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On the Cegrell Classes Associated to a Positive Closed Current

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Abstract The aim of this paper is to study the operator $(dd^c\centerdot)^q \wedge T$ on some classes of plurisubharmonic (psh) functions, which are not necessary bounded, where T is a positive closed current of bidimension (q, q) on an open set Ω of $\mathbb{C}^n$. The author introduces two classes $\mathcal{F}_p^T(\Omega)$ and $\mathcal{E}_p^T(\Omega)$ and shows first that they belong to the domain of definition of the operator $(dd^c\centerdot)^q \wedge T$. Then the author proves that all functions that belong to these classes are C_T-quasi-continuous and that the comparison principle is valid for them.

Keywords Positive closed current, Plurisubharmonic function, Capacity, Monge-Ampère Operator
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1 Introduction

Let Ω be a bounded open set of $\mathbb{C}^n$ and denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic (psh) functions on Ω. The definition of the complex Monge-Ampère operator $(dd^c\centerdot)^n$ on the set of psh functions was studied by Bedford and Taylor in [1], they proved that this operator is well defined on the set of locally bounded psh functions and they established the comparison principle to study the Dirichlet problem on $\mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$. The problem of extending its domain of definition has been treated by many other authors, in particular Cegrell introduced, between 1998 and 2004 (see [2–4]), a general class $\mathcal{E}(\Omega)$: The class of psh functions which are locally equal to decreasing limits of bounded psh functions vanishing on $\partial\Omega$ with bounded Monge-Ampère mass on Ω. He proved that the Monge-Ampère operator is well defined on $\mathcal{E}(\Omega)$ and that it is the largest domain of definition of $(dd^c\centerdot)^n$ if it is required to be continuous under decreasing sequences. The study of this class leads to many results such as the comparison principle, the convergence in capacity and the solvability of the Dirichlet problem. This paper continues the studies of plurisubharmonic functions and the complex Monge-Ampère operator associated to a positive closed current T.

Throughout this paper, we denote by T a positive closed current of bidimension (q, q) on Ω where $1 \le q \le n$. The operator $(dd^c\centerdot)^q \wedge T$ was studied by Dabbek and Elkhadhra [5] in the case of bounded psh functions. We will extend here the domain of definition of this operator to some classes of unbounded psh functions, and its different properties will be studied.

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In this paper we recall the classes $\mathcal{F}^T(\Omega)$ and $\mathcal{E}^T(\Omega)$ introduced by Hai and Dung in [7] where they proved that Monge-Ampère operator $(dd^c\boldsymbol{.})^q \wedge T$ is well defined. For such classes one of the results in [7] was cited with incomplete proof, so we state it here and give a completed proof (see Lemma 2.2).

In Section 2 we introduce the class $\mathcal{E}_p^T(\Omega)$ and show that the Monge-Ampère operator $(dd^c\boldsymbol{.})^q \wedge T$ is well defined on this class. Then we give some properties of the classes $\mathcal{E}_p^T(\Omega)$ and $\mathcal{F}^T(\Omega)$.

In Section 3 we prove that every function in $\mathcal{E}_p^T(\Omega)$ or in $\mathcal{F}^T(\Omega)$ is C_T-quasi-continuous; it means that it is continuous outside a subset of small C_T-capacity. The main tool will be an estimate of the growth of $C_T(\{u < -s\})$. Indeed we prove that

$$C_T(\{u < -s\}) = O\Big(\frac{1}{s^{p+q}}\Big) \quad \Big(\text{resp. } C_T(\{u < -s\}) = O\Big(\frac{1}{s^q}\Big)\Big)$$

for every $u \in \mathcal{E}_p^T(\Omega)$ (resp. $u \in \mathcal{F}^T(\Omega)$).

In Section 4, we give the main result of this article (see Theorem 4.1).

2 The Classes $\mathcal{E}_p^T(\Omega)$ and $\mathcal{F}_p^T(\Omega)$

2.1 Preliminary results

Throughout this paper, Ω will be a hyperconvex domain of $\mathbb{C}^n$, therefore it is open, bounded, connected and there exists $h \in \mathrm{PSH}^-(\Omega)$ such that for all $c < 0$, the set $\{z \in \Omega,\ h(z) < c\}$ is relatively compact in Ω where $\mathrm{PSH}^-(\Omega)$ is the set of negative psh functions. We introduce the class $\mathcal{E}_0^T(\Omega)$ associated to T, slightly different from the class $\mathcal{E}_0^T(\Omega)$ introduced in [7], as follows:

$$\mathcal{E}_0^T(\Omega) := \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega);\ \lim_{z \to \partial\Omega \cap \mathrm{Supp}\, T} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^q \wedge T < +\infty\Big\}.$$

Using the same proof as in [7], one can easily prove that this class is a convex cone and that for all $\psi \in \mathrm{PSH}^-(\Omega)$ and $\varphi \in \mathcal{E}_0^T(\Omega)$ the function $\max(\varphi, \psi) \in \mathcal{E}_0^T(\Omega)$.

In this section we will introduce new energy classes $\mathcal{E}_p^T(\Omega)$ and $\mathcal{F}_p^T(\Omega)$ similar to Cegrell's ones and we will prove that the Monge-Ampère operator is well defined on them.

Definition 2.1 *For every real $p \geq 1$ we define $\mathcal{E}_p^T(\Omega)$ as the set*

$$\mathcal{E}_p^T(\Omega) := \Big\{\varphi \in \mathrm{PSH}^-(\Omega);\ \exists\ \mathcal{E}_0^T(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j \geq 1} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^q \wedge T < +\infty\Big\}.$$

When the sequence $(\varphi_j)_j$ associated to φ can be chosen such that

$$\sup_{j \geq 1} \int_\Omega (dd^c\varphi_j)^q \wedge T < +\infty,$$

we say that $\varphi \in \mathcal{F}_p^T(\Omega)$.

It's easy to check that $\mathcal{E}_0^T(\Omega) \subset \mathcal{F}_p^T(\Omega) \subset \mathcal{E}_p^T(\Omega)$ and that using Hölder's inequality, one has $\mathcal{F}_{p_1}^T(\Omega) \subset \mathcal{F}_{p_2}^T(\Omega)$ for all $p_2 \leq p_1$.

We recall the following result which will be useful to prove some properties of our classes.

Theorem 2.1 (see [5]) *Suppose that $u, v \in \mathcal{E}_0^T(\Omega)$. If $p \geq 1$, then for every $0 \leq s \leq q$ one has*

$$\int_\Omega (-u)^p (dd^c u)^s \wedge (dd^c v)^{q-s} \wedge T$$
$$\leq D_{s,p} \Big(\int_\Omega (-u)^p (dd^c u)^q \wedge T \Big)^{\frac{p+s}{p+q}} \Big(\int_\Omega (-v)^p (dd^c v)^q \wedge T \Big)^{\frac{q-s}{p+q}},$$

where $D_{s,1} = \mathrm{e}^{(j+1)(q-j)}$ and $D_{s,p} = p^{\frac{(p+s)(q-s)}{p-1}}$, $p > 1$.

We prove firstly that these two classes inherit some properties of the energy class $\mathcal{E}_0^T(\Omega)$.

