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The Automorphism Group of a Finite *p*-Group with a Cyclic Frattini Subgroup*

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Abstract Let G be a finite p-group with a cyclic Frattini subgroup. In this paper, the automorphism group of G is determined.

Keywords Finite p-groups, Frattini subgroups, Automorphisms
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1 Introduction

In this paper, p always is a prime number, only finite groups will be considered. The terminologies and notations used are standard (cf. [1]).

Let G_1 and G_2 be any two groups, Z_1 and Z_2 be the centers of G_1 and G_2, respectively. Assume that Z_1 is isomorphic to Z_2, and $\theta : Z_1 \to Z_2$ is the isomorphic mapping, $G_1 * G_2$ is called the central product of G_1 and G_2 relative to Z_1, Z_2 and θ, that is, $G_1 * G_2$ is the quotient group of $G_1 \times G_2$ on the normal subgroup

$$\{(z_1, \theta(z_1)^{-1} \mid z_1 \in Z_1\}.$$

In particular, let G be any group, $Z \le \zeta G$, the central product $G * G$ is constructed by virtue of the identity mapping on Z. For any $l > 1$, G^{*l} is denoted by $G^{*(l-1)} * G$, and $G^{*1} := G$, $G^{*0} := 1$.

A finite p-group G is called extraspecial, if $G' = \operatorname{Frat} G = \zeta G$ and have order p. Winter [2] has given the automorphism group of an extraspecial p-group. When p is odd, Dietz [3] generalized the results of Winter, and determined the automorphism group of a finite p-group which is a central extension of a group with order p by an elementary abelian group.

In [1], a finite p-group G is called generalized extraspecial, if the center ζG of G is cyclic and the derived subgroup G' of G has order p. In [4], we determined the automorphism group of the generalized extraspecial p-group. Further, let G be the below central extension

$$1 \to \mathbb{Z}_{p^m} \to G \to \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \to 1,$$

and $|G'| \le p$. In [5], we determined the automorphism group of the finite p-group, which generalized the results of Winter and Dietz.

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Proposition 1.1 (cf. [5]) *Let p be an odd number, G be a finite p-group given by a central extension of the form*

$$1 \to \mathbb{Z}_{p^m} \to G \to \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \to 1,$$

and $|G'| = p$, where $m \geq 2$. Then $G = EA$, where E is a generalized extraspecial p-group, $A = \zeta G$, $E \cap A = \zeta E$. Suppose that $|E| = p^{2n+m}$, $|\zeta E| = p^m$ and $|A| = p^{m+l}$. Let $\mathrm{Aut}_f\, G = \{\ \alpha \in \mathrm{Aut}\, G \mid \alpha$ acts trivially on $\mathrm{Frat}\, G\}$. Then

(i) *If both E and A are of exponent p^m, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_{(p-1)p^{m-2}}$, and $\mathrm{Aut}_f\, G/K \cong \mathrm{Sp}(2n,p) \times (\mathrm{GL}(l,p) \ltimes (\mathbb{Z}_p)^l)$, where K is of order $p^{2n(l+1)+l+1}$.*

(ii) *If E and A are of exponent p^m and p^{m+1}, respectively, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_{(p-1)p^{m-1}}$, and $\mathrm{Aut}_f\, G/K \cong \mathrm{Sp}(2n,p) \times (\mathrm{GL}(l-1,p) \ltimes (\mathbb{Z}_p)^{l-1})$, where K is of order p^{2nl+l}.*

(iii) *If E and A are of exponent p^{m+1} and p^m, respectively, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_{(p-1)p^{m-1}}$, and $\mathrm{Aut}_f\, G/K \cong (I \rtimes \mathrm{Sp}(2n-2,p)) \times \mathrm{GL}(l,p)$, where I is an extraspecial p-group with order p^{2n-1} and K is of order $p^{2n(l+1)+l}$.*

Proposition 1.2 (cf. [5]) *Let G be a finite 2-group given by a central extension of the form*

$$1 \to \mathbb{Z}_{2^m} \to G \to \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \to 1,$$

and $|G'| = 2$, where $m \geq 2$. Then $G = EA$, where E is a generalized extraspecial 2-group, $A = \zeta G$, $E \cap A = \zeta E$. Suppose that $|E| = 2^{2n+m}$, $|\zeta E| = 2^m$ and $|A| = 2^{m+l}$. Let $\mathrm{Aut}_f\, G = \{\ \alpha \in \mathrm{Aut}\, G \mid \alpha$ acts trivially on $\mathrm{Frat} G\}$. Then

(i) *If both E and A are of exponent 2^m, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong 1$ $(m = 2)$ or $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-3}} (m \geq 3)$, and $\mathrm{Aut}_f\, G/K \cong \mathrm{Sp}(2n,2) \times (\mathrm{GL}(l,2) \ltimes (\mathbb{Z}_2)^l)$, where K is of order $2^{2n(l+1)+l+1}$.*

(ii) *If E and A are of exponent 2^m and 2^{m+1}, respectively, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$, and $\mathrm{Aut}_f\, G/K \cong \mathrm{Sp}(2n,2) \times (\mathrm{GL}(l-1,2) \ltimes (\mathbb{Z}_2)^{l-1})$, where K is of order 2^{2nl+l}.*

(iii) *If E and A are of exponent 2^{m+1} and 2^m, respectively, then $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$, and $\mathrm{Aut}_f\, G/K \cong (I \rtimes \mathrm{Sp}(2n-2,2)) \times \mathrm{GL}(l,2)$, where I is an elementary abelian 2-group with order 2^{2n-1} and K is of order $2^{2n(l+1)+l}$.*

In [6], the structure and the automorphism group of a finite p-group with a cyclic Frattini subgroup were studied. In this paper, by means of the results in [5], the automorphism group of a finite p-group with a cyclic Frattini subgroup is further determined. On the hand, if p is odd, or $p = 2$ and $\mathrm{Frat}\, G \leq \zeta G$, then G is a finite p-group which is a central extension of a cyclic group $\mathrm{Frat}\, G$ by an elementary abelian group and G' has order p by Lemma 1.2 and Lemma 1.3. According to Proposition 1.1 and Proposition 1.2, the automorphism group of G can be determined, on the other hand, if $p = 2$ and $\mathrm{Frat}\, G \nleq \zeta G$, we can obtain the below results.

In what follows, we are going to suppose that $|\mathrm{Frat}\, G| = p^m$ and R is an elementary abelian 2-group with rank r.

Theorem 1.1 *Let $G = R \times (D_8^{*n} * H)$, where $H = H_1$, H_2 or H_3, which are defined in Lemma 1.6. Let $C := C_G(\mathrm{Frat}\, G)$ and $\mathrm{Aut}_f\, G := \{\alpha \in \mathrm{Aut}\, G \mid \alpha$ acts trivially on $\mathrm{Frat}\, C\}$. Then*

(1) $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_2$(*if* $m = 2$), *or* $\mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$(*if* $m \geq 3$).

(2) $\mathrm{Aut}_f\, G/K \cong \mathrm{Sp}(2n,2) \times \mathrm{GL}(r,2) \ltimes (\mathbb{Z}_2)^r$, *where K is of order* $2^{(2n+2)(r+1)+m}$(if $H = H_1$ *or* H_3), *or* $2^{(2n+2)(r+1)+m-1}$(*if* $H = H_2$).

Theorem 1.2 *Let $G = R \times (D_8^{*n} * H)$, where $H = H_4$ or H_5, which are defined in Lemma 1.6. Let $C := C_G(\mathrm{Frat}\, G)$ and $\mathrm{Aut}_f\, G := \{\alpha \in \mathrm{Aut}\, G \mid \alpha$ acts trivially on $\mathrm{Frat}\, C\}$. Then*

(1) $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_2$ *(if* $m = 2$*), or* $\mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$ *(if* $m \geq 3$*).*

(2) $\operatorname{Aut}_f G/K \cong (I \rtimes \operatorname{Sp}(2n, 2)) \times \operatorname{GL}(r, 2)$*, where* I *is an elementary abelian 2-group with order* 2^{2n+1}*,* K *is of order* $2^{(2n+2)(r+1)+m+2r}$*.*

Theorem 1.3 *Let* $G = R \times (D_8^{*n} * H)$*, where* $H = H_6$ *or* H_7*, which are defined in Lemma* 1.6. *Let* $C := C_G(\operatorname{Frat} G)$ *and* $\operatorname{Aut}_f G := \{\alpha \in \operatorname{Aut} G \mid \alpha$ *acts trivially on* $\operatorname{Frat} C\}$*. Then*

(1) $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_2$ *(if* $m = 2$*), or* $\mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$ *(if* $m \geq 3$*).*

(2) $\operatorname{Aut}_f G/K \cong \operatorname{Sp}(2n, 2) \times (\operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^{2r})$*,* K *is of order* $2^{(2n+2)(r+2)+m-1}$*.*

Theorem 1.4 *Let* $G = R \times (D_8^{*n} * H)$*, where* $H = H_8$*, which is defined in Lemma* 1.6. *Let* $C := C_G(\operatorname{Frat} G)$ *and* $\operatorname{Aut}_f G := \{\alpha \in \operatorname{Aut} G \mid \alpha$ *acts trivially on* $\operatorname{Frat} C\}$*. Then*

(1) $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_2$ *(if* $m = 2$*), or* $\mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$ *(if* $m \geq 3$*).*

(2) $\operatorname{Aut}_f G/K \cong (I \rtimes \operatorname{Sp}(2n, 2)) \times (\operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^r)$*, where* I *is an elementary abelian 2-group with order* 2^{2n+1}*, and* K *is of order* $2^{(2n+2)(r+2)+2r+m+1}$*.*

According to the above theorems, let $r = 0$, then we can obtain the below conclusion in [6].

Corollary 1.1 (cf. [6]) *Let* $P = D_8^{*n} * H$*.*

(1) *If* $H = D_{2^{m+2}}$ *or* $H = Q_{2^{m+2}}$*, then* $|\operatorname{Aut} P| = 2^{(n+1)^2+2m} \prod_{i=1}^{n} (2^{2i} - 1)$*.*

(2) *If* $H = SD_{2^{m+2}}$*, then* $|\operatorname{Aut} P| = 2^{(n+1)^2+2m-1} \prod_{i=1}^{n} (2^{2i} - 1)$*.*

(3) *If* $H = D^+_{2^{m+3}}$ *or* $H = Q^+_{2^{m+3}}$*, then* $|\operatorname{Aut} P| = 2^{(n+2)^2+2m-2} \prod_{i=1}^{n} (2^{2i} - 1)$*.*

(4) *If* $H = D_{2^{m+2}} * C_4$ *or* $H = SD_{2^{m+2}} * C_4$*, then* $|\operatorname{Aut} P| = 2^{(n+2)^2+2m-2} \prod_{i=1}^{n} (2^{2i} - 1)$*.*

(5) *If* $H = D^+_{2^{m+3}} * C_4$*, then* $|\operatorname{Aut} P| = 2^{(n+3)^2+2m-4} \prod_{i=1}^{n} (2^{2i} - 1)$*.*

We need the following several lemmas in order to obtain the above theorems.

Lemma 1.1 (cf. [4]) *Let* G *be a generalized extraspecial* p*-group, then*

(i) $G/\zeta G$ *is an elementary abelian* p*-group.*

(ii) *Let* $G' = \langle c \rangle$*. For any two elements* $\overline{x} = x\zeta G$ *and* $\overline{y} = y\zeta G$ *of* $G/\zeta G$*, write* $[x, y] = c^r$ $(0 \leq r < p)$ *and* $f(\overline{x}, \overline{y}) = r$*, then* $G/\zeta G$ *becomes a nondegenerate symplectic space over* $\operatorname{GF}(p)$*.*

(iii) G *is a central product of some nonabelian subgroups* G_i *which satisfy both* $\zeta G_i = \zeta G$ *and* $|G_i/\zeta G_i| = p^2$*. Furthermore, let* $|G_i| = p^{m+2}$*, where* $m \geq 2$*, then* G_i *only has two types*:

$$M_m(p) = \langle x, y \mid x^{p^{m+1}} = y^p = 1, x^y = x^{1+p^m} \rangle$$

or

$$N_m(p) = \langle x, y, z \mid x^p = y^p = z^{p^m} = 1, [x, z] = [y, z] = 1, [x, y] = z^{p^{m-1}} \rangle.$$

Lemma 1.2 (cf. [6]) *Let* p *be odd and* G *be a nonabelian* p*-group. If* $\operatorname{Frat} G$ *is cyclic, then* $\operatorname{Frat} G$ *is a central subgroup.*

Lemma 1.3 *Let* G *be a nonabelian* p*-group. If* $\operatorname{Frat} G$ *is a cyclic and central subgroup, then* G' *is of order* p*.*

Proof Since G is a nonabelian p-group, G' is nontrivial, and is included in the cyclic Frattini subgroup $\operatorname{Frat} G$. Now we only need to prove that G' is of order p.

Since $G' \le \operatorname{Frat} G \le \zeta G$, for any $x, y \in G$, we have that

$$[x, y]^p = [x^p, y].$$

Moreover, since $x^p \in \operatorname{Frat} G \le \zeta G$, $[x^p, y] = 1$. Consequently, for any $x, y \in G$, we have that $[x, y]^p = 1$. The lemma is proved.

Lemma 1.4 (cf. [6]) *Let G be a nonabelian 2-group, $\Phi(G)$ be cyclic, $\operatorname{Frat} G \not\leq \zeta G$ and $|\operatorname{Frat} G| = 2^m$, then $m > 1$, and G is isomorphic to the direct product $R \times (D_8^{*n} * H)$, where R is an elementary abelian 2-group, $n \ge 0$, H is a nontrivial 2-group which is one of the following isomorphic types*:

$$D_{2^{m+2}}, Q_{2^{m+2}}, SD_{2^{m+2}}, D_{2^{m+2}} * C_4, SD_{2^{m+2}} * C_4, D^+_{2^{m+3}}, Q^+_{2^{m+3}}, D^+_{2^{m+3}} * C_4,$$

where

$$D^+_{2^{m+3}} := \langle x, y, z \mid x^2 = y^2 = z^{2^{m+1}} = 1, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}\rangle$$

and

$$Q^+_{2^{m+3}} := \langle x, y, z \mid x^2 = z^{2^{m+1}} = 1, y^2 = z^{2^m}, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}\rangle.$$

Lemma 1.5 (cf. [4]) *If $m \ge 3$, then*

$$a^{2^{m-2}} \equiv 1 \pmod{2^m}, \quad \text{where } a \text{ is an odd number},$$
$$3^{2^{m-3}} \not\equiv 1 \pmod{2^m}.$$

Lemma 1.6 *Let G be a nonabelian 2-group, $\Phi(G)$ be a cyclic group, and $\operatorname{Frat} G \not\leq \zeta G$, $|\operatorname{Frat} G| = 2^m$, then G is isomorphic to the direct product $R \times (D_8^{*n} * H)$, where R is an elementary abelian 2-group, $n \ge 0$, H is defined in Lemma 1.4. Further,*

(1) *If H is isomorphic to $D_{2^{m+2}}$, $SD_{2^{m+2}}$ or $Q_{2^{m+2}}$, then $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R$.*
(2) *If H is isomorphic to $D^+_{2^{m+3}}$ or $Q^+_{2^{m+3}}$, then $C_G(\operatorname{Frat} G) \cong N_m(2)^{*n} * M_m(2) \times R$.*
(3) *If H is isomorphic to $D_{2^{m+2}} * C_4$ or $SD_{2^{m+2}} * C_4$, then $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R \times \mathbb{Z}_2$.*
(4) *If H is isomorphic to $D^+_{2^{m+3}} * C_4$, then $C_G(\operatorname{Frat} G) \cong N_m(2)^{*n} * M_m(2) \times R \times \mathbb{Z}_2$.*

Proof Assume that $D_8^{*n} \cong \langle x_1, x_2\rangle * \langle x_3, x_4\rangle * \cdots * \langle x_{2n-1}, x_{2n}\rangle$.

