Chin. Ann. Math. Ser. B
40(4), 2019, 585–598
DOI: 10.1007/s11401-019-0153-5

Chinese Annals of Mathematics, Series B

Boundedness of Commutators of θ-Type Calderón-Zygmund Operators on Non-homogeneous Metric Measure Spaces*

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Abstract Let (X, d, μ) be a metric measure space satisfying both the upper doubling and the geometrically doubling conditions in the sense of Hytönen. In this paper, the authors obtain the boundedness of the commutators of θ-type Calderón-Zygmund operators with RBMO functions from $L^\infty(\mu)$ into RBMO(μ) and from $H^{1,\infty}_{\rm at}(\mu)$ into $L^1(\mu)$, respectively. As a consequence of these results, they establish the $L^p(\mu)$ boundedness of the commutators on the non-homogeneous metric spaces.

Keywords Non-homogeneous space, θ-Type Calderón-Zygmund operator, Commutator, RBMO(μ) space, $H^{1,\infty}_{\rm at}(\mu)$ space
2000 MR Subject Classification 42B20

1 Introduction and Preliminaries

The theory of Calderón-Zygmund operators and commutators plays an important role in harmonic analysis and partial differential equations. The theory of commutators were introduced in a general form by Calderón [2–3], in which Calderón showed that these kinds of operators are bounded on L^2. In [4], the authors proved that given a singular integral T with standard Calderón-Zygmund kernel, the operator $[b, T] = bT - Tb$ is bounded in $L^p, 1 < p < \infty$ if b is a BMO function, the converse implication is due to [12]. We refer also to [14], in which Pérez proved endpoint estimates for commutators of singular integrals with BMO functions.

Recently, many mathematicians pay attention to the study of non-doubling measure spaces. One of the most general settings to which Calderón-Zygmund theory extends naturally is the spaces of homogeneous type in the sense of Coifman and Weiss [5], Many results from real and harmonic analysis on Euclidean spaces have their natural extensions on these space(see, for example, [5–6, 8]). A metric space (X, d) equipped with a nonnegative Borel measure μ is called a space of homogeneous type if (X, d, μ) satisfies the following measure doubling condition that there exists a positive constant C_μ, depending on μ, such that for any ball

Manuscript received January 4, 2016. Revised April 16, 2017.
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*This work was supported by the National Natural Science Foundation of China (No. 11671414).

$B(x,r) = \{y \in X : d(x,y) < r\}$ with $x \in X$ and $r \in (0,\infty)$,

$$\mu(B(x,2r)) \le C_\mu \mu(B(x,r)). \tag{1.1}$$

This definition was introduced by Coifman and Weiss in [5]. We point out that d may also be assumed to be a quasi-metric. However, for the simplicity, in this paper, we always assume that d is a metric; see also [9]. The measure doubling condition (1.1) plays a key role in the classical theory of Calderón-Zygmund operators. However, many results on the classical Calderón-Zygmund theory have been proved still valid if the measure doubling condition is replaced by a weaker condition such as the polynomial growth condition; see, for example, [7, 15, 18]. To be precise, let $k \in (0,\infty)$, X be a metric space endowed with a metric d and a nonnegative "k-dimensional" Borel measure μ in the sense that there exists a positive constant C_0 such that for all $x \in X$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \le C_0 r^k. \tag{1.2}$$

Such a measure need not satisfy the doubling condition (1.1). In [15], Tolsa established Calderón-Zygmund theory for non doubling measures. Because the measures satisfying (1.2) are only different form, not more general than,the measures satisfying (1.1), the theory with this kinds of non-doubling measures is not in all respects a generalization of the corresponding theory of spaces of homogeneous type. Hytönen in [9] introduced a new class of metric measure spaces satisfying the so-called upper doubling and the geometrically doubling conditions (see Definitions 1.1–1.2 below). This new class of metric measure spaces includes both spaces of homogeneous type and metric spaces with the measures satisfying (1.2) as special cases. Recently, many results on the Calderón-Zygmund theory have been built on the non-homogeneous metric measure spaces (see [1, 10–11, 13, 18]). Let us mention that Bui and Duong [1] showed that if the Calderón-Zygmund operator is bounded on $L^2(\mu)$, the commutator of this operator with a function RBMO(μ) is bounded on $L^p(\mu)$,$1 < p < \infty$ on non-homogeneous metric measure space.

θ-type Calderón-Zygmund operator was introduced by Yabuta in [17], In [16], the authors studied the properties of θ-type Calderón-Zygmund operator and commutator. In this paper, we study the commutators of θ-type Calderón-Zygmund operator with RBMO function on non-homogeneous metric measure space. We show that this commutators is bounded from $L^\infty(\mu)$ into RBMO(μ) and from $H^{1,\infty}_{\rm at}(\mu)$ into $L^1(\mu)$, respectively. To state our main result, we recall some necessary notations which will be used in the proof of our main results. We start with the notion of the upper doubling and geometrically doubling metric measure space which introduced in [1, 9].

Definition 1.1 *A measure μ in the metric space (X,μ) is said to be an upper doubling measure if there exists a dominating function λ with the following properties*:

(i) $\lambda : X \times (0,\infty) \mapsto (0,\infty)$.