Theorem 2.2 *The classes $\mathcal{E}_p^T(\Omega)$ and $\mathcal{F}_p^T(\Omega)$ are convex cones.*

Proof It suffices to prove that $u+v \in \mathcal{E}_p^T(\Omega)$ for every $u, v \in \mathcal{E}_p^T(\Omega)$. Let $(u_j)_j$ and $(v_j)_j$ be two sequences that decrease to u and v respectively as in Definition 2.1. We want to estimate

$$\int_\Omega (-u_j - v_j)^p (dd^c(u_j + v_j))^q \wedge T.$$

Thanks to Minkowsky inequality, it is enough to estimate the following terms:

$$\int_\Omega (-u_j)^p (dd^c u_j)^s \wedge (dd^c v_j)^{q-s} \wedge T,$$
$$\int_\Omega (-v_j)^p (dd^c u_j)^s \wedge (dd^c v_j)^{q-s} \wedge T$$

for all $0 < s < q$. Using Theorem 2.1, we can estimate these two last terms by

$$\int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T \quad \text{and} \quad \int_\Omega (-v_j)^p (dd^c v_j)^q \wedge T.$$

As these sequences are uniformly bounded by the definition of $\mathcal{E}_p^T(\Omega)$, the result follows.

Proposition 2.1 *Let $u \in \mathcal{E}_p^T(\Omega)$ (resp. $\mathcal{F}_p^T(\Omega)$) and $v \in \mathrm{PSH}^-(\Omega)$. Then the function $w := \max(u, v)$ is in $\mathcal{E}_p^T(\Omega)$ (resp. in $\mathcal{F}_p^T(\Omega)$).*

Proof Let $(u_j)_j$ be a sequence that decreases to u as in Definition 2.1 and take $w_j := \max(u_j, v)$. The sequence (w_j) decreases to w. So it is enough to prove that

$$\sup_j \int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T < +\infty.$$

Thanks to Theorem 2.1, one has

$$\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \leq \int_\Omega (-u_j)^p (dd^c w_j)^q \wedge T$$
$$\leq D_{0,p} \Big(\int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T \Big)^{\frac{p}{p+q}} \Big(\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \Big)^{\frac{q}{p+q}}.$$

Therefore

$$\int_\Omega (-w_j)^p (dd^c w_j)^q \wedge T \leq D_{0,p}^{\frac{p+q}{p}} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.$$

The right-hand side is uniformly bounded because $u \in \mathcal{E}_p^T(\Omega)$ and the result follows.

The following theorem proves that the Monge-Ampère operator $(dd^c \boldsymbol{.})^q \wedge T$ is well defined on the new classes.

Theorem 2.3 *Let $u \in \mathcal{E}_p^T(\Omega)$ and let $(u_j)_j$ be a sequence of psh functions that decreases to u as in Definition 2.1. Then the sequence $((dd^c u_j)^q \wedge T)_j$ converges weakly to a positive measure μ and this limit is independent of the choice of the sequence $(u_j)_j$. We set $(dd^c u)^q \wedge T := \mu$.*

Proof Let $0 \leq \chi \in \mathcal{D}(\Omega)$, $\delta = \sup\{u_1(z);\ z \in \mathrm{Supp}\chi\}$ and $\varepsilon > 0$. There exists a sequence $(r_j)_j$ such that $0 < r_j < r_{j-1}$ and

$$r_j < \mathrm{dist}\Big(\Big\{u_j < \frac{\delta}{2}\Big\}, \Omega^c\Big).$$

Let

$$u_{r_j}(z) := \int_{\mathbb{B}} u_j(z + r_j\xi)\mathrm{d}V(\xi),$$

where $\mathrm{d}V$ is the normalized Lebesgue measure on the unit ball $\mathbb{B}$. Then one has

$$\Big|\int_\Omega \chi(dd^c u_{r_j})^q \wedge T - \chi(dd^c u_j)^q \wedge T\Big| < \varepsilon.$$

The function u_{r_j} is continuous, psh on $\{u_j < \frac{\delta}{2}\}$ and $u_j \leq u_{r_j}$ on Ω. Let $\widetilde{u}_j = \max(u_{r_j} + \delta, 2u_j)$. Then the sequence $(\widetilde{u}_j)_j$ decreases to a psh function $\widetilde{u}$ and $\widetilde{u}_j \in \mathcal{E}_0^T(\Omega)$ by Proposition 2.1. Furthermore, using the same technic of the previous proof, we obtain

$$\sup_{j\geq 1}\int_\Omega (-\widetilde{u}_j)^p (dd^c\widetilde{u}_j)^q \wedge T < +\infty.$$

The proof of the theorem will be complete if we show that

$$\lim_{j\to+\infty}\int_\Omega \chi(dd^c\widetilde{u}_j)^q \wedge T$$

exists.

Let h be an exhaustion function in $\mathcal{E}_0^T(\Omega)$. Then

$$\begin{aligned}\int_\Omega (-\widetilde{u})^p (dd^c h)^q \wedge T &= \lim_{j\to+\infty}\int_\Omega (-\widetilde{u}_j)^p (dd^c h)^q \wedge T\\ &\leq D_{0,p}\sup_{j\geq 1}\Big(\int_\Omega (-\widetilde{u}_j)^p (dd^c\widetilde{u}_j)^q \wedge T\Big)^{\frac{p}{p+q}}\Big(\int_\Omega (-h)^p (dd^c h)^q \wedge T\Big)^{\frac{q}{p+q}} < +\infty.\end{aligned}$$

Thanks to Dabbek-Elkhadhra [5], the sequence of measures $(dd^c \max(\widetilde{u}_j, -k))^q \wedge T$ converges weakly for every k. So it is enough to control

$$\Big|\int \chi(dd^c u_{r_j})^q \wedge T - \chi(dd^c \max(\widetilde{u}_j, -k))^q \wedge T\Big|.$$

Since $\widetilde{u}_j$ is continuous near $\mathrm{Supp}\chi$, we have

$$\begin{aligned}&\Big|\int \chi(dd^c u_j)^q \wedge T - \chi(dd^c \max(\widetilde{u}_j, -k))^q \wedge T\Big|\\ &= \Big|\int_{\{\widetilde{u}\leq -k\}} \chi(dd^c\widetilde{u}_j)^q \wedge T + \int_{\{\widetilde{u}> -k\}} \chi(dd^c\widetilde{u}_j)^q \wedge T\\ &\quad - \int_{\{\widetilde{u}\leq -k\}} \chi(dd^c \max(\widetilde{u}_j, -k))^q \wedge T - \int_{\{\widetilde{u}> -k\}} \chi(dd^c \max(\widetilde{u}_j, -k))^q \wedge T\Big|\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\{\widetilde{u}\leq -k\}} \chi (dd^c\widetilde{u}_j)^q\wedge T+\int_{\{\widetilde{u}\leq -k\}}\chi(dd^c\max(\widetilde{u}_j,-k))^q\wedge T\\
&\leq \frac{\sup\chi}{k^p}\int_{\{-\widetilde{u}\geq k\}} k^p[(dd^c\widetilde{u}_j)^q\wedge T+(dd^c\max(\widetilde{u}_j,-k))^q\wedge T]\\
&\leq \frac{\sup\chi}{k^p}\int_{\Omega}(-\widetilde{u})^p(dd^c\widetilde{u}_j)^q\wedge T+((-\max(\widetilde{u}_j,-k))^p dd^c\max(\widetilde{u}_j,-k))^q\wedge T\\
&\leq C\frac{\sup\chi}{k^p}\sup_{m\geq 1}\int_{\Omega}(-\widetilde{u}_m)^p(dd^c\widetilde{u}_m)^q\wedge T.
\end{aligned}$$

This completes the proof of the theorem.

