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Time-Consistent Asymptotic Exponential Arbitrage with Small Probable Maximum Loss

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Abstract Based on a concept of asymptotic exponential arbitrage proposed by Föllmer-Schachermayer, the author introduces a new formulation of asymptotic arbitrage with two main differences from the previous one: Firstly, the realising strategy does not depend on the maturity time while the previous one does, and secondly, the probable maximum loss is allowed to be small constant instead of a decreasing function of time. The main result gives a sufficient condition on stock prices for the existence of such asymptotic arbitrage. As a consequence, she gives a new proof of a conjecture of Föllmer and Schachermayer.

Keywords Asymptotic arbitrage, Time-consistent, Small probable maximum loss
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1 Formulations of Asymptotic Exponential Arbitrage

Let $(\Omega, \mathscr{F}, \mathbb{F}, P)$ be a filtered probability space where the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfies the usual conditions, and the discounted price process $S = (S_t)_{t\geq 0}$ initially be any $\mathbb{R}^d$-valued semimartingale. Based on a result of Schweizer [4], let us assume that the price process S has the form:

$$\mathrm{d}S_t = \mathrm{d}M_t + \mathrm{d}\langle M\rangle_t \lambda_t, \tag{1.1}$$

where M is a d-dimensional continuous local martingale with $M_0 = 0$, λ is a d-dimensional predictable process, the market price of risk, satisfying

$$\int_0^\infty \lambda^\top \,\mathrm{d}\langle M\rangle_t \lambda_t < \infty, \quad \text{a.s.}$$

The process $\langle \lambda \cdot M\rangle$ is called the mean-variance tradeoff.

Let $L(S)$ be the set of all predictable processes integrable with respect to S, and define for each $T > 0$ the set

$$\mathcal{H}^T := \{H \in L(S) \,|\, (H\cdot S)_t \geq -K \ \text{ for } t \in [0,T] \text{ and some } K \in \mathbb{R}_+\},$$

and in particular,

$$\mathcal{H}_0 := \{H \in L(S) \,|\, (H\cdot S)_t \geq -1, \ \forall\, t\}.$$

Clearly, $\mathcal{H}_0 \subset \mathcal{H}^T$ for any $T > 0$.

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1.1 Föllmer-Schachermayer's formulation

The following form of a long-term arbitrage was considered for the first time in Föllmer and Schachermayer [3].

Definition 1.1 (Asymptotic Exponential Arbitrage) *The process $S = (S_t)_{t\geq 0}$ allows asymptotic exponential arbitrage with exponentially decaying failure probability if there exist $0 < \widetilde{T} < \infty$ and constants $C, \kappa_1, \kappa_2 > 0$ such that for all $T \geq \widetilde{T}$, there is $H \in \mathcal{H}^T$ satisfying*

(a) $(H \cdot S)_T \geq -\mathrm{e}^{-\kappa_1 T}$ $\mathbb{P}$-*a.s.*;

(b) $\mathbb{P}[(H \cdot S)_T \leq \mathrm{e}^{\kappa_1 T}] \leq C\mathrm{e}^{-\kappa_2 T}$.

We should note that the choice of the realising strategy H depends on the maturity T. Föllmer and Schachermayer [3] showed how such a notion is related to large deviation estimates for the market price of risk. They derived the results for some concrete models (the geometric Ornstein-Uhlenbeck process and the Black-Scholes model), and suggested the following general result which has been proved by Du and Neufeld [2] by means of a time-change argument.

Theorem 1.1 (cf. [2]) *Let the filtration $\mathbb{F}$ be continuous in the sense that all local martingales are continuous. Assume that the market price of risk λ satisfies a large deviation estimate, i.e., there are constant $c_1, c_2 > 0$ such that*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big[\frac{1}{T}\langle \lambda \cdot M\rangle_T \leq c_1\Big] \leq -c_2. \tag{1.2}$$

Then S allows asymptotic exponential arbitrage.

1.2 Time-consistent asymptotic exponential arbitrage

In Föllmer-Schachermayer's formulation, the realizing strategies for asymptotic exponential arbitrage depend on the horizon, that means the strategies may change as T varies. From the practical point of view, we expect that the arbitrage-realizing strategy can be independent of the horizon, in other words, has time-consistence. As a cost, a constant but small probable maximum loss is permitted.