(ii) *For any fixed* $x \in X, r \mapsto \lambda(x,r)$ *is increasing.*

(iii) *There exists a constant* $C_\lambda > 0$ *such that* $\lambda(x,2r) \le C_\lambda \lambda(x,r)$ *for all* $x \in X, r > 0$.

(iv) *The inequality* $\mu(x,r) := \mu(B(x,r)) \le \lambda(x,r)$ *holds for all* $x \in X, r > 0$.

(v) *And* $\lambda(x,r) \approx \lambda(y,r)$ *for all* $r > 0, x, y \in X$ *and* $d(x,y) \le r$.

Obviously, a space of homogeneous type is a special case of upper doubling spaces, if we take the dominating function $\lambda(x,r) = \mu(B(x,r))$. On the other hand, a metric space (X,d,μ) satisfying the polynomial growth condition (1.2) is also an upper doubling measure space by taking $\lambda(x,r) = C_0 r^k$. We now recall the notion of the geometrically doubling space introduced in [9].

Definition 1.2 *A metric space* (X,d) *is called geometrically doubling if there exists some* $N_0 \in N = \{1, 2, \cdots\}$ *such that for any ball* $B(x,r) \subset X$ *, there exists a finite ball covering* $\{B(x_i, \frac{r}{2})\}_i$ *of* $B(x,r)$ *such that the cardinality of this covering is at most* N_0.

Let (X,d) be a metric space. In [9], Hytönen showed that the following statements are mutually equivalent:

(i) (X,d) is geometrically doubling.

(ii) For any $\varepsilon \in (0,1)$ and any ball $B(x,r) \subset X$, there exists a finite ball covering $\{B(x_i, \varepsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0 \varepsilon^{-n}$, where and in what follows, N_0 is as in Definition 1.2 and $n = \log_2 N_0$.

(iii) For every $\varepsilon \in (0,1)$, any ball $B(x,r) \subset X$ contains at most $N_0 \varepsilon^{-n}$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \varepsilon r)\}_i$.

(iv) There exists $M \in N$ such that any ball $B(x,r) \subset X$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^{M}$.

We now recall the coefficients $K_{B,Q}$ and $K'_{B,Q}$ for any two balls $B \subset Q$ which were introduced in [1]. For any two balls $B \subset Q$, let

$$K_{B,Q} = 1 + \int_{r_B \le d(x,x_B) \le r_Q} \frac{1}{\lambda(x_B, d(x,x_B))} \mathrm{d}\mu(x),$$

$$K'_{B,Q} = 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)},$$

where $N_{B,Q}$ is the smallest integer satisfying $6^{N_{B,Q}} r_B \ge r_Q$. If $\lambda(x, ar) = a^m \lambda(x,r)$ for all $x \in X$ and $a, r > 0$, it is not difficult to show that $K_{B,Q} \approx K'_{B,Q}$. However, in general, we only have $K_{B,Q} \le C K'_{B,Q}$. This definition is a variant of the definition in [15]. Similar to Lemma 2.1 in [15], the authors in [1] showed the following property.

Lemma 1.1 (i) *If* $Q \subset R \subset S$ *are balls in* X*, then*

$$\max\{K_{Q,R}, K_{R,S}\} \le K_{Q,S} \le C(K_{Q,R} + K_{R,S}).$$

(ii) *If* $Q \subset R$ *are of compatible size, then* $K_{Q,R} \le C$.

(iii) *If* $\alpha Q, \cdots, \alpha^{N-1} Q$ *are non* (α, β)*-doubling balls* $(\beta > C_\lambda^{\log_2 \alpha})$ *then* $K_{Q, \alpha^N Q} \le C$.

Throughout this paper, C always means a positive constant independent of the main parameters involved, but it may be different in different contents.

2 Main Results and Proofs

At first, based on the definition of θ-type Calderón-Zygmund operator in [17] , we define the θ-type Calderón-Zygmund operator on non-homogeneous metric measure spaces as follows.

Definition 2.1 *Let θ be a non-negative, non-decreasing function on $\mathbb{R}^+ = (0,\infty)$ satisfying*

$$\int_0^1 \frac{\theta(t)}{t}\mathrm{d}t < \infty. \tag{2.1}$$

A kernel $K(\cdot,\cdot) \in L^1_{\rm loc}(X \times X\backslash\{(x,y) : x = y\})$ is called a θ-type Calderón-Zygmund kernel if the following conditions hold

$$|K(x,y)| \le C \min\Big\{\frac{1}{\lambda(x,d(x,y))}, \frac{1}{\lambda(y,d(x,y))}\Big\} \tag{2.2}$$

and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\theta\Big(\frac{d(x,x')}{d(x,y)}\Big)\frac{1}{\lambda(x,d(x,y))}, \tag{2.3}$$

when $d(x,y) \ge 2d(x,x')$.

