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Nonexistence of Type II Blowup for Heat Equation with Exponential Nonlinearity*

Ruihong JI[1] Shan LI[2] Hui CHEN[1]

Abstract This paper deals with the blowup behavior of the radially symmetric solution of the nonlinear heat equation $u_t = \Delta u + \mathrm{e}^u$ in $\mathbb{R}^N$. The authors show the nonexistence of type II blowup under radial symmetric case in the lower supercritical range $3 \leq N \leq 9$, and give a sufficient condition for the occurrence of type I blowup. The result extends that of Fila and Pulkkinen (2008) in a finite ball to the whole space.

Keywords Nonlinear heat equation, Type II blowup, Exponential nonlinearity
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1 Introduction

In this paper, we consider the blowup phenomenon of the nonlinear heat equation

$$\begin{cases} u_t = \Delta u + \mathrm{e}^u, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with nonnegative, nontrivial initial data $u_0(x) \in L^\infty(\mathbb{R}^N)$.

Throughout this paper, we deal with radially symmetric solution $u(x,t) = U(r,t) = U(|x|,t)$, which satisfies the equation

$$U_t = U_{rr} + \frac{N-1}{r} U_r + \mathrm{e}^U. \tag{1.2}$$

The nonlinear reaction-diffusion model (1.1) with exponential nonlinearity is derived from the ignition model for a high activation energy thermal explosion of a solid fuel, which is important not only in combustion theory but also in other areas (see [1, 7–8, 23–24]). It is also interesting in differential geometry (see [14]) and other applications.

As we know, the solution of (1.1) will develop singularity in finite time, provided that the initial data is nonnegative and nontrivial. Namely, there exists a finite time $T < \infty$ such that

$$\limsup_{t \to T^-} \|u(\cdot,t)\|_{L^\infty} = \infty.$$

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[1]Geomathematics Key Laboratory of Sichuan Province, Chengdu University of Technology, Chengdu 610059, China. E-mail: jiruihong09@cdut.cn sirchenh@163.com
[2]Corresponding author. Business School, Sichuan University, Chengdu 610065, China.
E-mail: lishan@scu.edu.cn
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Here, T is called the blowup time.

Furthermore, another interesting and important problem is to investigate the singularity behavior near the blowup time. The simplest example of a blowup solution is $u(t) = -\ln(T-t)$. Another simple example is a self-similar blowup solution, which is given in the form

$$u(x,t) = -\ln(T-t) + \psi\Big(\frac{x-a}{\sqrt{T-t}}\Big),$$

where a is any point in $\mathbb{R}^N$ and $\psi(y)$ is a bounded solution of the equation

$$\Delta\psi - \frac{1}{2}y\cdot\nabla\psi + \mathrm{e}^{\psi} - 1 = 0 \quad \text{for } y \in \mathbb{R}^N.$$

Both of these two special solutions imply

$$\|u(\cdot,t)\|_{L^\infty} + \ln(T-t) < \infty.$$

We call the blowup is of type I if u satisfies

$$\limsup_{t\to T^-}(\|u(\cdot,t)\|_{L^\infty} + \ln(T-t)) < +\infty \tag{1.3}$$

and it is of type II otherwise. We would like to refer the important surveys [4, 16] in this topic.

Before stating the main issues in this paper, we will review the known results on another important nonlinear heat equation

$$u_t = \Delta u + |u|^{p-1}u, \quad p > 1, \tag{1.4}$$

which has been studied most extensively. Similarly, we say that the blowup is of type I provided that u satisfies

$$\limsup_{t\to T^-}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty} < \infty, \tag{1.5}$$

otherwise it is called type II blowup.

The Sobolev critical exponent $p_S = \frac{N+2}{N-2}$ for $N > 2$ and $p_S = \infty$ for $N \le 2$ plays a key role for studying the blowup profile. The singularity behavior is much better understood in subcritical case $p < p_S$, since the imbedding of Sobolev space $H^1(\Omega)$ in $L^p(\Omega)$ is valid, the functional analysis method and energy estimate can work here. It has been shown that only type I blowup can occur in subcritical case $p < p_S$ in [9–10].