Theorem 2.4 *If $u\in\mathcal{E}_1^T(\Omega)$, then*

$$\int_\Omega u(dd^cu)^q\wedge T>-\infty.$$

Moreover, if $v_j\in \mathrm{PSH}^-(\Omega)$ such that $(v_j)_j$ decreases to u, then

$$\int_\Omega v_j(dd^cv_j)^q\wedge T \text{ converges to } \int_\Omega u(dd^cu)^q\wedge T.$$

Proof Since $u\in\mathcal{E}_1^T(\Omega)$, there exists a sequence $(u_j)_j\subset\mathcal{E}_0^T$ such that

$$\lim_{j\to+\infty}u_j=u,\quad \alpha:=\sup_j\int -u_j(dd^cu_j)^q\wedge T<+\infty.$$

We then prove that

$$\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T=\int_\Omega u(dd^cu)^q\wedge T.$$

For every $k\geq j$ and $\varepsilon>0$, one has

$$\begin{aligned}
&\int_\Omega -u_j(dd^cu_j)^q\wedge T\\
&\leq \int_\Omega -u_j(dd^cu_k)^q\wedge T\\
&=\int_{\{u_j\geq-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T+\int_{\{u_j<-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{u_j\geq-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T\\
&=\int_{\{u_j\geq-\varepsilon\}}-\max(u_j,-\varepsilon)(dd^cu_k)^q\wedge T\\
&\leq\Big(\int_\Omega-\max(u_j,-\varepsilon)(dd^c\max(u_j,-\varepsilon))^q\wedge T\Big)^{\frac{1}{q+1}}\Big(\int_\Omega -u_k(dd^cu_k)^q\wedge T\Big)^{\frac{q}{q+1}}\\
&\leq\Big(\varepsilon\int_\Omega(dd^cu_j)^q\wedge T\Big)^{\frac{1}{q+1}}\alpha^{\frac{q}{q+1}}.
\end{aligned}$$

This goes to 0 when $\varepsilon\to0$. By Theorem 2.3 we obtain

$$\limsup_{k\to+\infty}\int_{\{u_j<-\varepsilon\}}-u_j(dd^cu_k)^q\wedge T\leq\int_\Omega -u_j(dd^cu)^q\wedge T.$$

Now since $-u_j$ is lower semi-continuous,

$$\liminf_{k\to+\infty}\int_\Omega -u_j(dd^cu_k)^q\wedge T \geq \int_\Omega -u_j(dd^cu)^q\wedge T.$$

Hence for all j,

$$\lim_{k\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T = \int_\Omega u_j(dd^cu)^q\wedge T.$$

It follows that

$$\begin{aligned}
&\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T\\
&\geq \lim_{j\to+\infty}\lim_{k\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T = \int_\Omega u(dd^cu)^q\wedge T\\
&\geq \limsup_{k\to+\infty}\int_\Omega u(dd^cu_k)^q\wedge T = \limsup_{k\to+\infty}\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_k)^q\wedge T\\
&\geq \lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T.
\end{aligned}$$

Thus

$$\lim_{j\to+\infty}\int_\Omega u_j(dd^cu_j)^q\wedge T = \int_\Omega u(dd^cu)^q\wedge T. \tag{2.1}$$

As $(v_k)_k$ decreases to u, we have $v_k\in\mathcal{E}_1^T(\Omega)$. It follows that

$$\int_\Omega \max(u_j,v_k)(dd^c\max(u_j,v_k))^q\wedge T \geq \int_\Omega u_j(dd^cu_j)^q\wedge T\geq -\alpha. \tag{2.2}$$

Moreover, $(\max(u_j,v_k))_{j\in\mathbb{N}}\subset\mathcal{E}_0^T(\Omega)$ and decreases to v_k so thanks to (2.1),

$$\lim_{j\to+\infty}\int_\Omega \max(u_j,v_k)(dd^c\max(u_j,v_k))^q\wedge T = \int_\Omega v_k(dd^cv_k)^q\wedge T. \tag{2.3}$$

By tending $j\to+\infty$, (2.1)–(2.3) give

$$\int_\Omega v_k(dd^cv_k)^q\wedge T\geq \int_\Omega u(dd^cu)^q\wedge T.$$

Thus

$$\liminf_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T\geq \int_\Omega u(dd^cu)^q\wedge T. \tag{2.4}$$

With the same reason, as $(\max(u_j,v_k))_{k\in\mathbb{N}}$ decreases to u_j, we have

$$\int_\Omega u_j(dd^cu_j)^q\wedge T\geq \limsup_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T.$$

Hence

$$\limsup_{k\to+\infty}\int_\Omega v_k(dd^cv_k)^q\wedge T\leq \int_\Omega u(dd^cu)^q\wedge T. \tag{2.5}$$

The result follows from (2.4)–(2.5).

Remark 2.1 We notice that if $u\in\mathcal{E}_1^T(\Omega)$ and $(u_j)_j$ is a decreasing sequence to u as in Definition 2.1, then the sequence $\left(\int_\Omega u_j(dd^cu_j)^q\wedge T\right)_j$ decreases to $\int_\Omega u(dd^cu)^q\wedge T$.

2.2 Comparison theorems

We recall two classes $\mathcal{E}^T(\Omega)$ and $\mathcal{F}^T(\Omega)$ introduced in [7] where the authors proved that the Monge-Ampère operator $(dd^c\centerdot)^q \wedge T$ is well defined on them.

Definition 2.2 *We say that $u \in \mathcal{F}^T(\Omega)$ if there exists a sequence $(u_j)_j \subset \mathcal{E}_0^T(\Omega)$ which decreases to u such that*

$$\sup_j \int_\Omega (dd^c u_j)^q \wedge T < +\infty.$$

We said that $u \in \mathcal{E}^T(\Omega)$ if for all $z \in \Omega$ there exists a neighborhood ω of z and a function $v \in \mathcal{F}^T(\Omega)$ such that $u = v$ on ω.

As a consequence, for every $p \geq 1$ one has $\mathcal{F}_p^T(\Omega) \subset \mathcal{F}^T(\Omega) \subset \mathcal{E}^T(\Omega)$, but there is no relationship between $\mathcal{E}_p^T(\Omega)$ and $\mathcal{E}^T(\Omega)$.

Lemma 2.1 *Let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ and let U be an open subset of Ω such that $u = v$ near ∂U. Then*

$$\int_U (dd^c u)^q \wedge T = \int_U (dd^c v)^q \wedge T.$$

Proof Let u_ε and v_ε be the usual regularizations of u and v respectively. Choose $U' \subset\subset U$ such that $u = v$ near $\partial U'$. If $\varepsilon > 0$ is small enough, one has $u_\varepsilon = v_\varepsilon$ near $\partial U'$, and if we take $\chi \in \mathcal{D}(U')$ with $\chi = 1$ near $\{u_\varepsilon \neq v_\varepsilon\}$, then $dd^c\chi = 0$ on $\{u_\varepsilon \neq v_\varepsilon\}$. So

$$\begin{aligned}\int_\Omega \chi (dd^c u_\varepsilon)^q \wedge T &= \int_\Omega u_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{q-1} \wedge T \\ &= \int_\Omega v_\varepsilon dd^c\chi \wedge (dd^c u_\varepsilon)^{q-1} \wedge T \\ &= \int_\Omega \chi (dd^c v_\varepsilon)^q \wedge T.\end{aligned}$$

Hence

$$\int_\Omega \chi (dd^c u)^q \wedge T = \int_\Omega \chi (dd^c v)^q \wedge T.$$

The result follows.

Corollary 2.1 *Let $u, v \in \mathcal{F}^T(\Omega)$. Assume that there exists an open subset U of Ω such that $u = v$ near ∂U. Then*

$$\int_U (dd^c u)^q \wedge T = \int_U (dd^c v)^q \wedge T.$$

Proof Let $u, v \in \mathcal{F}^T(\Omega)$ and $w \in \mathcal{E}_0^T(\Omega)$ such that $w(z) \neq 0$ for all z. Then $u_j := \max(u, jw)$ and $v_j = \max(v, jw)$ belong to $\mathcal{E}_0^T(\Omega)$ and they are equal on ∂U. The result follows from the previous lemma.

