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Pseudo Asymptotically Periodic Solutions for Volterra Difference Equations of Convolution Type*

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Abstract In this paper, the author studies the existence and uniqueness of discrete pseudo asymptotically periodic solutions for nonlinear Volterra difference equations of convolution type, where the nonlinear perturbation is considered as Lipschitz condition or non-Lipschitz case, respectively. The results are a consequence of application of different fixed point theorems, namely, the contraction mapping principle, the Leray-Schauder alternative theorem and Matkowski's fixed point technique.

Keywords Pseudo asymptotically periodic function, Volterra difference equations, Contraction mapping principle, Leray-Schauder alternative theorem
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1 Introduction

Besides its theoretical interest, the study of asymptotic ω-periodicity is of great importance in applications. Many contributions have been made to the study of existence of asymptotically ω-periodic solutions for differential equations (see [3, 7, 22, 31, 33] for more details). On the other hand, the notion of $\mathcal{S}$-asymptotic ω-periodicity, introduced by Henríquez et al. in [24–25], is related to and more general than that of asymptotic ω-periodicity. Since then, it has attracted the attention of many researchers (see [10, 15, 17, 27]). Particularly, for discrete $\mathcal{S}$-asymptotic ω-periodicity, the subject was studied in [2], where the authors discussed the existence of discrete $\mathcal{S}$-asymptotically ω-periodic solutions of semilinear difference equations with infinite delay. Recently, the concept of (continuous) pseudo $\mathcal{S}$-asymptotic ω-periodicity was introduced in [28] and some applications involving ordinary and partial differential equations were presented in [4, 12, 16, 23, 32]. This paper is a continuation of this study, which introduces the concept of discrete pseudo $\mathcal{S}$-asymptotic ω-periodicity and deals with its property.

In this paper, we study the existence and uniqueness of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic solutions of the Volterra difference equations of convolution type

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, Au(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$, $a(\cdot)$ is a summable function, A is a bounded linear operator on X and $f : \mathbb{Z}\times X \to X$ is a function bounded on bounded sets of X.

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Volterra difference equations arise in numerical discrete approximations of Volterra integral or integro-differential equations. The Volterra difference equations can be used in the modelling of real phenomena in economy and ecology, the theory of viscoelasticity and the study of optimal control problems (see [19–20]). The asymptotic behaviour of solutions of (1.1) is a classical subject of dynamical systems and operator theory. Many researchers have made important contributions to this topics, for example, almost periodicity (see [8, 30]), asymptotic almost periodicity (see [5, 29]), almost automorphy (see [1, 6, 11]), l^p-boundedness (see [9]), and $\mathcal{S}$-asymptotic ω-periodicity (see [2]). To our knowledge, there is no work reported in literature on pseudo $\mathcal{S}$-asymptotic ω-periodicity for (1.1). This is one of the key motivations of this study.

Motivated by the above mentioned papers, in this paper, we introduce a new class of functions called discrete pseudo $\mathcal{S}$-asymptotically ω-periodic functions, which generalize the notation of discrete $\mathcal{S}$-asymptotically ω-periodic functions. We systematically explore its properties in Banach spaces and discrete pseudo $\mathcal{S}$-asymptotic ω-periodicity of (1.1) when the nonlinear perturbation function f is considered as Lipschitz condition or non-Lipschitz case, respectively.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented, and the concept of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic functions is given. Sections 3 is divided into two parts. In Subsection 3.1, we investigate the existence and uniqueness of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic solution of (1.1) when f satisfies the Lipschitz condition. In Subsection 3.2, when f is non-Lipschitz, we explore the properties of solutions to the same equation.

2 Preliminaries and Basic Results

Let $(X, \|\cdot\|)$ be Banach space and $\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}, \mathbb{R}^+, \mathbb{C}$ stand for the sets of integers, nonnegative integers, real numbers, nonnegative real numbers, complex numbers, respectively. Let $B_r(X)$ be the closed ball with center at 0 and radius r in X. $\operatorname{card} E$ denotes the number of elements in any finite set $E \subset \mathbb{R}$. Let $v : \mathbb{Z}^+ \to \mathbb{C}$. If $\sum\limits_{k=0}^{\infty} |v(k)| < \infty$, we call that v is a summable function.

In order to facilitate the discussion below, we further introduce the following notations:

(1) $l^\infty(\mathbb{Z}, X) = \big\{x \mid \mathbb{Z} \to X : \|x\|_d = \sup\limits_{n\in\mathbb{Z}} \|x(n)\| < \infty\big\}$.

(2) $C_0(\mathbb{Z}, X) = \big\{x \in l^\infty(\mathbb{Z}, X) \mid \lim\limits_{|n|\to\infty} \|x(n)\| = 0\big\}$.

(3) $C_\omega(\mathbb{Z}, X) = \{x \in l^\infty(\mathbb{Z}, X) \mid x \text{ is } \omega\text{-periodic}\}$, where $\omega \in \mathbb{Z}\backslash\{0\}$.

(4) $L(X)$ denotes the space of bounded linear operators from X to X endowed with the operator topology.

(5) $\mathcal{UC}(\mathbb{Z} \times X, X)$ denotes the set of all functions $f : \mathbb{Z} \times X \to X$ satisfying that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$\|f(k, x) - f(k, y)\| \le \varepsilon$$

for all $k \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\| \le \delta$.

