac{1}{2}}P(\log P)^4}{\left(1+P^4\left|\alpha-\frac{a}{q}\right|\right)^{\frac{1}{2}}} \\ &\ll P^{1-\frac{1}{32}+\varepsilon} + \Psi(\alpha)^{\frac{1}{4}}\frac{q^{-\frac{1}{16}+\varepsilon}}{\left(1+P^4\left|\alpha-\frac{a}{q}\right|\right)^{\frac{1}{4}}} \\ &\ll P^{1-\frac{1}{32}+\varepsilon} + \Psi(\alpha)^{\frac{1}{4}}P(\log P)^4L^{-\frac{1}{18}}. \end{aligned} \tag{3.19}$$

We have

$$\begin{aligned} T &= \Big(\int_{\mathfrak{m}_0\bigcap\mathfrak{m}} + \int_{\mathfrak{m}_0\bigcap\mathfrak{M}}\Big)\Big|\prod_{j=1}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha \\ &= \mathrm{T}_1 + \mathrm{T}_2, \end{aligned} \tag{3.20}$$

As a consequence of (3.18)–(3.19), we get

$$\mathrm{T}_1 \ll (P^{1-\frac{1}{32}+\varepsilon})^2 \int_0^1 \Big|\prod_{j=2}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha \tag{3.21}$$

and

$$\mathrm{T}_2 \ll P^2(\log P)^8 L^{-\frac{1}{9}} \int_{\mathfrak{m}_0\bigcap\mathfrak{M}} \Psi(\alpha)^{\frac{1}{2}}\Big|\prod_{j=2}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha. \tag{3.22}$$

By Cauchy-Schwarz inequality and $g_6(\alpha) = g_7(\alpha)$, we have

$$\begin{aligned} &\int_{\mathfrak{m}_0\bigcap\mathfrak{M}} \Psi(\alpha)^{\frac{1}{2}}\Big|\prod_{j=2}^{7} g_j(\alpha)\Big|^2 \mathrm{d}\alpha \\ &= \int_{\mathfrak{m}_0\bigcap\mathfrak{M}} (\Psi(\alpha)^{\frac{1}{2}}|g_7(\alpha)|^{\frac{1}{3}}) \prod_{j=2}^{6}\Big(|g_j(\alpha)|\prod_{i=2}^{7}|g_i(\alpha)|\Big)^{\frac{1}{3}} \mathrm{d}\alpha \\ &\ll \Big(\int_{\mathfrak{M}} \Psi(\alpha)^3|g_6(\alpha)|^2\mathrm{d}\alpha\Big)^{\frac{1}{6}} \prod_{j=2}^{6}\Big(\int_0^1 \Big|g_j(\alpha)\prod_{i=2}^{7} g_i(\alpha)\Big|^2 \mathrm{d}\alpha\Big)^{\frac{1}{6}}. \end{aligned} \tag{3.23}$$

By (3.21) and Lemma 2.4, we get

$$\mathrm{T}_1 \ll P^{2-\frac{1}{16}+3\varepsilon}\prod_{j=2}^{7} P_j. \tag{3.24}$$

From (3.22)–(3.23), Lemmas 2.2 and 2.4–2.5, we have

$$\mathrm{T}_2 \ll \Big(\prod_{j=1}^{7} P_j\Big)^2 P^{-4}(\log P)^{10}L^{-\frac{1}{9}}. \tag{3.25}$$

Therefore by (3.17), (3.20) and (3.24)–(3.25), we have

$$T \ll \Big(\prod_{j=1}^{7} P_j\Big)^2 P^{-4}(\log P)^{10} L^{-\frac{1}{9}}$$

and

$$\begin{aligned} V &\ll P^8(\log x)^2\Big(\prod_{j=1}^{7} P_j\Big)^{-2}\Big(\prod_{j=1}^{7} P_j\Big)^2 P^{-4}(\log P)^{10} L^{-\frac{1}{8}} \\ &\ll P^4(\log P)^{12} L^{-\frac{1}{9}}. \end{aligned}$$

Take $B = 9(A+12)$ and Lemma 3.2 is proved.

Let $s(n)$ be the number of representations of n $(x < n \le 2x)$ as the sum of a square and two biquadrates of prime numbers. Let $k(x;l)$ denote the number of integers $n \equiv l \pmod{240}$ $(x < n \le 2x,\ l \in \{3, 51, 123, 171\})$ such that $s(n) > 0$. Here our aim is to find a lower bound for $k(x;l)$.