Definition 1.2 (Time-Consistent Asymptotic Exponential Arbitrage) *The process $S = (S_t)_{t\geq 0}$ allows time-consistent asymptotic exponential arbitrage if there exist $H \in \mathcal{H}_0$ and constants $T_0, C, \kappa_1, \kappa_2 > 0$ such that for all $T \geq T_0$,*

(a) $(H \cdot S)_T \geq -1$ $\mathbb{P}$-*a.s.*;

(b) $\mathbb{P}[(H \cdot S)_T \leq \mathrm{e}^{\kappa_1 T}] \leq C\mathrm{e}^{-\kappa_2 T}$.

This note aims to show that the condition (1.2) also suffices for time-consistent asymptotic exponential arbitrage.

Assumption 1.1 The martingale M in (1.1) is continuous, and $\mathcal{E}(-\alpha\lambda \cdot M)$ is a true martingale for each $\alpha > 0$.

It is worth noting that here we do not require the continuity of filtration. The main result of this note is as follows.

Theorem 1.2 *Let Assumption* 1.1 *and the large deviation estimate* (1.2) *be satisfied. Then S allows time-consistent asymptotic exponential arbitrage.*

We remark that a strategy realising time-consistent asymptotic exponential arbitrage naturally yields Föllmer–Schachermayer's asymptotic exponential arbitrage. Indeed, if H_t is a realising strategy for the former, then $\mathrm{e}^{-\frac{\kappa_1 T}{2}} H_t$ is the one for the latter, where T is the maturity time. Moreover, Theorem 2.1 below gives a precise form of realising strategies.

2 Proofs

Since $1 \ll \mathrm{e}^{\kappa_1 T}$ when T is large, a tiny adjustment of κ_1 gives

(a′) $1 + (H \cdot S)_T \geq 0$ $\mathbb{P}$-a.s.;

(b′) $\mathbb{P}[1 + (H \cdot S)_T \leq \mathrm{e}^{\kappa_1 T}] \leq C\mathrm{e}^{-\kappa_2 T}$.

Now let us denote

$$X_t = X_t(H) := 1 + (H \cdot S)_t, \quad H \in \mathcal{H}_0,$$

then we have the following lemma.

Lemma 2.1 *If there are $\kappa_1, \kappa_2 > 0$ such that*

$$\inf_{H \in \mathcal{H}_0} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}\Big[\frac{1}{T} \log X_T(H) \leq \kappa_1\Big] \leq -\kappa_2,$$

then S allows time-consistent asymptotic exponential arbitrage.

Proof Take $0 < \varepsilon < \kappa_2$. There exist $H^\varepsilon \in \mathcal{H}_0$ and $T_0 > 0$ such that for any $T \geq T_0$,

$$\frac{1}{T} \log \mathbb{P}\Big[\frac{1}{T} \log X_T(H^\varepsilon) \leq \kappa_1\Big] \leq -(\kappa_2 - \varepsilon),$$

that is

$$\mathbb{P}[X_T(H^\varepsilon) \leq \mathrm{e}^{\kappa_1 T}] \leq \mathrm{e}^{-(\kappa_2 - \varepsilon)T},$$

which concludes the result.

Let $\gamma < 0$. Chebyshev's inequality gives

$$\begin{aligned} \mathbb{P}\Big[\frac{1}{T} \log X_T(H) \leq \kappa\Big] &= \mathbb{P}[X_T(H) \leq \mathrm{e}^{\kappa T}] \\ &= \mathbb{P}[(X_T(H))^\gamma \geq \mathrm{e}^{\gamma \kappa T}] \\ &\leq \mathrm{e}^{-\gamma \kappa T} \mathbb{E}[(X_T(H))^\gamma], \end{aligned}$$

thus

$$\frac{1}{T} \log \mathbb{P}\Big[\frac{1}{T} \log X_T(H) \leq \kappa\Big] \leq \frac{1}{T} \log \mathbb{E}[(X_T(H))^\gamma] - \gamma\kappa.$$