A linear operator T is called the θ-type Calderón-Zygmund operator with kernel $K(\cdot,\cdot)$ satisfying (2.2) and (2.3) if for all $f \in L^\infty(\mu)$ with bounded support and $x \notin {\rm supp} f$,

$$Tf(x) = \int_X K(x,y)f(y)\mathrm{d}\mu(y). \tag{2.4}$$

The RBMO(μ) space for the general non-homogeneous space (X,μ) was introduced by Hytönen in [9], and studied systematically in [1]. Both of the authors also gave some characterizations of RBMO(μ) space.

Definition 2.2 *Given a ball $B \subset X$, let N be the smallest non-negative integer such that $\widetilde{B} = 6^N B$ is doubling (such a ball B exists due to Lemma 1.1). Let $\rho > 1$ be some fixed constant. We say that $f \in L^1_{\rm loc}(\mu)$ is in the* RBMO(μ) *if there exists some constant $C > 0$ such that for any ball B,*

$$\frac{1}{\mu(\rho B)}\int_B |f(x) - m_{\widetilde{B}}f|\mathrm{d}\mu(x) \le C \tag{2.5}$$

and for any two doubling balls Q and R such that $Q \subset R$,

$$|m_Q f - m_R f| \le CK_{Q,R}, \tag{2.6}$$

where $m_B f$ is the mean value of f over the ball B. Then we set

$$\|f\|_* = \inf\{C : (2.5)\ and\ (2.6)\ hold\}.$$

We need the following equivalent property of Definition 2.2 (see[1]).

Lemma 2.1 *For $f \in L^1_{\rm loc}(\mu)$, the following three are equivalent*:

(a) $f \in {\rm RBMO}(\mu)$.

(b) *There exists some constant C_b such that for any ball B,*

$$\frac{1}{\mu(6B)} \int_B |f(x) - m_B f| {\rm d}\mu(x) \le C_b \tag{2.7}$$

and for any two balls Q and R such that $Q \subset R$,

$$|m_Q f - m_R f| \le C_b K_{Q,R} \Big(\frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)} \Big). \tag{2.8}$$

(c) *There exists some constant C_c such that for any doubling ball B,*

$$\frac{1}{\mu(B)} \int_B |f(x) - m_B f| {\rm d}\mu(x) \le C_c \tag{2.9}$$

and

$$|m_Q f - m_R f| \le C_c K_{Q,R} \tag{2.10}$$

for any two doubling balls $Q \subset R$.

Definition 2.3 *Let the kernel K satisfy Definition* 2.1. *The commutator of the θ-type Calderón-Zygmund operator T with* RBMO *function is defined by*

$$[b, T] f(x) = b(x) T f(x) - T(bf)(x) = \int_X K(x, y)(b(x) - b(y)) f(y) {\rm d}y.$$

The main result of our paper is given as follows.

Theorem 2.1 *Let T be θ-type Calderón-Zygmund operator defined by* (2.4) *as above and T is bounded on $L^2(\mu)$. If $b \in {\rm RBMO}(\mu)$, then the commutator $[b, T]$ is bounded from $L^\infty(\mu)$ into* RBMO(μ) *on non-homogeneous space.*

Proof By characterization of RBMO(μ) given by (2.9)–(2.10) in Lemma 2.1, We only need to prove the following two estimations:

(i) There exists some constant C such that for any doubling ball B,

$$\int_B |([b, T] f - m_B([b, T] f) | {\rm d}\mu \le C \|f\|_{L^\infty(\mu)} \|b\|_* \mu(B). \tag{2.11}$$

(ii) For any two doubling balls Q and R with $Q \subset R$, one has

$$|m_Q([b, T] f) - m_R([b, T] f)| \le C K_{Q,R} \|f\|_{L^\infty(\mu)} \|b\|_*. \tag{2.12}$$

We first check (2.11). Let $\{b_B\}$ be a family of numbers, satisfying

$$\int_B |b - b_B| {\rm d}\mu \le 2\mu(6B) \|b\|_*$$

for balls B, and

$$|b_Q - b_R| \le 2 K_{Q,R} \|b\|_*$$

for balls $Q \subset R$. Denote $h_B = m_B(T((b-b_B)f\chi_{X\setminus 6B}))$, $f_1 = f\chi_{6B}$ and $f_2 = f\chi_{X\setminus 6B}$.

For any doubling ball B, we can write

$$\begin{aligned}
&\int_B |[b,T]f - m_B([b,T]f)|\mathrm{d}\mu \\
&= \int_B |(b-b_B)Tf - T((b-b_B)f_1) - T((b-b_B)f_2) + h_B - h_B - m_B([b,T]f)|\mathrm{d}\mu \\
&\le \int_B |(b-b_B)Tf|\mathrm{d}\mu + \int_B |T((b-b_B)f_1)|\mathrm{d}\mu \\
&\quad + \int_B |T((b-b_B)f_2) - h_B|\mathrm{d}\mu + \int_B |h_B + m_B([b,T]f)|\mathrm{d}\mu \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}$$

By Hölder's inequality, $L^2(\mu)$-boundedness of T and corollary 6.3 in [9], it follows

$$\begin{aligned}
\mathrm{I}_1 &\le \Big(\int_B |Tf|^2\mathrm{d}\mu\Big)^{\frac12}\Big(\int_B |b-b_B|^2\mathrm{d}\mu\Big)^{\frac12} \\
&\le C\|f\|_{L^\infty(\mu)}\mu(B)^{\frac12}\|b\|_*\mu(B)^{\frac12} \\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\mu(B).
\end{aligned}$$