(6) $\mathcal{UC}_k(\mathbb{Z} \times X, X)$ denotes the set of all functions $f : \mathbb{Z} \times X \to X$ satisfying that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$\|f(k, x) - f(k, y)\| \le L_f(k)\varepsilon$$

for all $k \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\| \le \delta$, where $L_f : \mathbb{Z} \to \mathbb{R}^+$ is a summable function.

First, we recall a useful compactness criterion. Let $h : \mathbb{Z} \to \mathbb{R}^+$ be a function such that $h(n) \geq 1$ for all $n \in \mathbb{Z}$, and $h(n) \to \infty$ as $|n| \to \infty$. Define

$$C_h^0(\mathbb{Z}, X) = \Big\{ \xi : \mathbb{Z} \to X \,\Big|\, \lim_{|n|\to\infty} \frac{\|\xi(n)\|}{h(n)} = 0 \Big\},$$

endowed with the form $\|\xi\|_h = \sup\limits_{n\in\mathbb{Z}} \frac{\|\xi(n)\|}{h(n)}$.

It is clear that $C_h^0(\mathbb{Z}, X)$ is a Banach space isometrically isomorphic with the space $C_0(\mathbb{Z}, X)$. According to a compactness criterion due to Cuevas and Pinto [14], one has the following result.

Lemma 2.1 (see [1]) *Let S be a subset of $C_h^0(\mathbb{Z}, X)$. Suppose that the following conditions are satisfied*:

(i) *The set $\mathcal{H}_n(S) = \big\{ \frac{u(n)}{h(n)} \mid u \in S \big\}$ is relatively compact in X for all $n \in \mathbb{Z}$.*

(ii) *S is weighted equiconvergent at $\pm\infty$, that is for every $\varepsilon > 0$, there exists a $T > 0$ such that $\|u(n)\| < \varepsilon h(n)$ for each $|n| \geq T$ for all $u \in S$.*

Then S is relatively compact in $C_h^0(\mathbb{Z}, X)$.

Now, we recall the so-called Matkowski's fixed point theorem (see [26]) and the Leray-Schauder alternative theorem (see [21]) which will be used in the sequel.

Theorem 2.1 (Matkowski's Fixed Point Theorem) *Let (X, d) be a complete metric space and $\mathcal{F} : X \to X$ be a map such that $d(\mathcal{F}x, \mathcal{F}y) \leq \Phi(d(x, y))$ for all $x, y \in X$, where $\Phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim\limits_{n\to\infty} \Phi^n(t) = 0$ for $t > 0$. Then $\mathcal{F}$ has a unique fixed point $z \in X$.*

Theorem 2.2 (Leray-Schauder Alternative Theorem) *Let D be a closed convex subset of X such that $0 \in D$. Let $\Gamma : D \to D$ be a completely continuous map. Then the set $\{x \in D : \ x = \lambda\Gamma(x), \ 0 < \lambda < 1\}$ is unbounded or the map Γ has a fixed point in D.*

Next, we give the concept of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic function.

Definition 2.1 *$f \in l^\infty(\mathbb{Z}, X)$ is called discrete asymptotically ω-periodic if there exist $g \in C_\omega(\mathbb{Z}, X)$ and $\varphi \in C_0(\mathbb{Z}, X)$ such that $f = g + \varphi$. The collection of those functions is denoted by $AP_\omega(\mathbb{Z}, X)$.*

Definition 2.2 *$f \in l^\infty(\mathbb{Z}, X)$ is called discrete $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega \in \mathbb{Z}\backslash\{0\}$ such that $\lim\limits_{|n|\to\infty} \|f(n+\omega) - f(n)\| = 0$. The collection of those functions is denoted by $\mathcal{S}AP_\omega(\mathbb{Z}, X)$.*

Definition 2.3 *A sequence $f \in l^\infty(\mathbb{Z}, X)$ is called discrete pseudo $\mathcal{S}$-asymptotically ω-periodic if there exists $\omega \in \mathbb{Z}\backslash\{0\}$ such that $\lim\limits_{n\to\infty} \frac{1}{2n} \sum\limits_{k=-n}^{n} \|f(k+\omega) - f(k)\| = 0$. Denote by $P\mathcal{S}AP_\omega(\mathbb{Z}, X)$ the set of such functions.*

It is easy to see that $P\mathcal{S}AP_\omega(\mathbb{Z}, X)$ is a Banach space when endowed with the norm $\|f\|_d := \sup\limits_{n\in\mathbb{Z}} \|f(n)\|$ and $\mathcal{S}AP_\omega(\mathbb{Z}, X) \subset P\mathcal{S}AP_\omega(\mathbb{Z}, X)$.

Definition 2.4 *A sequence $f \in l^\infty(\mathbb{Z} \times X, X)$ is called uniformly discrete pseudo $\mathcal{S}$-*

asymptotically ω-periodic on bounded sets of X if for every bounded subset $K \subseteq X$,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \sup_{x\in K} \|f(k+\omega, x) - f(k,x)\| = 0.$$

Denote by $PSAP_\omega(\mathbb{Z}\times X, X)$ the set of such functions.

Finally, we show some properties of $PSAP_\omega(\mathbb{Z}, X)$. We have the following results.

