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Approximate Forward Attractors of Non-Autonomous Dynamical Systems*

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Abstract In this paper the forward asymptotical behavior of non-autonomous dynamical systems and their attractors are investigated. Under general conditions, the authors show that every neighborhood of pullback attractor has forward attracting property.

Keywords Non-autonomous dynamical systems, Pullback attractors, Forward attractors, Uniform attractors, Approximate forward attractors
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1 Introduction

For an autonomous dynamical system, the global attractor, if exists, will capture all the asymptotic behavior in itself. As for non-autonomous dynamical systems, however, the situation is much more complicated. Three distinct notions of attraction: Pullback attraction, forward attraction and uniform attraction give rise to three distinct notions of attractor: Pullback attractor, forward attractor and uniform attractor. Simply speaking, a pullback or forward attractor is a family of nonempty compact subsets of phase space X driven by the base space Σ, which is invariant under a cocycle φ and attracting in the corresponding sense, and a uniform attractor is the minimal closed set of X that has uniformly attracting property. The notion of pullback attractor for non-autonomous dynamical systems can be regarded as a natural generalization of the concept of attractor for autonomous dynamical systems. The existence results have been obtained for numerous non-autonomous dynamical systems (see [6, 9–10, 19]). In contrast, forward attractor seems to be physically natural, but rarely exist mathematically. They will exist only in some very specific and restrictive situations (see [16–17]). The existence and the structure of uniform attractors have also been well-studied by many authors (see [11, 18, 22–23]). The "kernel sections" provide much finer information about the "structure" of the uniform attractor (see [6, 21]).

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Note that the existence of a compact uniform attractor for a non-autonomous dynamical system can imply the existence of a pullback attractor, but may not guarantee the existence of a forward attractor. More information on the relationship between the uniform attractor and the pullback attractor can be found in [4–5]. At the same time we need to be aware that pullback attractor and forward attractor are usually completely unrelated (see [1–3, 8, 12–14]). But if both the forward attractor and the pullback attractor exist, they will coincide. Indeed, as the biggest invariant compact set of φ, the forward attractor will contain all of the pullback ω-limit set of any bounded set in X. Thus it is natural to consider under what conditions the pullback attractor might have forward attracting property. One simple case in which the pullback attractor is uniformly pullback attracting, it must be a forward one. Besides, if the pullback attractor $\mathscr{A}(\sigma)$ is continuous in σ (in fact, it is usually only upper semicontinuous), the pullback attractor will deduce to a forward one (see [20]). Unfortunately, many facts have shown that a pullback attractor in general does not have forward attracting property (except for some particular cases such as [15, 24]). Now we are faced with a very natural question: How to illustrate the forward asymptotical behavior of a non-autonomous dynamical system in the case that it actually dose not have a forward attractor? This is a very realistic problem even for the most general non-autonomous system, since its many physical characteristics will related to its forward dynamics.

Assume the base space Σ is compact and the skew-product flow (φ, θ) is uniformly asymptotically compact, the authors in [20] have proved that, for any $\varepsilon > 0$, $A(\sigma)(\varepsilon) \triangleq \bigcup_{\rho(\sigma',\sigma)\leq\varepsilon} \mathscr{A}(\sigma')$, which is called the parametrically inflated pullback attractor, uniformly forward attracts any bounded subset $B \subset X$. However, $A(\sigma)(\varepsilon)$ is only negative invariant and may be much larger than $\mathscr{A}(\sigma)$. If one wants a better description of the forward dynamics of non-autonomous system, it is necessary to get more refined sets with forward attracting property. In this paper, we show that, under the same conditions as in [20], in any ε-neighborhood of $\mathscr{A}(\sigma)$ there exists a forward invariant subset $\mathscr{A}_\varepsilon(\sigma)$ of X which forward attracts any bounded set in phase space X uniformly in σ. We call $\mathscr{A}_\varepsilon(\sigma)$ an approximate global forward attractor of the skew-product flow (φ, θ). Moreover, we extend this result to the case when $\mathscr{A}(\sigma)$ only is a local pullback attractor.

This paper is organized as follows. In Section 2 we present some preliminary definitions, results and some examples to illustrate the relationship between the three types of attractors. In Section 3 we state and prove the main results. Finally, we give some examples to illustrate the main results in Section 4.

2 Skew-Product Flows and Their Attractors

In this section, we present some notions, preliminary definitions and examples (see [4] for more details).

A nonautonomous system consists of a "base flow" and a "cocycle semiflow" that is in some sense driven by the base flow. More precisely, the base flow consists of the base space Σ, which we take to be a metric space with metric ρ, and a group of continuous transformations $\{\theta_t\}_{t\in\mathbb{R}}$

from Σ into itself such that

(i) $\theta_0 = \mathrm{id}_\Sigma$,

(ii) $\theta_t\theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$,

(iii) $\theta_t\Sigma = \Sigma$ for all $t \in \mathbb{R}$.