Using the doubling property of ball B and the coefficient $K_{B,6B} \le C$, we have

$$\begin{aligned}
\mathrm{I}_2 &\le \Big(\int_B |T((b-b_B)f_1)|^2\mathrm{d}\mu\Big)^{\frac12}\Big(\int_B \mathrm{d}\mu\Big)^{\frac12} \\
&\le C\|(b-b_B)f_1\|_{L^2(\mu)}\mu\Big(\int_B \mathrm{d}\mu\Big)^{\frac12} \\
&\le C\|f\|_{L^\infty(\mu)}\mu(B)^{\frac12}\Big(\int_{6B} |b-b_B|^2\mathrm{d}\mu\Big)^{\frac12} \\
&\le C\|f\|_{L^\infty(\mu)}\mu(B)^{\frac12}\Big(\Big(\int_{6B} |b-b_{6B}|^2\mathrm{d}\mu\Big)^{\frac12} + \Big(\int_{6B} |b_B-b_{6B}|^2\mathrm{d}\mu\Big)^{\frac12}\Big) \\
&\le C\|f\|_{L^\infty(\mu)}\mu(B)^{\frac12}(\|b\|_*\mu(36B)^{\frac12} + K_{B,6B}\|b\|_*\mu(6B)^{\frac12}) \\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\mu(B)^{\frac12}\mu(36B)^{\frac12} \\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\mu(B).
\end{aligned}$$

In order to get the estimation of I_3, we need to estimate $|T((b-b_B)f_2) - h_B|$.

For $x, y \in B$, by the definition (2.1), we get

$$\begin{aligned}
&|T((b-b_B)f_2)(x) - T((b-b_B)f_2)(y)| \\
&= \Big|\int_{X\setminus 6B} K(x,z)(b(z)-b_B)f(z)\mathrm{d}\mu(z) - \int_{X\setminus 6B} K(y,z)(b(z)-b_B)f(z)\mathrm{d}\mu(z)\Big| \\
&\le C\|f\|_{L^\infty(\mu)}\int_{X\setminus 6B} |K(x,z)-K(y,z)||b(z)-b_B|\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)}\int_{X\setminus 6B} \theta\Big(\frac{d(x,y)}{d(x,z)}\Big)\frac{1}{\lambda(x,d(x,z))}|b(z)-b_B|\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)}\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB} \theta\Big(\frac{d(x,y)}{d(x,z)}\Big)\frac{1}{\lambda(x,d(x,z))}|b(z)-b_{6^{k+1}B}|\mathrm{d}\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&\quad + C\|f\|_{L^\infty(\mu)}\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\theta\Big(\frac{d(x,y)}{d(x,z)}\Big)\frac{1}{\lambda(x,d(x,z))}|b_B-b_{6^{k+1}B}|\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mu)}\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\theta(6^{-k})\frac{1}{\lambda(x,6^{k-1}r_B)}|b(z)-b_{6^{k+1}B}|\mathrm{d}\mu(z)\\
&\quad + C\|f\|_{L^\infty(\mu)}\|b\|_*\sum_{k=1}^{\infty}K_{B,6^{k+1}B}\int_{6^{k+1}B\setminus 6^kB}\theta(6^{-k})\frac{1}{\lambda(x,6^{k-1}r_B)}\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\Big(\sum_{k=1}^{\infty}\theta(6^{-k})\frac{\mu(6^{k+2}B)}{\lambda(x,6^{k+2}r_B)}+\sum_{k=1}^{\infty}K_{B,6^{k+1}B}\theta(6^{-k})\frac{\mu(6^{k+1}B)}{\lambda(x,6^{k+1}r_B)}\Big)\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*. \qquad (2.13)
\end{aligned}
$$

Here we have used the following inequality that

$$
\int_0^1\frac{\theta(t)}{t}\mathrm{d}t\ge\sum_{k=1}^{\infty}\int_{6^{-k}}^{6^{1-k}}\frac{\theta(6^{-k})}{6^{1-k}}\mathrm{d}t\ge C\sum_{k=1}^{\infty}\theta(6^{-k}),\quad \mu(6^{k+n}B)\le\lambda(x,6^{k+n}r_B).
$$

From the above estimate and the choice of h_B, we obtain

$$
\begin{aligned}
\mathrm{I}_3&=|T((b-b_B)f_2)(x)-h_B|\\
&=|T((b-b_B)f_2)(x)-m_B(T(b-b_B)f_2(y))|\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\mu(B)
\end{aligned}
$$

and

$$
\mathrm{I}_4\le|h_B+m_B([b,T]f)|\mu(B)\le C\int_B|[b,T]f+h_B|\mathrm{d}\mu\le C\|f\|_{L^\infty(\mu)}\|b\|_*\mu(B).
$$

Therefore we have (2.11).