Now we recall a result due to [7] and give a different proof.

Proposition 2.2 (see [7]) *For $u, v \in \mathcal{F}^T(\Omega)$ such that $u \leq v$ on Ω, one has*

$$\int_\Omega (dd^c v)^q \wedge T \leq \int_\Omega (dd^c u)^q \wedge T.$$

Proof Let $(u_j)_j$ and $(v_j)_j$ be the corresponding decreasing sequences to u and v respectively as in Definition 2.2. Replace v_j by $\max(u_j, v_j)$, we can assume that $u_j \leq v_j$ for all $j \in \mathbb{N}$. For $h \in \mathcal{E}_0^T(\Omega)$ and $\varepsilon > 0$ we have

$$\begin{aligned}\int_\Omega -h(dd^c v_j)^q \wedge T &\leq \int_\Omega -h(dd^c u_j)^q \wedge T \\ &\leq \int_\Omega -h(dd^c u)^q \wedge T + \limsup_{j\to+\infty} \int_{\{h>-\varepsilon\}} -h(dd^c u_j)^q \wedge T \\ &\leq \int_\Omega -h(dd^c u)^q \wedge T + \varepsilon \limsup_{j\to+\infty} \int_\Omega (dd^c u_j)^q \wedge T.\end{aligned}$$

By tending ε to 0 we obtain

$$\int_\Omega -h(dd^c v)^q \wedge T \leq \int_\Omega -h(dd^c u)^q \wedge T.$$

The result follows by choosing h decreasing to -1.

Lemma 2.2 *For all $u \in \mathcal{F}^T(\Omega)$, there exists a sequence $(u_j)_j \subset \mathcal{E}_0^T(\Omega) \cap \mathcal{C}(\overline{\Omega})$ that decreases to u.*

We claim that this lemma was cited in [7, Theorem 5.1] with incomplete proof. In fact the authors used a comparison theorem proved by Dabbek-Elkhadhra [5] only for bounded psh functions in $\mathcal{F}^T(\Omega)$ where functions are not in general bounded.

Proof We refer to Cegrell [3, Theorem 2.1] for the construction of the sequence $(u_j)_j$. It remains to show that

$$\int_\Omega (dd^c u_j)^q \wedge T < +\infty.$$

As $u_j \geq u$ then by Proposition 2.2 one has

$$\int_\Omega (dd^c u_j)^q \wedge T \leq \int_\Omega (dd^c u)^q \wedge T < +\infty.$$

3 C_T-Quasi-continuity

Now we establish the quasi-continuity of psh functions belonging to $\mathcal{F}^T(\Omega)$ and $\mathcal{E}_p^T(\Omega)$. We need to recall some notions given in [5] (see also [9]) about the capacity associated to T which is defined as

$$C_T(K, \Omega) = C_T(K) = \sup\Big\{ \int_K (dd^c v)^q \wedge T,\ v \in \mathrm{PSH}(\Omega, [-1, 0])\Big\}$$

for any compact subset K of Ω. If E is a subset of Ω, we define

$$C_T(E, \Omega) = \sup\{C_T(K),\ K \text{ is a compact subset of } E\}.$$

We refer to [5, 9] for the properties of this capacity.

Definition 3.1 (1) *A subset A of Ω is said to be T-pluripolar if $C_T(A, \Omega) = 0$.*

(2) *A psh function u is said to be quasi-continuous with respect to C_T, if for every $\varepsilon > 0$, there exists an open subset O_ε such that $C_T(O_\varepsilon, \Omega) < \varepsilon$ and u is continuous on $\Omega \setminus O_\varepsilon$.*

Proposition 3.1 *Let $u \in \mathcal{F}^T(\Omega)$. Then for every $s > 0$ one has*

$$s^q C_T(\{u \leq -s\}, \Omega) \leq \int_\Omega (dd^c u)^q \wedge T.$$

In particular, the set $\{u = -\infty\}$ is T-pluripolar.

Proof Let $(u_j)_j \subset \mathcal{E}_0^T(\Omega)$ be a decreasing sequence to u on Ω as in Definition 2.2. Take $s > 0$, $v \in \mathrm{PSH}(\Omega, [-1, 0])$ and let K be a compact subset in $\{u_j \leq -s\}$. Thanks to the comparison principle (for bounded psh functions), we have

$$\begin{aligned}\int_K (dd^c v)^q \wedge T &\leq \int_{\{s^{-1}u_j < v\}} (dd^c v)^q \wedge T \leq \frac{1}{s^q} \int_{\{s^{-1}u_j < v\}} (dd^c u_j)^q \wedge T \\ &\leq \frac{1}{s^q} \int_\Omega (dd^c u_j)^q \wedge T.\end{aligned}$$

It follows that

$$C_T(\{u_j \leq -s\}, \Omega) \leq \frac{1}{s^q} \int_\Omega (dd^c u_j)^q \wedge T.$$

By tending j to infinity, we obtain

$$C_T(\{u \leq -s\}, \Omega) \leq \frac{1}{s^q} \int_\Omega (dd^c u)^q \wedge T.$$

Corollary 3.1 *Every $u \in \mathcal{F}^T(\Omega)$ is C_T-quasi-continuous.*

Proof Let $u \in \mathcal{F}^T(\Omega)$ and $\varepsilon > 0$. Denote $B_u(t) := \{z \in \Omega;\ u(z) < t\}$, $t \leq 0$. By Proposition 3.1, there exists $s_\varepsilon \geq 1$ such that $C_T(B_u(-s_\varepsilon), \Omega) < \frac{\varepsilon}{2}$. The function $u_\varepsilon := \max(u, -s_\varepsilon)$ is bounded on Ω so thanks to Dabbek-Elkhadhra [5], there exists an open subset $\mathcal{O}$ in Ω such that $C_T(\mathcal{O}, \Omega) < \frac{\varepsilon}{2}$ and u_ε is continuous on $\Omega \setminus \mathcal{O}$. The result follows by taking $\mathcal{O}_\varepsilon = \mathcal{O} \cup B_u(-s_\varepsilon)$.

To study the C_T-quasi-continuity on $\mathcal{E}_p^T(\Omega)$, we will proceed as in the previous case.

Proposition 3.2 *Let $u \in \mathcal{E}_p^T(\Omega)$ and $(u_j)_j \subset \mathcal{E}_0^T(\Omega)$ decreases to u on Ω as in Definition* 2.1. *Then for every $s > 0$ one has*

$$s^{p+q} C_T(\{u \leq -2s\}, \Omega) \leq \sup_{j \geq 1} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.$$

In particular, the set $\{u = -\infty\}$ is T-pluripolar.

Proof Let $s > 0$, $v \in \mathrm{PSH}(\Omega, [-1, 0])$. Thanks to comparison principle (for bounded psh functions), we have

$$\begin{aligned}\int_{\{u_j \leq -2s\}} (dd^c v)^q \wedge T &\leq \int_{\{u_j < -s+sv\}} (dd^c v)^q \wedge T \leq \frac{1}{s^q} \int_{\{s^{-1}u_j < -1+v\}} (dd^c u_j)^q \wedge T \\ &\leq \frac{1}{s^{p+q}} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.\end{aligned}$$

It follows that

$$C_T(\{u_j \leq -2s\}, \Omega) \leq \frac{1}{s^{p+q}} \sup_{m \geq 1} \int_\Omega (-u_m)^p (dd^c u_m)^q \wedge T.$$

By tending j to infinity, we obtain

$$C_T(\{u \leq -2s\}, \Omega) \leq \frac{1}{s^{p+q}} \sup_{m \geq 1} \int_\Omega (-u_m)^p (dd^c u_m)^q \wedge T.$$

By the same argument as in Corollary 3.1 we can easily deduce the following result.