Let X be a complete metric space with metric $d(\cdot,\cdot)$. Let A and B be nonempty subsets of X. Denote the ε-neighborhood of A by $\mathcal{O}_\varepsilon(A) := \{x \in X \mid d(x, A) < \varepsilon\}$. The Hausdorff semidistance and the Hausdorff distance of A and B are defined, respectively, as

$$d_H(A, B) = \sup_{x\in A} d(x, B), \qquad \delta_H(A, B) = \max\{d_H(A, B),\ d_H(B, A)\}.$$

We also assign $d_H(\emptyset, B) = 0$.

The dynamics on the phase space (X, d) is given by a family of continuous mappings

$$\mathbb{R}^+ \times \Sigma \ni (t, \sigma) \to \varphi(t, \sigma) \in \mathcal{C}(X),$$

collectively "the cocycle", that satisfy

(i) $\varphi(0, \sigma) = \mathrm{id}_X$ for all $\sigma \in \Sigma$,

(ii) $\mathbb{R} \times \Sigma \ni (t, \sigma) \mapsto \varphi(t, \sigma)x \in X$ is continuous,

(iii) for all $t, s \geq 0$ and $\sigma \in \Sigma$,

$$\varphi(t + s, \sigma) = \varphi(t, \theta_s\sigma)\varphi(s, \sigma),$$

the "cocycle property".

In the paper, a "non-autonomous set" $A(\cdot)$ is a family of subsets of X indexed by $\sigma \in \Sigma$,

$$A(\cdot) = \{A(\sigma) : \sigma \in \Sigma\}.$$

We say $A(\cdot)$ is invariant under a skew-product flow (φ, θ) if

$$\varphi(t, \sigma)A(\sigma) = A(\theta_t\sigma)$$

for all $\sigma \in \Sigma,\ t \geq 0$. There are different types of "attractors" for non-autonomous systems.

Definition 2.1 *A family of compact sets $\mathscr{A}(\cdot)$ is called a global pullback (forward) attractor for a skew-product flow (φ, θ) if it is invariant and pullback (forward) attracts any bounded subset $B \subset X$, i.e.,*

$$\lim_{t\to\infty} d_H(\varphi(t, \theta_{-t}\sigma)B,\ \mathscr{A}(\sigma)) = 0, \qquad \lim_{t\to\infty} d_H(\varphi(t, \sigma)B,\ \mathscr{A}(\theta_t\sigma)) = 0.$$

The fundamental result gives a necessary and sufficient condition for the existence of pullback attractors of skew-product flows.

Proposition 2.1 *A skew-product flow (φ, θ) has a global pullback attractor $\mathscr{A}(\cdot)$ if and only if there exists a family of compact sets $K(\cdot)$ that pullback attracts every bounded subset of X.*

It should be emphasized that the pullback and forward dynamics are essentially independent.

Example 2.1 We consider a simple example on $\mathbb{R}$,

$$\begin{cases} \dot{x} = h(t)x, \\ x(s) = x_0, \end{cases}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a function. This equation has an explicit solution

$$x(t, s; x_0) = \mathrm{e}^{\int_s^t h(r)\mathrm{d}r} x_0. \tag{2.1}$$

We choose the base space $\Sigma = \mathbb{R}$ and the time shift $\{\theta_t\}_{t\in\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $\theta_t\sigma = \sigma + t$ as base flow. Let $\varphi(t, \sigma)x_0 := x(t + \sigma, \sigma; x_0)$, then the pair (φ, θ) is a skew-product flow. Then the following assertions hold:

(1) Let

$$h(t) = \begin{cases} \mathrm{e}^t - 1, & t < 0; \\ -\mathrm{e}^{-t} + 1, & t \geq 0. \end{cases}$$

It is clear that $h : \mathbb{R} \to \mathbb{R}$ is a continuous odd function and $\lim\limits_{t\to-\infty} h(t) = -1$ and $\lim\limits_{t\to\infty} h(t) = 1$. Observe that the solution (2.1) converges to 0 as $s \to -\infty$, while (2.1) goes to infinity as $t \to \infty$. This means that 0 is a pullback attractor but not a forward attractor.

(2) Let

$$h(t) = \begin{cases} \mathrm{e}^t + 1, & t < 0; \\ \mathrm{e}^{-t} - 1, & t \geq 0. \end{cases}$$

It is easy to verify that 0 turns out to be a forward attractor but not a pullback attractor anymore.

The uniform attractor is an important approach to study the asymptotic dynamics of a non-autonomous equation, which has been developed by Chepyzhov and Vishik [6].