Next we prove (2.12). For any $x\in Q, y\in\mathbb{R}$, we write

$$
\begin{aligned}
&\quad |[b,T]f(x)-[b,T]f(y)|\\
&=|(b(x)-b_Q)Tf(x)-T((b-b_Q)f)(x)-(b(y)-b_Q)Tf(y)+T((b-b_Q)f)(y)|\\
&\le|(b(x)-b_Q)Tf(x)|+|(b(y)-b_Q)Tf(y)|+|T((b-b_Q)f)(x)-T((b-b_Q)f)(y)|\\
&=\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3.
\end{aligned}
$$

With the argument similar to the estimate for I_1 it follows that

$$
m_Q(|(b-b_Q)Tf|)\le C\|f\|_{L^\infty(\mu)}\|b\|_*
$$

and

$$
\begin{aligned}
&\int_R|(b(y)-b_Q)Tf(y)|\mathrm{d}\mu(y)\\
&\le\Big(\int_R|Tf(y)|^2\mathrm{d}\mu(y)\Big)^{\frac12}\Big(\int_R|(b(y)-b_Q)|^2\mathrm{d}\mu(y)\Big)^{\frac12}\\
&\le C\|f\|_{L^\infty(\mu)}\mu(R)^{\frac12}\Big[\Big(\int_R|(b(y)-b_R)|^2\mathrm{d}\mu(y)\Big)^{\frac12}+\Big(\int_R|(b_R-b_Q)|^2\mathrm{d}\mu(y)\Big)^{\frac12}\Big]
\end{aligned}
$$

$$\le CK_{Q,R}\|b\|_*\|f\|_{L^\infty(\mu)}\mu(R)^{\frac{1}{2}}\mu(R)^{\frac{1}{2}}$$
$$\le CK_{Q,R}\|b\|_*\|f\|_{L^\infty(\mu)}\mu(R).$$

Therefore

$$m_R(|(b-b_Q)Tf|)\le CK_{Q,R}\|b\|_*\|f\|_{L^\infty(\mu)}.$$

We now estimate J_3, Let N be the first integer k such that $R\subset 6^kQ$. We denote $Q_R=6^{N+1}Q$. Then we write

$$\begin{aligned}
&|T((b-b_Q)f)(x)-T((b-b_Q)f)(y)|\\
&=|T((b-b_Q)f\chi_{X\setminus Q_R})(x)-T((b-b_Q)f\chi_{X\setminus Q_R})(y)|\\
&\quad+|T((b-b_Q)f\chi_{Q_R})(x)|-|T((b-b_Q)f\chi_{Q_R})(y)|\\
&=\mathrm{J}_{31}+\mathrm{J}_{32}+\mathrm{J}_{33}.
\end{aligned}$$

Similar to (2.13), we can show that

$$\begin{aligned}
\mathrm{J}_{31}&\le C\|f\|_{L^\infty(\mu)}\int_{X\setminus Q_R}|K(x,z)-K(y,z)||b(z)-b_Q|\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mu)}\int_{X\setminus Q_R}\theta\Big(\frac{d(x,y)}{d(x,z)}\Big)\frac{1}{\lambda(x,d(x,z))}|b(z)-b_Q|\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mu)}\sum_{k=1}^{\infty}\int_{6^{k+1}Q_R\setminus 6^kQ_R}\theta(6^{-k})\frac{1}{\lambda(x,6^{k-1}r_{Q_R})}|b(z)-b_{6^{k+1}Q_R}|\mathrm{d}\mu(z)\\
&\quad+C\|f\|_{L^\infty(\mu)}\|b\|_*K_{Q,6^{k+1}Q_R}\sum_{k=1}^{\infty}\int_{6^{k+1}Q_R\setminus 6^kQ_R}\theta(6^{-k})\frac{1}{\lambda(x,6^{k-1}r_{Q_R})}\mathrm{d}\mu(z)\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\Big(\sum_{k=1}^{\infty}\theta(6^{-k})\frac{\mu(6^{k+2}Q_R)}{\lambda(x,6^{k+2}r_{Q_R})}+\sum_{k=1}^{\infty}(K+K_{Q,R})\theta(6^{-k})\frac{\mu(6^{k+1}Q_R)}{\lambda(x,6^{k+1}r_{Q_R})}\Big)\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*\sum_{k=1}^{\infty}\theta(6^{-k})(K+K_{Q,R})\\
&\le C\|f\|_{L^\infty(\mu)}\|b\|_*K_{Q,R}.
\end{aligned}$$

Here we have used the following inequality

$$K_{Q,6^{k+1}Q_R}\le C(K_{Q,R}+K_{R,Q_R}+K_{Q_R,6^{k+1}Q_R})\le C(K_{Q,R}+K)$$

and the fact $r_{Q_R}\approx r_R$.