Lemma 2.2 *If $A \in L(X)$ and $u \in PSAP_\omega(\mathbb{Z}, X)$, then $Au \in PSAP_\omega(\mathbb{Z}, X)$.*

Lemma 2.3 *Let $f \in PSAP_\omega(\mathbb{Z}, X)$. Then $f(\cdot+\tau) \in PSAP_\omega(\mathbb{Z}, X)$ for all $\tau \in \mathbb{Z}$.*

Lemma 2.4 *Let $f \in l^\infty(\mathbb{Z}, X)$. Then $f \in PSAP_\omega(\mathbb{Z}, X)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\operatorname{card} E_f(n,\varepsilon)}{2n} = 0,$$

where $E_f(n,\varepsilon) = \{k \in [-n,n]\cap\mathbb{Z} \mid \|f(k+\omega) - f(k)\| \geq \varepsilon\}$.

The proof of Lemma 2.4 is similar to that of [18, Lemma 2.9]. Here we omit it.

Theorem 2.3 *Let $f : \mathbb{Z}\times X \to X$ be a function bounded on bounded sets of X. Assume that $f \in PSAP_\omega(\mathbb{Z}\times X, X) \cap \mathcal{UC}_k(\mathbb{Z}\times X, X)$. Then $\psi(\cdot) = f(\cdot, u(\cdot)) \in PSAP_\omega(\mathbb{Z}, X)$ if $u \in PSAP_\omega(\mathbb{Z}, X)$.*

Proof Since $f \in \mathcal{UC}_k(\mathbb{Z}\times X, X)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(k, u(k+\omega)) - f(k, u(k))\| \leq L_f(k)\varepsilon$ for all $k \in \mathbb{Z}$, $\|u(k+\omega) - u(k)\| \leq \delta$. Let $K = \overline{\{u(n) \mid n \in \mathbb{Z}\}}$. Then for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, $\frac{1}{2n}\sum\limits_{k=-n}^{n} \sup\limits_{x\in K} \|f(k+\omega, x) - f(k,x)\| \leq \varepsilon$. Denote

$$E_u(n,\delta) = \{k \in [-n,n]\cap\mathbb{Z} \mid \|u(k+\omega) - u(k)\| \geq \delta\}.$$

Then $\lim\limits_{n\to\infty} \frac{\operatorname{card} E_u(n,\delta)}{2n} = 0$ by Lemma 2.4. So

$$\begin{aligned}
&\frac{1}{2n}\sum_{k=-n}^{n} \|f(k, u(k+\omega)) - f(k, u(k))\| \\
={}& \frac{1}{2n}\sum_{k\in E_u(n,\delta)} \|f(k, u(k+\omega)) - f(k, u(k))\| \\
&+ \frac{1}{2n}\sum_{k\in([-n,n]\cap\mathbb{Z})\setminus E_u(n,\delta)} \|f(k, u(k+\omega)) - f(k, u(k))\| \\
\leq{}& 2\|\psi\|_d \frac{\operatorname{card} E_u(n,\delta)}{2n} + \frac{\|L_f\|_1}{2n}\varepsilon,
\end{aligned}$$

where $\|\psi\|_d = \sup\limits_{n\in\mathbb{Z}} \|\psi(n)\|$ and $\|L_f\|_1 = \sum\limits_{k\in\mathbb{Z}} L_f(k)$. For $n > N$, one has

$$\begin{aligned}
&\frac{1}{2n}\sum_{k=-n}^{n} \|f(k+\omega, u(k+\omega)) - f(k, u(k))\| \\
\leq{}& \frac{1}{2n}\sum_{k=-n}^{n} \|f(k+\omega, u(k+\omega)) - f(k, u(k+\omega))\| + \frac{1}{2n}\sum_{k=-n}^{n} \|f(k, u(k+\omega)) - f(k, u(k))\|
\end{aligned}$$

$$\leq \varepsilon + 2\|\psi\|_d \frac{\operatorname{card} E_u(n,\delta)}{2n} + \frac{\|L_f\|_1}{2n}\varepsilon,$$

which implies that $\psi(\cdot) \in PSAP_\omega(\mathbb{Z}, X)$.

Corollary 2.1 *Let $f : \mathbb{Z} \times X \to X$ be a function bounded on bounded sets of X. Assume that $f \in PSAP_\omega(\mathbb{Z} \times X, X)$ and satisfies the following Lipschitze type condition:*

$$\|f(k,u) - f(k,v)\| \leq L_f(k)\|u - v\|, \quad \forall k \in \mathbb{Z},\ u,v \in X,$$

where $L_f : \mathbb{Z} \to \mathbb{R}^+$ is a summable function. Then $f(\cdot, u(\cdot)) \in PSAP_\omega(\mathbb{Z}, X)$ if $u \in PSAP_\omega(\mathbb{Z}, X)$.

By making some revisions of the proof of Theorem 2.3, one get the following conclusions.

Theorem 2.4 *Let $f : \mathbb{Z} \times X \to X$ be a function bounded on bounded sets of X. Assume that $f \in PSAP_\omega(\mathbb{Z} \times X, X) \cap \mathcal{UC}(\mathbb{Z} \times X, X)$. Then $f(\cdot, u(\cdot)) \in PSAP_\omega(\mathbb{Z}, X)$ if $u \in PSAP_\omega(\mathbb{Z}, X)$.*

Corollary 2.2 *Let $f : \mathbb{Z} \times X \to X$ be a function bounded on bounded sets of X. Assume that $f \in PSAP_\omega(\mathbb{Z} \times X, X)$ and there exists a constant $L_f > 0$ such that*

$$\|f(k,u) - f(k,v)\| \leq L_f\|u - v\|, \quad \forall k \in \mathbb{Z},\ u,v \in X.$$

Then $f(\cdot, u(\cdot)) \in PSAP_\omega(\mathbb{Z}, X)$ if $u \in PSAP_\omega(\mathbb{Z}, X)$.