(1) Let $H_1 := H \cong D_{2^{m+2}}$, and $H_1 = \langle x, y \mid x^2 = y^{2^{m+1}} = 1, y^x = y^{-1}\rangle$, then $\zeta H_1 = \langle y^{2^m}\rangle$, $\operatorname{Frat} G = \langle y^2\rangle$, and

$$C_G(\operatorname{Frat} G) = \langle x_1, x_2, y\rangle * \langle x_3, x_4, y\rangle * \cdots * \langle x_{2n-1}, x_{2n}, y\rangle \times R.$$

Note that $\langle x_{2i-1}, x_{2i}, y\rangle \cong N_{m+1}(2)$, where $i = 1, 2, \cdots, n$. It follows that $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R$.

Let $H_2 := H \cong SD_{2^{m+2}}$, and $H_2 = \langle x, y \mid x^2 = y^{2^{m+1}} = 1, y^x = y^{-1+2^m}\rangle$. If $(y^k)^x = y^k$, where $0 \le k < 2^{m+1}$, then $y^{-k+2^m k} = y^k$. It follows that $2k - 2^m k \equiv 0 \pmod{2^{m+1}}$, which implies that $(1 - 2^{m-1})k \equiv 0 \pmod{2^m}$. Also $0 \le k < 2^{m+1}$, thus $k = 2^m$ and $\zeta H_2 = \langle y^{2^m}\rangle$. Consequently, $\operatorname{Frat} G = \langle y^2\rangle$. According to the results of H_1, we similarly have that $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R$.

Let $H_3 := H \cong Q_{2^{m+2}}$, and $H_3 = \langle x, y \mid x^4 = 1, y^{2^m} = x^2, y^x = y^{-1}\rangle$. Obviously, $\zeta H_3 = \langle y^{2^m}\rangle$, $\operatorname{Frat} G = \langle y^2\rangle$. According to the results of H_1, we similarly have that $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R$.

(2) Let $H_4 := H \cong D^+_{2^{m+3}}$, and

$$H_4 = \langle x, y, z \mid x^2 = y^2 = z^{2^{m+1}} = 1, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}\rangle.$$

Let $x^i y^j z^k \in \zeta H_4$, where $0 \le i < 2$, $0 \le j < 2$, $0 \le k < 2^{m+1}$, then $(x^i y^j z^k)^x = x^i y^j z^k$. It follows that $z^{2^m k+k} = z^k$, thus $2^m k \equiv 0 \pmod{2^{m+1}}$, that is $k \equiv 0 \pmod 2$. That $(x^i y^j z^k)^y = x^i y^j z^k$ implies that $z^{-k} = z^k$, thus $2k \equiv 0 \pmod{2^{m+1}}$, that is $k \equiv 0 \pmod{2^m}$. Since $(x^i y^j z^k)^z = x^i y^j z^k$, $(x^i)^z = x^i z^{-2^m i}$ and $(y^j)^z = y^j z^{(-1)^{j+1}+1}$, $-2^m i + (-1)^{j+1} + 1 \equiv 0 \pmod{2^{m+1}}$, which implies that $-2^m i + (-1)^{j+1} + 1 \equiv 0 \pmod{2^m}$. It follows that $(-1)^{j+1} + 1 \equiv 0 \pmod{2^m}$, thus $j = 0$. Consequently, $i = 0$. From the above, we have that $\zeta H_4 = \langle z^{2^m}\rangle$, and $\operatorname{Frat} H_4 = \langle z^2\rangle = \operatorname{Frat} G$. It follows that

$$C_G(\operatorname{Frat} G) = \langle x, z\rangle * \langle x_1, x_2, z^2\rangle * \langle x_3, x_4, z^2\rangle * \cdots * \langle x_{2n-1}, x_{2n}, z^2\rangle \times R.$$

Note that $\langle x, z\rangle \cong M_m(2)$, where $M_m(2)$ is defined in Lemma 1.1, thus $C_G(\operatorname{Frat} G) \cong M_m(2) * N_m(2)^{*n} \times R$.

Let $H_5 := H \cong Q^+_{2^{m+3}}$, and

$$H_5 = \langle x, y, z \mid x^2 = z^{2^{m+1}} = 1, y^2 = z^{2^m}, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}\rangle.$$

Let $x^i y^j z^k \in \zeta H_5$, where $0 \le i < 2$, $0 \le j < 4$ and $0 \le k < 2^{m+1}$, then $(x^i y^j z^k)^x = x^i y^j z^k$. It follows that $z^{2^m k+k} = z^k$, thus $2^m k \equiv 0 \pmod{2^{m+1}}$, that is $k \equiv 0 \pmod 2$. That $(x^i y^j z^k)^y = x^i y^j z^k$ implies that $z^{-k} = z^k$, thus $2k \equiv 0 \pmod{2^{m+1}}$, therefore $k \equiv 0 \pmod{2^m}$. Since $(x^i y^j z^k)^z = x^i y^j z^k$, $(x^i)^z = x^i z^{-2^m i}$ and $(y^j)^z = y^j z^{(-1)^{j+1}+1}$, $-2^m i + (-1)^{j+1} + 1 \equiv 0 \pmod{2^{m+1}}$, which implies that $-2^m i + (-1)^{j+1} + 1 \equiv 0 \pmod{2^m}$. It follows that $(-1)^{j+1} + 1 \equiv 0 \pmod{2^m}$, thus $j = 0$ or 2. Consequently, $i = 0$. From the above, we have that $\zeta H_5 = \langle z^{2^m}\rangle$, and $\operatorname{Frat} H_5 = \langle z^2\rangle = \operatorname{Frat} G$. According to the results of H_4, similarly, $C_G(\operatorname{Frat} G) \cong N_m(2)^{*n} * M_m(2) \times R$.

(3) Let $H_6 := H \cong D_{2^{m+2}} * C_4$, and

$$H_6 = \langle x, y, z \mid x^2 = y^{2^{m+1}} = 1, y^x = y^{-1}, z^2 = y^{2^m}, [x, z] = 1, [y, z] = 1\rangle.$$

It is easy to verify that $\zeta H_6 = \langle z\rangle$, $D_8^{*n} \cap H_6 = \langle z^2\rangle$ and $\operatorname{Frat} H_6 = \langle y^2\rangle$. It follows that

$$\begin{aligned} C_G(\operatorname{Frat} G) &= \langle x_1, x_2, x_3, x_4, \cdots, x_{2n-1}, x_{2n}, y, z\rangle \times R \\ &= \langle x_1, x_2, y\rangle * \langle x_3, x_4, y\rangle * \cdots * \langle x_{2n-1}, x_{2n}, y\rangle \times \langle zy^{2^{m-1}}\rangle \times R. \end{aligned}$$

Since

$$\langle x_{2i-1}, x_{2i}, y \mid x_{2i-1}^2 = x_{2i}^2 = y^{2^{m+1}} = 1, [x_{2i-1}, y] = 1 = [x_{2i}, y], [x_{2i-1}, x_{2i}] = y^{2^m}\rangle \cong N_{m+1}(2),$$

where $i = 1, 2, \cdots, n$, $\langle zy^{2^{m-1}}\rangle \cong \mathbb{Z}_2$. It follows that $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R \times \mathbb{Z}_2$.

Let $H_7 := H \cong SD_{2^{m+2}} * C_4$, and

$$H_7 = \langle x, y, z \mid x^2 = y^{2^{m+1}} = 1, y^x = y^{-1+2^m}, z^2 = y^{2^m}, [x, z] = 1, [y, z] = 1\rangle.$$

Obviously, $\zeta H_7 = \langle z\rangle$ and $\operatorname{Frat} H_7 = \langle y^2\rangle$. According to the results of H_6, we similarly have that $C_G(\operatorname{Frat} G) \cong N_{m+1}(2)^{*n} \times R \times \mathbb{Z}_2$.

(4) Let $H_8 := H \cong D^{+}_{2^{m+3}} * C_4$, and

$$H_8 = \langle x, y, z, u \mid x^2 = y^2 = z^{2^{m+1}} = 1, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}, u^2 = z^{2^m}, [x, u] = [y, u] = [z, u] = 1\rangle.$$

Obviously, $\zeta H_8 = \langle u\rangle$ and $\mathrm{Frat}\, H_8 = \langle z^2\rangle$. It follows that

$$\begin{aligned} C_G(\mathrm{Frat}\, G) &= \langle x_1, x_2, x_3, x_4, \cdots, x_{2n-1}, x_{2n}, x, z, u\rangle \times R \\ &= \langle x_1, x_2, z^2\rangle * \langle x_3, x_4, z^2\rangle * \cdots * \langle x_{2n-1}, x_{2n}, z^2\rangle * \langle x, z\rangle \times \langle uz^{2^{m-1}}\rangle \times R. \end{aligned}$$

Since

$$\langle x_{2i-1}, x_{2i}, z^2 \mid x_{2i-1}^2 = x_{2i}^2 = (z^2)^{2^m} = 1, [x_{2i-1}, z^2] = 1 = [x_{2i}, z^2], [x_{2i-1}, x_{2i}] = (z^2)^{2^{m-1}}\rangle \cong N_m(2),$$

where $i = 1, 2, \cdots, n$, $\langle x, z \mid x^2 = z^{2^{m+1}} = 1, z^x = z^{1+2^m}\rangle \cong M_m(2)$, $\langle uz^{2^{m-1}}\rangle \cong \mathbb{Z}_2$ and $C_G(\mathrm{Frat}\, G) \cong N_m(2)^{*n} * M_m(2) \times R \times \mathbb{Z}_2$.

2 Proof of Theorem 1.1

Since D_8^{*n} is an extraspecial 2-group, we may suppose that $x_1, x_2, \cdots, x_{2n-1}, x_{2n}, y^{2^m}$ are the generators of D_8^{*n}, which satisfy the following relations:

$$\begin{aligned} \zeta D_8^{*n} &= \langle y^{2^m}\rangle, \\ [x_{2i-1}, x_{2i}] &= y^{2^m}, \quad i = 1,\ 2,\ \cdots,\ n, \\ [x_{2i-1}, x_j] &= 1, \quad j \neq 2i, \\ [x_{2i}, x_k] &= 1, \quad k \neq 2i-1, \\ x_i^2 &= 1, \quad i = 1,\ 2,\ \cdots,\ n. \end{aligned}$$

According to (1) in Lemma 1.6, we have that

$$C = \langle x_1, x_2, y\rangle * \langle x_3, x_4, y\rangle * \cdots * \langle x_{2n-1}, x_{2n}, y\rangle \times R.$$

Let $\Phi : \mathrm{Aut}\, G \to \mathrm{Aut}(\mathrm{Frat}\, C)$ be a restriction homomorphism. Obviously, $\mathrm{Ker}\, \Phi = \mathrm{Aut}_f\, G \trianglelefteq \mathrm{Aut}\, G$. According to (1) in Lemma 1.6, $\mathrm{Frat}\, C = \langle y^2\rangle$.

Theorem 2.1

$$\mathrm{Im}\, \Phi \cong \begin{cases} \mathbb{Z}_2, & \textit{if } m = 2, \\ \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2, & \textit{if } m \geq 3. \end{cases}$$

Proof If $m = 2$, then $\mathrm{Frat}\, C \cong \mathbb{Z}_4$, thus $\mathrm{Aut}(\mathrm{Frat}\, C) \cong \mathbb{Z}_2$. Define a mapping:

$$\begin{aligned} \sigma_1 :\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}, \quad i = 1, 2, \cdots, n, \\ x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n, \\ z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r, \\ x &\mapsto x, \\ y &\mapsto y^3. \end{aligned}$$

It is easy to verify that σ_1 is an automorphism of G, which is of order 2. Since $\Phi(\sigma_1)(y^2) = (y^2)^3$ and $\Phi(\sigma_1)^2(y^2) = y^2$, $\mathrm{Aut}(\mathrm{Frat}\, C) = \langle \Phi(\sigma_1)\rangle$. It follows that $\mathrm{Aut}\, G = \mathrm{Aut}_f\, G \rtimes \langle \sigma_1 \rangle$.

If $m \geq 3$, then $\mathbb{Z}_{2^m}^* = \langle \upsilon_1\rangle \times \langle \upsilon_2 \rangle$, where $\upsilon_1 = 3$ and $\upsilon_2 = 2^m - 1$. By Lemma 1.5, we have that the orders of υ_1 and υ_2 are 2^{m-2} and 2, respectively. Define a mapping:

$$\begin{aligned} \sigma_2:\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}^{2^m-1}, \quad i = 1, 2, \cdots, n, \\ x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n, \\ z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r, \\ x &\mapsto x, \\ y &\mapsto y^{2^m-1}. \end{aligned}$$

It is easy to verify that σ_1 and σ_2 are commutative automorphisms each other and their orders are 2^{m-1} and 2, respectively.

Take any $\alpha \in \mathrm{Aut}\, G$, then $\alpha(y^2) = y^{2s_1}$, where $s_1 \in \mathbb{Z}_{2^m}^*$. Hence there exist $0 \leq t_1 < 2^{m-2}$ and $0 \leq t_2 < 2$ such that $\upsilon_1^{t_1}\upsilon_2^{t_2} \equiv s_1^{-1} \pmod{2^m}$. Since

$$\begin{aligned} \sigma_1^{t_1}\sigma_2^{t_2}\alpha(y^2) &= \sigma_1^{t_1}\sigma_2^{t_2}(y^{2s_1}) = \sigma_1^{t_1}(\sigma_2^{t_2}(y))^{2s_1} = \sigma_1^{t_1}(y^{\upsilon_2^{t_2}})^{2s_1} \\ &= (\sigma_1^{t_1}(y))^{2\upsilon_2^{t_2}s_1} = (y^{2\upsilon_1^{t_1}\upsilon_2^{t_2}})^{s_1} = y^{2s_1^{-1}s_1} = y^2, \end{aligned}$$

$\sigma_1^{t_1}\sigma_2^{t_2}\alpha \in \mathrm{Aut}_f\, G$. Consequently, $\mathrm{Aut}\, G = \langle \sigma_1,\ \sigma_2\rangle \mathrm{Aut}_f\, G$.

We claim that $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = 1$. In fact, let $\sigma_1^{w_1} = \sigma_2^{w_2} \in \langle\sigma_1\rangle \cap \langle\sigma_2\rangle$, where $w_1, w_2 \in \mathbb{Z}$, then

$$y^{2\upsilon_1^{w_1}} = \sigma_1^{w_1}(y^2) = \sigma_2^{w_2}(y^2) = y^{2\upsilon_2^{w_2}},$$

which implies that $\upsilon_1^{w_1} \equiv \upsilon_2^{w_2} \pmod{2^m}$, thus $w_1 \equiv 0 \pmod{2^{m-2}}$ and $w_2 \equiv 0 \pmod 2$. It follows that $\sigma_1^{w_1} = \sigma_2^{w_2} = 1$.