For J_{32}, we have

$$\begin{aligned}
\mathrm{J}_{32}&=|T((b-b_Q)f\chi_{Q_R})(x)|\le\int_{Q_R}|K(x,y)(b(y)-b_Q)f(y)|\mathrm{d}\mu(y)\\
&\le C\|f\|_{L^\infty(\mu)}\int_{Q_R}\frac{1}{\lambda(x,d(x,y))}|b(y)-b_Q|\mathrm{d}\mu(y)\\
&\le C\|f\|_{L^\infty(\mu)}\frac{1}{\lambda(x,y)}\Big(\int_{Q_R}|b(y)-b_{Q_R}|\mathrm{d}\mu(y)+\int_{Q_R}|b_Q-b_{Q_R}|\mathrm{d}\mu(y)\Big)
\end{aligned}$$

$$\leq C\|f\|_{L^\infty(\mu)}\frac{1}{\lambda(x,r_{Q_R})}\|b\|_*K_{Q,Q_R}\mu(6Q_R)$$
$$\leq C\|f\|_{L^\infty(\mu)}\|b\|_*K_{Q,R}.$$

The estimation of J_{33} is similar. Hence $J_3 \leq C\|f\|_{L^\infty(\mu)}\|b\|_*K_{Q,R}$. We proved (2.12) and complete the proof of Theorem 2.1.

Now we are going to show that if a θ-type Calderón-Zygmund operator is bounded on $L^2(\mu)$, then the commutators of θ-type Calderón-Zygmund operator with RBMO function is bounded from $H^{1,\infty}_{\mathrm{at}}(\mu)$ into $L^1(\mu)$. Before stating our results, we first recall some definitions.

Definition 2.4 *Let $\rho > 1$. A function $a \in L^1_{\mathrm{loc}}(\mu)$ is called a atomic block if*

(1) *there exists some balls B such that* $\mathrm{supp}(a) \subset B$*;*

(2) $\int_X a\mathrm{d}(\mu) = 0$*;*

(3) *there are functions a_j supported on balls $B_j \subset B$ and numbers $\lambda_j \in R$ such that*

$$a = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the sum converges in $L^1(\mu)$, and $\|a_j\|_{L^\infty(\mu)} \leq (\mu(\rho B_j)K_{B_j,B})^{-1}$ and the constant $K_{B_j,B}$ being given in the paragraph before Lemma 1.1.

We denote $|a|_{H^{1,\infty}_{\mathrm{at}}(\mu)} = \sum\limits_{j=1}^{\infty} |\lambda_j|$. We say that $f \in H^{1,\infty}_{\mathrm{at}}(\mu)$ if there are atomic blocks a_i such that

$$f = \sum_{i=1}^{\infty} a_i$$

with $\sum\limits_{i=1}^{\infty} |a_i|_{H^{1,\infty}_{\mathrm{at}}(\mu)} < \infty$. The $H^{1,\infty}_{\mathrm{at}}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,\infty}_{\mathrm{at}}(\mu)} = \inf \sum_{i=1}^{\infty} |a_i|_{H^{1,\infty}_{\mathrm{at}}(\mu)},$$

where the infimum is taken over all the possible decompositions of f in atomic blocks. It was proved in [1] that the atomic Hardy space $H^{1,\infty}_{\mathrm{at}}(\mu)$ is independent of the choice of ρ. Here we choose $\rho = 6$.

Theorem 2.2 *Assume that θ-type Calderón-Zygmund operator defined by* (2.4) *as above and T is bounded on $L^2(\mu)$, then the commutators of T with* RBMO *function is bounded from $H^{1,\infty}_{\mathrm{at}}(\mu)$ into $L^1(\mu)$ on non-homogeneous space.*

Proof It is enough to show that

$$\int_X |[b,T]a(x)|\mathrm{d}\mu(x) \leq C\|a\|_{H^{1,\infty}_{\mathrm{at}}(\mu)}$$

for any atomic block a with $\mathrm{supp}(a) \subset B$, $a = \sum\limits_j \lambda_j a_j$, where the a_j is function satisfying property of the definition of atomic block.

For a atomic block $a(x)$, we can write

$$\begin{aligned}\int_X |[b,T]a(x)|\mathrm{d}\mu(x) &\le \int_{6B} |[b,T]a(x)|\mathrm{d}\mu(x) + \int_{X\setminus 6B} |[b,T]a(x)|\mathrm{d}\mu(x)\\ &= \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

First we will estimate I_1.

In fact

$$\begin{aligned}\mathrm{I}_1 &= \int_{6B} |[b,T]a(x)|\mathrm{d}\mu(x) \le \sum_j |\lambda_j| \int_{6B} |[b,T]a_j(x)|\mathrm{d}\mu(x)\\ &\le \sum_j |\lambda_j| \int_{6B\setminus 6B_j} |[b,T]a_j(x)|\mathrm{d}\mu(x) + \sum_j |\lambda_j| \int_{6B_j} |[b,T]a_j(x)|\mathrm{d}\mu(x)\\ &= \mathrm{I}_{11} + \mathrm{I}_{12}.\end{aligned}$$

Since

$$[b,T]a_j(x) = (b - m_{B_j}b)Ta_j(x) - T((b - m_{B_j}b)a_j)(x),$$

we write

$$\begin{aligned}\mathrm{I}_{12} &= \sum_j |\lambda_j| \int_{6B_j} |[b,T]a_j(x)|\mathrm{d}\mu(x)\\ &\le \sum_j |\lambda_j| \int_{6B_j} |b - m_{B_j}b||Ta_j(x)|\mathrm{d}\mu(x) + \sum_j |\lambda_j| \int_{6B_j} |T((b - m_{B_j}b)a_j)(x)|\mathrm{d}\mu(x).\end{aligned}$$