A first example of type II blowup has been constructed by Herrero and Vázquez for $p > p_{JL}$ in [11–12], where p_{JL} is the so-called Joseph-Lundgren critical exponent as

$$p_{JL} = \begin{cases} \dfrac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}, & \text{if } N > 10, \\ \infty, & \text{if } N \le 10. \end{cases}$$

It was known to characterize the spectral properties of the singular stationary solution in [13]. Later on, Mizoguchi used a brief and clean proof to give a series works on the existence of type II blowup for $p > p_{JL}$ (see [19–21]).

In the recent significant work [17], Matano and Merle characterized the nonexistence of type II blowup in lower supercritical case $p_S < p < p_{JL}$ both in a finite ball and in $\mathbb{R}^N$. In the latter case, sign conditions are additionally satisfied, i.e., $|u(r,t_0) - \varphi^*(r)|$ and $u_t(r,t_0)$ change sign at most finitely many times over $[0,\infty)$ for some $t_0 \in [0,T)$, where

$$\varphi^*(r) = c^* r^{-\frac{2}{p-1}} \quad \text{with} \quad (c^*)^{p-1} = \frac{2}{p-1}\Big(N - 2 - \frac{2}{p-1}\Big),$$

which is the singularity solution of the stationary equation of (1.4). Furthermore, they improved their results in the case of $\mathbb{R}^N$ by eliminating restrict condition on $u_t(r,t_0)$ in [18]. Mizoguchi [22] obtained the nonexistence of type II blowup in $\mathbb{R}^N$ under the condition that $u_0(r)$ is nonincreasing with respect to r for $p_S < p < p_{JL}$.

However, there are few works to investigate the singularity behavior to the exponential equation (1.1). In the pioneering work [6] in this topic, the nonexistence of type II blowup was shown for bounded domain provided that u, u_t are always nonnegative. The fundamental issue of whether there exists type II blowup to the exponential equation in (1.1) remains outstanding open. As pointed out by Matano in the end of his famous survey [16]: "$\cdots$, we are led to speculate that every (radially symmetric) blow-up for (1.1) in the range $3 \le N \le 9$ is of type I. $\cdots$. This question is still open." We have realized that the range $3 \le N \le 9$ for (1.1) looks much like the lower supercritical range $p_S < p < p_{JL}$ for (1.4) in [13]. Fila and Pulkkinen [5] got a sufficient condition for type I blowup for equation (1.1) with $u(0,t) = \max\limits_{B_R} u(\cdot,t)$ in a finite ball. In this paper, we will give a rigorous proof to show the nonexistence of type II blowup in lower supercritical range for (1.1) in $\mathbb{R}^N$. Though the result is the same as that in finite ball, there is some technical difficulty when using zero number theory for $\mathbb{R}^N$, and we resolve it in Lemma 3.1. The main result is stated as follows.

Theorem 1.1 (Nonexistence of Type II Blowup in $\mathbb{R}^N$) *Let $3 \le N \le 9$ and u be a solution of (1.1) with radially symmetric nonincreasing initial data $u_0 \in L^\infty(\mathbb{R}^N)$. Suppose that u blows up in $L^\infty(\mathbb{R}^N)$ at $t = T$ for some $0 < T < +\infty$. Then u exhibits type I blowup provided that the blowup set $B(u_0)$ is not $[0,\infty)$.*

Remark 1.1 The proof of Theorem 1.1 is based on the intersection comparison method, which has been used extensively to solve the same problem for equation (1.4) in [16–18, 20, 22]. Thus, we have to restrict the solution to be radially symmetric.

Remark 1.2 The nonexistence type II blowup result in this paper coincides with the one for the nonlinear heat equation (1.4), obtained in [22] for lower supercritical range.

This paper is organized as follows. In Section 2, we will give some known results including zero number properties, the behavior of stationary solutions and so on. In Section 3, we will give the proof of Theorem 1.1.

2 Preliminary

First, we introduce some important notations.