Corollary 3.2 *Every function in $\mathcal{E}_p^T(\Omega)$ is C_T-quasi-continuous.*

Now we need a first version of the comparison principle where one of the functions will be unbounded. This result was proved in [5] for bounded functions.

Theorem 3.1 *Let $u \in \mathcal{F}^T(\Omega)$ and $v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\liminf_{z \to \partial\Omega \cap \mathrm{Supp} T} [u(z) - v(z)] \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^q \wedge T \leq \int_{\{u<v\}} (dd^c u)^q \wedge T.$$

Proof Firstly we assume that u and v are continuous on a neighborhood W of Supp$T$. Without loss of generality, we can assume that $u < v$ on W and $u = v$ on ∂W. Let $v_\varepsilon := \max(u, v - \varepsilon)$. Then one has $v_\varepsilon = u$ on ∂W and

$$\int_{\{u<v\}} (dd^c v_\varepsilon)^q \wedge T = \int_{\{u<v\}} (dd^c u)^q \wedge T.$$

Since the family of measures $(dd^c v_\varepsilon)^q \wedge T$ converges weakly to $(dd^c u)^q \wedge T$ as $\varepsilon \to 0$, then we obtain

$$\int_{\{u<v\}} (dd^c v)^q \wedge T = \int_{\{u<v\}} (dd^c u)^q \wedge T.$$

Let us now treat the general case. Replace u by $u + \delta$ if necessary, we can assume that $\liminf(u - v) \geq 2\delta$. So there exists an open subset $\mathcal{O} \subset\subset \Omega$ such that $u(z) \geq v(z) + \delta$ for all $z \in \Omega \setminus \mathcal{O}$. Let $(u_k)_k$ and $(v_j)_j$ be two smooth sequences of psh functions which decrease respectively to u and v on a neighborhood of $\overline{\mathcal{O}}$ such that $u_k \geq v_j$ on $\partial\overline{\mathcal{O}} \cap \mathrm{Supp}T$ for $j \geq k$. Using the previous argument we obtain

$$\int_{\{u_k<v_j\}} (dd^c v_j)^q \wedge T = \int_{\{u_k<v_j\}} (dd^c u_k)^q \wedge T.$$

For $\varepsilon > 0$, there exists an open subset G of Ω such that $C_T(G, \Omega) < \varepsilon$ and u, v are continuous on $\Omega \setminus G$. We can write $v = \varphi + \psi$ where φ is continuous on Ω and $\psi = 0$ on $\Omega \setminus G$. Take $U := \{u_k < \varphi\}$ then

$$\int_U (dd^c v)^q \wedge T \leq \lim_{j \to +\infty} \int_U (dd^c v_j)^q \wedge T.$$

Since $U \cup G = \{u_k < v\} \cup G$, we have

$$\begin{aligned}
&\int_{\{u_k<v\}} (dd^c v)^q \wedge T \\
&\leq \int_U (dd^c v)^q \wedge T + \int_G (dd^c v)^q \wedge T
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{j\to+\infty}\int_U (dd^c v_j)^q\wedge T+\int_G (dd^c v)^q\wedge T\\
&\le \lim_{j\to+\infty}\Big(\int_{\{u_k<v_j\}}(dd^c v_j)^q\wedge T+\int_G (dd^c v_j)^q\wedge T\Big)+\int_G (dd^c v)^q\wedge T\\
&\le \lim_{j\to+\infty}\int_{\{u_k<v_j\}}(dd^c v_j)^q\wedge T+2\varepsilon\|v\|_\infty^q\\
&\le \lim_{j\to+\infty}\int_{\{u_k<v_j\}}(dd^c u_k)^q\wedge T+2\varepsilon\|v\|_\infty^q.
\end{aligned}
$$

Now as $\{u_k<v_j\}\downarrow\{u_k\le v\}$, $\{u_k<v\}\uparrow\{u<v\}$, we have

$$
\int_{\{u<v\}}(dd^c v)^q\wedge T\le \lim_{k\to+\infty}\int_{\{u_k\le v\}}(dd^c u_k)^q\wedge T+2\varepsilon\|v\|_\infty^q.
$$

The continuity of u and v on $\Omega\setminus G$ gives that $\{u\le v\}\setminus G$ is a closed subset of Ω. It follows that

$$
\int_{\{u\le v\}\setminus G}(dd^c u)^q\wedge T\ge \lim_{k\to+\infty}\int_{\{u\le v\}\setminus G}(dd^c u_k)^q\wedge T.
$$

Thus

$$
\begin{aligned}
\int_{\{u\le v\}}(dd^c u)^q\wedge T &\ge \int_{\{u\le v\}\setminus G}(dd^c u)^q\wedge T\\
&\ge \lim_{k\to+\infty}\int_{\{u\le v\}\setminus G}(dd^c u_k)^q\wedge T\\
&\ge \lim_{k\to+\infty}\Big(\int_{\{u_k<v\}}(dd^c u_k)^q\wedge T-\int_G (dd^c u_k)^q\wedge T\Big)\\
&\ge \lim_{k\to+\infty}\int_{\{u_k<v\}}(dd^c u_k)^q\wedge T-\varepsilon\|v\|_\infty^q.
\end{aligned}
$$

So

$$
\int_{\{u<v\}}(dd^c v)^q\wedge T\le \int_{\{u\le v\}}(dd^c u)^q\wedge T+3\varepsilon\|v\|_\infty^q.
$$

By tending ε to 0, we obtain

$$
\int_{\{u<v\}}(dd^c v)^q\wedge T\le \int_{\{u\le v\}}(dd^c u)^q\wedge T.
$$

As $\{u+\rho<v\}\uparrow\{u<v\}$ and $\{u+\rho\le v\}\uparrow\{u<v\}$ when $\rho\searrow 0$, the desired inequality follows by replacing u by $u+\rho$.

Recall that the Lelong-Demailly number of T with respect to a psh function φ is defined as the limit $\nu(T,\varphi):=\lim\limits_{t\to-\infty}\nu(T,\varphi,t)$ where

$$
\nu(T,\varphi,t)=\int_{B_\varphi(t)} T\wedge(dd^c\varphi)^q,\quad t<0\,.
$$

The following result was proved in [6] but the author used Stokes formula where a regularity condition on φ was required.