By using Hölder's inequality and the boundedness of T on $L^2(\mu)$, we obtain

$$\begin{aligned}&\int_{6B_j} |b - m_{B_j}b||Ta_j(x)|\mathrm{d}\mu(x)\\ &\le \Big(\int_{6B_j} |b - m_{B_j}b|^2\mathrm{d}\mu(x)\Big)^{\frac{1}{2}}\Big(\int_{6B_j} |Ta_j(x)|^2\mathrm{d}\mu(x)\Big)^{\frac{1}{2}}\\ &\le \|a\|_{L^\infty(\mu)}\mu(6B_j)^{\frac{1}{2}}\Big[\Big(\int_{6B_j} |b - m_{6B_j}b|^2\mathrm{d}\mu(x)\Big)^{\frac{1}{2}} + \Big(\int_{6B_j} |m_{B_j} - m_{6B_j}b|^2\mathrm{d}\mu(x)\Big)^{\frac{1}{2}}\Big]\\ &\le \|a\|_{L^\infty(\mu)}\mu(6B_j)^{\frac{1}{2}}(C\mu(36B_j)^{\frac{1}{2}} + CK_{B_j,6B_j}\mu(6B_j)^{\frac{1}{2}})\\ &\le C.\end{aligned}$$

Similarly

$$\int_{6B_j} |T((b - m_{B_j}b)a_j)(x)|\mathrm{d}\mu(x) \le \Big(\int_{6B_j} \big|(b - m_{B_j}b)a_j(x)\big|^2\mathrm{d}\mu(x)\Big)^{\frac{1}{2}} \le C.$$

Thus

$$\mathrm{I}_{12} \le C\sum_j |\lambda_j| \le C\|a\|_{H^{1,\infty}_{\mathrm{at}}(\mu)}.$$

For I_{11}, we write

$$\begin{aligned}\mathrm{I}_{11} &\leq \sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^kB_j} |[b,T]a_j(x)|\mathrm{d}\mu(x)\\ &\leq \sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^kB_j} |b-m_{B_j}b||Ta_j(x)|\mathrm{d}\mu(x)\\ &\quad+\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^kB_j} |T((b-m_{B_j}b)a_j)(x)|\mathrm{d}\mu(x)\\ &=\mathrm{I}_{111}+\mathrm{I}_{112}.\end{aligned}$$

Let x_j be the center of ball B_j. According to the definition of atomic block, we obtain

$$\begin{aligned}\mathrm{I}_{111} &\leq \sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^kB_j} \int_{B_j} |b(x)-m_{B_j}b||K(x,y)-K(x,x_j)|\\ &\quad\cdot |a_j(y)|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^kB_j} \int_{B_j} |b(x)-m_{B_j}b|\theta\Big(\frac{d(y,x_j)}{d(x,x_j)}\Big)\\ &\quad\cdot \frac{1}{\lambda(x,d(x,x_j))}|a_j(y)|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\ &\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{B_j} |a_j(y)|\mathrm{d}\mu(y)\theta(6^{-k})\frac{1}{\lambda(x_j,6^{k-1}r_{B_j})}\\ &\quad\cdot \int_{6^{k+1}B_j\setminus 6^kB_j} |b(x)-m_{B_j}b|\mathrm{d}\mu(x).\end{aligned}$$

Since (2.6)–(2.7) and Lemma 1.1,

$$\begin{aligned}&\int_{6^{k+1}B_j\setminus 6^kB_j} |b(x)-m_{B_j}b|\mathrm{d}\mu(x)\\ &\leq \int_{6^{k+1}B_j\setminus 6^kB_j} |b(x)-m_{6^{k+1}B_j}b|\mathrm{d}\mu(x)+\int_{6^{k+1}B_j\setminus 6^kB_j} |m_{B_j}b-m_{6^{k+1}B_j}b|\mathrm{d}\mu(x)\\ &\leq C\mu(6^{k+2}B_j)+CK_{B_j,6^{k+1}B_j}\mu(6^{k+1}B_j)\\ &\leq CK_{B_j,B}\mu(6^{k+2}B_j).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{I}_{111} &\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \|a_j\|_{L^\infty(\mu)}\mu(B_j)\theta(6^{-k})K_{B_j,B}\\ &\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \theta(6^{-k})\|a_j\|_{L^\infty(\mu)}\mu(6B_j)K_{B_j,B}\\ &\leq C\sum_j |\lambda_j|\end{aligned}$$

$$\leq C\|a\|_{H^{1,\infty}_{\rm at}(\mu)}$$

and

$$\begin{aligned}
\mathrm{I}_{112} &\leq \sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^k B_j} \Big(\int_{B_j} K(x,y)((b-m_{B_j}b)a_j)(y)\mathrm{d}\mu(y)\Big)\mathrm{d}\mu(x)\\
&\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} \int_{6^{k+1}B_j\setminus 6^k B_j} \frac{1}{\lambda(x,d(x,y))}\|a_j\|_{L^\infty(\mu)}\mu(6B_j)\mathrm{d}\mu(x)\\
&\leq C\sum_j |\lambda_j| \sum_{k=1}^{N_{B_j,B}} K_{B_j,B}^{-1}\\
&\leq C\|a\|_{H^{1,\infty}_{\rm at}(\mu)}.
\end{aligned}$$