For a function $f \not\equiv 0$ on $[0,b]$ with $0 < b < \infty$, let $\mathcal{Z}_{[0,b]}(f)$ be the supremum over all j such that there exist $0 \le r_1 < r_2 < \cdots < r_{j+1} < b$ with $f(r_i) \cdot f(r_{i+1}) < 0$ for $i = 1, 2, \cdots, j$. We denote by $\mathcal{Z}(f)$ for simplicity if $b = \infty$.

We introduce a so-called Sturm-type theorem which has been established in [2].

Lemma 2.1 (Zero Number Properties) *Let $V(r,t) := v(x,t)$ with $r = |x|$ be a smooth, radially symmetric solution of the linear equation*

$$v_t = \Delta v + a(|x|,t)v \quad \text{for } |x| < R,\ t \in (t_1, t_2),$$

where $0 < R < \infty$, $-\infty \le t_1 < t_2 \le \infty$, and $a(r,t)$ is continuous on $[0,R] \times (t_1,t_2)$. Assume that $V(r,t)$ is not identically equal to 0 and satisfies either of the following boundary conditions:

$$\text{(a)}\ \ V(R,t) \equiv 0, \quad t \in (t_1,t_2), \quad \text{(b)}\ \ V(R,t) \neq 0, \quad t \in (t_1,t_2).$$

Then the following statements hold:

(i) $\mathcal{Z}_{[0,R]}(V(\cdot,t))$ *is finite for any $t \in (t_1,t_2)$,*

(ii) $\mathcal{Z}_{[0,R]}(V(\cdot,t))$ *is monotone nonincreasing with respect to t,*

(ii) *if $V_r(r^*,t^*) = V(r^*,t^*) = 0$ for some $r^* \in [0,R]$, $t^* \in (t_1,t_2)$, then*

$$\mathcal{Z}_{[0,R]}(V(\cdot,t)) > \mathcal{Z}_{[0,R]}(V(\cdot,s)), \quad t_1 < t < t^* < s < t_2.$$

It is easy to see that there exists a singular stationary solution of (1.2), denoted by

$$\varphi_\infty(r) = \ln \frac{2(N-2)}{r^2} \quad \text{for } N \ge 3.$$

Next, we show some important properties of the stationary solution which play a crucial role to establish the nonexistence of type II blowup in lower supercritical range $3 \le N \le 9$.

Lemma 2.2 (see [13, 15]) *For $a > 0$, let φ_a be a positive solution of*

$$\begin{cases} \varphi''(r) + \dfrac{N-1}{r}\varphi'(r) + \mathrm{e}^{\varphi(r)} = 0 \quad \text{in } (0,\infty), \\ \varphi'(0) = 0, \quad \varphi(0) = a. \end{cases} \tag{2.1}$$

If $3 \le N \le 9$, then $\mathcal{Z}(\varphi_a - \varphi_\infty) = \infty$.

A direct computation gives that

$$\varphi_a(r) = a - 1 + \varphi_1(\mathrm{e}^{\frac{a-1}{2}} r). \tag{2.2}$$

3 Proof of the Main Results

We will give the proof of Theorem 1.1 in this section. The proof is separated into four steps. First, we prove that the possible type II blowup can occur only at the origin. Second, we will show that $\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty)$ is finite. Furthermore, we prove that any type II blowup solution converges to a stationary solution of (1.1). Finally by constructing a contradiction we get the desired result.

Step 1 Now we give the fact that the possible type II blowup can occur only at the origin. For a solution u to problem (1.1) and $T > 0$, set

$$\omega(y,s) = u(r,t) + \log(T-t), \quad y = \frac{r}{\sqrt{T-t}}, \quad s = -\log(T-t).$$

Then ω satisfies

$$\omega_s = w_{yy} + \frac{N-1}{y}\omega_y - \frac{y}{2}\omega_y + \mathrm{e}^{\omega} - 1 \quad \text{in } (0,\infty) \times (s_T, \infty), \tag{3.1}$$

where $s_T = -\log T$.