If $\sigma_1^{u_1}\sigma_2^{u_2} \in \langle\sigma_1,\ \sigma_2\rangle \cap \mathrm{Aut}_f\, G$, where $0 \leq u_1 < 2^{m-1}$ and $0 \leq u_2 < 2$, then $y^2 = \sigma_1^{u_1}\sigma_2^{u_2}(y^2) = y^{2\upsilon_1^{u_1}\upsilon_2^{u_2}}$, which implies that $\upsilon_1^{u_1}\upsilon_2^{u_2} \equiv 1 \pmod{2^m}$, thus $u_1 \equiv 0 \pmod{2^{m-2}}$ and $u_2 \equiv 0 \pmod 2$. It is easy to verify that $\sigma_1^{2^{m-2}} \in \mathrm{Aut}_f\, G$, thus $\langle\sigma_1,\ \sigma_2\rangle \cap \mathrm{Aut}_f\, G = \langle\sigma_1^{2^{m-2}}\rangle$. It follows that $\mathrm{Aut}\, G/\mathrm{Aut}_f\, G \cong \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$.

The theorem is proved.

Let $\Psi_1 : \mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C)$, $\Psi_2 : \mathrm{Aut}_f\, G \to \mathrm{Aut}(C/\zeta C)$ and

$$\Psi_3 : \mathrm{Aut}_f\, G \to \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C)$$

be the natural induced homomorphisms. From this, we may obtain the below homomorphic mapping

$$\begin{aligned} \Psi:\ & \mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C) \times \mathrm{Aut}(C/\zeta C) \times \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C), \\ & \alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)). \end{aligned}$$

Since $G/C = \langle xC\rangle \cong \mathbb{Z}_2$, $\mathrm{Im}\,\Psi_1 = \mathrm{Aut}(G/C) = 1$.

Since $\zeta C = \langle y\rangle \times R$, we may define the inner product as follows:

$$f(\overline{a}, \overline{b}) = t, \quad \text{where } \overline{a} = a\zeta C,\ \overline{b} = b\zeta C,\ a, b \in C \text{ and } [a, b] = (y^{2^m})^t,\ 0 \leq t < 2.$$

From this, $C/\zeta C$ can become a nondegenerate symplectic space over GF(2).

Take any $\alpha \in \mathrm{Aut}_f\, G$, then $[\alpha(a), \alpha(b)] = \alpha[a, b] = [a, b]$, thus, for any $\overline{a} = a\zeta C, \overline{b} = b\zeta C \in C/\zeta C$, we have that

$$f(\Psi_2(\alpha)(\overline{a}), \Psi_2(\alpha)(\overline{b})) = f(\overline{\alpha(a)}, \overline{\alpha(b)}) = f(\overline{a}, \overline{b}),$$

therefore $\Psi_2(\alpha) \in \mathrm{Sp}(2n, 2)$. Consequently, $\Psi_2(\mathrm{Aut}_f\, G) \le \mathrm{Sp}(2n, 2)$. From the above, Ψ is the homomorphic mapping as follows:

$$\begin{aligned}\Psi:\ &\mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C) \times \mathrm{Sp}(2n, 2) \times \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C),\\ &\alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).\end{aligned}$$

Theorem 2.2 $\mathrm{Im}\, \Psi_2 = \mathrm{Sp}(2n, 2)$.

Proof Take any $T \in \mathrm{Sp}(2n, 2)$, let (a_{ik}) be the matrix of T relative to a basis $\{x_i\zeta C,\ i = 1, 2, \cdots, 2n\}$ of $C/\zeta C$. Define a mapping

$$\begin{aligned}\phi:\ &G \to G,\\ x^c\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^d \mapsto\ & x^c\Big(\prod_{i=1}^{2n}\Big(\prod_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^{d'},\end{aligned}$$

where $0 \le a_i < 2$, $i = 1, 2, \cdots, 2n$, $0 \le b_j < 2$, $j = 1, 2, \cdots, r$, $0 \le c < 2$, $0 \le d < 2^{m+1}$, $d' \equiv d + \sum\limits_{i=1}^{2n} 2^{m-1} a_i\Big(\sum\limits_{j=1}^{n}(a_{i,2j-1} \cdot a_{i,2j})\Big) \pmod{2^{m+1}}$.

Note that (a_{ik}) is a nonsingular matrix. It is easy to verify ϕ is a bijection. Therefore, ϕ is an automorphism of G if and only if ϕ preserves multiplications. By the definition of ϕ, we have

(1) $\phi(x_i^{a_i}) = \Big(\prod\limits_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i} y^{\sum_{j=1}^{n}(a_{i,2j-1}\cdot a_{i,2j})2^{m-1}a_i} = \Big[\Big(\prod\limits_{k=1}^{2n} x_k^{a_{ik}}\Big) y^{\sum_{j=1}^{n}(a_{i,2j-1}\cdot a_{i,2j})2^{m-1}}\Big]^{a_i}$
$= \phi(x_i)^{a_i}$.

(2)
$$\begin{aligned}\phi\Big[x^c\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^d\Big] &= x^c\Big[\prod_{i=1}^{2n}\Big(\prod_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^{d'}\\ &= x^c\Big[\prod_{i=1}^{2n}\Big(\prod_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^{d+\sum_{i=1}^{2n} 2^{m-1}a_i(\sum_{j=1}^{n} a_{i,2j-1}\cdot a_{i,2j})}\\ &= x^c\Big[\prod_{i=1}^{2n}\Big(\Big(\prod_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i} y^{(\sum_{j=1}^{n} a_{i,2j-1}\cdot a_{i,2j})2^{m-1}a_i}\Big)\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^d\\ &= x^c\Big[\prod_{i=1}^{2n}\phi(x_i)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^d.\end{aligned}$$

(3) $\phi(x) = x$.

(4) $\phi(z_j) = z_j$, $j = 1, 2, \cdots, r$.

(5) For any $\overline{a} = a\zeta C, \overline{b} = b\zeta C \in C/\zeta C$, $f(\overline{\phi(a)}, \overline{\phi(b)}) = f(T(\overline{a}), T(\overline{b})) = f(\overline{a}, \overline{b})$, thus $[\phi(a), \phi(b)] = [a, b]$.

We call the above ϕ the induced mapping of G by T.

Claim 2.1 If $\phi(x_i)^2 = 1$, $i = 1, 2, \cdots, 2n$, then $\phi \in \mathrm{Aut}_f\, G$.

In fact, let $\phi(x_i)^2 = 1$, where $i = 1, 2, \cdots, 2n$. For any $g_1,\ g_2 \in G$, we have

$$g_1 = x^{c_1}\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^{d_1}, \quad g_2 = x^{c_2}\Big(\prod_{i=1}^{2n} x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b'_j}\Big)y^{d_2}$$

and

$$\begin{aligned} g_1g_2 &= x^{c_1}\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)y^{d_1}x^{c_2}\Big(\prod_{i=1}^{2n} x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b'_j}\Big)y^{d_2} \\ &= x^{c_1+c_2}\Big(\prod_{i=1}^{2n} x_i^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n-1}\prod_{t=k+1}^{2n}[x_t^{a_t}, x_k^{a'_k}]\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^{d_2+(-1)^{c_2}d_1} \\ &= x^{c_1+c_2}\Big(\prod_{i=1}^{2n} x_i^{a_i+a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^{e}, \end{aligned}$$

where $y^e = (\prod\limits_{k=1}^{2n-1}\prod\limits_{t=k+1}^{2n}[x_t^{a_t}, x_k^{a'_k}])y^{d_2+(-1)^{c_2}d_1}$ and $0 \le e < 2^{m+1}$.

Let $c_1+c_2 = c+2c'$, $a_i+a'_i = t_i+2s_i$, $b_j+b'_j = t'_j+2s'_j$, where $0 \le c, t_i, t'_j < 2$, $c', s_i, s'_j \in \mathbb{Z}$, $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, r$, then

$$\begin{aligned} \phi(g_1g_2) &= \phi\Big[x^{c_1+c_2}\Big(\prod_{i=1}^{2n} x_i^{a_i+a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^e\Big] = \phi\Big[x^{c+2c'}\Big(\prod_{i=1}^{2n} x_i^{t_i+2s_i}\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j+2s'_j}\Big)y^e\Big] \\ &= \phi\Big[x^{c}\Big(\prod_{i=1}^{2n} x_i^{t_i}\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j}\Big)y^e\Big] = x^c\Big(\prod_{i=1}^{2n} \phi(x_i)^{t_i}\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j}\Big)y^e, \end{aligned}$$

$$\begin{aligned} \phi(g_1)\phi(g_2) &= x^{c_1}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a_i}\Big)y^{d_1}x^{c_2}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^{d_2} \\ &= x^{c_1+c_2}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n-1}\prod_{t=k+1}^{2n}[\phi(x_t)^{a_t}, \phi(x_k)^{a'_k}]\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^{d_2+(-1)^{c_2}d_1} \\ &= x^{c_1+c_2}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n-1}\prod_{t=k+1}^{2n}[\phi(x_t^{a_t}), \phi(x_k^{a'_k})]\Big)\Big(\prod_{j=1}^{r} z_j^{b_j+b'_j}\Big)y^{d_2+(-1)^{c_2}d_1} \\ &= x^{c_1+c_2}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n-1}\prod_{t=k+1}^{2n}[x_t^{a_t}, x_k^{a'_k}]\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j}\Big)y^{d_2+(-1)^{c_2}d_1} \\ &= x^{c}\Big(\prod_{i=1}^{2n} \phi(x_i)^{a_i+a'_i}\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j}\Big)y^e = x^c\Big(\prod_{i=1}^{2n} \phi(x_i)^{t_i}\Big)\Big(\prod_{j=1}^{r} z_j^{t'_j}\Big)y^e = \phi(g_1g_2). \end{aligned}$$

Hence $\phi \in \mathrm{Aut}\, G$. Also since $\phi(y) = y$, $\phi \in \mathrm{Aut}_f\, G$.

The claim is proved.

For $i = 1, 2, \cdots, 2n$, we have

$$\phi(x_i)^2 = \Big[\Big(\prod_{j=1}^{2n} x_j^{a_{ij}}\Big)y^{\sum\limits_{j=1}^{n}(a_{i,2j-1}a_{i,2j})2^{m-1}}\Big]^2 = \Big[\prod_{j=1}^{n}(x_{2j-1}^{a_{i,2j-1}}x_{2j}^{a_{i,2j}})^2\Big]y^{\sum\limits_{j=1}^{n}(a_{i,2j-1}a_{i,2j})2^{m}}$$

$$= \Big[\prod_{j=1}^{n}(x_{2j-1}^{2a_{i,2j-1}} x_{2j}^{2a_{i,2j}} y^{2^m a_{i,2j-1}a_{i,2j}})\Big] y^{\sum\limits_{j=1}^{n}(a_{i,2j-1}a_{i,2j})2^m}$$

$$= y^{\sum\limits_{j=1}^{n}(a_{i,2j-1}a_{i,2j})2^m} y^{\sum\limits_{j=1}^{n}(a_{i,2j-1}a_{i,2j})2^m} = 1.$$

By Claim 2.1, the induced mapping ϕ by T is an automorphism of G, and $\Psi_1(\phi) = T$. Consequently, $\operatorname{Im}\Psi_1 = \operatorname{Sp}(2n, 2)$.

The theorem is proved.

Theorem 2.3 *$\operatorname{Im}\Psi_3 \cong \operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^r$.*

Proof Let

$$\mathscr{A} := \left\{ \begin{pmatrix} A_{11} & 0 \\ A_{21} & 1 \end{pmatrix} \in \operatorname{GL}(r+1, 2) \right\},$$

where A_{11} is a $r \times r$ matrix, A_{21} is a $1 \times r$ matrix. It is easy to verify that $\mathscr{A} \leq \operatorname{GL}(r+1, 2)$. For convenience, we may let $z_{r+1} := y$.

Take any $\alpha \in \operatorname{Aut}_f G$. Let (a_{jk}) be the matrix of $\Psi_3(\alpha)$ relative to a basis $\{z_j \operatorname{Frat} C, j = 1, 2, \cdots, r+1\}$ of $\zeta C/\operatorname{Frat} C$.

Let (a_{jk}) be the partitioned matrix as follows:

$$(a_{jk}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \operatorname{GL}(r+1, 2),$$

where A_{11}, A_{12}, A_{21} and A_{22} are $r \times r$, $r \times 1$, $1 \times r$ and 1×1 matrices, respectively.

Since $\Psi_3(\alpha)(\overline{z}_j) = \prod\limits_{k=1}^{r+1} \overline{z}_k^{a_{jk}}$, where $j = 1, 2, \cdots, r$, there exists $0 \leq a_j < 2^m$ such that $\alpha(z_j) = (\prod\limits_{k=1}^{r+1} z_k^{a_{jk}}) y^{2a_j}$.

Since $z_j^2 = 1$ for $j = 1, 2, \cdots, r$,

$$1 = \alpha(z_j^2) = \alpha(z_j)^2 = \Big(\prod_{k=1}^{r+1} z_k^{2a_{jk}}\Big) y^{2^2 a_j} = y^{2a_{j,r+1} + 2^2 a_j},$$

thus $a_{j,r+1} + 2a_j \equiv 0 \pmod{2^m}$. But $m > 1$ and $0 \leq a_{j,r+1} < 2$, consequently, for $j = 1, 2, \cdots, r$, we have $a_{j,r+1} = 0$, that is $A_{12} = 0$.

Since

$$y^2 = z_{r+1}^2 = \alpha(z_{r+1}^2) = \alpha(z_{r+1})^2 = \Big(\prod_{k=1}^{r+1} z_k^{2a_{r+1,k}}\Big) y^{2^2 a_{r+1}}$$
$$= z_{r+1}^{2a_{r+1,r+1} + 2^2 a_{r+1}} = (y^2)^{a_{r+1,r+1} + 2a_{r+1}},$$

$a_{r+1,r+1} + 2a_{r+1} \equiv 1 \pmod{2^m}$. But $m > 1$ and $0 \leq a_{r+1,r+1} < 2$, thus $a_{r+1,r+1} = 1$, that is $A_{22} = 1$.

Conversely, for $\begin{pmatrix} B_{11} & 0 \\ B_{21} & 1 \end{pmatrix} = (b_{jk}) \in \mathscr{A}$, define a mapping:

$$\begin{aligned} \delta:\ & G \to G, \\ & x \mapsto x, \\ & x_i \mapsto x_i, \quad i = 1, 2, \cdots, 2n, \\ & z_j \mapsto \prod_{k=1}^{r+1} z_k^{b_{jk}}, \quad j = 1, 2, \cdots, r+1. \end{aligned}$$

It is easy to verify that $\delta \in \operatorname{Aut} G$. Since

$$\delta(y^2) = \delta(y)^2 = \Big(\prod_{k=1}^{r} z_k^{b_{r+1,k}} y\Big)^2 = y^2,$$

$\delta \in \operatorname{Aut}_f G$ and the matrix of $\Psi_2(\delta)$ is (b_{jk}) relative to a basis $\{z_j \operatorname{Frat} C, j = 1, 2, \cdots, r+1\}$ of $\zeta C/\operatorname{Frat} C$. Hence $\operatorname{Im} \Psi_2 \cong \mathscr{A}$. Also since $\mathscr{A} \cong \operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^r$, we have that $\Psi_2(\operatorname{Aut}_f G) \cong \operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^r$.

The theorem is proved.

Theorem 2.4 (1) *If $H = H_1$ or H_3, then* $\operatorname{Ker} \Psi$ *is a 2-group with order* $2^{(2n+2)(r+1)+m}$.
(2) *If $H = H_2$, then* $\operatorname{Ker} \Psi$ *is a 2-group with order* $2^{(2n+2)(r+1)+m-1}$.

Proof Since $\operatorname{Ker} \Psi$ acts trivially on all factors of the series $G \geq C \geq \zeta C \geq \operatorname{Frat} C \geq 1$, $\operatorname{Ker} \Psi$ is a 2-group.