Definition 2.2 (see [6–7]) *A closed set $\mathscr{A}_\Sigma \subset X$ is called a global uniform attractor for a skew-product flow (φ, θ) if it is the minimal closed set that uniform attracts any bounded subset $B \subset X$, i.e.,*

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t, \sigma)B, \mathscr{A}_\Sigma) \Big) = 0.$$

Let $\mathscr{A}(\cdot)$ be a global pullback attractor. We say $\mathscr{A}(\cdot)$ is uniformly pullback attracting if for any bounded subset $B \subset X$,

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t, \theta_{-t}\sigma)B,\ \mathscr{A}(\sigma)) \Big) = 0.$$

Obviously, $\mathscr{A}(\cdot)$ is uniformly pullback attracting if and only if it is uniformly forward attracting, i.e., for any bounded set $B \subset X$,

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t, \sigma)B, \mathscr{A}(\theta_t\sigma)) \Big) = 0.$$

In other words, if a global pullback attractor $\mathscr{A}(\cdot)$ is uniformly pullback attracting, then it is also a global forward attractor. Furthermore, $\overline{\bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma)}^X$ is a global uniform attractor.

An example of uniform attractors is given below.

Example 2.2 We consider the following Bernoulli equation on $\mathbb{R}^+$:

$$\begin{cases} \dot{x} = -x - h(t)x^2, \\ x(s) = x_0, \end{cases} \tag{2.2}$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is bounded and continuous.

This equation has an explicit solution

$$x(t, s; x_0) = \frac{\mathrm{e}^{-t}}{x_0^{-1}\mathrm{e}^{-s} + \int_s^t \mathrm{e}^{-r} h(r)\mathrm{d}r}.$$

The solution converges to 0 as $s \to -\infty$ or $t \to \infty$, which shows 0 is a global pullback attractor and a global forward attractor. It is worth noting that 0 is also a global uniform attractor. Indeed, (2.2) implies that $\dot{x} \le -x$ and this shows $|x(t, s; x_0)| \le \mathrm{e}^{-(t-s)}|x_0|$, then the result follows immediately.

Given a skew-product flow (φ, θ), we can define an associated autonomous semigroup S on $\mathbb{X} := X \times \Sigma$ by setting

$$S(t)(x, \sigma) = (\varphi(t, \sigma), \theta_t\sigma), \quad t \ge 0.$$

The group property of θ and the cocycle property of φ ensure that S satisfies the semigroup property

$$S(t + s)(x, \sigma) = S(t)S(s)(x, \sigma), \quad t, s \ge 0.$$

It is well known that an autonomous semigroup S has a global attractor $\mathbb{A}$ if and only if it has a compact attracting set $K \subset X$, i.e., for any bounded set B,

$$\lim_{t\to\infty} d_H(S(t)B,\, K) = 0.$$

A skew-product flow (φ, θ) is uniformly asymptotically compact if there exists a compact set $K \subset X$ such that

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t, \sigma)B,\, K) \Big) = 0$$

for every bounded subset B of X.

Proposition 2.2 (see [4–5]) *Let (φ, θ) be a skew-product flow and S be the associated semigroup on $\mathbb{X}$. Assume (φ, θ) is uniformly asymptotically compact. Then the following statements hold.*

(1) *For the skew-product flow (φ, θ), there exists a global uniform attractor $\mathscr{A}_\Sigma$ and a global pullback attractor $\mathscr{A}(\cdot)$. Moreover,*

$$\mathscr{A}_\Sigma \supseteq \bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma).$$

(2) *Suppose further that Σ is compact. Then*

$$\mathscr{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma)$$

and S has a global attractor $\mathbb{A}$ with

$$\mathbb{A} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{A}(\sigma).$$

Remark 2.1 Under assumptions of Proposition 2.2, the global pullback attractor $\mathscr{A}(\sigma)$ is upper semicontinuous in σ in the sense of Hausdorff distance. If $\mathscr{A}(\sigma)$ is also lower semicontinuous in σ, then the global pullback attractor $\mathscr{A}(\cdot)$ is also a global forward attractor.

Since uniform attractors are not required to be invariant, the existence of uniform attractors may not imply the existence of forward attractors.

Example 2.3 Consider one linear problem on $\mathbb{R}$:

$$\begin{cases} \dot{x} = -x + \sin t, \\ x(s) = x_0. \end{cases}$$

The equation can be solved explicitly, with solution

$$x(t,s;x_0) = \Big(x_0 - \frac{1}{2}(\sin s - \cos s)\Big)\mathrm{e}^{-(t-s)} + \frac{1}{2}(\sin t - \cos t). \tag{2.3}$$

It is clear that $x(t) = \frac{1}{2}(\sin t - \cos t)$, $t \in \mathbb{R}$, is a bounded solution. From (2.3), we have

$$\Big|x(t,s;x_0) - \frac{1}{2}(\sin t - \cos t)\Big| \le \Big(|x_0| + \frac{\sqrt{2}}{2}\Big)\mathrm{e}^{-(t-s)},$$

which shows $\frac{1}{2}(\sin t - \cos t)$ is a global uniform forward attractor. Since $\bigcup\limits_{t\in\mathbb{R}} \big(\frac{1}{2}(\sin t - \cos t)\big) = \big[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big]$, then $\big[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big]$ is a uniform attractor.