Proposition 3.1 *Suppose that* $u_0(r)$ *is nonincreasing with* r*. Let* u *be a radially symmetric solution of* (1.1) *with blowup time* T *and* ω *be the corresponding solution of* (3.1)*. Then the possible type* II *blowup must occur at* $r = 0$.

Proof Let λ_R and ϕ_R be respectively the first eigenvalue and eigenfunction of

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R \end{cases}$$

with $\phi_R > 0$ in B_R and normalized by $\int_{B_R} \phi_R \mathrm{d}x = 1$. For any $R > 0$,

$$\lambda_R = \frac{\lambda_1}{R^2} \quad \text{and} \quad \phi_R(r) = \frac{1}{R^N}\phi_1\Big(\frac{r}{R}\Big) \quad \text{for } 0 < r < R.$$

It is easy to see that $\phi_R(r)$ is nonincreasing with respect to $r = |x|$. Since the assumption $u_0(r)$ is nonincreasing with r that implies $\omega_y(y,s) \le 0$ in $[0,\infty) \times [s_T,\infty)$, we have

$$\omega_s \ge y^{1-N}(\omega_y y^{N-1})_y + \mathrm{e}^{\omega} - 1 \quad \text{in } (0,\infty) \times (s_T,\infty). \tag{3.2}$$

Multiplying (3.2) by $\phi_R y^{N-1}$ and integrating over $[0,R]$ yield that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\int_0^R \omega\phi_R y^{N-1}\mathrm{d}y &\ge -\int_0^R \omega_y(\phi_R)_y y^{N-1}\mathrm{d}y + \int_0^R (\mathrm{e}^{\omega}-1)\phi_R y^{N-1}\mathrm{d}y \\ &\ge -\lambda_R \int_0^R \omega\phi_R y^{N-1}\mathrm{d}y + \int_0^R (\mathrm{e}^{\omega}-1)\phi_R y^{N-1}\mathrm{d}y \\ &\ge -\lambda_R \int_0^R \omega\phi_R y^{N-1}\mathrm{d}y + \mathrm{e}^{\int_0^R \omega\phi_R y^{N-1}\mathrm{d}y} - 1 \end{aligned}$$

for $s \ge s_T$. Here we have used Jensen's inequality. Putting

$$W(s) = \int_0^R \omega\phi_R y^{N-1}\mathrm{d}y \quad \text{for } s \ge s_T,$$

we get

$$W'(s) \ge -\lambda_R W(s) + \mathrm{e}^{W(s)} - 1 \quad \text{for } s \ge s_T.$$

Since $W(s)$ exists globally in s, there holds

$$\mathrm{e}^{W(s)} \le \lambda_R W(s) + 1 \quad \text{for } s \ge s_T,$$

which yields the uniform upper bound of $W(s)$,

$$W(s) \leq C$$

for some constant $C > 0$ depending only on R.

Suppose that $u(r,t)$ blows up at some $r_0 > 0$ and it is of type II. Then there exists a sequence $t_n \to T$ such that

$$u(r_0, t_n) + \log(T - t_n) \to +\infty, \quad n \to \infty.$$

Since y_n, s_n are sufficiently large for $n \to \infty$, $\omega(y_n, s_n)$ is also sufficiently large. Then for fixed sufficiently large n, we can choose some $k \in (0,1)$ such that $R_1 = \frac{y_n}{k} \in \{y_n\}_1^\infty$ and $\omega(R_1, s_n')$ large, where s_n' corresponds to $\frac{y_n}{k}$. Moreover, $\omega(y,s)$ is nonincreasing with respect to y, then we have $\omega(y,s) > 0$ for $0 < y < R_1$. This together with the uniform upper bound of $W(s)$ implies

$$\omega(y_n)\frac{k^N}{N}\phi_1(k) \leq \int_0^{y_n} \omega(y,s)\phi_R y^{N-1}\mathrm{d}y \leq \int_0^{R_1} \omega(y,s)\phi_R y^{N-1}\mathrm{d}y \leq C.$$

Hence, taking limit $n \to \infty$ leads to a contradiction.

Step 2 $\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty)$ is finite.

