Chin. Ann. Math.
38B(3), 2017, 759–786
DOI: 10.1007/s11401-017-1094-5

**Chinese Annals of Mathematics, Series B**


# Quasi-periodic Solutions for the Derivative Nonlinear Schrödinger Equation with Finitely Differentiable Nonlinearities*

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**Abstract** The authors are concerned with a class of derivative nonlinear Schrödinger equation

$$\mathbf{i}u_t + u_{xx} + \mathbf{i}\epsilon f(u, \overline{u}, \omega t)u_x = 0, \quad (t, x) \in \mathbb{R} \times [0, \pi],$$

subject to Dirichlet boundary condition, where the nonlinearity $f(z_1, z_2, \phi)$ is merely finitely differentiable with respect to all variables rather than analytic and quasi-periodically forced in time. By developing a smoothing and approximation theory, the existence of many quasi-periodic solutions of the above equation is proved.

**Keywords** Derivative NLS, KAM theory, Newton iterative scheme, Reduction theory, Quasi-periodic solutions, Smoothing techniques
**2000 MR Subject Classification** 37K55, 35B15, 35J10, 35Q40, 35Q55

## 1 Introduction

In this paper we prove the existence of quasi-periodic solutions of the derivative nonlinear Schrödinger (DNLS for short) equation

$$\mathbf{i}u_t + u_{xx} + \mathbf{i}\epsilon f(u, \overline{u}, \omega t)u_x = 0 \tag{1.1}$$

subject to Dirichlet boundary conditions $u(t, 0) = 0 = u(t, \pi)$, $-\infty < t < +\infty$, where the nonlinearity $f(z_1, z_2, \phi)$ is differentiable for finite times with $z_1, z_2, \phi$ and quasi-periodic in time with frequency vector $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$. Moreover, $f(z_1, z_2, \phi) \in C^{\overline{p}}(\mathbb{C} \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$ for some $\overline{p} \in \mathbb{R}^+$ large enough with

$$\overline{f(u, \overline{u}, \phi)} = f(u, \overline{u}, \phi), \quad f(-u, -\overline{u}, \phi) = -f(u, \overline{u}, \phi), \ \ \phi \in \mathbb{T}^n.$$

The same as in [18], introducing the inner product in a suitable phase space, for example, the usual Sobolev space $H_0^2([0, \pi])$

$$\langle u, v\rangle = \mathrm{Re} \int_0^\pi u\overline{v}\mathrm{d}x,$$

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*This work was supported by the National Natural Science Foundation of China (No. 11201292), Shanghai Natural Science Foundation (No. 12ZR1444300) and the Key Discipline “Applied Mathematics” of Shanghai Second Polytechnic University (No. XXKZD1304).

then (1.1) can be written in the Hamiltonian form

$$\dot{u} = -\mathbf{i}\nabla H, \tag{1.2}$$

$$H(u,\overline{u},\phi) = \frac{1}{2}\int_0^\pi |u_x|^2\mathrm{d}x + \frac{1}{2}\int_0^\pi g(u,\overline{u},\phi)u_x\mathrm{d}x \tag{1.3}$$

with the "Reality" condition $\overline{H(u,\overline{u},\phi)} = H(u,\overline{u},\phi)$ for $\phi \in \mathbb{T}^n$, where the gradient of $H$ is defined with respect to $\langle\cdot,\cdot\rangle$ and $g(z_1,z_2,\phi) = -\mathbf{i}\int_0^{z_2} f(z_1,\zeta,\phi)\mathrm{d}\zeta$.