Take any $\alpha \in \operatorname{Ker} \Psi$, let α be an automorphism as follows:

$$\begin{aligned} \alpha:\ & G \to G, \\ & x \mapsto x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big), \\ & x_i \mapsto x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big), \quad i = 1, 2, \cdots, 2n, \\ & z_k \mapsto z_k y^{2c_k}, \quad k = 1, 2, \cdots, r+1, \\ & y^2 \mapsto y^2, \end{aligned}$$

where $z_{r+1} = y$, $0 \leq a_i < 2$, $0 \leq b_j < 2$, $0 \leq b_{r+1} < 2^{m+1}$, $0 \leq a_{ij} < 2$, $0 \leq a_{i,r+1} < 2^{m+1}$, $0 \leq c_k < 2^m$, $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, r$, $k = 1, 2, \cdots, r+1$.

Since $\alpha(x_i)^2 = 1$, where $i = 1, 2, \cdots, 2n$, $1 = \big(x_i\big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\big)\big)^2 = y^{2a_{i,r+1}}$. Hence $a_{i,r+1} \equiv 0 \pmod{2^m}$. Consequently, $a_{i,r+1} = 0$ or 2^m.

Since $\alpha(x)$ and $\alpha(x_i)$ are commutative each other,

$$\begin{aligned} 1 &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big), x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big)\Big] = \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big) y^{b_{r+1}}, x_i y^{a_{i,r+1}}\Big] \\ &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big) y^{b_{r+1}}, y^{a_{i,r+1}}\Big]\Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big) y^{b_{r+1}}, x_i\Big]^{y^{a_{i,r+1}}} = [x, y^{a_{i,r+1}}]\Big[\prod_{i=1}^{2n} x_i^{a_i}, x_i\Big]. \end{aligned}$$

If $H=H_1$ or H_3, then $[x,y^{a_{i,r+1}}]=y^{2a_{i,r+1}}=1$. If $H=H_2$, then $[x,y^{a_{i,r+1}}]=y^{2a_{i,r+1}-2^m a_{i,r+1}}$ $=1$. In a word, $\big[\prod\limits_{i=1}^{2n} x_i^{a_i}, x_i\big]=1$. If i is odd, we can let $i=2l-1$, where $l=1,2,\cdots,n$, then $y^{2^m a_{2l}}=1$, which implies that $a_{2l}=0$. If i is even, we can let $i=2l$, where $l=1,2,\cdots,n$, then $y^{2^m a_{2l-1}}=1$, which implies that $a_{2l-1}=0$. Consequently, for $i=1,2,\cdots,2n$, we have that $a_i=0$.

Since $\alpha(x)$ and $\alpha(z_k)$ are commutative each other, where $k=1,2,\cdots,r$,

$$1=\Big[x\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big), z_k y^{2c_k}\Big]=[xy^{b_{r+1}}, y^{2c_k}]=[x,y^{2c_k}].$$

If $H=H_1$ or H_3, then $y^{4c_k}=1$. If $H=H_2$, then $1=[x,y^{2c_k}]=y^{4c_k-2^{m+1}c_k}=y^{4c_k}$. In a word, $c_k\equiv 0 \pmod{2^{m-1}}$, which implies that $c_k=0$ or 2^{m-1}. Also since $\alpha(y^2)=y^2$, $y^2=(y^{1+2c_{r+1}})^2=y^{2+4c_{r+1}}$, which implies that $c_{r+1}\equiv 0 \pmod{2^{m-1}}$, thus $c_{r+1}=0$ or 2^{m-1}. Consequently, for $k=1,2,\cdots,r+1$, we have that $c_k=0$ or 2^{m-1}.

Since $\alpha(z_k)^2=1$, where $k=1,2,\cdots,r$, $1=(z_k y^{2c_k})^2=y^{4c_k}$, which implies that $c_k\equiv 0$ $\pmod{2^{m-1}}$, thus $c_k=0$ or 2^{m-1}.

If $H=H_1$ or H_3, then $\alpha(x)^2=\big(x\big(\prod\limits_{j=1}^{r+1} z_j^{b_j}\big)\big)^2=(xy^{b_{r+1}})^2=1$, which has no effect on the parameters of α. If $H=H_2$, then $\alpha(x)^2=\big(x\big(\prod\limits_{j=1}^{r+1} z_j^{b_j}\big)\big)^2=(xy^{b_{r+1}})^2=y^{2^m b_{r+1}}$, thus $b_{r+1}\equiv 0$ $\pmod 2$.

It is easy to verify other generated relations have no effect on the parameters of α.

In conclusion, α is an automorphism as follows:

$$\begin{aligned}\alpha:\ & G\to G,\\ & x\mapsto x\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big),\\ & x_i\mapsto x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big),\quad i=1,2,\cdots,2n,\\ & z_k\mapsto z_k y^{2c_k},\quad k=1,2,\cdots,r+1,\end{aligned}$$

where $z_{r+1}=y$, $0\le b_j<2$, $0\le a_{ij}<2$, $a_{i,r+1}=0$ or 2^m, $c_k=0$ or 2^{m-1}, $i=1,2,\cdots,2n$, $j=1,2,\cdots,r$, $k=1,2,\cdots,r+1$, $0\le b_{r+1}<2^{m+1}$(if $H=H_1$ or H_3); $b_{r+1}\equiv 0 \pmod 2$(if $H=H_2$).

Conversely, if α is an automorphism of G, which satisfies the above conditions, then $\alpha\in \operatorname{Ker}\Psi$. Hence, if $H=H_1$ or H_3, then $|\operatorname{Ker}\Psi|=2^{(2n+2)(r+1)+m}$; if $H=H_2$, then $|\operatorname{Ker}\Psi|=2^{(2n+2)(r+1)+m-1}$.

The theorem is proved.

3 Proof of Theorem 1.2

For convenience, we may let $x_3,x_4,\cdots,x_{2n+1},x_{2n+2},z^{2^m}$ be the generators of D_8^{*n}, which satisfy the following conditions:

$$\zeta D_8^{*n}=\langle z^{2^m}\rangle,$$

$$\begin{aligned}
[x_{2i-1}, x_{2i}] &= z^{2^m}, \quad i = 2,\ 3,\ \cdots,\ n,\\
[x_{2i-1}, x_j] &= 1, \quad j \neq 2i,\\
[x_{2i}, x_k] &= 1, \quad k \neq 2i-1,\\
x_i^2 &= 1, \quad i = 2,\ 3,\ \cdots,\ n
\end{aligned}$$

According to (2) in Lemma 1.6, we have that

$$C = \langle x_1, x_2\rangle * \langle x_3, x_4, z^2\rangle * \langle x_5, x_6, z^2\rangle * \cdots * \langle x_{2n+1}, x_{2n+2}, z^2\rangle \times R \cong M_m(2) * N_m(2)^{*n} \times R,$$

where $x_1 := z$, $x_2 := x$.

For convenience, we sometimes adopt the notations in Theorem 1.1.

Let $\Phi : \operatorname{Aut} G \to \operatorname{Aut}(\operatorname{Frat} C)$ be the restriction homomorphism. Clearly, $\operatorname{Ker}\Phi = \operatorname{Aut}_f G \trianglelefteq \operatorname{Aut} G$. According to (2) in Lemma 1.6, we have that $\operatorname{Frat} C = \langle z^2\rangle = \operatorname{Frat} G \cong \mathbb{Z}_{2^m}$.

Theorem 3.1

$$\operatorname{Im}\Phi \cong \begin{cases} \mathbb{Z}_2, & \text{if } m = 2,\\ \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2, & \text{if } m \geq 3. \end{cases}$$

Proof If $m = 2$, then $\operatorname{Frat} C \cong \mathbb{Z}_4$, thus $\operatorname{Aut}(\operatorname{Frat} C) \cong \mathrm{Z}_2$. Define a mapping:

$$\begin{aligned}
\sigma_3 :\ & G \to G,\\
x_{2i-1} &\mapsto x_{2i-1}^3, \quad i = 1, 2, \cdots, n+1,\\
x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n+1,\\
z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r,\\
y &\mapsto y.
\end{aligned}$$

It is easy to verify that σ_3 is an automorphism of G, which is of order 2. Since $\Phi(\sigma_3)(z^2) = (z^2)^3$ and $\Phi(\sigma_3)^2(z^2) = z^2$, $\operatorname{Aut}(\operatorname{Frat} C) = \langle\Phi(\sigma_3)\rangle$. Consequently, $\operatorname{Aut} G = \operatorname{Aut}_f G \rtimes \langle\sigma_3\rangle$.

If $m \geq 3$, then $\mathbb{Z}_{2^m}^* = \langle\upsilon_1\rangle \times \langle\upsilon_2\rangle$, where $\upsilon_1 = 3$ and $\upsilon_2 = 2^m - 1$ and their orders are 2^{m-2} and 2 by Lemma 1.5, respectively. Define a mapping:

$$\begin{aligned}
\sigma_4 :\ & G \to G,\\
x_{2i-1} &\mapsto x_{2i-1}^{2^m-1}, \quad i = 1, 2, \cdots, n+1,\\
x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n+1,\\
z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r,\\
y &\mapsto y.
\end{aligned}$$

It is easy to verify that σ_3 and σ_4 are commutative automorphisms each other and their orders are 2^{m-1} and 2, respectively.

According to the argument in Theorem 2.1, we similarly have that $\operatorname{Aut} G = \langle\sigma_3,\ \sigma_4\rangle \operatorname{Aut}_f G$, and $\langle\sigma_3,\ \sigma_4\rangle \cap \operatorname{Aut}_f G = \langle\sigma_3^{2^{m-2}}\rangle$. Consequently, $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$.

The theorem is proved.

Let

$$\begin{aligned}
\Psi_1 &: \operatorname{Aut}_f G \to \operatorname{Aut}(G/C),\\
\Psi_2 &: \operatorname{Aut}_f G \to \operatorname{Aut}(C/\zeta C),
\end{aligned}$$

$$\Psi_3 : \mathrm{Aut}_f\, G \to \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C)$$

be the natural induced homomorphisms. Hence we may define the below homomorphic mapping:

$$\Psi: \ \mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C) \times \mathrm{Aut}(C/\zeta C) \times \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C),$$
$$\alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).$$

Since $G/C = \langle yC \rangle \cong \mathbb{Z}_2$, $\mathrm{Im}\,\Psi_1 = \mathrm{Aut}(G/C) = 1$.

Since $\zeta C = \langle z^2 \rangle \times R$, we may define the inner product as follows:

$$f(\overline{a}, \overline{b}) = t, \text{ where } \overline{a} = a\zeta C,\ \overline{b} = b\zeta C,\ a, b \in C \text{ and } [a, b] = (z^{2^m})^t,\ 0 \le t < 2.$$

From this, $C/\zeta C$ can become a nondegenerate symplectic space over GF(2).

For any $\alpha \in \mathrm{Aut}_f\, G$, $[\alpha(a), \alpha(b)] = \alpha[a, b] = [a, b]$, thus, for any $\overline{a} = a\zeta C, \overline{b} = b\zeta C \in C/\zeta C$, we have

$$f(\Psi_2(\alpha)(\overline{a}), \Psi_2(\alpha)(\overline{b})) = f(\overline{\alpha(a)}, \overline{\alpha(b)}) = f(\overline{a}, \overline{b}),$$

therefore $\Psi_2(\alpha) \in \mathrm{Sp}(2n, 2)$. Consequently, $\Psi_2(\mathrm{Aut}_f\, G) \le \mathrm{Sp}(2n, 2)$. In a word, Ψ is a homomorphic mapping as follows:

$$\Psi: \ \mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C) \times \mathrm{Sp}(2n, 2) \times \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C),$$
$$\alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).$$

Theorem 3.2 $\mathrm{Im}\,\Psi_2 = I \rtimes \mathrm{Sp}(2n, 2)$, *where I is an elementary abelian 2-group with order* 2^{2n+1}.

Proof Let $\mathscr{B} := \{T \in \mathrm{Sp}(2n+2, 2) \mid$ the first column and second row of the matrix of T are $(1, 0, \cdots, 0)^{\mathrm{T}}$ and $(0, 1, 0, \cdots, 0)$ relative to a basis $x_1\zeta C, x_2\zeta C, \cdots, x_{2n+2}\zeta C$ of $C/\zeta C$, respectively$\}$.

Take any $T \in \mathscr{B}$, let (a_{ik}) be the matrix of T relative to a basis $\{x_i\zeta C, i = 1, 2, \cdots, 2n+2\}$ of $C/\zeta C$. Define a mapping:

$$\phi: \ G \to G,$$
$$y^c \Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big) z^{2d} \mapsto (yx^t)^c \Big(\prod_{i=1}^{2n+2} \Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big) z^{2d'},$$

where $0 \le a_i < 2$, $i = 1, 2, \cdots, 2n+2$, $0 \le b_j < 2$, $j = 1, 2, \cdots, r$, $0 \le c < 2$, $0 \le d < 2^m$, $d' \equiv d + \sum_{i=1}^{2n+2} 2^{m-2} a_i \big(\sum_{k=1}^{n+1} (a_{i,2k-1} \cdot a_{i,2k})\big) \pmod{2^m}$, $t = 0$ $\big($if $\sum_{k=1}^{n+1} (a_{1,2k-1} \cdot a_{1,2k}) \equiv 0 \pmod 2\big)$ or $t = 1$ $\big($if $\sum_{k=1}^{n+1} (a_{1,2k-1} \cdot a_{1,2k}) \equiv 1 \pmod 2\big)$.

Note that (a_{ik}) is a nonsingular matrix. It is easy to verify ϕ is a bijection. Therefore, ϕ is an automorphism of G if and only if ϕ preserves multiplications. By the definition of ϕ, we have

(1)

$$\phi(x_i^{a_i}) = \Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i} z^{\sum_{k=1}^{n+1} (a_{i,2k-1} a_{i,2k}) 2^{m-1} a_i}$$

$$= \Big[\Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big) z^{\sum_{k=1}^{n+1}(a_{i,2k-1}a_{i,2k})2^{m-1}}\Big]^{a_i} = \phi(x_i)^{a_i}.$$

(2)

$$\phi\Big[y^c\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2d}\Big] = (yx^t)^c\Big[\prod_{i=1}^{2n+2}\Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2d'}$$

$$= (yx^t)^c\Big[\prod_{i=1}^{2n+2}\Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2d+\sum_{i=1}^{2n+2}2^{m-1}a_i(\sum_{k=1}^{n+1}a_{i,2k-1}a_{i,2k})}$$

$$= (yx^t)^c\Big[\prod_{i=1}^{2n+2}\Big(\Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i} z^{\sum_{k=1}^{n+1}(a_{i,2k-1}a_{i,2k})2^{m-1}a_i}\Big)\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2d}$$

$$= (yx^t)^c\Big[\prod_{i=1}^{2n+2}\phi(x_i)^{a_i}\Big]\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2d}.$$

(3) $\phi(z_j) = z_j, j = 1, 2, \cdots, r.$

(4) $\phi(y) = yx^t.$

(5) $\phi(z^2) = z^2.$

(6) For $\overline{a} = a\zeta C, \overline{b} = b\zeta C \in C/\zeta C$, $f(\overline{\phi(a)}, \overline{\phi(b)}) = f(T(\overline{a}), T(\overline{b})) = f(\overline{a}, \overline{b})$, thus $[\phi(a), \phi(b)] = [a, b]$.