Let $T := \inf\big\{t \ge s : |x(t,s;x_0)| \le \frac{\sqrt{2}}{2}\big\}$. Now we construct a new system: When $s \le t < T$, $\dot{x} = -x + \sin t$, when $t \ge T$, $\dot{x} = x - x^3$. Then the new system possesses a uniform attractor in $[-1, 1]$, and its every solution goes to -1, 0 or 1 as $t \to \infty$. However, none of -1, 0 and 1 is invariant for the system. It follows that the new system dose not have a forward attractor.

3 Approximate Forward Attractors

In this section, the base space Σ is assumed to be compact. We first state and prove a basic lemma.

Lemma 3.1 *Suppose that a skew-product flow (φ, θ) is uniformly asymptotically compact and $\mathscr{A}(\cdot)$ is a global pullback attractor of (φ, θ). Then for any $\varepsilon > 0$, there exists a non-autonomous set $\mathscr{A}_\varepsilon(\cdot) \subset X$ such that for each $\sigma \in \Sigma$,*

$$\mathscr{A}(\sigma) \subset \mathscr{A}_\varepsilon(\sigma) \subset \mathcal{O}_\varepsilon(\mathscr{A}(\sigma)),$$

and the mapping $\sigma \to \mathscr{A}_\varepsilon(\sigma)$ is continuous in the sense of Hausdorff distance.

Proof Since the base space Σ is compact and the skew-product flow (φ, θ) is uniformly asymptotically compact, by Proposition 2.2, (φ, θ) has a global pullback attractor $\mathscr{A}(\cdot)$ and a compact global uniform attractor $\mathscr{A}_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma)$, and the associated semigroup S has a global attractor

$$\mathbb{A} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{A}(\sigma).$$

Take any bounded open set $D \subset X$ with $\mathscr{A}_\Sigma \subset D$. Then $\mathcal{D} \triangleq \Sigma \times D$ is a bounded neighborhood of $\mathbb{A}$, so for any $\varepsilon > 0$ there is a $T > 0$ independent of $\sigma \in \Sigma$ such that

$$\mathbb{A} \subset S(T)\mathcal{D} \subset \mathcal{O}_\varepsilon(\mathbb{A}),$$

where $\mathcal{O}_\varepsilon(\mathbb{A})$ is the ε-neighborhood of $\mathbb{A}$ in $X \times \Sigma$ in the metric

$$d_1((\sigma_1, x_1), (\sigma_2, x_2)) = \sqrt{\rho^2(\sigma_1, \sigma_2) + d^2(x_1, x_2)}.$$

We denote by $\mathscr{A}_\varepsilon(\sigma) \subset X$ the section of $S(T)\mathcal{D}$ over $\sigma \in \Sigma$, i.e.,

$$S(T)\mathcal{D} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{A}_\varepsilon(\sigma). \tag{3.1}$$

It is clear that

$$\mathscr{A}_\varepsilon(\sigma) = \varphi(T, \theta_{-T}\sigma)D, \quad \mathscr{A}(\sigma) \subset \mathscr{A}_\varepsilon(\sigma) \subset \mathcal{O}_\varepsilon(\mathscr{A}(\sigma)).$$

For any fixed $T > 0$, the continuity of $\sigma \to \varphi(T, \theta_{-T}\sigma)$ implies $\sigma \to \mathscr{A}_\varepsilon(\sigma)$ is continuous in the sense of Hausdorff distance. This completes the proof.

Remark 3.1 (1) If $S(T)\mathcal{D}$ in the proof of Lemma 3.1 is replaced by $\bigcup_{t\geq T} S(t)\mathcal{D}$, then $\mathscr{A}_\varepsilon(\sigma)$ is forward invariant.

(2) If $A(\sigma)$ is singleton for each $\sigma \in \Sigma$, then $\sigma \to A(\sigma)$ is continuous, hence we can take $\mathscr{A}_\varepsilon(\sigma) = A(\sigma)$.

In general, a pullback attractor of a skew-flow dose not have froward attracting property. We show in the following theorem that every neighborhood of a pullback attractor contains a set which froward attracts every bounded set of phase space.

Theorem 3.1 *Suppose that a skew-product flow (φ, θ) is uniformly asymptotically compact and $\mathscr{A}(\cdot)$ is a global pullback attractor of (φ, θ). Then for any $\varepsilon > 0$, $\mathscr{A}_\varepsilon(\cdot)$ (obtained in Lemma* 3.1 *) forward attracts every bounded set B of X uniformly in σ, i.e.,*

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} d_H(\varphi(t, \sigma)B, \mathscr{A}_\varepsilon(\theta_t\sigma)) = 0. \tag{3.2}$$

Remark 3.2 Since for any $\varepsilon > 0$, there is a $\mathscr{A}_\varepsilon(\cdot)$ such that $\mathscr{A}(\sigma) \subset \mathscr{A}_\varepsilon(\sigma) \subset \mathcal{O}_\varepsilon(\mathscr{A}(\sigma))$ for each $\sigma \in \Sigma$, then $\mathscr{A}_\varepsilon(\cdot)$ is called an approximate global forward attractor of the skew-product flow (φ, θ).