KAM theory is a powerful tool to deal with the existence of periodic, quasi-periodic or almost periodic solutions of partial differential equations (PDEs for short) under small perturbations. The first KAM results for PDEs have been obtained for 1-d semi-linear Schrödinger and wave equations by Kuksin [14], Craig-Wayne [10, 29], see the references therein. For PDEs in higher space dimension, the theory has been more recently extended by Bourgain [8], Eliasson-Kuksin [11], Berti-Bolle [5], and Geng-Xu-You [12]. For unbounded perturbations, the first KAM results have been proved by Kuksin [15–16] and Kappeler-Pöschel [13] for KdV equation (see also [7]), and more recently by Liu-Yuan [17–19], Zhang-Gao-Yuan [32] for derivative NLS equation, Baldi-Berti-Montalto [1] for the Hamiltonian quasi-linear perturbations of the KdV equation, and Berti-Biasco-Procesi [2–3] for derivative NLW equation.

However, the results mentioned above require the analyticity of the perturbations of the PDEs to overcome the well-known "loss of regularity" problem. By shrinking the width of the angle variables, one can estimate the solutions of the homological equations and obtain the convergence of the KAM iterative procedure. For dynamical systems with differentiable perturbations, it is clear that after finitely many steps all derivatives are exhausted which leads to the failure of the KAM iteration. To cope with this difficulty, the primary approach is due to Moser [20–21], which extended the classical KAM theory for nearly integrable Hamiltonian systems under real-analytic perturbations, to smooth category. The main idea exploited by Moser is to use a smoothing operator, and re-insert enough regularity into the problem at every Newton iterative step in order to compensate the loss of regularity. A closely related approach was given by Nash [24] researching the embedding problem of compact Riemannian manifolds. In [21], Moser first proved the existence of the invariant curves for area preserving annulus mappings satisfying the monotone twist property which corresponds to the Hamiltonian system case in "one and a half" degrees of freedom. The number of derivative of the perturbation is required to be $\ell > 333$, which was later reduced by Rüssmann to $\ell > 5$ in [26]. For the Hamiltonian case we refer to [23, 25].

The KAM theory in Moser [20–21] dealt with the persistence of maximal-dimensional invariant tori in the context of smooth category. It is natural to ask whether lower-dimensional tori can be persisted or not. By exploiting a technique following [23], Chierchia-Qian [9] considered the existence of lower-dimensional elliptic tori of any dimension between one and the number of degrees of freedom for the nearly integrable Hamiltonian system with finitely differentiable perturbation. The framework of this method is mainly based on an approximation of the differentiable functions with analytic ones. Zhang [31] proved the existence of the lower-dimensional invariant tori for the reversible system with finite degrees of freedom under finitely differentiable perturbation. For infinite dimensional Hamiltonian systems, the research just began in the last few years, the main results were given by Berti-Bolle-Procesi [4–6]. By using a Nash-Moser iterative scheme in scales of the Sobolev functions space, they got the existence of quasi-periodic solutions which has Sobolev regularity both in time and space for PDEs with bounded perturbations, such as the NLS and NLW for any spatial dimension.

The perturbation of (1.1) is finitely differentiable. What is more important, the answer is unbounded. Thus (1.1) is excluded by the above approach. The aim of the present paper is to construct a large amount of quasi-periodic solutions of small amplitude for the derivative NLS equation (1.1). More precisely, in the following we consider a class of "vector" derivative NLS equations:

$$\begin{cases} \mathbf{i}u_t + u_{xx} + \mathbf{i}\epsilon f(u,\overline{u},\omega t)u_x = 0, \\ \mathbf{i}\overline{u}_t - \overline{u}_{xx} + \mathbf{i}\epsilon \overline{f(u,\overline{u},\omega t)}\overline{u}_x = 0, \end{cases} \qquad (t,x) \in \mathbb{R} \times [0,\pi] \tag{1.4}$$

subject to Dirichlet boundary condition $u(t,0) = 0 = u(t,\pi)$, where the nonlinearities are quasi-periodic in time with frequency $\omega \in \mathbb{R}^n$ and $f(z_1,z_2,\phi) \in C^{\overline{p}}(\mathbb{C}\times\mathbb{C}\times\mathbb{T}^n;\ \mathbb{C})$ for some $\overline{p} \in \mathbb{R}^+$ large enough and the second equation is the formal complex conjugation of the first one.