Hence

$$\mathrm{I}_1 = \int_{6B} |[b,T]a(x)|\mathrm{d}\mu(x) \leq C\|a\|_{H^{1,\infty}_{\rm at}(\mu)}.$$

Now, for I_2, we can write

$$\mathrm{I}_2 = [b,T]a(x) = (b-m_Bb)Ta(x) - T((b-m_Bb)a)(x).$$

Thus

$$\begin{aligned}
\mathrm{I}_2 &\leq \int_{X\setminus 6B} |b-m_Bb||Ta(x)|\mathrm{d}\mu(x) + \int_{X\setminus 6B} |T((b-m_Bb)a)(x)|\mathrm{d}\mu(x)\\
&= \mathrm{I}_{21}+\mathrm{I}_{22}.
\end{aligned}$$

Let x_B be the center of ball B and r be the radius of ball B. Then

$$\begin{aligned}
\mathrm{I}_{21} &\leq \int_{X\setminus 6B} |b(x)-m_Bb| \int_B |K(x,y)-K(x,x_B)||a(y)|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\
&\leq C\int_B |a(y)| \sum_{k=1}^{\infty} \int_{6^{k+1}B\setminus 6^k B} \theta\Big(\frac{d(y,x_B)}{d(x,x_B)}\Big)\frac{1}{\lambda(x,d(x,x_B))}|b(x)-m_Bb|\mathrm{d}\mu(x)\mathrm{d}\mu(y).
\end{aligned}$$

Similarly estimation of I_{111}, we can show that

$$\mathrm{I}_{21} = \int_{X\setminus 6B} |b-m_Bb||Ta(x)|\mathrm{d}\mu(x) \leq C\|a\|_{H^{1,\infty}_{\rm at}(\mu)}.$$

For I_{22}, we get

$$\begin{aligned}
\mathrm{I}_{22} &\leq \int_{X\setminus 6B} \Big|\int_B (K(x,y)-K(x,x_B))(b(y)-m_Bb)a(y)\mathrm{d}\mu(y)\Big|\mathrm{d}\mu(x)\\
&\leq C\sum_{k=1}^{\infty} \int_{6^{k+1}B\setminus 6^k B} \int_B \theta\Big(\frac{d(y,x_B)}{d(x,x_B)}\Big)\frac{1}{\lambda(x,d(x,x_B))}|b(y)-m_Bb||a(y)|\mathrm{d}\mu(y)\mathrm{d}\mu(x)\\
&\leq C\sum_{k=1}^{\infty} \theta(6^{-k}) \int_B |b(y)-m_Bb||a(y)|\mathrm{d}\mu(y)
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_j |\lambda_j| \int_{B_j} |b(y)-m_B b||a_j(y)|\mathrm{d}\mu(y)\\
&\le C\sum_j |\lambda_j|\|a_j\|_{L^\infty(\mu)}\Big(\int_{B_j}|b(y)-m_{B_j}b|\mathrm{d}\mu(y)+\int_{B_j}|m_B b-m_{B_j}b|\mathrm{d}\mu(y)\Big)\\
&\le C\sum_j |\lambda_j|\mu(6B_j)^{-1}K_{B_j,B}^{-1}\mu(6B_j)K_{B_j,B}\\
&\le C\sum_j |\lambda_j|.
\end{aligned}$$

Therefore $\mathrm{I}_{22}\le C\|a\|_{H^{1,\infty}_{\mathrm{at}}(\mu)}$. Then we obtain $\mathrm{I}_2\le C\|a\|_{H^{1,\infty}_{\mathrm{at}}(\mu)}$. The theorem is proved.

Remark 2.1 In [1], Bui and Duong showed that the space $\mathrm{RBMO}(\mu)$ is embedded in the dual space of $H^{1,\infty}_{\mathrm{at}}$, That is $\mathrm{RBMO}(\mu)\subset(H^{1,\infty}_{\mathrm{at}})^*$. So we can not obtain the above result by duality.

In [1], Bui and Duong established the following interpolation theorem.

Theorem 2.3 *Let T be a linear operator which is bounded from $H^{1,\infty}_{\mathrm{at}}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into* $\mathrm{RBMO}(\mu)$. *Then T extends boundedly to $L^p(\mu)$ for all $1<p<\infty$.*

By using Theorem 2.3, as a consequence of Theorem 2.1 and Theorem 2.2, we immediately obtain the following theorem.

Theorem 2.4 *Let T be θ-type Calderón-Zygmund operator defined by* (2.4) *as above and $b\in\mathrm{RBMO}(\mu)$. Then the commutator $[b,T]$ can be extended to a bounded operator on $L^p(\mu)$ for all* $(1<p<\infty)$.

Remark 2.2 The classical theorem of the boundedness of the commutator on $L^p(\mu)$ space was obtained by using the pointwise estimate for commutator, see the papers [4,14]. However, our method is different. We derive the $L^p(\mu)$ boundedness from the endpoint estimates on the non-homogeneous metric spaces.

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