We quote the following result which was established in [3].

Proposition 3.2 *For positive constants R and T, let $Q_{R,T} = (\mathbb{R}^N \setminus B_R) \times [0,T]$. Assume that u satisfies*

$$|\Delta u + u_t| \leq M(|u| + |\nabla u|) \quad \text{in } Q_{R,T}$$

and

$$|u(x,t)| \leq M \exp(M|x|^2) \quad \text{in } Q_{R,T}$$

for some constant $M > 0$. If $u(x,0) = 0$ for any $x \in \mathbb{R}^N \setminus B_R$, then u vanishes identically in $Q_{R,T}$.

Lemma 3.1 *Assuming that $u_0(r)$ is nonincreasing with r and $B(u_0) \neq [0,\infty)$. Let u be a radially symmetric solution of the Cauchy problem* (1.2) *which blows up at $t = T$. Then there exist $\delta_0 > 0$, $R > 0$, $t_0 \in [0,T)$ such that*

$$|u(R,t) - \varphi_\infty(R)| \geq \delta_0 \quad \text{for } t \in [t_0, T).$$

Proof Let $p(r,t) = u(r,t) - \varphi_\infty(r)$ in $(0,\infty) \times [0,T)$, which satisfies

$$p_t = p_{rr} + \frac{N-1}{r}p_r + \frac{2(N-2)}{r^2}(\mathrm{e}^p - 1) \quad \text{in } (0,\infty) \times (0,T). \tag{3.3}$$

Due to the assumption that $B(u_0) \neq [0,\infty)$, there exist some positive constants C and K such that

$$u(r,t) \leq C \quad \text{in } [K,\infty) \times [0,T),$$

which implies

$$\frac{2(N-2)}{r^2}(\mathrm{e}^p - 1) \leq Cp \quad \text{in } [K,\infty) \times [0,T).$$

It follows from (3.3) that $p(r,t)$ can not blowup at any time in $[K,\infty)$. Let $p(r,T) = \lim_{t\to T} p(r,t)$ for $r \geq K$. Change the variable $t' = T - t$ and denote $p_1(r,t') = p(r,t)$ such that

$$|\Delta p_1 + p_{1t}| \leq M(|p_1| + |\nabla p_1|) \quad \text{for some } M > 0 \text{ in } [K,\infty) \times [0,T).$$

Thanks to Proposition 3.2 there exists $R \geq K$ such that

$$p_1(R,0) = p(R,T) \neq 0.$$

Then there exists a $t_0 \in [0,T)$ such that

(i) if $p(R,T) > 0$, then $p(R,t) \geq \frac{1}{2}p(R,T)$ for $t \in [t_0,T)$,

(ii) if $p(R,T) < 0$, then $p(R,t) \leq \frac{1}{2}p(R,T)$ for $t \in [t_0,T)$,

which implies the conclusion by choosing $\delta_0 = \frac{1}{2}\big|p(R,T)\big|$.

Hence, we can conclude that for the whole domain $\mathbb{R}^N$, there exists a $R > 0$ such that $\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty(r))$ is finite for $t \in [t_0,T)$.

Step 3 We assume that the blowup is of type II. We will show that there exists a time sequence $t_n \to T$ such that, after a suitable space-time rescaling, the solution $u(r,t_n)$ converges to a stationary solution of (1.1) in $\mathbb{R}^N$ for $N \geq 3$.

Lemma 3.2 *Suppose that $u_0(r)$ is nonincreasing with r and $B(u_0) \neq [0,\infty)$. Let u be a solution of (1.2) blowing up at $t = T$. Then there exists $R > 0$ and $t_0 \in [0,T)$ such that $u_t(R,t) \neq 0$ for $t \in [t_0,T)$.*

The proof of Lemma 3.4 follows from the similar argument in Lemma 3.3 replacing $p(r,t)$ by $u_t(r,t)$, hence we omit it here.