Proof of Theorem 3.1 Suppose that (3.2) is not true. Then for some $\delta > 0$ there are sequences $t_n \in \mathbb{R}^+$, $x_n \in B$ and $\sigma_n \in \Sigma$ with $t_n \to \infty$ as $n \to \infty$ such that

$$d(\varphi(t_n, \sigma_n)x_n, \mathscr{A}_\varepsilon(\theta_{t_n}\sigma_n)) \geq \delta. \tag{3.3}$$

Since Σ is compact, we can assume that the sequence $\{\theta_{t_n}\sigma_n\}$ is convergent as $n \to \infty$. Suppose

$$\theta_{t_n}\sigma_n \to \sigma_0, \quad n \to \infty. \tag{3.4}$$

By the continuity of $\mathscr{A}_\varepsilon(\sigma)$ in σ, for n sufficiently large, we have

$$d_H(\mathscr{A}_\varepsilon(\sigma_0), \mathscr{A}_\varepsilon(\theta_{t_n}\sigma_n)) < \frac{\delta}{2}. \tag{3.5}$$

On the other hand, since $\mathbb{A} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{A}(\sigma)$ is the global attractor of the semigroup S, then we can suppose that $\varphi(t_n, \sigma_n)x_n \to x_0$, $n \to \infty$. From (3.4), one knows $x_0 \in \mathscr{A}(\sigma_0)$.

We conclude from (3.3) that

$$\delta \le d(\varphi(t_n, \sigma_n)x_n, \mathscr{A}_\varepsilon(\theta_{t_n}\sigma_n)) \le d(\varphi(t_n, \sigma_n)x_n, \mathscr{A}_\varepsilon(\sigma_0)) + d(\mathscr{A}_\varepsilon(\sigma_0), \mathscr{A}_\varepsilon(\theta_{t_n}\sigma_n)).$$

This together with (3.5) show that

$$d(x_0, \mathscr{A}_\varepsilon(\sigma_0)) \ge \frac{\delta}{2},$$

which is a contradiction.

Corollary 3.1 *Suppose that $\mathscr{A}(\cdot)$ is a global pullback attractor of a skew-product flow (φ, θ) and $\mathscr{A}(\cdot)$ is uniformly pullback attracting and $\bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma)$ is precompact. Then $\mathscr{A}_\varepsilon(\cdot)$ (obtained in Lemma* 3.1) *is a global approximate forward attractor of (φ, θ).*

Proof By Theorem 15.8 of the reference [4], we know that $\mathbb{A} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathscr{A}(\sigma)$ is the global attractor of S, then $\bigcup_{\sigma\in\Sigma} \mathscr{A}(\sigma)$ is compact and uniformly attracts every bounded set B of X, i.e., the skew-product flow (φ, θ) is uniformly asymptotically compact. It is now an immediate corollary of Theorem 3.1.

We now extend Theorem 3.1 to the case of local pullback attractors. The following are efficient definitions and results.

Definition 3.1 *Let (φ, θ) be a skew-product flow. A family of compact sets $\mathcal{A}(\cdot)$ is called a local pullback attractor if it is invariant and pullback attracts one of its neighborhoods.*

An example of the local pullback attractors is given in the following.

Example 3.1 Consider the following system on $\mathbb{R}$,

$$\dot{x} = x - x^2.$$

Observe that $\{1\}$ is a local attractor, but $\{0\}$ is not a local attractor.

The pullback behavior of non-autonomous model

$$\dot{x} = x - h(t)x^2$$

with $x(s) = x_0$ and $h(t) > 0$ for all $t \in \mathbb{R}$ is qualitatively similar. Indeed, we can obtain the explicit solution

$$x(t, s; x_0) = \frac{\mathrm{e}^t}{x_0^{-1}\mathrm{e}^s + \int_s^t \mathrm{e}^r h(r)\mathrm{d}r},$$

provided that the integral

$$\int_{-\infty}^{0} \mathrm{e}^r h(r)\mathrm{d}r$$

converges. Thus,

(1) for any $x_0 > 0$, there is a unique pullback attracting global solution given by

$$x^*(t) = \frac{\mathrm{e}^t}{\displaystyle\int_{-\infty}^{t} \mathrm{e}^r h(r)\mathrm{d}r};$$

(2) for any $x_0 < 0$, every solution $x(t, s; x_0)$ will blow up in finite time.

This shows that $x^*(t)$ is a local pullback attractor which pullback attracts every bounded set of $[0, \infty)$.

Comparing to the notion of global uniformly asymptotically compact, we have the following definition.