We have the following theorem.

**Theorem 1.1** *Suppose that the nonlinearities $f$ and $\overline{f}$ are finitely differentiable with $\overline{p} > 100(1+\rho)(3n+2\tau+1)+3+p$, where $\rho$, $p$ and $\tau$ are positive constants which will be defined below, and $\Pi \subset \mathbb{R}^n$ is a compact set of positive Lebesgue measure. Then there exists a small constant $\epsilon^* > 0$ such that for $|\epsilon| < \epsilon^*$, a Cantor set $\Pi_\epsilon \subset \Pi$ with $\mathrm{Meas}(\Pi \setminus \Pi_\epsilon) \to 0$ as $\epsilon \to 0$, and for arbitrary $\omega \in \Pi_\epsilon$, (1.4) possesses a quasi-periodic solution of frequency $\omega$ with small amplitude.*

The main ideas of our proof consist of a smoothing technique elaborated in [23] and Newton iterative scheme. Note that (1.4) is a Hamiltonian PDE, however, we do not use its Hamiltonian structure explicitly. Instead, we write (1.4) into an abstract nonlinear equation

$$\mathbf{i}\dot{\zeta} - \Lambda\zeta + F(\zeta,\omega t) = 0 \tag{1.5}$$

(see Theorem 2.1), of which we try to construct the quasi-periodic solutions by Newton's method. The essential element of Newton's method is to find the approximate solution by solving the linearized equation of the original equation. Therefore, at each Newton iteration, we solve the linearized equation of (1.5)

$$\mathbf{i}\dot{\eta} - (\Lambda + P(\omega t))\eta + F(\omega t) = 0. \tag{1.6}$$

Moreover, we need to prove the convergence of the iterative process. In order to solve (1.6), we need to estimate the inverse of $(\Lambda + P(\omega t))^{-1}$, which is "big" due to the unboundedness of the perturbation. Hence we do not solve the linearized equation (1.6) directly but do the KAM type reduction first. Luckily, as we discuss (1.4) under Dirichlet boundary condition, the frequencies are simple, the reduction process is feasible (see Lemma 3.2 for the details). The Hamiltonian structure guarantees the reality of $\Lambda$ which is necessary in Theorem 1.4 in Liu-Yuan [17]. In fact we get a new system after the reduction

$$\mathbf{i}\dot{\varphi} - (\widehat{\Lambda} + \widehat{P}(\omega t))\varphi + \widehat{F}(\omega t) = 0, \tag{1.7}$$

where $\widehat{P}(\omega t)$ is much smaller and can be treated as perturbation. Thus we just need to find the solution for the linearized equation

$$\mathbf{i}\dot{\varphi} - \widehat{\Lambda}\varphi + \widehat{F}(\omega t) = 0, \tag{1.8}$$

where $\widehat{\Lambda}$ denotes a diagonal matrix close to $\Lambda$ in some sense (see Lemma 4.1 for the details).

We remark that $P(\omega t), F(\omega t)$ and $\widehat{F}(\omega t)$ are only differentiable with respect to $t$. The main difficulty during the whole procedure is the phenomenon of "loss of regularity". To overcome this, we use an approximation theorem which is closely related to the classical theorem due to Jackson, of which the fundamental observation is that the qualitative property of differentiability of a function can be characterized in terms of quantitative estimates for an approximating sequence of analytic functions. Then we can solve the linearized equation in analytic category with good estimates to guarantee the convergence of the Newton iterative process.

This paper is organized as follows: In Section 2, we rewrite the derivative NLS equation (1.4) in infinite coordinates, and this new equation will be our starting point for the following discussion. What's more, we list a similar theory to the approximation theory of Jackson, Moser and Zehnder, which will be used as the basis of our smoothing technique. In Section 3, a KAM type reduction lemma will be proven with finitely differentiable unbounded perturbation. In Section 4, we describe the solving procedure of the linearized equation at each step and the iterative process in details. Finally some technical lemmas are exhibited in Section 5.