Taking limits and inferiors we have

$$\begin{aligned} &\inf_{H \in \mathcal{H}_0} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}\Big[\frac{1}{T} \log X_T(H) \leq \kappa\Big] \\ &\leq \inf_{\gamma < 0} \Big\{ \inf_{H \in \mathcal{H}_0} \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}[(X_T(H))^\gamma] - \gamma\kappa \Big\}. \end{aligned} \tag{2.1}$$

Therefore S allows asymptotic exponential arbitrage provided the right-hand side is negative, and the original problem is converted to bound from above the value function of a long-term risk-sensitive control problem:

$$\inf_{H\in\mathcal{H}_0}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}[(X_T(H))^\gamma],\quad \gamma<0.$$

We consider an investor trading in the above market. More specifically, the wealth process, denoted by $X=X^{(\pi)}$, starting from 1, satisfies

$$X^{(\pi)}=\mathcal{E}(\pi\cdot M+\pi\cdot\langle M\rangle\lambda),$$

where π denotes the strategy and $\mathcal{E}$ the stochastic exponential.

Define

$$V(\gamma,T)=\inf_{\pi\in\mathcal{A}}\log\mathbb{E}[(X_T^{(\pi)})^\gamma],\quad \gamma<0,\ T>0$$

and

$$\chi(\gamma)=\inf_{\pi\in\mathcal{A}}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}[(X_T^{(\pi)})^\gamma],\quad \gamma<0, \tag{2.2}$$

where $\mathcal{A}$ denotes the admissible set containing strategy π such that $\pi\in L(M)$. Such an utility-based optimal investment problem has been addressed in numerous literature.

Here we are going not to solve $V(\gamma,T)$ explicitly, but to give an appropriate upper bound for it.

To this end, we select the following time-consistent strategy:

$$\pi_t^\star=\frac{\lambda_t}{1-\gamma}. \tag{2.3}$$

Then

$$\begin{aligned}V(\gamma,T)&\le\log\mathbb{E}[(X_T^{(\pi^\star)})^\gamma]\\&=\log\mathbb{E}\exp\Big[-\frac{\gamma}{\gamma-1}(\lambda\cdot M)_T-\frac{\gamma(2\gamma-1)}{2(\gamma-1)^2}\langle\lambda\cdot M\rangle_T\Big]\\&=\log\mathbb{E}\exp\Big[-\beta(\lambda\cdot M)_T-\frac{\beta+\beta^2}{2}\langle\lambda\cdot M\rangle_T\Big],\end{aligned}$$

where, for simplicity, we have denote

$$\beta:=\frac{\gamma}{\gamma-1}\in(0,1).$$

Thus from Hölder's inequality, for $1<p\le 1+\frac{1}{\beta}$,

$$\begin{aligned}\exp(V(\gamma,T))&\le\Big\{\mathbb{E}\Big[\exp(-\frac{p}{p-1}\frac{\beta+(1-p)\beta^2}{2}\langle\lambda\cdot M\rangle_T)\Big]\Big\}^{\frac{p-1}{p}}\{\mathbb{E}[\mathcal{E}(-p\beta\lambda\cdot M)_T]\}^{\frac{1}{p}}\\&=\Big\{\mathbb{E}\Big[\exp\Big(-\frac{p}{p-1}\frac{\beta+(1-p)\beta^2}{2}\langle\lambda\cdot M\rangle_T)\Big]\Big\}^{\frac{p-1}{p}}<\infty.\end{aligned}$$

Thus

$$V(\gamma,T)\le\inf_{1<p\le1+\frac{1}{\beta}}\frac{p-1}{p}\log\mathbb{E}\Big[\exp\Big(-\frac{p}{p-1}\frac{\beta-(p-1)\beta^2}{2}\langle\lambda\cdot M\rangle_T\Big)\Big].$$

To get the exact infimum is not easy, so we take, for simplicity,

$$p = 2,$$

then

$$\begin{aligned} V(\gamma, T) &\le \frac{1}{2}\log \mathbb{E}[\exp(-\beta(1-\beta)\langle \lambda \cdot M\rangle_T)] \\ &= \frac{1}{2}\log \mathbb{E}\exp\Big[\frac{\gamma}{(1-\gamma)^2}\langle \lambda \cdot M\rangle_T\Big]. \end{aligned}$$

Since the strategy $\pi^\star$ is independent of T, we have

$$\chi(\gamma) \le \frac{1}{2}\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{E}\exp\Big[\frac{\gamma}{(1-\gamma)^2}\langle \lambda \cdot M\rangle_T\Big].$$

To sum up, we have proved the following proposition.