Lemma 2.5 *Let $v : \mathbb{Z}^+ \to \mathbb{C}$ be a summable function. Then $\Psi(\cdot) \in PSAP_\omega(\mathbb{Z}, X)$ if $u \in PSAP_\omega(\mathbb{Z}, X)$, where $\Psi(k) = \sum_{l=-\infty}^{k} |v(k-l)|u(l)$, $k \in \mathbb{Z}$.*

Proof Note that

$$\|\Psi(k)\| \leq \sum_{l=0}^{\infty} |v(l)|\|u(k-l)\| \leq \|u\|_d \sum_{l=0}^{\infty} |v(l)| < \infty,$$

$$\|\Psi(k+\omega) - \Psi(k)\| \leq \sum_{l=-\infty}^{k} |v(k-l)|\|u(l+\omega) - u(l)\| = \sum_{l=0}^{\infty} |v(l)|\|u(k-l+\omega) - u(k-l)\|.$$

Then

$$\begin{aligned} \frac{1}{2n}\sum_{k=-n}^{n} \|\Psi(k+\omega) - \Psi(k)\| &\leq \frac{1}{2n}\sum_{k=-n}^{n}\sum_{l=0}^{\infty} |v(l)|\|u(k-l+\omega) - u(k-l)\| \\ &= \sum_{l=0}^{\infty} |v(l)|\Big(\frac{1}{2n}\sum_{k=-n}^{n} \|u(k-l+\omega) - u(k-l)\|\Big). \end{aligned}$$

By Lemma 2.3 and Lebesgue dominated convergence theorem, one has $\Psi(\cdot) \in PSAP_\omega(\mathbb{Z}, X)$.

3 Volterra Difference Equation

This section is devoted to establish some sufficient criteria for the existence and uniqueness of $PSAP_\omega$ solutions of (1.1).

Consider the linear Volterra difference equations

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n), \quad n \in \mathbb{Z}, \tag{3.1}$$

where $\lambda \in \mathbb{C}$, $a(\cdot)$ is a summable function.

For a given $\lambda \in \mathbb{C}$, let $s(\lambda, k) \in \mathbb{C}$ be the solution of the difference equation

$$\begin{aligned} s(\lambda, k+1) &= \lambda \sum_{j=0}^{k} a(k-j) s(\lambda, j), \quad k = 0, 1, 2, \cdots, \\ s(\lambda, 0) &= 1. \end{aligned} \tag{3.2}$$

In this case, $s(\lambda, k)$ is called the fundamental solution to (3.1) generated by $a(\cdot)$. We define the set

$$\Omega_s = \Big\{ \lambda \in \mathbb{C} \,\Big|\, \|s(\lambda, \cdot)\|_1 := \sum_{k=0}^{\infty} |s(\lambda, k)| < \infty \Big\}.$$

By [11], if $\lambda \in \Omega_s$, the solution to (3.1) is given by

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k).$$

To establish our results, we introduce the following conditions:

(H1) $\lambda \in \Omega_s$, $A \in L(X)$.

(H2) $f \in PSAP_\omega(\mathbb{Z} \times X, X)$.

(H3$_1$) There exists a constant $L_f > 0$ such that

$$\|f(k, u) - f(k, v)\| \le L_f \|u - v\|, \quad \forall k \in \mathbb{Z}, \ u, v \in X.$$

(H3$_2$) There exists a linear nondecreasing function $\Phi : [0, \infty) \to [0, \infty)$ and f satisfies

$$\|f(k, u) - f(k, v)\| \le \Phi(\|u - v\|), \quad \forall k \in \mathbb{Z}, \ u, v \in X.$$

(H3$_3$) f satisfies the following Lipschitze type condition:

$$\|f(k, u) - f(k, v)\| \le L_f(k) \|u - v\|, \quad \forall k \in \mathbb{Z}, \ u, v \in X,$$

where $L_f : \mathbb{Z} \to \mathbb{R}^+$ is a summable function.

(H3$_4$) f satisfies the locally Lipschitze condition, that is, for each $\sigma > 0$, $k \in \mathbb{Z}$ and $u, v \in X$ with $\|u\| \le \sigma$, $\|v\| \le \sigma$, one has

$$\|f(k, u) - f(k, v)\| \le L_f(\sigma) \|u - v\|,$$

where $L_f : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

(H4) $f \in \mathcal{UC}_k(\mathbb{Z} \times X, X)$ or $f \in \mathcal{UC}(\mathbb{Z} \times X, X)$.

3.1 Lipschitz case

In this subsection, we study the existence and uniqueness of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic solution of (1.1) when the perturbation f satisfies the Lipschitz condition.

Theorem 3.1 *Assume that* (H1), (H2), (H3$_1$) *hold and* $L_f \|A\| |s(\lambda, \cdot)|_1 < 1$. *Then* (1.1) *has a unique solution* $u \in PSAP_\omega(\mathbb{Z}, X)$ *which is given by*

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k, Au(k)). \tag{3.3}$$

Proof Similar as the proof in [11], it can be shown that $u(\cdot)$ given by (3.3) is the solution to (1.1).