Definition 3.2 *A skew-product flow (φ, θ) is called local uniformly asymptotically compact if there exists a compact set $K \subset X$ and a set B which is a neighborhood of K such that*

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t,\sigma)B,\, K) \Big) = 0.$$

Lemma 3.2 *Assume that a skew-product flow (φ, θ) is local uniformly asymptotically compact. Let S be the associated semigroup on $\mathbb{X}$. Then there exists a local pullback attractor $\mathcal{A}(\cdot)$ of (φ, θ) and S has a local attractor $\mathbb{A}_{\mathrm{loc}}$ such that*

$$\mathbb{A}_{\mathrm{loc}} = \bigcup_{\sigma\in\Sigma} \{\sigma\} \times \mathcal{A}(\sigma).$$

Proof Since the skew-product flow (φ, θ) is local uniformly asymptotically compact, there exists a compact set $K \subset X$ and a set B which is a neighborhood of K such that

$$\lim_{t\to\infty} \Big(\sup_{\sigma\in\Sigma} d_H(\varphi(t,\sigma)B,\, K) \Big) = 0. \tag{3.6}$$

Take $\mathbb{K} = K \times \Sigma$ and $\mathbb{B} = B \times \Sigma$, then $\mathbb{B}$ is a neighborhood of $\mathbb{K}$ and $\mathbb{K}$ is compact since K and Σ are both compact. Moreover,

$$S(t)\mathbb{B} = \Big[\bigcup_{\sigma\in\Sigma} \varphi(t,\sigma)B \Big] \times \Sigma.$$

It follows that

$$d_H(S(t)\mathbb{B},\, \mathbb{K}) \le \sup_{\sigma\in\Sigma} d_H(\varphi(t,\sigma)B,\, K) = \sup_{\sigma\in\Sigma} d_H(\varphi(t,\theta_{-t}\sigma)B,\, K),$$

whence K pullback attracts B. Hence the compact set $\mathbb{K}$ attracts its neighborhood $\mathbb{B}$ under S, so S has a local attractor $\mathcal{A} \subset \mathbb{K}$, which is the maximal invariant set in $\mathbb{B}$.

On the other hand, (3.6) shows that for each $\sigma \in \Sigma$,

$$\lim_{t\to\infty} d_H(\varphi(t,\theta_{-t}\sigma)B,\, K) = 0,$$

hence the skew-product flow (φ, θ) has local pullback attractor in B, denoted by $\mathcal{A}(\cdot)$, which is the maximal invariant set in B. It follows that

$$\bigcup_{\sigma \in \Sigma} \{\sigma\} \times \mathcal{A}(\sigma)$$

is the maximal invariant set in $\mathbb{B}$, which means

$$\mathbb{A}_{\mathrm{loc}} = \bigcup_{\sigma \in \Sigma} \{\sigma\} \times \mathcal{A}(\sigma).$$

The proof is completed.

Theorem 3.2 *Suppose that a skew-product flow (φ, θ) is local uniformly asymptotically compact and $\mathcal{A}(\cdot)$ is the corresponding local pullback attractor of (φ, θ). Then for any $\varepsilon > 0$ and $\sigma \in \Sigma$, there exists a set $\mathcal{A}_\varepsilon(\sigma) \subset X$ which is a neighborhood of $\mathcal{A}(\sigma)$ such that $\mathcal{A}_\varepsilon(\sigma) \subset \mathcal{O}_\varepsilon(\mathcal{A}(\sigma))$ and $\mathcal{A}_\varepsilon(\cdot)$ forward attracts one of its neighborhoods uniformly in σ. We call $\mathcal{A}_\varepsilon(\cdot)$ an approximate local forward attractor.*

Proof From Lemma 3.2, one knows that $\mathbb{A}_{\mathrm{loc}} = \bigcup_{\sigma \in \Sigma} \{\sigma\} \times \mathcal{A}(\sigma)$ is a local attractor of the associated semigroup S, hence it attracts one of its neighborhoods $\mathbb{B} \subset X \times \Sigma$, it follows that for any $\varepsilon > 0$ there is a $T > 0$ such that

$$S(T)\mathbb{B} \subset \mathcal{O}_\varepsilon(\mathbb{A}_{\mathrm{loc}}),$$

where $\mathcal{O}_\varepsilon(\mathbb{A}_{\mathrm{loc}})$ is the ε-neighborhood of $\mathbb{A}_{\mathrm{loc}}$ in $X \times \Sigma$ with the metric

$$d_1((\sigma_1, x_1), (\sigma_2, x_2)) = \sqrt{\rho^2(\sigma_1, \sigma_2) + d^2(x_1, x_2)}.$$

We denote by $\mathcal{A}_\varepsilon(\sigma)$ the section of $S(T)\mathbb{B}$ over $\sigma \in \Sigma$, i.e.,

$$S(T)\mathbb{B} = \bigcup_{\sigma \in \Sigma} \{\sigma\} \times \mathcal{A}_\varepsilon(\sigma).$$

Similar to the proof of Lemma 3.1, we know that

$$\mathcal{A}(\sigma) \subset \mathcal{A}_\varepsilon(\sigma) \subset \mathcal{O}_\varepsilon(\mathcal{A}(\sigma))$$

and $\mathcal{A}_\varepsilon(\sigma)$ is continuous in σ.