Theorem 3.2 *Let $\varphi \in \mathcal{F}^T(\Omega)$ such that e^φ is continuous on Ω. Then for every $s, t > 0$ one has*

$$s^q C_T(B_\varphi(-t-s), \Omega) \le \nu(T, \varphi, -t) \le (s+t)^q C_T(B_\varphi(-t), \Omega). \tag{3.1}$$

In particular,

$$\nu(T, \varphi) = \int_{\{\varphi=-\infty\}} T \wedge (dd^c\varphi)^q = \lim_{t\to+\infty} t^q C_T(B_\varphi(-t), \Omega).$$

Proof Let $t, s > 0$ and $v \in \mathrm{PSH}(\Omega, [-1, 0])$. For $\varepsilon > 0$, we set $v_\varepsilon = \max\left(v, \frac{\varphi+t+\varepsilon}{s}\right)$. Thanks to Theorem 3.1 we have

$$\begin{aligned}
\int_{B_\varphi(-t-s-\varepsilon)} T \wedge (dd^c v)^q &= \int_{B_\varphi(-t-s-\varepsilon)} T \wedge (dd^c v_\varepsilon)^q \\
&\le \int_{\{\varphi<-t+sv-\varepsilon\}} T \wedge (dd^c v_\varepsilon)^q \\
&\le \frac{1}{s^q} \int_{\{\varphi<-t+sv-\varepsilon\}} T \wedge (dd^c\varphi)^q \\
&\le \frac{1}{s^q} \int_{B_\varphi(-t)} T \wedge (dd^c\varphi)^q.
\end{aligned}$$

By passing to the supremum over all $v \in \mathrm{PSH}(\Omega, [-1, 0])$, we obtain the following estimate:

$$s^q C_T(B_\varphi(-s-t-\varepsilon), \Omega) \le \nu(T, \varphi, -t).$$

By passing to the limit when $\varepsilon \to 0$, the left inequality in (3.1) is obtained. However, for the right inequality, we remark that the function $\psi = \max\left(\frac{\varphi}{s+t}, -1\right)$ is psh and satisfies $-1 \le \psi \le 0$ on Ω. So by Corollary 2.1 and using the fact that $\psi > -1$ near $\partial B_\varphi(-t)$ we obtain

$$\begin{aligned}
\int_{B_\varphi(-t)} T \wedge (dd^c\varphi)^q &= (s+t)^q \int_{B_\varphi(-t)} T \wedge (dd^c\psi)^q \\
&\le (s+t)^q C_T(B_\varphi(-t), \Omega).
\end{aligned}$$

Thus the right inequality in (3.1) follows.

By the right inequality in (3.1), we have

$$\nu(T, \varphi) = \lim_{t\to+\infty} \nu(T, \varphi, -t) \le \lim_{t\to+\infty} \frac{(s+t)^q}{t^q} t^q C_T(B_\varphi(-t), \Omega) = \lim_{t\to+\infty} t^q C_T(B_\varphi(-t), \Omega).$$

If we take $\alpha > 1$ and $s = \alpha t$ in the left inequality in (3.1), we obtain

$$\begin{aligned}
\nu(T, \varphi) = \lim_{t\to+\infty} \nu(T, \varphi, -t) &\ge \lim_{t\to+\infty} \frac{\alpha^q}{(1+\alpha)^q} (1+\alpha)^q t^q C_T(B_\varphi(-(1+\alpha)t), \Omega) \\
&= \left(\frac{\alpha}{1+\alpha}\right)^q \lim_{t\to+\infty} t^q C_T(B_\varphi(-t), \Omega).
\end{aligned}$$

The result follows by letting $\alpha \to +\infty$.

Remark 3.1 By combining Proposition 3.2 and Theorem 3.2, it is easy to check that if $\varphi \in \mathcal{F}_p^T(\Omega)$ such that e^φ is continuous on Ω then $\nu(T, \varphi) = 0$.

4 Main Result

The aim of this part is to prove the following main result.

Theorem 4.1 (Comparison Principle) *Let $u \in \mathcal{F}^T(\Omega)$ and $v \in \mathcal{E}^T(\Omega)$. Then*

$$\int_{\{u<v\}} (dd^c v)^q \wedge T \le \int_{\{u<v\}\cup\{u=v=-\infty\}} (dd^c u)^q \wedge T.$$

Before presenting the proof, we give some corollaries.

4.1 Consequences of Theorem 4.1

Corollary 4.1 *Let $u, v \in \mathcal{F}_p^T(\Omega)$ such that e^u is continuous on Ω. Then*

$$\int_{\{u<v\}} (dd^c v)^q \wedge T \le \int_{\{u<v\}} (dd^c u)^q \wedge T.$$

Proof Thanks to the comparison principle, we have

$$\int_{\{u<v\}} (dd^c v)^q \wedge T \le \int_{\{u<v\}\cup\{u=v=-\infty\}} (dd^c u)^q \wedge T \le \int_{\{u<v\}} (dd^c u)^q \wedge T + \nu(T, u).$$

The result follows by the fact that $\nu(T, u) = 0$ because $u \in \mathcal{F}_p^T(\Omega)$.

Corollary 4.2 *Let $u \in \mathcal{F}^T(\Omega)$ and $v \in \mathcal{F}_p^T(\Omega)$ such that e^v is continuous on Ω. We assume that*

$$(dd^c u)^q \wedge T \le (dd^c v)^q \wedge T.$$

Then $C_T(\{u < v\}, \Omega) = 0$.

Proof Assume $C_T(\{u < v\}, \Omega) > 0$. Then there exists $\psi \in \mathrm{PSH}(\Omega, [0, 1])$ such that

$$\int_{\{u<v\}} (dd^c \psi)^q \wedge T > 0.$$

For $\varepsilon > 0$ small enough, one has $v + \varepsilon\psi \in \mathcal{F}^T(\Omega)$ so thanks to the comparison principle,

$$\begin{aligned}\int_{\{u<v+\varepsilon\psi\}} (dd^c (v + \varepsilon\psi))^q \wedge T &\le \int_{\{u<v+\varepsilon\psi\}\cup\{u=v=-\infty\}} (dd^c u)^q \wedge T \\ &\le \int_{\{u<v+\varepsilon\psi\}\cup\{u=v=-\infty\}} (dd^c v)^q \wedge T \\ &\le \int_{\{u<v+\varepsilon\psi\}} (dd^c v)^q \wedge T + \nu(T, v).\end{aligned}$$

Hence

$$\varepsilon^q \int_{\{u<v\}} (dd^c \psi)^q \wedge T + \int_{\{u<v+\varepsilon\psi\}} (dd^c v)^q \wedge T \le \int_{\{u<v+\varepsilon\psi\}} (dd^c v)^q \wedge T,$$

which is absurd.

4.2 Proof of Theorem 4.1

To prove this main result, we shall use similar Xing's inequalities (see [10–11] for more details), generalized to $\mathcal{E}^T(\Omega)$. We start by recalling the following lemma.

Lemma 4.1 (see [7]) *Let S be a positive closed current of bidimension $(1,1)$ on Ω and $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$. Assume that $u \leq v$ on Ω and*

$$\lim_{z \to \partial\Omega} [u(z) - v(z)] = 0.$$

Then one has

$$\int_\Omega (v-u)^k dd^c w \wedge S \leq k \int_\Omega (1-w)(v-u)^{k-1} dd^c u \wedge S$$

for all $k \geq 1$ and $w \in \mathrm{PSH}(\Omega, [0,1])$.

Lemma 4.2 *Let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $u \leq v$ on Ω and*

$$\lim_{z \to \partial\Omega} [u(z) - v(z)] = 0.$$

Then one has

$$\frac{1}{q!} \int_\Omega (v-u)^q dd^c w_1 \wedge \cdots \wedge dd^c w_q \wedge T + \int_\Omega (r - w_1)(dd^c v)^q \wedge T \leq \int_\Omega (r-w_1)(dd^c u)^q \wedge T$$

for every $r \geq 1$ and $w_1, \cdots, w_q \in \mathrm{PSH}(\Omega, [0,1])$.