Define the operator $\mathcal{F} : PSAP_\omega(\mathbb{Z}, X) \to PSAP_\omega(\mathbb{Z}, X)$ by

$$(\mathcal{F}u)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n-1-k) f(k, Au(k)). \tag{3.4}$$

Since $u \in PSAP_\omega(\mathbb{Z}, X)$ and (H3$_1$) holds, $f(\cdot, Au(\cdot)) \in PSAP_\omega(\mathbb{Z}, X)$ by Lemma 2.2 and Corollary 2.2. By Lemma 2.5, $\mathcal{F}u \in PSAP_\omega(\mathbb{Z}, X)$. Hence $\mathcal{F}$ is well defined.

For $u, v \in PSAP_\omega(\mathbb{Z}, X)$,

$$\begin{aligned}\|\mathcal{F}u - \mathcal{F}v\|_d &\le \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\| \\ &\le L_f \|A\| \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|u(k) - v(k)\| \\ &\le L_f \|A\| |s(\lambda, \cdot)|_1 \|u - v\|_d.\end{aligned}$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $u \in PSAP_\omega(\mathbb{Z}, X)$, which is the unique $PSAP_\omega$ solution to (1.1).

Example 3.1 For $a(k) = p^k$, where $|p| < 1$, after a calculation using in (3.2) the unilateral-Z transform, we have $s(\lambda, k) = \lambda(\lambda + p)^{k-1}$, $k \ge 1$, and define

$$\mathbb{D}(-p, 1) := \{z \in \mathbb{C} \mid |z + p| < 1\} \subseteq \Omega_s.$$

Consider the following difference equation:

$$u(n+1) = \lambda \sum_{k=-\infty}^{n} p^{n-k} u(k) + \mu g(k) u(k), \quad n \in \mathbb{Z}, \tag{3.5}$$

where $|p| < 1$, $\lambda \in \mathbb{D}(-p, 1)$, $g \in PSAP_\omega(\mathbb{Z}, X)$. It is easy to see that (H1), (H2), (H3$_1$) hold with $L_f = |\mu| \|g\|_d$. By Theorem 3.1, if $|\lambda| |\mu| \|g\|_d \sum\limits_{k=0}^{\infty} |\lambda + p|^{k-1} < 1$, then (3.5) has a unique solution $u \in PSAP_\omega(\mathbb{Z}, X)$.

Theorem 3.2 *Assume that* (H1), (H2), (H3$_2$) *hold. Then* (1.1) *has a unique solution* $u \in PSAP_\omega(\mathbb{Z}, X)$ *if* $(\|A\| |s(\lambda, \cdot)|_1 \Phi)^n(t) \to 0$ *as* $n \to \infty$ *for each* $t > 0$.

Proof Define the operator $\mathcal{F}$ as in (3.4), so $\mathcal{F}$ is well defined. For $u, v \in PSAP_\omega(\mathbb{Z}, X)$, one has

$$\begin{aligned}\|(\mathcal{F}u)(n) - (\mathcal{F}v)(n)\| &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\| \\ &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \Phi(\|Au(k) - Av(k)\|) \\ &\le \|A\| |s(\lambda, \cdot)|_1 \Phi(\|u(k) - v(k)\|).\end{aligned}$$

Since $(\|A\| |s(\lambda, \cdot)|_1 \Phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$, by Theorem 2.1, $\mathcal{F}$ has a unique fixed point $u \in PSAP_\omega(\mathbb{Z}, X)$, which is the unique $PSAP_\omega$ solution to (1.1).

Theorem 3.3 *Assume that* (H1), (H2), (H3$_3$) *hold. Then* (1.1) *has a unique solution* $u \in PSAP_\omega(\mathbb{Z}, X)$.

Proof Define the operator $\mathcal{F}$ as in (3.4), and $\mathcal{F}$ is well defined by Corollary 2.1 and Lemma 2.5. For $u, v \in PSAP_\omega(\mathbb{Z}, X)$, one has

$$\begin{aligned}\|(\mathcal{F}u)(n) - (\mathcal{F}v)(n)\| &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\| \\ &\le \|A\| \sum_{k=-\infty}^{n-1} L_f(k)|s(\lambda, n-1-k)| \|u(k) - v(k)\| \\ &\le \|A\| |s(\lambda, \cdot)|_\infty \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big) \|u - v\|_d,\end{aligned}$$

where $|s(\lambda, \cdot)|_\infty = \sup\limits_{n \in \mathbb{Z}} |s(\lambda, n)|$.

Similarly, by [13, Lemma 3.2], one has

$$\begin{aligned}\|(\mathcal{F}^2u)(n) - (\mathcal{F}^2v)(n)\| &\le \|A\| \sum_{k=-\infty}^{n-1} L_f(k)|s(\lambda, n-1-k)| \|(\mathcal{F}u)(k) - (\mathcal{F}v)(k)\| \\ &\le (\|A\| |s(\lambda, \cdot)|_\infty)^2 \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big(\sum_{j=-\infty}^{k-1} L_f(j) \Big) \Big) \|u - v\|_d, \\ &\le \frac{(\|A\| |s(\lambda, \cdot)|_\infty)^2}{2!} \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big)^2 \|u - v\|_d.\end{aligned}$$

By the method of mathematical induction, we have

$$\|(\mathcal{F}^nu)(n) - (\mathcal{F}^nv)(n)\| \le \frac{(\|A\| |s(\lambda, \cdot)|_\infty)^n}{n!} \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big)^n \|u - v\|_d.$$

Moreover, since L_f is a summable function, defining $\|L_f\|_1 := \sum\limits_{k \in \mathbb{Z}} L_f(k)$, one has

$$\|(\mathcal{F}^nu)(n) - (\mathcal{F}^nv)(n)\| \le \frac{(\|A\| |s(\lambda, \cdot)|_\infty \|L_f\|_1)^n}{n!} \|u - v\|_d,$$

which implies that $\|(\mathcal{F}^nu) - (\mathcal{F}^nv)\|_d \le \frac{(\|A\| |s(\lambda, \cdot)|_\infty \|L_f\|_1)^n}{n!} \|u - v\|_d$. For sufficiently large n, we have $\frac{(\|A\| |s(\lambda, \cdot)|_\infty \|L_f\|_1)^n}{n!} < 1$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PSAP_\omega(\mathbb{Z}, X)$, which is the unique $PSAP_\omega$ solution to (1.1).