Lemma 3.2 implies immediately that there exists an $R > 0$ such that

$$\mathcal{Z}_{[0,R]}\big(u_t\big) < \infty \quad \text{for } t \in (t_0,T).$$

Moreover, since $u(0,t) = \|u(\cdot,t)\|_{L^\infty}$ and u blows up in finite time T, there exists a $t_1 \in [0,T)$ such that

$$u_t(0,t) > 0 \quad \text{in } [t_1,T). \tag{3.4}$$

The following idea is inspired by [17] to show the similar result for (1.4). Let $M(t) = \|u(t)\|_{L^\infty}$ for $t \in [0,T)$. It follows from (3.4) that

$$0 \leq M'(t) \leq \mathrm{e}^{M(t)} \quad \text{in } [t_1,T), \tag{3.5}$$

which gives

$$-1 \leq \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-M(t)}) \leq 0 \quad \text{in } [t_1,T). \tag{3.6}$$

Integrating (3.6) from t to T yields

$$\mathrm{e}^{M(t)} \geq \frac{1}{e(T-t)}. \tag{3.7}$$

Since $M(t)\to\infty$ as $t\to T$, we obtain

$$\mathrm{e}^{-M(t)}\to 0 \quad \text{as } t\to T.$$

Let

$$\tau=\int_0^t \mathrm{e}^{M(s)-1}\mathrm{d}s$$

and one has

$$\tau\ge\int_0^t \frac{1}{\mathrm{e}(T-s)}\mathrm{d}s\ge\frac{1}{\mathrm{e}}\ln\frac{T}{T-t}\to\infty \quad \text{as } t\to T$$

due to the facts (3.6)–(3.7). Denoting $\rho(\tau)=\mathrm{e}^{-M(t)+1}$, we get

$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\mathrm{e}^{M(t)-1}=\frac{1}{\rho(\tau)} \quad \text{and} \quad \mathrm{d}t=\rho(\tau)\mathrm{d}\tau,$$

thus

$$T-t=\int_\tau^\infty \rho(s)\mathrm{d}s.$$

Set

$$v(\eta,\tau)=u(r,t)-M(t)+1 \quad \text{and} \quad \eta=r\mathrm{e}^{\frac{M(t)-1}{2}}.$$

Then v satisfies

$$v_\tau=v_{\eta\eta}+\frac{N-1}{\eta}v_\eta+\mathrm{e}^v-\sigma(\tau)\Big(\frac{\eta}{2}v_\eta+1\Big) \quad \text{in } (0,\infty)\times(0,\infty), \tag{3.8}$$

where

$$\sigma(\tau)=-\frac{\rho'(\tau)}{\rho(\tau)}=M'(t)\mathrm{e}^{-M(t)+1}. \tag{3.9}$$

It follows from (3.5) that

$$0\le\sigma(\tau)\le\mathrm{e} \quad \text{in } (0,\infty). \tag{3.10}$$

Set

$$\xi(\tau)=\frac{\rho(\tau)}{\displaystyle\int_\tau^\infty \rho(s)\mathrm{d}s}=\frac{1}{\mathrm{e}^{M(t)-1+\ln(T-t)}}.$$

Since u undergoes type II blowup at $t=T$, there exists $\{\tau_n\}$ with $\tau_n\to\infty$ as $n\to\infty$ such that $\xi(\tau_n)\to 0$ as $n\to\infty$. A direct calculation yields

$$\sigma(\tau)=-\frac{\xi_\tau}{\xi}+\xi \quad \text{in } (0,\infty), \tag{3.11}$$

which together with (3.9)–(3.10) leads

$$-\mathrm{e}\xi\le\xi_\tau\le\xi^2 \quad \text{in } (0,\infty). \tag{3.12}$$

The following idea borrows from [17, Lemma 4.2]. For convenience to the readers, we sketch the proof as follows.