Proposition 2.1 *Under Assumption* 1.1, *the wealth process* $X = X^{(\pi^\star)}$ *realized by the strategy*:

$$\pi_t^\star = \frac{\lambda_t}{1-\gamma}, \quad \gamma < 0$$

satisfies

$$\begin{aligned} \log \mathbb{E}[(X_T^{(\pi^\star)})^\gamma] &\le \frac{1}{2}\log \mathbb{E}\exp\Big[\frac{\gamma}{(1-\gamma)^2}\langle \lambda \cdot M\rangle_T\Big] \\ &\le \frac{1}{2}\log \mathbb{E}\exp(\gamma\langle \pi^\star \cdot M\rangle_T) \end{aligned} \tag{2.4}$$

for each $T > 0$. *Consequently, the function* $\chi(\cdot)$ *define in* (2.2) *satisfies*

$$\chi(\gamma) \le \frac{1}{2}\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{E}\exp\Big[\frac{\gamma}{(1-\gamma)^2}\langle \lambda \cdot M\rangle_T\Big]. \tag{2.5}$$

By means of Varadhan's integral lemma (cf. [1, Theorem 4.3.1]), we have the following lemma.

Lemma 2.2 *Suppose that* $\{T^{-1}\langle \lambda \cdot M\rangle_T\}$ *satisfies a large deviation principle with a rate function* $I(x)$. *Then*

$$\chi(\gamma) \le -\frac{1}{2}\inf_{x>0}\Big\{I(x) - \frac{\gamma x}{(1-\gamma)^2}\Big\}, \quad \gamma < 0. \tag{2.6}$$

Recalling Lemma 2.1 and relation (2.1), we have actually proved the following theorem.

Theorem 2.1 *Let Assumption* 1.1 *be satisfied. Suppose that* $\{T^{-1}\langle \lambda\cdot M\rangle_T\}$ *satisfies a large deviation principle with a rate function* $I(x)$. *Then the wealth process* $X. = X.(H^\star)$ *realized by*

$$H_t^\star = \frac{X_t\lambda_t}{1-\gamma} \tag{2.7}$$

satisfies

$$\begin{aligned} &\limsup_{T\to\infty}\frac{1}{T}\log \mathbb{P}\Big[\frac{1}{T}\log X_T \le \kappa\Big] \\ &\le -\sup_{\gamma<0}\Big\{\gamma\kappa + \frac{1}{2}\inf_{x>0}\Big\{I(x) - \frac{\gamma x}{(1-\gamma)^2}\Big\}\Big\}. \end{aligned} \tag{2.8}$$

Consequently, S allows time-consistent asymptotic exponential arbitrage provided the right-hand side of (2.8) *is negative for some* $\kappa > 0$.

In fact, the condition (1.2) is sufficient to ensure time-consistent asymptotic exponential arbitrage of S.

Proof of Theorem 1.2 Let $X_\cdot = X_\cdot(H^\star)$ be the process defined in Theorem 2.1. For $K > 0$,we have

$$\mathbb{E}[\exp(-K\langle\lambda\cdot M\rangle_T)] \le \mathrm{e}^{-Kc_1T}\,\mathbb{P}[\langle\lambda\cdot M\rangle_T > c_1T] + \mathbb{P}[\langle\lambda\cdot M\rangle_T \le c_1T],$$

thus by (1.2),

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}[\exp(-K\langle\lambda\cdot M\rangle_T)] \le -\min\{Kc_1, c_2\}.$$

Recalling (2.4), we gain

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{E}[(X_T)^\gamma] \le -\frac{1}{2}\min\Big\{\frac{-\gamma c_1}{(1-\gamma)^2}, c_2\Big\},$$

which along with (2.1) yields

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}[X_T \le \mathrm{e}^{\kappa T}] \le -\sup_{\gamma<0}\Big\{\gamma\kappa + \frac{1}{2}\min\Big\{\frac{-\gamma c_1}{(1-\gamma)^2}, c_2\Big\}\Big\}.$$

A proper choice of κ can ensure the negativeness of the right-hand side. The proof of Theorem 1.2 is complete.

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