Next, consider with the local condition on the perturbation f, we have the following result.

Theorem 3.4 *Assume that* (H1), (H2), (H3$_4$) *hold, and if there exists* $r > 0$ *such that*

$$|s(\lambda, \cdot)|_1 \Big(L_f(\|A\| r) \|A\| + \frac{\|f(\cdot, 0)\|_d}{r} \Big) < 1, \tag{3.6}$$

then (1.1) *has a unique solution* $u \in B_r(PSAP_\omega(\mathbb{Z}, X))$.

Proof Let u be in $B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$ and define

$$\mathcal{F}: B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X)) \to B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$$

by

$$(\mathcal{F}u)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n-1-k) f(k, Au(k)).$$

Similar as the proof of Theorem 3.1, $\mathcal{F}u \in P\mathcal{S}AP_\omega(\mathbb{Z},X)$. Let $u \in B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$. One has

$$\begin{aligned}
\|\mathcal{F}u)(n)\| &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k,0)\| + \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k,0)\| \\
&\le L_f(\|A\|r) \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|Au\|_d + |s(\lambda,\cdot)|_1 \|f(\cdot,0)\|_d \\
&\le |s(\lambda,\cdot)|_1 \Big(L_f(\|A\|r)\|A\| + \frac{\|f(\cdot,0)\|_d}{r} \Big) r \le r.
\end{aligned}$$

Hence $\mathcal{F}u \in B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$ and $\mathcal{F}$ is well defined.

Moreover, for $u, v \in B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$,

$$\begin{aligned}
\|\mathcal{F}u - \mathcal{F}v\|_d &\le \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\| \\
&\le L_f(\|A\|r) \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|Au(k) - Av(k)\| \\
&\le L_f(\|A\|r)\|A\| |s(\lambda,\cdot)|_1 \|u - v\|_d.
\end{aligned}$$

By (3.6), $L_f(\|A\|r)\|A\||s(\lambda,\cdot)|_1 < 1$. It follows that $\mathcal{F}$ is a contraction on $B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $B_r(P\mathcal{S}AP_\omega(\mathbb{Z},X))$, which is the unique $P\mathcal{S}AP_\omega$ solution to (1.1).

3.2 Non-Lipschitz case

In this subsection, we study the existence of discrete pseudo $\mathcal{S}$-asymptotically ω-periodic solution of (1.1) when the perturbation f is a non-Lipschitz nonlinearity.

Theorem 3.5 *Assume that* (H1), (H2), (H4) *hold and the following conditions are satisfied*:

(A1) *There are nondecreasing function* $W: \mathbb{R}^+ \to \mathbb{R}^+$ *and a function* $M: \mathbb{Z} \to \mathbb{R}^+$ *such that* $\|f(k,x)\| \le M(k)W(\|x\|)$ *for all* $k \in \mathbb{Z}$, $x \in X$.

(A2) *For each* $\nu > 0$, $\lim_{|n|\to\infty} \frac{1}{h(n+1)} \sum_{k=-\infty}^{n} |s(\lambda, n-k)| M(k) W(\nu h(k)) = 0$, *where* h *is given by Lemma* 2.1.

(A3) *For each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every* $u, v \in C_h^0(\mathbb{Z},X)$, $\|u - v\|_h \le \delta$ *implies that*

$$\sum_{k=-\infty}^{n} |s(\lambda, n-k)| \|f(k, Au(k)) - f(k, Av(k))\| \le \varepsilon$$

for all $n \in \mathbb{Z}$.

(A4) *For all* $a, b \in \mathbb{Z}$, $a \le b$, $\sigma > 0$, *the set* $\{f(k,x) \mid a \le k \le b,\ \|x\| \le \sigma\}$ *is relatively compact in* X.

(A5) $\liminf\limits_{r\to\infty} \frac{r}{\widetilde{\beta}(r)} > 1$, *where* $\widetilde{\beta}(r) = \sup\limits_{n\in\mathbb{Z}} \big(\frac{1}{h(n+1)} \sum\limits_{k=-\infty}^{n} |s(\lambda, n-k)| M(k) W(r\|A\| h(k))\big)$.

Then (1.1) *has a solution* $u \in PSAP_\omega(\mathbb{Z}, X)$.

Proof Define $\Gamma : C_h^0(\mathbb{Z}, X) \to C_h^0(\mathbb{Z}, X)$ by

$$(\Gamma u)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n-1-k) f(k, Au(k)).$$

Next, we will prove that Γ has a fixed point in $PSAP_\omega(\mathbb{Z}, X)$. We divide the proof into several steps.