Lemma 3.3 *For any $k > 0$ there exists $\{a_n\}$ with $a_n \to \infty$ as $n \to \infty$ such that*

$$\max_{\tau \in [a_n, a_n+k]} |\xi(\tau)| \to 0 \quad \text{as } n \to \infty \tag{3.13}$$

and

$$\int_{a_n}^{a_n+k} \sigma(\tau)\mathrm{d}\tau \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Proof Integrating (3.11) from a to $a + k$ yields

$$\int_a^{a+k} \sigma(\tau)\mathrm{d}\tau = \ln \frac{\xi(a)}{\xi(a+k)} + \int_a^{a+k} \xi(\tau)\mathrm{d}\tau \quad \text{for } a > 0. \tag{3.15}$$

Case 1 $\xi(\tau) \to 0$ as $\tau \to \infty$. The statement (3.13) is trivial. Furthermore, suppose that (3.14) does not hold. Then there exists $\delta_0 > 0$ such that

$$\int_a^{a+k} \sigma(\tau)\mathrm{d}\tau \geq \delta_0 \quad \text{for } a \gg 1. \tag{3.16}$$

From (3.7), we have

$$\int_0^\infty \xi(\tau)\mathrm{d}\tau < \infty, \tag{3.17}$$

which contradicts the fact

$$\int_0^\infty \xi(\tau)\mathrm{d}\tau = \Big[-\ln \int_\tau^\infty \rho(s)\mathrm{d}s \Big]_0^\infty = \infty.$$

Case 2 $\xi(\tau) \nrightarrow 0$ as $\tau \to \infty$. There exists $\delta_1 > 0$, $\{\tau_n\}$ and $\{\widetilde{\tau}_n\}$ satisfying

$$\widetilde{\tau}_1 < \tau_1 < \widetilde{\tau}_2 < \tau_2 < \cdots < \widetilde{\tau}_n < \tau_n < \cdots \to \infty \quad \text{as } n \to \infty,$$

such that

$$\xi(\widetilde{\tau}_n) \geq \delta_1 \quad \text{for} \quad n = 1, 2, \cdots, \quad \text{and} \quad \xi(\tau_n) \to 0 \quad \text{as } n \to \infty.$$

Since $\xi(\tau)$ varies slowly near $\tau = \tau_n$ for $n \gg 1$ by (3.12), there exist $\{a_n\}$, $\{b_n\}$ with $a_n \in (\widetilde{\tau}_n, \tau_n)$, $b_n \in (\tau_n, \widetilde{\tau}_{n+1})$ and $b_n - a_n = k$ for $n \gg 1$ such that

$$\xi(a_n) = \xi(b_n) = \max_{\tau \in [a_n, b_n]} \xi(\tau) \to 0 \quad \text{as } n \to \infty.$$

It follows from (3.15) that

$$\int_{a_n}^{a_n+k} \sigma(\tau)\mathrm{d}\tau = \ln \frac{\xi(a_n)}{\xi(b_n)} + \int_{a_n}^{b_n} \xi(\tau)\mathrm{d}\tau = \int_{a_n}^{b_n} \xi(\tau)\mathrm{d}\tau \to 0 \quad \text{as } n \to \infty,$$

which is nothing but (3.14). This completes the proof.

In view of (3.10) and Lemma 3.3, we obtain

$$\sigma(\tau + a_n) \to 0 \quad \text{in } L^1(0, k) \quad \text{as } n \to \infty.$$

Therefore, we can get the following result by the diagonalization argument.

Lemma 3.4 *There exist $\{a_n\}$ and $\{k_n\}$ with $a_n \to \infty$ and $k_n \to \infty$ as $n \to \infty$ such that*

$$\max_{\tau \in [a_n, a_n+k_n]} |\xi(\tau)| \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{a_n}^{a_n+k_n} \sigma(\tau)\mathrm{d}\tau \to 0 \quad \text{as } n \to \infty.$$

Moreover,

$$\sigma(\tau + a_n + k_n) \to 0 \quad \text{a.e. } \tau \in (-\infty, 0] \quad \text{as } n \to \infty.$$

Lemma 3.5 *There exists $\{\tau_n\}$ with $\tau_n \to \infty$ as $n \to \infty$ and a solution φ_1 of (2.1) such that*

$$v(\eta, \tau_n) \to \varphi_1(\eta) \quad \text{locally uniformly in } [0, \infty) \quad \text{as } n \to \infty.$$