Lemma 3.3 *There is an absolute constant $C > 0$ such that*

$$k(x;l) \gg \frac{x}{(\log x)^C},$$

where $l \in \{3, 51, 123, 171\}$.

Proof By Cauchy-Schwarz inequality, we have

$$\Big(\sum_{\substack{x<n\le 2x \\ n\equiv l \pmod{240}}} s(n)\Big)^2 \le k(x;l) \sum_{\substack{x<n\le 2x \\ n\equiv l \pmod{240}}} s^2(n). \tag{3.26}$$

By noting $(l-2, 240) = 1$ and Dirichlet Theorem of prime numbers, we have

$$\sum_{\substack{x<n\le 2x \\ n\equiv l \pmod{240}}} s(n) \ge \sum_{\substack{x<p_1^2+p_2^4+p_3^4\le 2x \\ p_1\equiv\sqrt{l-2} \pmod{240}}} 1 \gg x(\log x)^{-3}. \tag{3.27}$$

It follows from (3.26)–(3.27) that

$$\begin{aligned} k(x;l) &\gg x^2(\log x)^{-6}\Big(\sum_{\substack{x<n\le 2x \\ n\equiv l \pmod{240}}} s^2(n)\Big)^{-1} \\ &\gg x^2(\log x)^{-6}\Big(\sum_{x<n\le 2x} s^2(n)\Big)^{-1}. \end{aligned} \tag{3.28}$$

Write

$$H(x) = \sum_{x<n\le 2x} s^2(n).$$

Then $H(x)$ is equal to the number of solutions of equation

$$x < p_1^2 + p_2^4 + p_3^4 = p_4^2 + p_5^4 + p_6^4 \le 2x. \tag{3.29}$$

Write

$$H(x) = H_1(x) + 2H_2(x), \tag{3.30}$$

where $H_1(x)$ is the contribution to $H(x)$ of those solutions of (3.29) for which $p_1 = p_4$ and $H_2(x)$ is the contribution for which $p_1 > p_4$.

By [5, Theorem 4], we have

$$H_1(x) \ll x^{\frac{1}{2}} \int_0^1 \Big| \sum_{1 \le n \le 4P} e(\alpha n^4) \Big|^4 \mathrm{d}\alpha \ll x(\log x)^{C_1}, \tag{3.31}$$

where C_1 is a positive constant.

Let $f(n)$ denote the number of representations of n as the form

$$n = x_1^4 + x_2^4 - x_3^4 - x_4^4$$

with $x_j \le (4P)^{\frac{1}{4}}$ $(1 \le j \le 4)$. From [5, Theorem 3], we have

$$H_2(x) \le \sum_{1 \le n \le x} \tau(n) f(n) \ll x(\log x)^{C_2}, \tag{3.32}$$

where C_2 is a positive constant.

Take $C = \max\{C_1, C_2\} + 6$. Lemma 3.3 follows from (3.28) and (3.30)–(3.32).

4 Proof of Theorem 1.1

Take $A = 2C$ in Lemma 3.2, where C is the constant in Lemma 3.3. Let $h \in \{10, 58, 130, 178\}$, $h \equiv N \pmod{240}$ and

$$\mathfrak{A}(h) = \Big\{ n \mid n = N - p_1^2 - p_2^4 - p_3^4,\ p_1^2 \equiv h - 9 \pmod{240},\ p_1^2 + p_2^4 + p_3^4 \le \frac{N}{2} \Big\}.$$

For $n \in \mathfrak{A}(h)\left(\frac{N}{2} < n \le N\right)$, we have $n \equiv 7 \pmod{240}$, $N - n \equiv h - 7 \pmod{240}$ and $s(N - n) > 0$. Hence

$$k\Big(\frac{N}{2}, h - 7\Big) = \mathrm{card}(\mathfrak{A}(h)).$$

By Lemma 3.3, we have

$$\mathrm{card}(\mathfrak{A}(h)) \gg N(\log N)^{-C}. \tag{4.1}$$

Then by Lemma 3.2, we get

$$\sum_{\substack{n \in \mathfrak{A}(h) \\ n \text{ satisfies (3.1)}}} 1 \ll N(\log N)^{-2C}. \tag{4.2}$$

Upon comparison of (4.1)–(4.2), we know that every sufficiently large integer N satisfying the congruence condition $N \equiv h \pmod{240}$, $h \in \{10, 58, 130, 178\}$ may be represented as the sum of one square and nine fourth powers of prime numbers. The theorem is proved.

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