(i) For $u \in C_h^0(\mathbb{Z}, X)$, by (A1), one has

$$\begin{aligned}\|(\Gamma u)(n)\| &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| M(k) W(\|A\| \|u(k)\|) \\ &\le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| M(k) W(\|A\| \|u\|_h h(k)),\end{aligned}$$

whence

$$\frac{\|(\Gamma u)(n)\|}{h(n)} \le \frac{1}{h(n)} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| M(k) W(\|A\| \|u\|_h h(k)).$$

It follows from (A2) that Γ is well defined.

(ii) Γ is continuous. In fact, for each $\varepsilon > 0$, by (A3), there exists $\delta > 0$, such that for $u, v \in C_h^0(\mathbb{Z}, X)$, $\|u - v\|_h \le \delta$, one has

$$\|\Gamma u - \Gamma v\| \le \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \|f(k, Au(k)) - f(k, Av(k))\|.$$

Taking into account that $h(n) \ge 1$, by (A3), one has $\frac{\|\Gamma u - \Gamma v\|}{h(n)} \le \varepsilon$, which implies that $\|\Gamma u - \Gamma v\|_h \le \varepsilon$. Hence Γ is continuous.

(iii) Γ is complete continuous. Let $V = \Gamma(B_r(C_h^0(\mathbb{Z}, X)))$ and $v = \Gamma(u)$ for $u \in B_r(C_h^0(\mathbb{Z}, X))$. Initially, we prove that $\mathcal{H}_n(V) := \big\{\frac{v(n)}{h(n)} : v \in V\big\}$ is relatively compact in X for each $n \in \mathbb{Z}$. By (A2), for $\varepsilon > 0$, we can choose $l \in \mathbb{Z}^+$ such that

$$\frac{1}{h(n)} \sum_{k=l}^{\infty} |s(\lambda, k)| M(n-1-k) W(r\|A\| h(n-1-k)) \le \varepsilon.$$

Since $v = \Gamma(u)$ for $u \in B_r(C_h^0(\mathbb{Z}, X))$, we have

$$v(n) = \sum_{k=0}^{l-1} s(\lambda, k) f(n-1-k, Au(n-1-k)) + \sum_{k=l}^{\infty} s(\lambda, k) f(n-1-k, Au(n-1-k)),$$

so

$$\frac{v(n)}{h(n)} = \frac{l}{h(n)} \Big(\frac{1}{l} \sum_{k=0}^{l-1} s(\lambda, k) f(n-1-k, Au(n-1-k)) \Big)$$

$$+\frac{1}{h(n)}\sum_{k=l}^{\infty}s(\lambda,k)f(n-1-k,Au(n-1-k)).$$

Note that

$$\begin{aligned}&\frac{1}{h(n)}\Big\|\sum_{k=l}^{\infty}s(\lambda,k)f(n-1-k,Au(n-1-k))\Big\|\\ &\le\frac{1}{h(n)}\sum_{k=l}^{\infty}|s(\lambda,k)|M(n-1-k)W(\|A\|\|u\|_h h(n-1-k))\\ &\le\frac{1}{h(n)}\sum_{k=l}^{\infty}|s(\lambda,k)|M(n-1-k)W(r\|A\|h(n-1-k))\le\varepsilon.\end{aligned}$$

So

$$\frac{v(n)}{h(n)}\in\frac{l}{h(n)}\overline{\mathrm{co}(K)}+B_\varepsilon(X),$$

where $\overline{\mathrm{co}(K)}$ denotes the convex hull of K and

$$K=\bigcup_{k=0}^{l-1}\{s(\lambda,k)f(\xi,x)\mid\xi\in[n-l,n-1]\cap\mathbb{Z},\|x\|\le R\},$$

where $R=r\max_{\xi\in[n-l,n-1]\cap\mathbb{Z}}h(\xi)$, and K is relatively compact by (A4). Since $\mathcal{H}_n(V)\subseteq\frac{l}{h(n)}\overline{\mathrm{co}(K)}+B_\varepsilon(X)$, we infer that $\mathcal{H}_n(V)$ is relatively compact in X for all $n\in\mathbb{Z}$.

Next, we show that V is weighted equiconvergent at $\pm\infty$. In fact,

$$\frac{\|v(n)\|}{h(n)}\le\frac{1}{h(n)}\sum_{k=-\infty}^{n-1}|s(\lambda,n-1-k)|M(k)W(r\|A\|h(k)),$$

so $\frac{\|v(n)\|}{h(n)}\to0$ as $|n|\to\infty$ and this convergence is independent of $u\in B_r(C_h^0(\mathbb{Z},X))$. Hence V satisfies Lemma 2.1(i)–(ii), which completes the proof that V is a relatively compact set in $C_h^0(\mathbb{Z},X)$.

(iv) If $u^\lambda\in C_h^0(\mathbb{Z},X)$ is a solution of the equation $u^\lambda=\lambda\Gamma(u^\lambda)$ for some $0<\lambda<1$, then

$$\|u^\lambda(n)\|\le\sum_{k=-\infty}^{n-1}|s(\lambda,n-1-k)|M(k)W(\|A\|\|u^\lambda\|_h h(k))\le h(n)\widetilde{\beta}(\|u^\lambda\|_h).$$

Hence, one has

$$\frac{\|u^\lambda\|_h}{\widetilde{\beta}(\|u^\lambda\|_h)}\le1,$$

and by (A5), we conclude that the set $\{u^\lambda:u^\lambda=\lambda\Gamma(u^\lambda),\lambda\in(0,1)\}$ is bounded.