Namely, there exists $\{t_n\}$ with $t_n \to T$ as $n \to \infty$ such that

$$u(\eta \mathrm{e}^{-\frac{M(t_n)-1}{2}}, t_n) - M(t_n) + 1 \to \varphi_1(\eta) \quad \text{locally uniformly in } [0, \infty) \quad \text{as } n \to \infty.$$

Proof Let a_n and k_n be as in Lemma 3.4. Set $\overline{k}_n = \frac{k_n}{3}$ and $\tau_n = a_n + 2\overline{k}_n$, and denote

$$v_n(\eta, \tau) = v(\eta, \tau + \tau_n) \quad \text{for } \eta \geq 0 \text{ and } \tau \in [-2\overline{k}_n, \overline{k}_n].$$

Then v_n satisfies

$$v_\tau = v_{\eta\eta} + \frac{N-1}{\eta} v_\eta + \mathrm{e}^v - \sigma(\tau + \tau_n)\Big(\frac{\eta}{2} v_\eta + 1\Big) \quad \text{in } (0, \infty) \times (0, \infty). \tag{3.18}$$

Obviously, $\|v_n(\cdot, \tau)\|_{L^\infty} = 1$ for $\tau \in [-2\overline{k}_n, \overline{k}_n]$. Since $\sigma(\tau + \tau_n)$ is uniformly bounded from (3.10), there exists $V \in C^2\big([0, \infty) \times \mathbb{R}\big)$ such that

$$v_n(\eta, \tau) \to V(\eta, \tau) \quad \text{locally uniformly in } [0, \infty) \times \mathbb{R} \quad \text{as } n \to \infty$$

by the parabolic regularity theory. Then V satisfies

$$V_\tau = V_{\eta\eta} + \frac{N-1}{\eta} V_\eta + \mathrm{e}^V \quad \text{in } (0, \infty) \times \mathbb{R}. \tag{3.19}$$

Differentiating (3.19) with respect to τ yields

$$(V_\tau)_\tau = (V_\tau)_{\eta\eta} + \frac{N-1}{\eta} (V_\tau)_\eta + \mathrm{e}^V V_\tau \quad \text{in } (0, \infty) \times \mathbb{R}.$$

Since $V(0, \tau) \equiv 1$ and $V_\eta(0, \tau) \equiv 0$, it is trivial that $V_\tau(0, \tau) = (V_\tau)_\eta(0, \tau) = 0$ for $\tau \in \mathbb{R}$. This together with [17, Corollary 2.9] yields that $V_\tau(\eta, \tau) = 0$ in $[0, \infty) \times \mathbb{R}$, which implies that $V(\eta, \tau) = \varphi_1(\eta)$ for $\tau \in \mathbb{R}$.

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1 Suppose on the contrary. We assume that there exists a solution u to the problem (1.1) which exhibits type II blow up at $t = T$. It follows from Lemma 3.1 that there exists some positive integer m such that for $R > 0$ in Lemma 3.1,

$$\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty(r)) \leq m, \quad t \in [t_0, T). \tag{3.20}$$

With the aid of Lemmas 2.2–3.5, we get

$$\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty(r)) = \mathcal{Z}_{[0,R]}(v(r\mathrm{e}^{\frac{M(t)-1}{2}}, \tau) + M(t) - 1 - \varphi_\infty(r)).$$

Moreover, thanks to the fact that $\varphi_a - \varphi_\infty$ are all simple zeros for any $a > 0$ (see [15]), we have

$$\begin{aligned}\mathcal{Z}_{[0,R]}(u(r,t) - \varphi_\infty(r)) &= \mathcal{Z}_{[0,R]}(\varphi_1(r\mathrm{e}^{\frac{M(t)-1}{2}}) + M(t) - 1 - \varphi_\infty(r)) \\ &= \mathcal{Z}_{[0,\infty)}(\varphi_1(\eta) - \varphi_\infty(\eta)) \\ &> m \quad \text{for } t \to T.\end{aligned}$$

This leads a contradiction to (3.20), and it completes the proof.

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