(7)

$$[\phi(x_1), \phi(y)] = [\phi(z), \phi(y)] = \Big[z\Big(\prod_{k=2}^{2n+2} x_k^{a_{1k}}\Big)z^{\sum_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k})2^{m-1}}, yx^t\Big]$$

$$= \Big[z\Big(\prod_{k=2}^{2n+2} x_k^{a_{1k}}\Big)z^{\sum_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k})2^{m-1}}, x^t\Big]\Big[z\Big(\prod_{k=2}^{2n+2} x_k^{a_{1k}}\Big)z^{\sum_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k})2^{m-1}}, y\Big]^{x^t}$$

$$= [z, x^t][z, y][z^{\sum_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k})2^{m-1}}, y]$$

$$= z^{2^m t} z^{-\sum_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k})2^m}[z, y] = [z, y] = [x_1, y].$$

Note that

$$\phi(x_1)^2 = \Big(\prod_{j=1}^{2n+2} x_j^{a_{1j}}\Big)^2 z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m} = \Big[\prod_{j=1}^{n+1}(x_{2j-1}^{a_{1,2j-1}}x_{2j}^{a_{1,2j}})^2\Big]z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m}$$

$$= \Big[\prod_{j=1}^{n+1}(x_{2j-1}^{2a_{1,2j-1}}x_{2j}^{2a_{1,2j}}z^{2^m a_{1,2j-1}a_{1,2j}})\Big]z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m}$$

$$= [x_1^{2a_{11}}z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m}]z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m} = x_1^2,$$

and for any $i = 2, 3, \cdots, 2n+2$, we have that

$$\phi(x_i)^2 = \Big(\prod_{j=1}^{2n+2} x_j^{a_{ij}}\Big)^2 z^{(\sum_{j=1}^{n+1}a_{i,2j-1}a_{i,2j})2^m} = \Big[\prod_{j=1}^{n+1}(x_{2j-1}^{a_{i,2j-1}}x_{2j}^{a_{i,2j}})^2\Big]z^{(\sum_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m}$$

$$= \Big[\prod_{j=1}^{n+1}(x_{2j-1}^{2a_{i,2j-1}}x_{2j}^{2a_{i,2j}}z^{2^m a_{i,2j-1}a_{i,2j}})\Big]z^{(\sum\limits_{j=1}^{n+1}a_{i,2j-1}a_{1,2j})2^m}$$

$$= z^{(\sum\limits_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m}z^{(\sum\limits_{j=1}^{n+1}a_{1,2j-1}a_{1,2j})2^m} = 1.$$

For $g_1,\ g_2 \in G$,

$$g_1 = y^{c_1}\Big(\prod_{i=1}^{2n+2}x_i^{a_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j}\Big)z^{2d_1}, \quad g_2 = y^{c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b'_j}\Big)z^{2d_2},$$

we have that

$$\begin{aligned} g_1g_2 &= y^{c_1}\Big(\prod_{i=1}^{2n+2}x_i^{a_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j}\Big)z^{2d_1}y^{c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b'_j}\Big)z^{2d_2}\\ &= y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j}\Big)z^{2d_1}[x_1^{a_1},y^{c_2}][x_1^{a_1},y^{c_2},x_2^{a_2}][z^{2d_1},y^{c_2}]\Big(\prod_{i=1}^{2n+2}x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b'_j}\Big)z^{2d_2}\\ &= y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j}\Big)z^{2d_1}\Big(\prod_{i=1}^{2n+2}x_i^{a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b'_j}\Big)[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]\\ &\quad\cdot[x_1^{a_1},y^{c_2},x_2^{a'_2}][z^{2d_1},y^{c_2},x_2^{a'_2}]z^{2d_2}\\ &= y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n+1}\prod_{t=k+1}^{2n+2}[x_t^{a_t},x_k^{a'_k}]\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b'_j}\Big)[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]\\ &\quad\cdot[z^{-a_1+(-1)^{c_2}a_1},x_2^{a'_2}][z^{2d_1+(-1)^{c_2}2d_1},x_2^{a'_2}]z^{2(d_1+d_2)}\\ &= y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i+a'_i}\Big)\Big(\prod_{k=1}^{2n+1}\prod_{t=k+1}^{2n+2}[x_t^{a_t},x_k^{a'_k}]\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b'_j}\Big)[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]z^{2(d_1+d_2)}\\ &= y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i+a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b'_j}\Big)z^e, \end{aligned}$$

where $z^e = (\prod\limits_{k=1}^{2n+1}\prod\limits_{t=k+1}^{2n+2}[x_t^{a_t},x_k^{a'_k}])[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]z^{2(d_1+d_2)}$, $0 \le e < 2^{m+1}$.

Let $c_1+c_2 = c+2c'$, $a_i+a'_i = t_i+2s_i$, $b_j+b'_j = t'_j+2s'_j$, $2s_1+e \equiv e_1 \pmod{2^{m+1}}$, where $0 \le c, t_i, t'_j < 2$, $c', s_i, s'_j \in \mathbb{Z}$, $0 \le e_1 < 2^{m+1}$, $i = 1,2,\cdots,2n$, $j = 1,2,\cdots,r$, then

$$\begin{aligned} \phi(g_1g_2) &= \phi\Big[y^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}x_i^{a_i+a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b'_j}\Big)z^e\Big]\\ &= \phi\Big[y^{c+2c'}\Big(\prod_{i=1}^{2n+2}x_i^{t_i+2s_i}\Big)\Big(\prod_{j=1}^{r}z_j^{t'_j+2s'_j}\Big)z^e\Big]\\ &= \phi\Big[y^{c}\Big(\prod_{i=1}^{2n+2}x_i^{t_i}\Big)\Big(\prod_{j=1}^{r}z_j^{t'_j}\Big)z^{e+2s_1}\Big] = (yx^t)^c\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{t_i}\Big)\Big(\prod_{j=1}^{r}z_j^{t'_j}\Big)z^{e_1}, \end{aligned}$$

$$\phi(g_1)\phi(g_2) = (yx^t)^{c_1}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i}\Big)z^{2d_1}(yx^t)^{c_2}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a'_i}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b'_j}\Big)z^{2d_2}$$

$$= (yx^t)^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i}\Big)z^{2d_1}[\phi(x_1)^{a_1},(yx^t)^{c_2}]$$

$$\cdot[\phi(x_1)^{a_1},(yx^t)^{c_2},\phi(x_2)^{a_2}][z^{2d_1},(yx^t)^{c_2}]\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i'}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b_j'}\Big)z^{2d_2}$$

$$= (yx^t)^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i}\Big)z^{2d_1}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i'}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b_j'}\Big)$$

$$\cdot[\phi(x_1)^{a_1},\phi(y)^{c_2}][z^{2d_1},(yx^t)^{c_2}][\phi(x_1^{a_1}),(yx^t)^{c_2},\phi(x_2)^{a_2'}][z^{2d_1},(yx^t)^{c_2},\phi(x_2)^{a_2'}]z^{2d_2}$$

$$= (yx^t)^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i}\Big)z^{2d_1}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i'}\Big)\Big(\prod_{j=1}^{r}z_j^{b_j+b_j'}\Big)[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]z^{2d_2}$$

$$= (yx^t)^{c_1+c_2}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{a_i+a_i'}\Big)\Big(\prod_{k=1}^{2n+1}\prod_{t=k+1}^{2n+2}[\phi(x_t)^{a_t},\phi(x_k)^{a_k'}]\Big)$$

$$\cdot\Big(\prod_{j=1}^{r}z_j^{b_j+b_j'}\Big)[x_1^{a_1},y^{c_2}][z^{2d_1},y^{c_2}]z^{2(d_1+d_2)}$$

$$= (yx^t)^{c}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{t_i}\Big)\Big(\prod_{j=1}^{r}z_j^{t_j'}\Big)z^{e+2s_1}$$

$$= (yx^t)^{c}\Big(\prod_{i=1}^{2n+2}\phi(x_i)^{t_i}\Big)\Big(\prod_{j=1}^{r}z_j^{t_j'}\Big)z^{e_1} = \phi(g_1g_2),$$

therefore $\phi\in \operatorname{Aut}G$. Also since $\phi(z^2)=z^2$, $\phi\in\operatorname{Aut}_f G$ and $\Psi_2(\phi)=T$.

Conversely, take any $\varphi\in\operatorname{Aut}_f G$. Let $\Psi_2(\varphi)=T\in\mathrm{Sp}(2n+2,2)$, the matrix of T be (a_{ij}) relative to a basis $\{x_i\zeta C, i=1,2,\cdots,2n+2\}$ of $C/\zeta C$, $\varphi(x_i)=\big(\prod_{k=1}^{2n+2}x_k^{a_{ik}}\big)\big(\prod_{j=1}^{r}z_j^{b_{ij}}\big)z^{2d_i}$, where $0\le b_{ik}<2$, $i=1,2,\cdots,2n+2$, $0\le d_i<2^m$.

Since

$$z^2=\varphi(z^2)=\varphi(x_1^2)=\varphi(x_1)^2=\Big[\Big(\prod_{k=1}^{2n+2}x_k^{a_{1k}}\Big)\Big(\prod_{j=1}^{r}z_j^{b_{1j}}\Big)z^{2d_1}\Big]^2$$

$$=\Big[\prod_{k=1}^{n+1}(x_{2k-1}^{a_{1,2k-1}}x_{2k}^{a_{1,2k}})^2\Big]\Big(\prod_{j=1}^{r}z_j^{2b_{1j}}\Big)z^{4d_1}$$

$$=\Big[\prod_{k=1}^{n+1}(x_{2k-1}^{2a_{1,2k-1}}x_{2k}^{2a_{1,2k}}z^{2^m(a_{1,2k-1}a_{1,2k})})\Big]\Big(\prod_{j=1}^{r}z_j^{2b_{1j}}\Big)z^{4d_1}$$

$$=x_1^{2a_{11}}z^{(\sum_{k=1}^{n+1}2^m(a_{1,2k-1}a_{1,2k}))+4d_1}=z^{2a_{11}+4d_1'},$$

where $d_1'=\big(\sum_{k=1}^{n+1}2^{m-2}(a_{1,2k-1}a_{1,2k})\big)+d_1$, $a_{11}+2d_1'\equiv 1 \pmod{2^m}$. From this, we have $a_{11}\equiv 1 \pmod 2$, thus $a_{11}=1$.

For $i=2,\cdots,2n+2$,

$$1=\varphi(x_i^2)=\varphi(x_i)^2=\Big[\Big(\prod_{k=1}^{2n+2}x_k^{a_{ik}}\Big)\Big(\prod_{j=1}^{r}z_j^{b_{ij}}\Big)z^{2d_i}\Big]^2=\Big[\prod_{k=1}^{n+1}(x_{2k-1}^{a_{i,2k-1}}x_{2k}^{a_{i,2k}})^2\Big]\Big(\prod_{j=1}^{r}z_j^{2b_{ij}}\Big)z^{4d_i}$$

$$= \Big[\prod_{k=1}^{n+1} (x_{2k-1}^{2a_{i,2k-1}} x_{2k}^{2a_{i,2k}} z^{2^m(a_{i,2k-1}a_{i,2k})}) \Big] \Big(\prod_{j=1}^{r} z_j^{2b_{ij}} \Big) z^{4d_i}$$
$$= x_1^{2a_{i1}} z^{(\sum\limits_{k=1}^{n+1} 2^m(a_{i,2k-1}a_{i,2k}))+4d_i} = z^{2a_{i1}+4d_i'},$$

where $d_i' = \big(\sum\limits_{k=1}^{n+1} 2^{m-2}(a_{i,2k-1}a_{i,2k}) \big) + d_i$, therefore $a_{i1} + 2d_1' \equiv 0 \pmod{2^m}$. From this, $a_{i1} \equiv 0 \pmod 2$, thus $a_{i1} = 0$.

According to the results in [2], $\Psi_2(\varphi) = T \in \mathscr{B} \cong I \rtimes \mathrm{Sp}(2n, 2)$, where I is an elementary abelian 2-group with order 2^{2n+1}.

The theorem is proved.

Theorem 3.3 $\mathrm{Im}\,\Psi_3 \cong \mathrm{GL}(r, 2)$.

Proof Since $\mathrm{Frat}\,C = \langle z^2 \rangle$, $\{z_j \mathrm{Frat}\,C, j = 1, 2, \cdots, r\}$ is a basis of $\zeta C/\mathrm{Frat}\,C$. It follows that $\zeta C/\mathrm{Frat}\,C$ is a linear space over $\mathrm{GF}(2)$ with dimension r, which implies that $\mathrm{Im}\,\Psi_3$ can be embedded in $\mathrm{GL}(r, 2)$.

Conversely, for any $(d_{jk})_{r\times r} \in \mathrm{GL}(r, 2)$, we may define a mapping:

$$\begin{aligned} \delta_1 :\ & G \to G, \\ & y \mapsto y, \\ & x_i \mapsto x_i, \quad i = 1, 2, \cdots, 2n+2, \\ & z_j \mapsto \prod_{k=1}^{r} z_k^{b_{jk}}, \quad j = 1, 2, \cdots, r. \end{aligned}$$

It is easy to verify that $\delta_1 \in \mathrm{Aut}_f\, G$, and the matrix of $\Psi_2(\delta_1)$ is (b_{jk}) relative to a basis $\{z_j \mathrm{Frat}\,C, j = 1, 2, \cdots, r\}$ of $\zeta C/\mathrm{Frat}\,C$. Consequently, $\Psi_2(\mathrm{Aut}_f\, G) \cong \mathrm{GL}(r, 2)$.

The theorem is proved.

Theorem 3.4 $\mathrm{Ker}\,\Psi$ *is a 2-group with order* $2^{(2n+2)(r+1)+m+2r}$.

Proof Since $\mathrm{Ker}\,\Psi$ acts trivially on the factors of the series $G \geq C \geq \zeta C \geq \mathrm{Frat}\,C \geq 1$, thus $\mathrm{Ker}\,\Psi$ is a 2-group.

Take any $\alpha \in \mathrm{Ker}\,\Psi$, let α be an automorphism as follows:

$$\begin{aligned} \alpha :\ & G \to G, \\ & y \mapsto y \Big(\prod_{i=1}^{2n+2} x_i^{a_i} \Big) \Big(\prod_{j=1}^{r} z_j^{b_j} \Big) z^{2a}, \\ & x_i \mapsto x_i \Big(\prod_{j=1}^{r} z_j^{a_{ij}} \Big) z^{2c_i}, \quad i = 1, 2, \cdots, 2n+2, \\ & z_j \mapsto z_j z^{2d_j}, \quad j = 1, 2, \cdots, r, \\ & z^2 \mapsto z^2, \end{aligned}$$

where $0 \leq a_i < 2$, $0 \leq b_j < 2$, $0 \leq a < 2^m$, $0 \leq a_{ij} < 2$, $0 \leq c_i < 2^m$, $0 \leq d_j < 2^m$, $i = 1, 2, \cdots, 2n+2$, $j = 1, 2, \cdots, r$.

Since $\alpha(z)^2 = z^2$, $z^2 = \big(z (\prod\limits_{j=1}^{r} z_j^{a_{1j}}) z^{2c_1} \big)^2 = z^{2+4c_1}$, which implies that $c_1 = 0$ or 2^{m-1}.

Since $\alpha(x_i)^2 = 1$, where $i = 2, \cdots, 2n+2$, $1 = \big(x_i(\prod_{j=1}^{r} z_j^{a_{ij}})z^{2c_i}\big)^2 = z^{4c_i}$, which implies that $c_i \equiv 0 \pmod{2^{m-1}}$, consequently, $c_i = 0$ or 2^{m-1}.