Proof Let $K \subset\subset \Omega$ and assume that $u = v$ on $\Omega \setminus K$. Using Lemma 4.1 we obtain

$$\begin{aligned}
&\int_\Omega (v-u)^q dd^c w_1 \wedge \cdots \wedge dd^c w_q \wedge T \\
&\leq q \int_\Omega (v-u)^{q-1} dd^c w_1 \wedge \cdots \wedge dd^c w_{q-1} \wedge dd^c u \wedge T \\
&\vdots \\
&\leq q! \int_\Omega (v-u) dd^c w_1 \wedge (dd^c u)^{q-1} \wedge T \\
&\leq q! \int_\Omega (w_1 - r) dd^c (v-u) \wedge \Big(\sum_{i=0}^{q-1} (dd^c u)^i \wedge (dd^c v)^{q-i-1} \Big) \wedge T \\
&= q! \int_\Omega (r - w_1) dd^c (u-v) \wedge \Big(\sum_{i=0}^{q-1} (dd^c u)^i \wedge (dd^c v)^{q-i-1} \Big) \wedge T \\
&= q! \int_\Omega (r-w_1)((dd^c u)^q - (dd^c v)^q) \wedge T.
\end{aligned}$$

In the general case, for every $\varepsilon > 0$ we set $v_\epsilon = \max(u, v - \varepsilon)$. Then $v_\epsilon \nearrow v$ on Ω and satisfies $v_\epsilon = u$ on $\Omega \setminus K$ for some $K \subset\subset \Omega$. Hence

$$\frac{1}{q!} \int_\Omega (v_\varepsilon - u)^q dd^c w_1 \wedge \cdots \wedge dd^c w_q \wedge T + \int_\Omega (r-w_1)(dd^c v_\varepsilon)^q \wedge T \leq \int_\Omega (r-w_1)(dd^c u)^q \wedge T.$$

Since $v_\varepsilon - u \nearrow v - u$, the family of measures $(dd^c v_\varepsilon)^q \wedge T$ converges weakly to $(dd^c v)^q \wedge T$ as $\varepsilon \searrow 0$ and the function $r - w_1$ is lower semicontinuous. Then by letting $\varepsilon \searrow 0$, we obtain the desired inequality.

Proposition 4.1 *Let $r \geq 1$ and $w \in \mathrm{PSH}(\Omega, [0,1])$.*

(1) *For every $u, v \in \mathcal{F}^T(\Omega)$ such that $u \leq v$ on Ω one has*

$$\frac{1}{q!}\int_\Omega (v-u)^q (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v)^q \wedge T \leq \int_\Omega (r-w)(dd^c u)^q \wedge T. \tag{4.1}$$

(2) *Furthermore, (4.1) holds for $u, v \in \mathcal{E}^T(\Omega)$ such that $u \leq v$ on Ω and $u = v$ on $\Omega \setminus K$ for some $K \subset\subset \Omega$.*

Proof (1) Let $u, v \in \mathcal{F}^T(\Omega)$ and $u_m, v_j \in \mathcal{E}_0^T(\Omega)$ which decrease to u and v respectively as in Definition 2.2. Replacing v_j by $\max(u_j, v_j)$ we may assume that $u_j \leq v_j$ for $j \geq 1$. By Lemma 4.2 we have for $m \geq j \geq 1$,

$$\frac{1}{q!}\int_\Omega (v_j-u_m)^q \wedge (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v_j)^q \wedge T \leq \int_\Omega (r-w)(dd^c u_m)^q \wedge T.$$

By approximating w by a sequence of continuous psh functions vanishing on $\partial\Omega$ (see [3]) and using Proposition 2.2, we obtain that when $m \to +\infty$,

$$\frac{1}{q!}\int_\Omega (v_j-u)^q \wedge (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v_j)^q \wedge T \leq \int_\Omega (r-w)(dd^c u)^q \wedge T.$$

Since $r-w$ is lower semi-continuous, we have

$$\lim_{j\to\infty}\int_\Omega (r-w)(dd^c v_j)^q \wedge T \geq \int_\Omega (r-w)(dd^c v)^q \wedge T.$$

Hence by tending $j \to +\infty$, we obtain the result.

(2) Let G and W be open subsets of Ω such that $K \subset\subset G \subset\subset W \subset\subset \Omega$. There exists $\widetilde{v} \in \mathcal{F}^T(\Omega)$ such that $\widetilde{v} \geq v$ on Ω and $\widetilde{v} = v$ on W. Let $\widetilde{u}$ be such that $\widetilde{u} = u$ on G and $\widetilde{u} = \widetilde{v}$ either. Since $u = v = \widetilde{v}$ on $W \setminus K$, we have $\widetilde{u} \in \mathrm{PSH}^-(\Omega)$. It follows that $\widetilde{u} \in \mathcal{F}^T(\Omega)$, $\widetilde{u} \leq \widetilde{v}$ and $\widetilde{u} = u$ on W.

Using (1) we obtain

$$\frac{1}{q!}\int_\Omega (\widetilde{v}-\widetilde{u})^q \wedge (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c \widetilde{v})^q \wedge T \leq \int_\Omega (r-w)(dd^c \widetilde{u})^q \wedge T.$$

As $\widetilde{v} = \widetilde{u}$ on $\Omega \setminus G$, we have

$$\frac{1}{q!}\int_W (\widetilde{v}-\widetilde{u})^q \wedge (dd^c w)^q \wedge T + \int_W (r-w)(dd^c \widetilde{v})^q \wedge T \leq \int_W (r-w)(dd^c \widetilde{u})^q \wedge T.$$

Now since $\widetilde{u} = u$, $\widetilde{v} = v$ and $u = v$ on $\Omega \setminus K$, we obtain

$$\frac{1}{q!}\int_\Omega (v-u)^q \wedge (dd^c w)^q \wedge T + \int_\Omega (r-w)(dd^c v)^q \wedge T \leq \int_\Omega (r-w)(dd^c u)^q \wedge T.$$

Remark 4.1 If we take $w = 0$ and $r = 1$ in Proposition 4.1, we obtain another proof of Proposition 2.2.

The following inequality is a generalization of Theorem 4.1 in [8] since we shall prove it for an arbitrary positive closed current T.

Theorem 4.2 *Let $u, w_1, \cdots, w_{q-1} \in \mathcal{F}^T(\Omega)$ and $v \in \mathrm{PSH}^-(\Omega)$. If we set $S = dd^c w_1 \wedge \cdots \wedge dd^c w_{q-1}$, then*

$$dd^c \max(u,v) \wedge T \wedge S_{|\{u>v\}} = dd^c u \wedge T \wedge S_{|\{u>v\}}.$$

Proof We prove the theorem in two steps. First we assume that $v \equiv a < 0$. Thanks to Lemma 2.2, there exist $u_j, w_{k,j} \in \mathcal{E}_0^T(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $(u_j)_j$ decreases to u and $(w_{k,j})_j$ decreases to w_k for each $1 \leq k \leq q-1$. Since $\{u_j > a\}$ is open, one has

$$dd^c \max(u_j, a) \wedge T \wedge S^j_{|\{u_j>a\}} = dd^c u_j \wedge T \wedge S^j_{|\{u_j>a\}},$$

where $S^j = dd^c w_{1,j} \wedge \cdots \wedge dd^c w_{q-1,j}$. As $\{u > a\} \subset \{u_j > a\}$ we obtain

$$dd^c \max(u_j, a) \wedge T \wedge S^j_{|\{u>a\}} = dd^c u_j \wedge T \wedge S^j_{|\{u>a\}}.$$

It follows from [7] that

$$\max(u-a,0) dd^c \max(u_j,a) \wedge T \wedge S^j \underset{j\to+\infty}{\longrightarrow} \max(u-a,0) dd^c \max(u,a) \wedge T \wedge S,$$

$$\max(u-a,0) dd^c u_j \wedge T \wedge S^j \underset{j\to+\infty}{\longrightarrow} \max(u-a,0) dd^c u \wedge T \wedge S.$$

Hence

$$\max(u-a,0)[dd^c \max(u,a) \wedge T \wedge S - dd^c u \wedge T \wedge S] = 0.$$