## 2 The Basic Modes

We study (1.4) on some suitable phase space, for example, the usual Sobolev space $H_0^p([0,\pi])$. We rewrite it in infinitely many coordinates by making the ansatz

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \quad \phi_j(x) = \sqrt{\frac{2}{\pi}}\sin jx, \quad j \geq 1. \tag{2.1}$$

The coordinates are taken from the Hilbert space $\ell^p$ of all complex-valued sequences $q = (q_1, q_2, \cdots)$ with $\|q\|_p^2 = \sum_{j\geq 1} |q_j|^2 j^{2p} < \infty$. We fix $p > \frac{3}{2}$ later. Then (1.4) can be written as

$$\begin{cases} \mathbf{i}\dot{q} - Aq + \epsilon\widetilde{f}(q,\overline{q},\omega t) = 0, \\ \mathbf{i}\dot{\overline{q}} + A\overline{q} + \epsilon\widetilde{g}(q,\overline{q},\omega t) = 0, \end{cases} \tag{2.2}$$

where $A = \mathrm{diag}(\lambda_i : i \geq 1)$ with $\lambda_i = i^2$ and $\widetilde{f}(q,\overline{q},\omega t) = -\overline{\widetilde{g}(q,\overline{q},\omega t)}$. In the following we consider $q, \overline{q}$ to be independent, and (1.4) equals to (2.2) when the bar means the complex conjugate. We investigate the regularity of the nonlinear vector field first. In fact, we have the following observation.

**Lemma 2.1** *The nonlinear vector field $(\widetilde{f}(q,\overline{q},\phi), \widetilde{g}(q,\overline{q},\phi))^{\mathrm{T}}$ defines a finitely differentiable map from $\mathcal{O} \times \mathbb{T}^n$ into $\ell^{p-1} \times \ell^{p-1}$, where $\mathcal{O}$ denotes some small neighborhood of the origin in $\ell^p \times \ell^p$. To be more precise, for any $\phi \in \mathbb{T}^n$, $\mathbb{N} \ni \ell < \overline{p} - p$ and $k = (k_1, \cdots, k_n)$ with $0 \leq |k| := |k_1| + \cdots + |k_n| \leq \ell$,*

$$\Big(\frac{\partial^k \widetilde{f}}{\partial \phi^k}(q,\overline{q},\phi), \frac{\partial^k \widetilde{g}}{\partial \phi^k}(q,\overline{q},\phi)\Big)^{\mathrm{T}}$$

$$:= \Big(\frac{\partial^{|k|}\widetilde{f}}{\partial\phi_1^{k_1}\cdots\partial\phi_n^{k_n}}, \frac{\partial^{|k|}\widetilde{g}}{\partial\phi_1^{k_1}\cdots\partial\phi_n^{k_n}}\Big)^{\mathrm{T}} \in C^{\ell-|k|}(\mathcal{O};\ \ell^{p-1}\times\ell^{p-1}), \tag{2.3}$$

*where $T$ means the transpose of a vector.*

**Proof** Set $\xi = (u,\overline{u})^{\mathrm{T}} \in \mathbf{H}^p := H_0^p \times H_0^p$. Introduce a map $F$ defined on $\mathbf{H}^p \times \mathbb{T}^n$ with

$$F(\xi,\phi)(x) = (F_1(\xi,\phi)(x), F_2(\xi,\phi)(x))^{\mathrm{T}} = (\mathbf{i}f(u,\overline{u},\phi)u_x, \mathbf{i}\overline{f}(u,\overline{u},\phi)\overline{u}_x)^{\mathrm{T}}. \tag{2.4}$$

We will prove that there exists some neighborhood $\mathcal{U}$ of the origin in $\mathbf{H}^p$ such that for any $\phi \in \mathbb{T}^n$ and $0 \le |k| \le \ell$, $\frac{\partial^k F}{\partial\phi^k} \in C^{\ell-|k|}(\mathcal{U};\ \mathbf{H}^{p-1})$. Then (2.3) follows directly.