Since $\alpha(y)$ is commutative with $\alpha(x_i)$, where $i = 3, 4, \cdots, 2n+2$,

$$\begin{aligned} 1 &= \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2a}, x_i\Big(\prod_{j=1}^{r} z_j^{a_{ij}}\Big)z^{2c_i}\Big] = \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i z^{2c_i}\Big] \\ &= \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), z^{2c_i}\Big]\Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big]^{z^{2c_i}} \\ &= [y, z^{2c_i}]^{x_1^{a_1}x_2^{a_2}}[x_1^{a_1}x_2^{a_2}, z^{2c_i}]\Big[\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big] = z^{4c_i}\Big[\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big]. \end{aligned}$$

Note that $4c_i \equiv 0 \pmod{2^{m+1}}$. If i is odd, we can suppose that $i = 2l+1$, where $l = 1, 2, \cdots, n$, then $z^{2^m a_{2l+2}} = z^{4c_{2l+1}+2^m a_{2l+2}} = 1$, which implies that $a_{2l+2} = 0$; if i is even, we can suppose that $i = 2l$, where $l = 2, \cdots, n+1$, then $z^{2^m a_{2l-1}} = 1$, which implies that $a_{2l-1} = 0$. In a word, for $i = 3, 4, \cdots, 2n+2$, we have that $a_i = 0$.

Since $\alpha(z)^{-2} = [\alpha(z), \alpha(y)]$,

$$z^{-2-4c_1} = \Big(z\Big(\prod_{j=1}^{r} z_j^{a_{1j}}\Big)z^{2c_1}\Big)^{-2} = \Big[z\Big(\prod_{j=1}^{r} z_j^{a_{1j}}\Big)z^{2c_1}, yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2a}\Big] = z^{2^m a_2 - 2 - 4c_1},$$

which implies that $a_2 = 0$.

Since $\alpha(x)$ is commutative with $\alpha(y)$,

$$1 = \Big[x_2\Big(\prod_{j=1}^{r} z_j^{a_{2j}}\Big)z^{2c_2}, yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2a}\Big] = z^{2^m a_1 - 4c_2}.$$

Also since $c_2 = 2^{m-1}$ or 0, we have that $a_1 = 0$.

Since $\alpha(y)$ is commutative with $\alpha(z_j)$, where $j = 1, 2, \cdots, r$,

$$1 = \Big[y\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2a}, z_j z^{2d_j}\Big] = z^{4d_j},$$

which implies that $d_j = 0$ or 2^{m-1}.

Since $\alpha(z_j)^2 = (z_j z^{2d_j})^2 = z^{4d_j}$, where $j = 1, 2, \cdots, r$, $d_j = 0$ or 2^{m-1}.

It is easy to verify generated relations of H_4 and H_5 have no effect on the parameters of α.

In conclusion, α is an automorphism as follows:

$$\begin{aligned} \alpha:\ & G \to G, \\ & y \mapsto y\Big(\prod_{j=1}^{r} z_j^{b_j}\Big)z^{2a}, \\ & x_i \mapsto x_i\Big(\prod_{j=1}^{r} z_j^{a_{ij}}\Big)z^{2c_i}, \quad i = 1, 2, \cdots, 2n+2, \\ & z_j \mapsto z_j z^{2d_j}, \quad j = 1, 2, \cdots, r, \end{aligned}$$

where $0 \le b_j < 2$, $0 \le a < 2^m$, $0 \le a_{ij} < 2$, $c_i = 0$ or 2^{m-1}, $d_j = 0$ or 2^{m-1}, $i = 1, 2, \cdots, 2n+2$, $j = 1, 2, \cdots, r$.

Conversely, if α is an automorphism of G, which satisfies the above conditions, then $\alpha \in \operatorname{Ker}\Psi$. It follows that $|\operatorname{Ker}\Psi| = 2^{(2n+2)(r+1)+m+2r}$.

The theorem is proved.

4 Proof of Theorem 1.3

Since D_8^{*n} is an extraspecial 2-group, we can suppose that $x_1, x_2, \cdots, x_{2n-1}, x_{2n}, y^{2^m}$ are the generators of D_8^{*n}, which satisfy the following conditions:

$$\begin{aligned} \zeta D_8^{*n} &= \langle y^{2^m} \rangle, \\ [x_{2i-1}, x_{2i}] &= y^{2^m}, \quad i = 1,\ 2,\ \cdots,\ n, \\ [x_{2i-1}, x_j] &= 1, \quad j =\neq 2i, \\ [x_{2i}, x_k] &= 1, \quad k \neq 2i-1, \\ x_i^2 &= 1, \quad i = 1,\ 2,\ \cdots,\ n. \end{aligned}$$

According to (3) in Lemma 1.6,

$$C = \langle x_1, x_2, y \rangle * \langle x_3, x_4, y \rangle * \cdots * \langle x_{2n-1}, x_{2n}, y \rangle \times \langle z y^{2^{m-1}} \rangle \times R.$$

For convenience, we may let $z_{r+1} := z y^{2^{m-1}}$, then $[z_{r+1}, x] = y^{2^m}$. Let $R_1 := R \times \langle z_{r+1} \rangle$.

Let $\Phi : \operatorname{Aut} G \to \operatorname{Aut}(\operatorname{Frat} C)$ be the restriction homomorphism. Obviously, $\operatorname{Ker}\Phi = \operatorname{Aut}_f G \trianglelefteq \operatorname{Aut} G$. According to (3) in Lemma 1.6, $\operatorname{Frat} C = \langle y^2 \rangle$.

Theorem 4.1

$$\operatorname{Im}\Phi \cong \begin{cases} \mathbb{Z}_2, & \text{if } m = 2, \\ \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2, & \text{if } m \ge 3. \end{cases}$$

Proof If $m = 2$, then $\operatorname{Frat} C \cong \mathbb{Z}_4$, therefore $\operatorname{Aut}(\operatorname{Frat} C) \cong \mathrm{Z}_2$. Define a mapping:

$$\begin{aligned} \sigma_5 :\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}^3, \quad i = 1, 2, \cdots, n, \\ x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n, \\ z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r+1, \\ x &\mapsto x, \\ y &\mapsto y^3. \end{aligned}$$

It is easy to verify that σ_5 is an automorphism of G with order 2. Since $\Phi(\sigma_5)(y^2) = (y^2)^3$ and $\Phi(\sigma_5)^2(y^2) = y^2$, $\operatorname{Aut}(\operatorname{Frat} C) = \langle \Phi(\sigma_5) \rangle$. It follows that $\operatorname{Aut} G = \operatorname{Aut}_f G \rtimes \langle \sigma_5 \rangle$.

If $m \ge 3$, then $\mathbb{Z}_{2^m}^* = \langle \upsilon_1 \rangle \times \langle \upsilon_2 \rangle$, where $\upsilon_1 = 3$ and $\upsilon_2 = 2^m - 1$. By Lemma 1.5, the orders of υ_1 and υ_2 are 2^{m-2} and 2, respectively. Define a mapping:

$$\begin{aligned} \sigma_6 :\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}^{2^m-1}, \quad i = 1, 2, \cdots, n, \\ x_{2i} &\mapsto x_{2i}, \quad i = 1, 2, \cdots, n, \\ z_j &\mapsto z_j, \quad j = 1, 2, \cdots, r+1, \\ x &\mapsto x^{2^m-1}, \\ y &\mapsto y^{2^m-1}. \end{aligned}$$

It is easy to verify σ_5 and σ_6 are the commutative automorphisms of G each other and their orders are 2^{m-1} and 2, respectively.

According to the argument in Theorem 2.1, we similarly have that $\operatorname{Aut} G = \langle\sigma_5,\ \sigma_6\rangle \operatorname{Aut}_f G$, $\langle\sigma_5,\ \sigma_6\rangle \cap \operatorname{Aut}_f G = \langle\sigma_5^{2^{m-2}}\rangle$, thus $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$.

The theorem is proved.

Let $\Psi_1 : \operatorname{Aut}_f G \to \operatorname{Aut}(G/C)$, $\Psi_2 : \operatorname{Aut}_f G \to \operatorname{Aut}(C/\zeta C)$ and $\Psi_3 : \operatorname{Aut}_f G \to \operatorname{Aut}(\zeta C/\operatorname{Frat} C)$ be the natural induced homomorphisms. Define a homomorphic mapping:

$$\begin{aligned}\Psi :\ \operatorname{Aut}_f G &\to \operatorname{Aut}(G/C) \times \operatorname{Aut}(C/\zeta C) \times \operatorname{Aut}(\zeta C/\operatorname{Frat} C),\\ \alpha &\mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).\end{aligned}$$

Since $G/C = \langle xC\rangle \cong \mathbb{Z}_2$, $\operatorname{Im}\Psi_1 = \operatorname{Aut}(G/C) = 1$.

Since $\zeta C = \langle y\rangle \times R_1$, we may define the inner product as follows:

$$f(\overline{a}, \overline{b}) = t, \text{ where } \overline{a} = a\zeta C,\ \overline{b} = b\zeta C,\ a, b \in C \text{ and } [a, b] = (y^{2^m})^t,\ 0 \le t < 2.$$

From this, $C/\zeta C$ can become a nondegenerate symplectic space over GF(2). For $\alpha \in \operatorname{Aut}_f G$, $[\alpha(a), \alpha(b)] = \alpha[a, b] = [a, b]$, thus, for any $\overline{a} = a\zeta C, \overline{b} = b\zeta C \in C/\zeta C$, we have that

$$f(\Psi_2(\alpha)(\overline{a}), \Psi_2(\alpha)(\overline{b})) = f(\overline{\alpha(a)}, \overline{\alpha(b)}) = f(\overline{a}, \overline{b}),$$

therefore $\Psi_2(\alpha) \in \operatorname{Sp}(2n, 2)$. Hence $\Psi_2(\operatorname{Aut}_f G) \le \operatorname{Sp}(2n, 2)$. It follows that Ψ is a homomorphism as follows:

$$\begin{aligned}\Psi :\ \operatorname{Aut}_f G &\to \operatorname{Aut}(G/C) \times \operatorname{Sp}(2n, 2) \times \operatorname{Aut}(\zeta C/\operatorname{Frat} C),\\ \alpha &\mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).\end{aligned}$$

Theorem 4.2 $\operatorname{Im}\Psi_2 = \operatorname{Sp}(2n, 2)$.

Proof Take any $T \in \operatorname{Sp}(2n, 2)$, let (a_{ik}) be the matrix of T relative to a basis $\{x_i\zeta C,\ i = 1, 2, \cdots, 2n\}$ of $C/\zeta C$. Define a mapping:

$$\phi :\ G \to G,$$
$$x^c\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)y^d \mapsto x^c\Big(\prod_{i=1}^{2n}\Big(\prod_{k=1}^{2n} x_k^{a_{ik}}\Big)^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)y^{d'},$$

where $0 \le a_i < 2$, $i = 1, 2, \cdots, 2n$, $0 \le b_j < 2$, $j = 1, 2, \cdots, r+1$, $0 \le c < 2$, $0 \le d < 2^{m+1}$, $d' \equiv d + \sum\limits_{i=1}^{2n} 2^{m-1}a_i\big(\sum\limits_{j=1}^{n}(a_{i,2j-1} \cdot a_{i,2j})\big) \pmod{2^{m+1}}$.

Note that (a_{ik}) is a nonsingular matrix. It is easy to verify ϕ is a bijection. Therefore, ϕ is an automorphism of G if and only if ϕ preserves multiplications.

According to the argument in Theorem 2.2, we similarly have that $\operatorname{Im}\Psi_1 = \operatorname{Sp}(2n, 2)$.

The theorem is proved.

Theorem 4.3 $\operatorname{Im}\Psi_3 \cong \operatorname{GL}(r, 2) \ltimes (\mathbb{Z}_2)^{2r}$.

Proof Let

$$\mathscr{C} := \left\{\begin{pmatrix} A_{11} & 0 \\ A_{21} & I_2 \end{pmatrix} \in \operatorname{GL}(r+2, 2)\right\},$$

where A_{11} is a $r \times r$ matrix, A_{21} is a $2 \times r$ matrix, I_2 is a 2×2 identity matrix. It is easy to verify that $\mathscr{A} \le \mathrm{GL}(r+2,2)$. For convenience, we may let $z_{r+2} := y$.

Take any $\alpha \in \mathrm{Aut}_f\, G$. Let (a_{jk}) be the $(r+2)\times(r+2)$ matrix of $\Psi_3(\alpha)$ relative to a basis $\{z_j \mathrm{Frat}\, C, j = 1, 2, \cdots, r+2\}$ of $\zeta C/\mathrm{Frat}\, C$.

Let (a_{jk}) be the partitioned matrix as follows:

$$(a_{jk}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{GL}(r+2,2),$$

where A_{11}, A_{12}, A_{21} and A_{22} are $r \times r$, $r \times 2$, $2 \times r$ and 2×2 matrices, respectively.

Since $\Psi_3(\alpha)(\overline{z}_j) = \prod\limits_{k=1}^{r+2} \overline{z}_k^{a_{jk}}$, where $j = 1, 2, \cdots, r+2$, there exists $0 \le a_j < 2^m$ such that $\alpha(z_j) = (\prod\limits_{k=1}^{r+2} z_k^{a_{jk}}) y^{2a_j}$.

For $j = 1, 2, \cdots, r+1$, $z_j^2 = 1$, thus

$$1 = \alpha(z_j^2) = \alpha(z_j)^2 = \Big(\prod_{k=1}^{r+2} z_k^{2a_{jk}}\Big) y^{2^2 a_j} = y^{2a_{j,r+2} + 2^2 a_j}.$$

Hence $a_{j,r+2} + 2a_j \equiv 0 \pmod{2^m}$. But $m > 1$ and $0 \le a_{j,r+2} < 2$, then, for $j = 1, 2, \cdots, r+1$, $a_{j,r+2} = 0$, $a_j = 0$ or 2^{m-1}.

Since

$$\begin{aligned} y^2 = z_{r+2}^2 = \alpha(z_{r+2}^2) = \alpha(z_{r+2})^2 &= \Big(\prod_{k=1}^{r+2} z_k^{2a_{r+2,k}}\Big) y^{2^2 a_{r+2}} \\ = z_{r+2}^{2a_{r+2,r+2} + 2^2 a_{r+2}} &= (y^2)^{a_{r+2,r+2} + 2a_{r+2}}, \end{aligned}$$

$a_{r+2,r+2} + 2a_{r+2} \equiv 1 \pmod{2^m}$. But $m > 1$ and $0 \le a_{r+2,r+2} < 2$, thus $a_{r+2,r+2} = 1$, $a_{r+2} = 0$ or 2^{m-1}.

Let $\alpha(x) = x(\prod\limits_{i=1}^{2n} x_i^{a_i})(\prod\limits_{j=1}^{r+1} z_j^{b_j}) y^d$, where $0 \le a_i < 2$, $0 \le b_j < 2$, $0 \le c < 2$, $0 \le d < 2^{m+1}$, $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, r+1$. Then, for any $j = 1, 2, \cdots, r$,

$$\begin{aligned} 1 = [\alpha(x), \alpha(z_j)] &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big) y^d, \Big(\prod_{k=1}^{r+2} z_k^{a_{jk}}\Big) y^{2a_j}\Big] \\ &= [x, z_{r+1}^{a_{j,r+1}} y^{2a_j}] = [x, z_{r+1}^{a_{j,r+1}}] = y^{2^m a_{j,r+1}}. \end{aligned}$$

From this, $2^m a_{j,r+1} \equiv 0 \pmod{2^{m+1}}$, thus $a_{j,r+1} = 0$.

Since

$$\begin{aligned} y^{2^m} &= \alpha(y^{2^m}) = [\alpha(x), \alpha(z_{r+1})] \\ &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big) y^d, \Big(\prod_{k=1}^{r+2} z_k^{a_{r+1,k}}\Big) y^{2a_{r+1}}\Big] = [x, z_{r+1}^{a_{r+1,r+1}} y^{2a_{r+1}}] = y^{2^m a_{r+1,r+1}}, \end{aligned}$$

$2^m a_{r+1,r+1} \equiv 1 \pmod{2^{m+1}}$. Thus $a_{r+1,r+1} = 1$.