Denote by $B(\sigma)$ the section of $\mathbb{B}$ over $\sigma \in \Sigma$, then $B(\cdot)$ is a neighborhood of $\mathcal{A}(\cdot)$. Similar to the proof of Theorem 3.1, we have $\mathcal{A}_\varepsilon(\cdot)$ forward attracts $B(\cdot)$.

4 Applications

When the base space of a skew-product flow is compact, from Theorem 3.1, we know that the skew-product flow is uniformly asymptotically compact, then for any $\varepsilon > 0$ it will have an approximate global forward attractor $\mathscr{A}_\varepsilon(\cdot)$. In this section, we give two examples to illustrate the result.

4.1 Delay differential equations

Let $p : \mathbb{R} \to \mathbb{R}^n$ be a continuous, bounded, and be periodic, quasi-periodic or almost periodic function. Then the hull $H(p) = \overline{\{p(\tau + \cdot); \tau \in \mathbb{R}\}}$ of p is compact in the metric $d(p_1, p_2) = \sup_{s\in\mathbb{R}} d(p_1(s), p_2(s))$. Moreover, the shift operator $\theta_t : H(p) \to H(p)$ defined for each $t \in \mathbb{R}$ by $\theta(p)(\cdot) = p(t + \cdot)$ forms a continuous dynamical system on the compact metric space $H(p)$.

We now consider a delay differential equation on $\mathbb{R}^n$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = f_1(x(t)) + f_2(x(t-r)) + p(t), & t > 0, \\ x(t) = \psi(t), & -r \le t \le 0, \end{cases} \tag{4.1}$$

where $r > 0$ is the delay, the initial condition ψ is specified in the space $X = C^0([-r, 0] : \mathbb{R}^n)$ with the usual norm $\|\cdot\|$, $p \in H(p)$. For a function $x \in X$, the notation x_s denotes the function in X given by $x_s(t) = x(s + t)$, $t \in [-r, 0]$ and makes sense for any $0 \le s \le T$.

Assume that the system is sufficiently regular such that (4.1) possesses a unique solution, so we can define a continuous cocycle $\varphi(t, p)$ on X, which gives the solution at time t when $x_0 = \psi$, via

$$\varphi(t, p)\psi = x_t(\cdot, p; 0, \psi), \quad p \in H(p).$$

Lemma 4.1 *Suppose that $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the dissipativity condition*

$$\langle f_1(x), x\rangle \le -\alpha_0|x|^2 + \beta_0$$

for all $x \in \mathbb{R}^n$ and that $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded, i.e., there exists $K \ge 0$ such that

$$|f_2(x)| \le K$$

for all $x \in \mathbb{R}^n$. Then the skew-product flow (φ, θ) induced by (4.1) *is uniformly asymptotically compact on $X \times H(p)$.*

Proof Taking inner product (4.1) by $x(t)$ we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 &\le 2\langle x(t), f_1(x(t))\rangle + 2\langle x(t), f_2(x(t-r))\rangle + 2\langle x(t), p(t)\rangle \\ &\le -2\alpha_0|x(t)|^2 + 2\beta_0 + 2(K + M_p)|x(t)| \\ &\le -\alpha_0|x(t)|^2 + 2\beta_0 + \frac{(K + M_p)^2}{\alpha_0}, \end{aligned}$$

where $M_p > 0$ is a constant such that $\|p\| \le M_p$ and is independent of p.

Gronwall's lemma shows

$$|x(t)|^2 \le |x(0)|^2 \mathrm{e}^{-\alpha_0 t} + \frac{2\beta}{\alpha_0} + \frac{(K + M_p)^2}{\alpha_0^2}.$$

It follows that

$$\|\varphi(t, p)\psi\| = \sup_{\tau\in[t-r,t]} |x(t, p; 0, \psi)|^2$$

$$\le \sup_{\theta\in[-r,0]} |\psi(\theta)|^2 \mathrm{e}^{-\alpha_0(t-r)} + \frac{2\beta}{\alpha_0} + \frac{(K+M_p)^2}{\alpha_0^2}$$
$$= \|\psi\|^2 \mathrm{e}^{-\alpha_0 c(t-r)} + \frac{2\beta}{\alpha_0} + \frac{(K+M_p)^2}{\alpha_0^2},$$

which shows the skew-product flow (φ,θ) is uniformly dissipative on $X\times H(p)$. Then by Arzelà-Ascoli theorem, we know that (φ,θ) is uniformly asymptotically compact on $X\times H(p)$. The proof of the lemma is complete.

Apply Theorem 3.1 to the above delay differential equations, one obtains the following theorem.