(v) It follows from Theorems 2.3–2.4 and Lemma 2.5 that $\Gamma(PSAP_\omega(\mathbb{Z},X))\subseteq PSAP_\omega(\mathbb{Z},X)$. Similar to the proof of (iv), we claim that there exists $r_0>0$ such that $\Gamma(B_{r_0}(C_h^0(\mathbb{Z},X)))\subseteq B_{r_0}(C_h^0(\mathbb{Z},X))$. Consequently, we infer that

$$\Gamma(B_{r_0}(C_h^0(\mathbb{Z},X))\cap PSAP_\omega(\mathbb{Z},X))\subset B_{r_0}(C_h^0(\mathbb{Z},X))\cap PSAP_\omega(\mathbb{Z},X).$$

Hence we derive the following conclusion:

$$\Gamma(\overline{B_{r_0}(C_h^0(\mathbb{Z},X))\cap PSAP_\omega(\mathbb{Z},X)}^{C_h^0(\mathbb{Z},X)})\subseteq\overline{\Gamma(B_{r_0}(C_h^0(\mathbb{Z},X))\cap PSAP_\omega(\mathbb{Z},X))}^{C_h^0(\mathbb{Z},X)}$$

$$\subseteq \overline{B_{r_0}(C_h^0(\mathbb{Z},X)) \cap PSAP_\omega(\mathbb{Z},X)}^{C_h^0(\mathbb{Z},X)},$$

where $\overline{B}^{C_h^0(\mathbb{Z},X)}$ denotes the closure of a set B in the space $C_h^0(\mathbb{Z},X)$. Consider the operator

$$\Gamma : \overline{B_{r_0}(C_h^0(\mathbb{Z},X)) \cap PSAP_\omega(\mathbb{Z},X)}^{C_h^0(\mathbb{Z},X)} \to \overline{B_{r_0}(C_h^0(\mathbb{Z},X)) \cap PSAP_\omega(\mathbb{Z},X)}^{C_h^0(\mathbb{Z},X)}.$$

By (i)–(iii), we see that Γ is completely continuous. Applying (iv) and Theorem 2.2, we deduce that Γ has a fixed point $u \in \overline{B_{r_0}(C_h^0(\mathbb{Z},X)) \cap PSAP_\omega(\mathbb{Z},X)}^{C_h^0(\mathbb{Z},X)}$.

Let u_n be a sequence in $B_{r_0}(C_h^0(\mathbb{Z},X)) \cap PSAP_\omega(\mathbb{Z},X)$ such that it converges to u in the norm $C_h^0(\mathbb{Z},X)$. For $\varepsilon > 0$, let $\delta > 0$ be the constant in (A3). There exists $n_0 \in \mathbb{Z}^+$ such that $\|u_n - u\|_h \le \delta$ for all $n \ge n_0$. For $n \ge n_0$,

$$\|\Gamma u_n - \Gamma u\|_d \le \sup_{m\in\mathbb{Z}} \sum_{k=-\infty}^{m-1} |s(\lambda, m-1-k)| \|f(k, Au_n(k)) - f(k, Au(k))\| \le \varepsilon,$$

which implies that $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $\mathbb{Z}$. Whence $u \in PSAP_\omega(\mathbb{Z},X)$.

Corollary 3.1 *Assume that* (H1)–(H2) *hold and the following conditions are satisfied*

(a) $f(k,0) = q(k)$ *for* $k \in \mathbb{Z}$.

(b) f *satisfies the Hölder type condition*

$$\|f(k,u) - f(k,v)\| \le C_1\|u-v\|^\alpha, \quad u,v \in X,\ k \in \mathbb{Z}.$$

where $0 < \alpha < 1$, $C_1 > 0$ *is a constant.*

(c) *For all* $a,b \in \mathbb{Z}$, $a \le b$, $\sigma > 0$, *the set* $\{f(k,x) : a \le k \le b, \|x\| \le \sigma\}$ *is relatively compact in* X.

Then (1.1) *has a solution* $u \in PSAP_\omega(\mathbb{Z},X)$.

Proof By (c), it is easy to see that (A4) holds. Let $C_0 = \|q\|_d$, $M(\cdot) = 1$ and $W(\xi) = C_0 + C_1\xi^\alpha$. Then (A1) is satisfied. Take a function h such that $\sup_{n\in\mathbb{Z}}\big(\sum_{k=-\infty}^{n} |s(\lambda, n-k)| h(k)^\alpha\big) := C_2 < \infty$. It is not difficult to see that (A2) is satisfied. To verify (A3), note that for each $\varepsilon > 0$, there exists $0 < \delta < \frac{1}{\|A\|}\big(\frac{\varepsilon}{C_1C_2}\big)^{\frac{1}{\alpha}}$, such that for every $u,v \in C_h^0(\mathbb{Z},X)$, $\|u-v\|_h \le \delta$ implies that

$$\begin{aligned}&\sum_{k=-\infty}^{n} |s(\lambda,n-k)| \|f(k,Au(k)) - f(k,Av(k))\| \\ &\le C_1\|A\|^\alpha \sum_{k=-\infty}^{n} |s(\lambda,n-k)| \|u(k)-v(k)\|^\alpha \\ &\le C_1\|A\|^\alpha \sum_{k=-\infty}^{n} |s(\lambda,n-k)| h(k)^\alpha \|u-v\|_h^\alpha \le \varepsilon\end{aligned}$$

for all $n \in \mathbb{Z}$. Moreover, (A5) can be easily verified using the definition of W. By Theorem 3.5, (1.1) has a solution $u \in PSAP_\omega(\mathbb{Z},X)$.

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