If $H = H_6$, then

$$y^2 = \alpha(y^2) = [\alpha(x), \alpha(z_{r+2})]$$

$$= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)y^d, \Big(\prod_{k=1}^{r+2} z_k^{a_{r+2,k}}\Big)y^{2a_{r+2}}\Big] = [x, z_{r+1}^{a_{r+2,r+1}}y] = y^{2+2^m a_{r+2,r+1}},$$

which implies that $2 + 2^m a_{r+2,r+1} \equiv 2 \pmod{2^{m+1}}$, therefore $a_{r+2,r+1} = 0$; if $H = H_7$, then

$$\begin{aligned} y^{2-2^m} &= \alpha(y^{2-2^m}) = [\alpha(x), \alpha(z_{r+2})] \\ &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)y^d, \Big(\prod_{k=1}^{r+2} z_k^{a_{r+2,k}}\Big)y^{2a_{r+2}}\Big] = [x, z_{r+1}^{a_{r+2,r+1}}y] = y^{2-2^m+2^m a_{r+2,r+1}}, \end{aligned}$$

which implies that $2^m a_{r+2,r+1} \equiv 0 \pmod{2^{m+1}}$, therefore $a_{r+2,r+1} = 0$.

Conversely, for any $\begin{pmatrix} B_{11} & 0 \\ B_{21} & I_2 \end{pmatrix} = (b_{jk}) \in \mathscr{C}$. Define a mapping:

$$\begin{aligned} \delta_2 :\ & G \to G, \\ & x \mapsto x, \\ & x_i \mapsto x_i, \quad i = 1, 2, \cdots, 2n, \\ & z_j \mapsto \prod_{k=1}^{r+2} z_k^{b_{jk}}, \quad j = 1, 2, \cdots, r+2. \end{aligned}$$

It is easy to verify that $\delta_2 \in \operatorname{Aut} G$. Also since

$$\delta_2(y^2) = \delta(y)^2 = \Big(\prod_{k=1}^{r} z_k^{b_{r+2,k}} y\Big)^2 = y^2,$$

$\delta_2 \in \operatorname{Aut}_f G$, and the matrix of $\Psi_2(\delta_2)$ is (b_{jk}) relative to a basis $\{z_j \operatorname{Frat} C, j = 1, 2, \cdots, r+2\}$ of $\zeta C/\operatorname{Frat} C$. Thus $\operatorname{Im} \Psi_2 \cong \mathscr{C}$. Also since $\mathscr{C} \cong \mathrm{GL}(r, 2) \ltimes (\mathbb{Z}_2)^{2r}$, $\Psi_2(\operatorname{Aut}_f G) \cong \mathrm{GL}(r, 2) \ltimes (\mathbb{Z}_2)^{2r}$.

The theorem is proved.

Theorem 4.4 $\operatorname{Ker} \Psi$ *is a 2-group with order* $2^{(2n+2)(r+2)+m-1}$.

Proof Since $\operatorname{Ker} \Psi$ acts trivially on the factors of the series $G \geq C \geq \zeta C \geq \operatorname{Frat} C \geq 1$, $\operatorname{Ker} \Psi$ is a 2-group.

Take any $\alpha \in \operatorname{Ker} \Psi$. Let

$$\begin{aligned} \alpha :\ & G \to G, \\ & x \mapsto x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2} z_j^{b_j}\Big), \\ & x_i \mapsto x_i\Big(\prod_{j=1}^{r+2} z_j^{a_{ij}}\Big), \quad i = 1, 2, \cdots, 2n, \\ & z_k \mapsto z_k y^{2c_k}, \quad k = 1, 2, \cdots, r+2, \\ & y^2 \mapsto y^2, \end{aligned}$$

where $z_{r+2} = y$, $0 \leq a_i < 2$, $0 \leq b_j < 2$, $0 \leq b_{r+2} < 2^{m+1}$, $0 \leq a_{ij} < 2$, $0 \leq a_{i,r+2} < 2^{m+1}$, $0 \leq c_k < 2^m$, $i = 1, 2, \cdots, 2n$, $j = 1, 2, \cdots, r+1$, $k = 1, 2, \cdots, r+2$.

Since $\alpha(x_i)^2 = 1$, where $i = 1, 2, \cdots, 2n$, $1 = \big(x_i\big(\prod_{j=1}^{r+2} z_j^{a_{ij}}\big)\big)^2 = y^{2a_{i,r+2}}$, which implies that $a_{i,r+2} \equiv 0 \pmod{2^m}$, that is $a_{i,r+2} = 0$ or 2^m.

Since $\alpha(x)$ is commutative with $\alpha(x_i)$,

$$\begin{aligned}1 &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2} z_j^{b_j}\Big), x_i\Big(\prod_{j=1}^{r+2} z_j^{a_{ij}}\Big)\Big] = \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big), x_i z_{r+1}^{a_{i,r+1}} y^{a_{i,r+2}}\Big]\\ &= \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big), z_{r+1}^{a_{i,r+1}} y^{a_{i,r+2}}\Big]\Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big), x_i\Big]^{z_{r+1}^{a_{i,r+1}} y^{a_{i,r+2}}}\\ &= [x, z_{r+1}^{a_{i,r+1}} y^{a_{i,r+2}}]\Big[\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big), x_i\Big]\\ &= [x, z_{r+1}^{a_{i,r+1}}]\Big[\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big), x_i\Big].\end{aligned}$$

If i is odd, let $i = 2l - 1$, where $l = 1, 2, \cdots, n$, then $y^{2^m(a_{2l-1,r+1}+a_{2l})} = 1$, which implies that $a_{2l-1,r+1} + a_{2l} \equiv 0 \pmod 2$. If i is even, let $i = 2l$, where $l = 1, 2, \cdots, n$, then $y^{2^m(a_{2l,r+1}+a_{2l-1})} = 1$, which implies that $a_{2l,r+1} + a_{2l-1} \equiv 0 \pmod 2$.

Since $\alpha(x)$ is commutative with $\alpha(z_k)$, where $k = 1, 2, \cdots, r$,

$$1 = \Big[x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2} z_j^{b_j}\Big), z_k y^{2c_k}\Big] = [x, y^{2c_k}].$$

If $H = H_6$ or $H = H_7$, then $y^{4c_k} = 1$, thus $c_k = 0$ or 2^{m-1}. Also since

$$y^{2^m} = [\alpha(x), \alpha(z_{r+1})] = [x, z_{r+1}y^{2c_{r+1}}] = y^{2^m+4c_{r+1}},$$

$y^{2^m} = y^{2^m+4c_{r+1}}$, which implies that $4c_{r+1} \equiv 0 \pmod{2^{m+1}}$, that is $c_{r+2} = 0$ or 2^{m-1}. If $H = H_6$,

$$y^2 = [\alpha(x), \alpha(z_{r+2})] = [x, yy^{2c_{r+2}}] = y^{2+4c_{r+2}},$$

thus $4c_{r+2} \equiv 0 \pmod{2^{m+1}}$, that is $c_{r+2} = 0$ or 2^{m-1}. If $H = H_7$,

$$y^{2-2^m} = [\alpha(x), \alpha(z_{r+2})] = [x, yy^{2c_{r+2}}] = y^{2-2^m+4c_{r+2}},$$

thus $4c_{r+2} \equiv 0 \pmod{2^{m+1}}$, that is $c_{r+2} = 0$ or 2^{m-1}. In conclusion, for $k = 1, 2, \cdots, r+2$, $c_k = 0$ or 2^{m-1}.

If $H = H_6$,

$$1 = \alpha(x)^2 = \Big(x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2} z_j^{b_j}\Big)\Big)^2 = y^{2^m(b_{r+1}+\sum_{l=1}^{n} a_{2l-1}a_{2l})},$$

thus $b_{r+1} + \sum\limits_{l=1}^{n} a_{2l-1}a_{2l} \equiv 0 \pmod 2$; if $H = H_7$,

$$1 = \alpha(x)^2 = \Big(x\Big(\prod_{i=1}^{2n} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2} z_j^{b_j}\Big)\Big)^2 = y^{2^m(b_{r+1}+b_{r+2}+\sum_{l=1}^{n} a_{2l-1}a_{2l})},$$

thus $b_{r+1} + b_{r+2} + \sum\limits_{l=1}^{n} a_{2l-1}a_{2l} \equiv 0 \pmod 2$.

For $k=1,2,\cdots,r+1$, $1=\alpha(z_k)^2=z_k^2y^{2c_k}=y^{4c_k}$, thus $4c_k\equiv 0 \pmod{2^{m+1}}$, which implies that $c_k=0$ or 2^{m-1}. Also since $y^2=\alpha(y^2)=(y^{1+c_{r+2}})^2=y^{2+4c_{r+2}}$, $4c_{r+2}\equiv 0 \pmod{2^{m+1}}$, which implies that $c_{r+2}=0$ or 2^{m-1}.

It is easy to verify other generated relations of H_6 and H_7 which have no effect on the parameters of α.

In conclusion, α is an automorphism as follows:

$$\begin{aligned}\alpha:\ & G\to G,\\ & x\mapsto x\Big(\prod_{i=1}^{2n}x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+2}z_j^{b_j}\Big),\\ & x_i\mapsto x_i\Big(\prod_{j=1}^{r+2}z_j^{a_{ij}}\Big),\quad i=1,2,\cdots,2n,\\ & z_k\mapsto z_ky^{2c_k},\quad k=1,2,\cdots,r+2,\end{aligned}$$

where $z_{r+2}=y$, $0\le b_j<2$, $0\le a_{ij}<2$, $b_{r+1}+\sum_{l=1}^{n}a_{2l-1}a_{2l}\equiv 0 \pmod 2$ (if $H=H_6$) or $b_{r+1}+b_{r+2}+\sum_{l=1}^{n}a_{2l-1}a_{2l}\equiv 0 \pmod 2$ (if $H=H_7$), $0\le b_{r+2}<2^{m+1}$, $a_{2l-1,r+1}+a_{2l}\equiv 0 \pmod 2$, $a_{2l,r+1}+a_{2l-1}\equiv 0 \pmod 2$, $a_{i,r+2}=0$ or 2^m, $c_k=0$ or 2^{m-1}, $i=1,2,\cdots,2n$, $j=1,2,\cdots,r$, $k=1,2,\cdots,r+2$, $l=1,2,\cdots,n$.

Conversely, if α is an automorphism of G, which satisfies the above conditions, then $\alpha\in\operatorname{Ker}\Psi$. It follows that $|\operatorname{Ker}\Psi|=2^{(2n+2)(r+2)+m-1}$.

The theorem is proved.

5 Proof of Theorem 1.4

For convenience, we may suppose that $x_3,x_4,\cdots,x_{2n+1},x_{2n+2},z^{2^m}$ are the generators of D_8^{*n}, which satisfy the following conditions:

$$\begin{aligned}\zeta D_8^{*n}&=\langle z^{2^m}\rangle,\\ [x_{2i-1},x_{2i}]&=z^{2^m},\quad i=2,\ 3,\ \cdots,\ n,\\ [x_{2i-1},x_j]&=1,\quad j\ne 2i,\\ [x_{2i},x_k]&=1,\quad k\ne 2i-1,\\ x_i^2&=1,\quad i=2,\ 3,\ \cdots,\ n.\end{aligned}$$

According to (4) in Lemma 1.6,

$$C=\langle x_1,x_2\rangle*\langle x_3,x_4,z^2\rangle*\langle x_5,x_6,z^2\rangle*\cdots*\langle x_{2n+1},x_{2n+2},z^2\rangle\times R\cong M_m(2)*N_m(2)^{*n}\times R,$$

where $x_1:=z$, $x_2:=x$.

Let $\Phi:\operatorname{Aut}G\to\operatorname{Aut}(\operatorname{Frat}C)$ be the restriction homomorphism. Obviously, $\operatorname{Ker}\Phi=\operatorname{Aut}_f G\trianglelefteq\operatorname{Aut}G$. According to (4) in Lemma 1.6, $\operatorname{Frat}C=\langle z^2\rangle=\operatorname{Frat}G\cong\mathbb{Z}_{2^m}$.

Theorem 5.1

$$\operatorname{Im}\Phi\cong\begin{cases}\mathbb{Z}_2, & \text{if } m=2,\\ \mathbb{Z}_{2^{m-2}}\times\mathbb{Z}_2, & \text{if } m\ge 3.\end{cases}$$

Proof If $m=2$, then $\operatorname{Frat} C \cong \mathbb{Z}_4$, thus $\operatorname{Aut}(\operatorname{Frat} C) \cong \mathrm{Z}_2$. Define a mapping:

$$\begin{aligned} \sigma_7:\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}^3, \quad i=1,2,\cdots,n+1, \\ x_{2i} &\mapsto x_{2i}, \quad i=1,2,\cdots,n+1, \\ z_j &\mapsto z_j, \quad j=1,2,\cdots,r, \\ y &\mapsto y, \\ u &\mapsto u^3. \end{aligned}$$

It is easy to verify that σ_7 is an automorphism of G, which is of order 2. Since $\Phi(\sigma_7)(z^2)=(z^2)^7$ and $\Phi(\sigma_7)^2(z^2)=z^2$, $\operatorname{Aut}(\operatorname{Frat} C)=\langle\Phi(\sigma_7)\rangle$. It follows that $\operatorname{Aut} G=\operatorname{Aut}_f G \rtimes \langle\sigma_7\rangle$.

If $m \geq 3$, then $\mathbb{Z}_{2^m}^*=\langle v_1\rangle \times \langle v_2\rangle$, where $v_1=3$ and $v_2=2^m-1$ and their orders are 2^{m-2} and 2 by Lemma 1.5, respectively. Define a mapping:

$$\begin{aligned} \sigma_8:\ & G \to G, \\ x_{2i-1} &\mapsto x_{2i-1}^{2^m-1}, \quad i=1,2,\cdots,n+1, \\ x_{2i} &\mapsto x_{2i}, \quad i=1,2,\cdots,n+1, \\ z_j &\mapsto z_j, \quad j=1,2,\cdots,r, \\ y &\mapsto y, \\ u &\mapsto u^{2^m-1}. \end{aligned}$$

It is easy to verify that σ_7 and σ_8 are the commutative automorphisms of G each other and their orders are 2^{m-1} and 2, respectively.

By means of the argument in Theorem 2.1, we similarly have that $\operatorname{Aut} G=\langle\sigma_7,\ \sigma_8\rangle \operatorname{Aut}_f G$, and $\langle\sigma_7,\ \sigma_8\rangle \cap \operatorname{Aut}_f G=\langle\sigma_7^{2^{m-2}}\rangle$. It follows that $\operatorname{Aut} G/\operatorname{Aut}_f G \cong \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2$.

The theorem is proved.

Let

$$\begin{aligned} \Psi_1 &: \operatorname{Aut}_f G \to \operatorname{Aut}(G/C), \\ \Psi_2 &: \operatorname{Aut}_f G \to \operatorname{Aut}(C/\zeta C) \\ \Psi_3 &: \operatorname{Aut}_f G \to \operatorname{Aut}(\zeta C/\operatorname{Frat} C) \end{aligned}$$

be the natural induced homomorphisms. From this, we can obtain the below homomorphism:

$$\begin{aligned} \Psi:\ & \operatorname{Aut}_f G \to \operatorname{Aut}(G/C) \times \operatorname{Aut}(C/\zeta C) \times \operatorname{Aut}(\zeta C/\operatorname{Frat} C), \\ & \alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)). \end{aligned}$$

Since $G/C=\langle yC\rangle \cong \mathbb{Z}_2$, $\operatorname{Im}\Psi_1=\operatorname{Aut}(G/C)=1$.