Theorem 4.1 *Let (φ,θ) be the skew-product flow induced by (4.1) on $X\times H(p)$ and let $\mathscr{A}(\cdot)$ be the global pullback attractor. Then for any $\varepsilon>0$ there is a set $\mathscr{A}_\varepsilon(\cdot)$ such that for each $p\in H(p)$, $\mathscr{A}(p)\subset\mathscr{A}_\varepsilon(p)\subset\mathcal{O}_\varepsilon(\mathscr{A}(p))$, and $\mathscr{A}_\varepsilon(\cdot)$ forward attracts every bounded set $B\subset X$ uniformly on $p\in H(p)$.*

4.2 A class of evolutionary equations in infinite-dimensional Banach spaces

Let X be a Banach space with metric $\|\cdot\|$. Let A be sectorial operator on X. Assume $\mathrm{Re}\,\sigma(A)\ge\delta>0$, then for $t>0$,

$$\|\mathrm{e}^{-At}\|\le C\mathrm{e}^{-\delta t},\quad \|A^\alpha \mathrm{e}^{-At}\|\le Ct^{-\alpha}\mathrm{e}^{-\delta t},\quad 0\le\alpha<1,$$

for some constant C. We can define for each $\alpha\ge0$, $X^\alpha=D(A^\alpha)$ with the graph norm $\|x\|_\alpha=\|A^\alpha x\|$, $x\in X^\alpha$. Assume in addition that A has compact resolvent, then one has a compact imbedding $X^\beta\hookrightarrow X^\alpha$, $0\le\alpha<\beta<1$.

Let $p:\mathbb{R}\to X^\alpha$ be continuous, bounded, and uniformly continuous on $\mathbb{R}$. We define the hull of p as follows:

$$H(p)=\overline{\{p(\tau+\cdot);\tau\in\mathbb{R}\}},$$

where the closure is taken with respect to some metric such that $H(p)$ is compact, for example:

$$d(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n(f,g)}{1+p_n(f,g)}$$

where $p_n(f,g)=\sup_{t\in[-n,n]}\|f(t)-g(t)\|_\alpha$. Define the shift operator $\theta_t:H(p)\to H(p)$, $t\in\mathbb{R}$, as $\theta_t p=p(t+\cdot)$, then θ_t is continuous.

Now we consider the nonlinear equation

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}+Ax=f(x)+p(t), & t>t_0,\\ x(t_0)=x_0,\end{cases}\tag{4.2}$$

where we assume $f:X^\alpha\to X$ is bounded. Then this system is sufficiently regular to ensure the existence and the uniqueness of mild solution $x(t,t_0;x_0)$, $t\ge t_0$, which satisfies

$$x(t)=\mathrm{e}^{-A(t-t_0)}x_0+\int_{t_0}^{t}\mathrm{e}^{-A(t-s)}[f(x(s))+p(s)]\mathrm{d}s.\tag{4.3}$$

$x(t, t_0; x_0)$ generates a continuous cocycle φ on X^α by $\varphi(t,p)x_0 := x(t, 0; x_0)$, $t \geq 0$. The cocycle φ is driven by the shift operator θ_t on $H(p)$, i.e., (φ, θ) is skew-product flow on $X^\alpha \times H(p)$.

Lemma 4.2 *The skew-product flow (φ, θ) induced by (4.3) is uniformly asymptotically compact on $X^\alpha \times H(p)$.*

Proof If $\alpha < \beta < 1$, $X^\beta \subset X^\alpha$ has compact inclusion. Without loss of generality, we suppose $\|f(x(t, t_0; x_0))\| \leq C$, $t \geq t_0$ and $\sup\limits_{t\in\mathbb{R}} \|p(t)\| \leq C$ for some constant C. Hence

$$\|x(t, t_0, x_0)\|_\beta \leq M(t-t_0)^{-(\beta-\alpha)} \mathrm{e}^{-\delta(t-t_0)} \|x_0\|_\alpha + 2MC \int_{t_0}^{t} (t-s)^{-\beta} \mathrm{e}^{-\delta(t-s)} \mathrm{d}s.$$

Let $C_\infty := 2MC \int_0^\infty u^{-\beta} \mathrm{e}^{-\delta u} \mathrm{d}u$ and $K := \{x \in X : \|x\|_\beta \leq C_\infty\}$. Then $K \subset X^\alpha$ is compact and

$$\lim_{t\to\infty} \Big(\sup_{p\in H(p)} \sup_{x_0 \in B} d_H(x(t; t_0, x_0),\, K) \Big) = 0$$

for every bounded subset B of X^α, i.e., (φ, θ) is uniformly asymptotically compact on $X^\alpha \times H(p)$.

By Theorem 3.1, we immediately have the following theorem.

Theorem 4.2 *Let (φ, θ) be the skew-product flow induced by (4.2) on $X^\alpha \times H(p)$ and $\mathscr{A}(\cdot)$ be the global pullback attractor. Then for any $\varepsilon > 0$ there is an approximate global forward attractor $\mathscr{A}_\varepsilon(\cdot)$. Specifically, for each $p \in H(p)$, $\mathscr{A}(p) \subset \mathscr{A}_\varepsilon(p) \subset \mathcal{O}_\varepsilon(\mathscr{A}(p))$ and $\mathscr{A}_\varepsilon(\cdot)$ forward attracts every bounded set $B \subset X^\alpha$ uniformly on $p \in H(p)$.*

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