So

$$dd^c \max(u,a) \wedge T \wedge S = dd^c u \wedge T \wedge S \quad \text{on } \{u > a\}.$$

Now assume $v \in \mathrm{PSH}^-(\Omega)$. Since $\{u > v\} = \bigcup\limits_{a \in \mathbb{Q}^-} \{u > a > v\}$, it suffices to show that

$$dd^c \max(u,v) \wedge T \wedge S = dd^c u \wedge T \wedge S \quad \text{on } \{u > a > v\}$$

for all $a \in \mathbb{Q}^-$. As $\max(u,v) \in \mathcal{F}^T(\Omega)$, by the first step, we have

$$\begin{aligned} dd^c \max(u,v) \wedge T \wedge S_{|\{\max(u,v)>a\}} &= dd^c \max(\max(u,v),a) \wedge T \wedge S_{|\{\max(u,v)>a\}} \\ &= dd^c \max(u,v,a) \wedge T \wedge S_{|\{\max(u,v)>a\}}, \\ dd^c u \wedge T \wedge S_{|\{u>a\}} &= dd^c \max(u,a) \wedge T \wedge S_{|\{v>a\}}. \end{aligned}$$

The fact that $\max(u,v,a) = \max(u,a)$ on the open set $\{a > v\}$ gives

$$dd^c \max(u,v,a) \wedge T \wedge S_{|\{a>v\}} = dd^c \max(u,a) \wedge T \wedge S_{|\{a>v\}}.$$

As $\{u > a > v\}$ is contained in $\{u > a\}$, in $\{\max(u,v) > v\}$ and in $\{a > v\}$, then by combining the last equalities we obtain

$$dd^c \max(u,v) \wedge T \wedge S_{|\{u>a>v\}} = dd^c \max(u,a) \wedge T \wedge S_{|\{u>a>v\}}.$$

We can now prove an inequality analogous to Demailly's one. Such inequality was proved in [8] for the extremal case $q = n$.

Proposition 4.2 (1) *Let $u, v \in \mathcal{F}^T(\Omega)$ such that $(dd^c u)^q \wedge T(\{u = v = -\infty\}) = 0$. Then*

$$(dd^c \max(u, v))^q \wedge T \geq 1\!\!1_{\{u \geq v\}}(dd^c u)^q \wedge T + 1\!\!1_{\{u<v\}}(dd^c v)^q \wedge T.$$

(2) *Let μ be a positive measure vanishing on all pluripolar sets of Ω and $u, v \in \mathcal{E}^T(\Omega)$ such that $(dd^c u)^q \wedge T \geq \mu$, $(dd^c v)^q \wedge T \geq \mu$. Then $(dd^c max(u, v))^q \wedge T \geq \mu$.*

Proof (1) For each $\epsilon > 0$, put $A_\epsilon = \{u = v - \epsilon\} \setminus \{u = v = -\infty\}$. Since $A_\epsilon \cap A_\delta = \emptyset$ for $\epsilon \neq \delta$, there exists $\epsilon_j \searrow 0$ such that $(dd^c u)^q \wedge T(A_{\epsilon_j}) = 0$ for $j \geq 1$. On the other hand, since $(dd^c u)^q \wedge T(\{u = v = -\infty\}) = 0$ we have $(dd^c u)^q \wedge T(\{u = v - \epsilon_j\}) = 0$ for $j \geq 1$. Using Theorem 4.2 it follows that

$$\begin{aligned}
&(dd^c \max(u, v - \epsilon_j))^q \wedge (dd^c w)^q \wedge T \\
&\geq (dd^c \max(u, v - \epsilon_j))^q \wedge T_{|\{u > v - \epsilon_j\}} + (dd^c \max(u, v - \epsilon_j))^q \wedge T_{|\{u < v - \epsilon_j\}} \\
&= (dd^c u)^q \wedge T_{|\{u > v - \epsilon_j\}} + (dd^c v)^q \wedge T_{|\{u < v - \epsilon_j\}} \\
&= 1\!\!1_{\{u \geq v - \epsilon_j\}}(dd^c u)^q \wedge T + 1\!\!1_{\{u < v - \epsilon_j\}}(dd^c v)^q \wedge T \\
&\geq 1\!\!1_{\{u \geq v\}}(dd^c u)^q \wedge T + 1\!\!1_{\{u < v - \epsilon_j\}}(dd^c v)^q \wedge T.
\end{aligned}$$

Letting $j \to +\infty$ and by Theorem 2.3, we get

$$(dd^c \max(u, v))^q \wedge T \geq 1\!\!1_{\{u \geq v\}}(dd^c u)^q \wedge T + 1\!\!1_{\{u<v\}}(dd^c u)^q \wedge T$$

because $\max(u, v - \epsilon_j) \nearrow \max(u, v)$ and $1\!\!1_{\{u<v-\epsilon_j\}} \nearrow 1\!\!1_{\{u<v\}}$ as $j \to +\infty$.

(2) Argument is similar to that of (1).

Proposition 4.3 *Let $u \in \mathcal{F}^T(\Omega)$, $v \in \mathcal{E}^T(\Omega)$. Then*

$$\begin{aligned}
&\frac{1}{q!}\int_{\{u<v\}} (v-u)^q \wedge (dd^c w)^q \wedge T + \int_{\{u<v\}} (r-w)(dd^c v)^q \wedge T \\
&\leq \int_{\{u<v\}\cup\{u=v=-\infty\}} (r-w)(dd^c u)^q \wedge T
\end{aligned}$$

for $w \in \mathrm{PSH}(\Omega, [0, 1])$ and all $r \geq 1$.

Proof Let $\varepsilon > 0$ and set $\widetilde{v} = \max(u, v - \varepsilon)$. By (4.1) we have

$$\begin{aligned}
&\frac{1}{q!}\int_\Omega (\widetilde{v} - u)^q \wedge (dd^c w)^q \wedge T + \int_\Omega (r - w)(dd^c \widetilde{v})^q \wedge T \\
&\leq \int_\Omega (r - w)(dd^c u)^q \wedge T.
\end{aligned}$$

Since $\{u < \widetilde{v}\} = \{u < v - \varepsilon\}$, thanks to Theorem 4.2, we haves

$$\begin{aligned}
&\frac{1}{q!}\int_{\{u<v-\varepsilon\}} (v - \varepsilon - u)^q \wedge (dd^c w)^q \wedge T + \int_{\{u\leq v-\varepsilon\}} (r - w)(dd^c v)^q \wedge T \\
&\leq \int_{\{u\leq v-\varepsilon\}} (r - w)(dd^c u)^q \wedge T.
\end{aligned}$$

As $\{u \leq v - \varepsilon\} \subset \{u < v\} \cup \{u = v = -\infty\}$, one has

$$\frac{1}{q!}\int_{\{u<v-\varepsilon\}} (v - \varepsilon - u)^q \wedge (dd^c w)^q \wedge T + \int_{\{u\leq v-\varepsilon\}} (r - w)(dd^c v)^q \wedge T$$

$$\leq \int_{\{u\leq v\}\cup\{u=v=-\infty\}} (r-w)(dd^c u)^q \wedge T.$$

Letting $\varepsilon \to 0$ we obtain

$$\frac{1}{q!}\int_{\{u<v\}} (v-u)^q \wedge (dd^c w)^q \wedge T + \int_{\{u<v\}} (r-w)(dd^c v)^q \wedge T$$
$$\leq \int_{\{u<v\}\cup\{u=v=-\infty\}} (r-w)(dd^c u)^q \wedge T.$$

To conclude the proof of Theorem 4.1, it suffices to take $w = 0$ and $r = 1$ in the previous proposition.

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