Since $\zeta C=(\langle z^2\rangle \times R)\cdot\langle u\rangle$, we can define the inner product as follows:

$$f(\overline{a}, \overline{b})=t, \text{ where } \overline{a}=a\zeta C,\ \overline{b}=b\zeta C,\ a, b \in C \text{ and } [a, b]=(z^{2^m})^t,\ 0 \leq t<2.$$

Hence $C/\zeta C$ can become a nondegenerated symplectic space over GF(2). For any $\alpha \in \operatorname{Aut}_f G$, $[\alpha(a), \alpha(b)]=\alpha[a, b]=[a, b]$, then, for any $\overline{a}=a\zeta C, \overline{b}=b\zeta C \in C/\zeta C$,

$$f(\Psi_2(\alpha)(\overline{a}), \Psi_2(\alpha)(\overline{b}))=f(\overline{\alpha(a)}, \overline{\alpha(b)})=f(\overline{a}, \overline{b}),$$

therefore $\Psi_2(\alpha) \in \mathrm{Sp}(2n,2)$. Thus $\Psi_2(\mathrm{Aut}_f\, G) \le \mathrm{Sp}(2n,2)$. In a word, Ψ is a homomorphism as follows:

$$\Psi:\ \mathrm{Aut}_f\, G \to \mathrm{Aut}(G/C) \times \mathrm{Sp}(2n,2) \times \mathrm{Aut}(\zeta C/\mathrm{Frat}\, C),$$
$$\alpha \mapsto (\Psi_1(\alpha), \Psi_2(\alpha), \Psi_3(\alpha)).$$

Theorem 5.2 $\mathrm{Im}\,\Psi_2 = I \rtimes \mathrm{Sp}(2n,2)$, *where I is an elementary abelian 2-group with order* 2^{2n+1}.

Proof Let $\mathscr{D} := \{T \in \mathrm{Sp}(2n+2,2) \mid$ the first column and second row of the matrix of T are $(1,0,\cdots,0)^{\mathrm{T}}$ and $(0,1,0,\cdots,0)$ relative to a basis $x_1\zeta C, x_2\zeta C, \cdots, x_{2n+2}\zeta C$ of $C/\zeta C$, respectively$\}$.

Take any $T \in \mathscr{D}$. Let (a_{ik}) be the matrix of T relative to a basis $\{x_i\zeta C,\ i = 1,2,\cdots,2n+2\}$ of $C/\zeta C$. Define a mapping:

$$\phi:\ G \to G,$$
$$y^c\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2d} \mapsto (yx^t)^c\Big(\prod_{i=1}^{2n+2}\Big(\prod_{k=1}^{2n+2} x_k^{a_{ik}}\Big)^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2d'},$$

where $z_{r+1} := u$, $0 \le a_i < 2$, $i = 1,2,\cdots,2n+2$, $0 \le b_j < 2$, $j = 1,2,\cdots,r+1$, $0 \le c < 2$, $0 \le d < 2^m$, $d' \equiv d + \sum\limits_{i=1}^{2n+2} 2^{m-2}a_i\big(\sum\limits_{k=1}^{n+1}(a_{i,2k-1}\cdot a_{i,2k})\big) \pmod{2^m}$, $t = 0$ $\big($if $\sum\limits_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k}) \equiv 0 \pmod 2\big)$ or $t = 1$ $\big($if $\sum\limits_{k=1}^{n+1}(a_{1,2k-1}a_{1,2k}) \equiv 1 \pmod 2\big)$.

Note that (a_{ik}) is a nonsingular matrix. It is easy to verify ϕ is a bijection. Therefore, ϕ is an automorphism of G if and only if ϕ preserves multiplications.

According to the argument in Theorem 3.2, we similarly have that $\mathrm{Im}\,\Psi_2 = \mathscr{D} = I \rtimes \mathrm{Sp}(2n,2)$, where I is an elementary abelian 2-group with order 2^{2n+1}.

The theorem is proved.

Theorem 5.3 $\mathrm{Im}\,\Psi_3 \cong \mathrm{GL}(r,2) \ltimes (\mathbb{Z}_2)^r$.

Proof For convenience, let $z_{r+1} := uz^{2^{m-1}}$, then $\zeta C = R \times \langle z_{r+1}\rangle \times \langle z^2\rangle$, and $H_8 = \langle x, y, z, z_{r+1} \mid x^2 = y^2 = z^{2^{m+1}} = z_{r+1}^2 = 1, y^x = y, z^x = z^{2^m+1}, z^y = z^{-1}, [x, z_{r+1}] = 1 = [z, z_{r+1}], [y, z_{r+1}] = z^{2^m}\rangle$.

Since $\mathrm{Frat}\, C = \langle z^2\rangle$, $\{z_j\mathrm{Frat}\, C, j = 1,2,\cdots,r+1\}$ is a basis of $\zeta C/\mathrm{Frat}\, C$ and $\zeta C/\mathrm{Frat}\, C$ is a linear space over $\mathrm{GF}(2)$ with the dimension $r+1$. Hence $\mathrm{Im}\,\Psi_3$ can be embedded in $\mathrm{GL}(r+1,2)$.

Let

$$\mathscr{H} := \left\{\begin{pmatrix} H_{11} & 0 \\ H_{21} & 1 \end{pmatrix} \in \mathrm{GL}(r+1,2)\right\},$$

where H_{11} is a $r \times r$ matrix, H_{21} is a $1 \times r$ matrix. It is easy to verify that $\mathscr{H} \le \mathrm{GL}(r+1,2)$.

For any $\alpha \in \mathrm{Aut}_f\, G$, let (h_{jk}) be the matrix of $\Psi_3(\alpha)$ relative to a basis $\{z_j\mathrm{Frat}\, C, j = 1,2,\cdots,r+1\}$ of $\zeta C/\mathrm{Frat}\, C$.

Let (h_{jk}) be the partitioned matrix as follows:

$$(h_{jk}) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \in \mathrm{GL}(r+1,2),$$

where H_{11}, H_{12}, H_{21} and H_{22} are $r\times r$, $r\times 1$, $1\times r$ and 1×1 matrices, respectively.

Since $\Psi_3(\alpha)(\overline{z}_j)=\prod_{k=1}^{r+1}\overline{z}_k^{h_{jk}}$, there exists $0\le h_j<2^m$ such that $\alpha(z_j)=\big(\prod_{k=1}^{r+1} z_k^{h_{jk}}\big)z^{2h_j}$.

For $j=1,2,\cdots,r+1$, $1=\alpha(z_j)^2=z^{4h_j}$, thus $4h_j\equiv 0 \pmod{2^{m+1}}$.

Let $\alpha(y)=yy_1$, where $y_1\in C$. Since $\alpha(y)$ is commutative with $\alpha(z_j)$ for $j=1,2,\cdots,r$,

$$1=\Big[yy_1,\Big(\prod_{k=1}^{r+1} z_k^{h_{jk}}\Big)z^{2h_j}\Big]=[y,z^{2h_j}][y,z_{r+1}^{h_{j,r+1}}]^{z^{2h_j}}=z^{4h_j+2^m h_{j,r+1}}.$$

Hence $h_{j,r+1}=0$, that is $H_{12}=0$. Since

$$z^{2^m}=\Big[yy_1,\Big(\prod_{k=1}^{r+1} z_k^{h_{r+1,k}}\Big)z^{2h_{r+1}}\Big]=[y,z_{r+1}^{h_{r+1,r+1}}z^{2h_{r+1}}]=z^{4h_{r+1}+2^m h_{r+1,r+1}},$$

$h_{r+1,r+1}=1$, that is $H_{22}=1$.

Conversely, for any $\begin{pmatrix} H_{11} & 0\\ H_{21} & 1\end{pmatrix}=(h_{jk})\in\mathscr{H}$, define a mapping:

$$\begin{aligned}\delta_3:\ & G\to G,\\ & y\mapsto y,\\ & x_i\mapsto x_i,\quad i=1,2,\cdots,2n+2,\\ & z_j\mapsto\prod_{k=1}^{r+1} z_k^{b_{jk}},\quad j==1,2,\cdots,r+1.\end{aligned}$$

It is easy to verify that $\delta_3\in\operatorname{Aut}_f G$, and the matrix of $\Psi_2(\delta_3)$ is (b_{jk}) relative to a basis $\{z_j\operatorname{Frat}C, j=1,2,\cdots,r+1\}$ of $\zeta C/\operatorname{Frat}C$. Hence $\operatorname{Im}\Psi_2\cong\mathscr{H}$. Also since $\mathscr{H}\cong\mathrm{GL}(r,2)\ltimes(\mathbb{Z}_2)^r$, $\Psi_2(\operatorname{Aut}_f G)\cong\mathrm{GL}(r,2)\ltimes(\mathbb{Z}_2)^r$.

The theorem is proved.

Theorem 5.4 $\operatorname{Ker}\Psi$ *is a* 2*-group with order* $2^{(2n+2)(r+2)+2r+m+1}$.

Proof For convenience, let $z_{r+1}:=uz^{2^{m-1}}$.

Since $\operatorname{Ker}\Psi$ acts trivially on the factors of the series $G\ge C\ge\zeta C\ge\operatorname{Frat}C\ge 1$, $\operatorname{Ker}\Psi$ is a 2-group.

For any $\alpha\in\operatorname{Ker}\Psi$, let

$$\begin{aligned}\alpha:\ & G\to G,\\ & y\mapsto y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a},\\ & x_i\mapsto x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big)z^{2c_i},\quad i=1,2,\cdots,2n+2,\\ & z_j\mapsto z_jz^{2d_j},\quad j=1,2,\cdots,r+1,\\ & z^2\mapsto z^2,\end{aligned}$$

where $0\le a_i<2$, $0\le b_j<2$, $0\le a<2^m$, $0\le a_{ij}<2$, $0\le c_i<2^m$, $0\le d_j<2^m$, $i=1,2,\cdots,2n+2$, $j=1,2,\cdots,r+1$.

Since $\alpha(z)^2 = z^2$, $z^2 = \big(z(\prod_{j=1}^{r+1} z_j^{a_{1j}})z^{2c_1}\big)^2 = z^{2+4c_1}$, which implies that $c_1 = 0$ or 2^{m-1}.

Since $\alpha(x_i)^2 = 1$, where $i = 2, \cdots, 2n+2$, $1 = \big(x_i(\prod_{j=1}^{r+1} z_j^{a_{ij}})z^{2c_i}\big)^2 = z^{4c_i}$, which implies that $c_i \equiv 0 \pmod{2^{m-1}}$, that is $c_i = 0$ or 2^{m-1}.

Since $\alpha(y)$ is commutative with $\alpha(x_i)$, where $i = 3, 4, \cdots, 2n+2$,

$$\begin{aligned} 1 &= \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}, x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big)z^{2c_i}\Big] = \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i z_{r+1}^{a_{i,r+1}} z^{2c_i}\Big] \\ &= \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), z_{r+1}^{a_{i,r+1}} z^{2c_i}\Big]\Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big]^{z_{r+1}^{a_{i,r+1}} z^{2c_i}} \\ &= [y, z^{2c_i}][y, z_{r+1}^{a_{i,r+1}}]\Big[\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big] \\ &= z^{4c_i} z^{2^m a_{i,r+1}}\Big[\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big), x_i\Big]. \end{aligned}$$

Note that $4c_i \equiv 0 \pmod{2^{m+1}}$. If i is odd, let $i = 2j-1$, where $j = 2, \cdots, n+1$, then $z^{2^m(a_{2j-1,r+1}+a_{2j})} = 1$, which implies that $a_{2j-1,r+1} + a_{2j} \equiv 0 \pmod 2$; if i is even, let $i = 2j$, where $j = 2, \cdots, n+1$, then $z^{2^m(a_{2j,r+1}+a_{2j-1})} = 1$, which implies that $a_{2j,r+1} + a_{2j-1} \equiv 0 \pmod 2$.

Since $\alpha(x)$ is commutative with $\alpha(y)$,

$$1 = \Big[x_2\Big(\prod_{j=1}^{r+1} z_j^{a_{2j}}\Big)z^{2c_2}, yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}\Big] = z^{2^m(a_1+a_{2,r+1})-4c_2}.$$

Also since $c_2 = 0$ or 2^{m-1}, $a_1 + a_{2,r+1} \equiv 0 \pmod 2$.

Since $\alpha(z)^{-2} = [\alpha(z), \alpha(y)]$,

$$\begin{aligned} z^{-2-4c_1} &= \Big(z\Big(\prod_{j=1}^{r+1} z_j^{a_{1j}}\Big)z^{2c_1}\Big)^{-2} = \Big[z\Big(\prod_{j=1}^{r+1} z_j^{a_{1j}}\Big)z^{2c_1}, yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}\Big] \\ &= z^{2^m a_2 - 2 - 4c_1 + 2^m a_{1,r+1}}, \end{aligned}$$

which implies that $a_2 + a_{1,r+1} \equiv 0 \pmod 2$.

Since

$$1 = \alpha(y)^2 = \Big[y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}\Big]^2 = z_{r+1}^c,$$

where $c := 2^m\big(b_{r+1} + \sum_{j=1}^{n+1} a_{2j-1}a_{2j}\big)$, $b_{r+1} + \sum_{j=1}^{n+1} a_{2j-1}a_{2j} \equiv 0 \pmod 2$.

Since $\alpha(y)$ is commutative with $\alpha(z_j)$, where $j = 1, 2, \cdots, r$,

$$1 = \Big[yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}, z_j z^{2d_j}\Big] = z^{4d_j},$$

which implies that $d_j = 0$ or 2^{m-1}. Since

$$z^{2^m} = \alpha(z^{2^m}) = \alpha(z)^{2^m} = [\alpha(y), \alpha(z_{r+1})] = \Big[yz^{a_1}x^{a_2}\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a}, z_{r+1}z^{2d_{r+1}}\Big]$$
$$= z^{2^m+4d_{r+1}},$$

$d_{r+1} = 0$ or 2^{m-1}.

Since $1 = \alpha(z_j)^2 = (z_j z^{2d_j})^2 = z^{4d_j}$, where $j = 1, 2, \cdots, r+1$, $d_j = 0$ or 2^{m-1}.

It is easy to verify other generated relations of H_8 have effect on the parameters of α.

In conclusion, α is an automorphism as follows:

$$\begin{aligned}\alpha:\ & G \to G,\\ & y \mapsto y\Big(\prod_{i=1}^{2n+2} x_i^{a_i}\Big)\Big(\prod_{j=1}^{r+1} z_j^{b_j}\Big)z^{2a},\\ & x_i \mapsto x_i\Big(\prod_{j=1}^{r+1} z_j^{a_{ij}}\Big)z^{2c_i}, \quad i = 1, 2, \cdots, 2n+2,\\ & z_j \mapsto z_j z^{2d_j}, \quad j = 1, 2, \cdots, r+1,\end{aligned}$$

where $a_{2j-1,r+1} + a_{2j} \equiv 0 \pmod 2$, $a_{2j,r+1} + a_{2j-1} \equiv 0 \pmod 2$, $b_{r+1} + \sum_{j=1}^{n+1} a_{2j-1}a_{2j} \equiv 0 \pmod 2$, $0 \le b_j < 2$, $0 \le a < 2^m$, $0 \le a_{ij} < 2$, $c_i = 0$ or 2^{m-1}, $d_j = 0$ or 2^{m-1}, $i = 1, 2, \cdots, 2n+2$, $j = 1, 2, \cdots, r+1$.

Conversely, if α is an automorphism of G, which satisfies the above conditions, then $\alpha \in \operatorname{Ker}\Psi$. Hence $|\operatorname{Ker}\Psi| = 2^{(2n+2)(r+2)+2r+m+1}$.

The theorem is proved.

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