From the assumption that $\overline{p} > p+\ell$ and $0 \le |k| \le \ell$, we can find some $\mathcal{N}$ being the neighborhood of the origin in $\mathbb{C}^2$ such that for $a+b \le p-1$, all the derivatives $\frac{\partial^k}{\partial\phi^k}\frac{\partial^a}{\partial z_1^a}\frac{\partial^b}{\partial z_2^b}f(z_1,z_2,\phi)$ are bounded on $\mathcal{N}\times\mathbb{T}^n$. Then we set $\mathcal{U}$ denoting the neighborhood of the origin in $\mathbf{H}^p$ of which the element $(u,\overline{u})$ has graphs lying in $\mathcal{N}$. We will show that for any $\xi\in\mathcal{U}$, $\phi\in\mathbb{T}^n$, $\frac{\partial^k F}{\partial\phi^k}(\xi,\phi) \in \mathbf{H}^{p-1}$. Using the chain rule we can write

$$\frac{\mathrm{d}^{p-1}}{\mathrm{d}x^{p-1}}\frac{\partial^k F_1}{\partial\phi^k}(\xi,\phi)(x) = \mathbf{i}\sum_{*}\frac{\partial^{k+a+b}f}{\partial\phi^k\partial z_1^a\partial z_2^b}\frac{\mathrm{d}^{i_1}u}{\mathrm{d}x^{i_1}}\cdots\frac{\mathrm{d}^{i_a}u}{\mathrm{d}x^{i_a}}\frac{\mathrm{d}^{j_1}\overline{u}}{\mathrm{d}x^{j_1}}\cdots\frac{\mathrm{d}^{j_b}\overline{u}}{\mathrm{d}x^{j_b}}\frac{\mathrm{d}^m u_x}{\mathrm{d}x^m},$$

where $*$ represents $i_1+\cdots+i_a+j_1+\cdots+j_b+m = p-1$. For $\xi\in\mathcal{U}$, we have $\|u\|_{L^\infty}, \|\overline{u}\|_{L^\infty} \le C$ and $\big\|\frac{\partial^{k+a+b}f(u,\overline{u},\phi)}{\partial\phi^k\partial z_1^a\partial z_2^b}\big\|_{L^\infty} \le C$, which is due to the fact that $p+\ell<\overline{p}$. Thus we can get the estimate

$$\begin{aligned}\Big\|\frac{\partial^k F_1}{\partial\phi^k}(\xi,\phi)\Big\|_{p-1} &:= \Big\|\frac{\mathrm{d}^{p-1}}{\mathrm{d}x^{p-1}}\frac{\partial^k F_1}{\partial\phi^k}(\xi,\phi)(x)\Big\|_{L^2}\\ &\le C\sum_{*}\Big\|\frac{\mathrm{d}^{i_1}u}{\mathrm{d}x^{i_1}}\|_{L^2}\cdots\|\frac{\mathrm{d}^{i_a}u}{\mathrm{d}x^{i_a}}\Big\|_{L^2}\Big\|\frac{\mathrm{d}^{j_1}\overline{u}}{\mathrm{d}x^{j_1}}\Big\|_{L^2}\cdots\Big\|\frac{\mathrm{d}^{j_b}\overline{u}}{\mathrm{d}x^{j_b}}\Big\|_{L^2}\Big\|\frac{\mathrm{d}^m u_x}{\mathrm{d}x^m}\Big\|_{L^2}\\ &\le C\sum_{*}\|u\|_{i_1}\cdots\|u\|_{i_a}\|\overline{u}\|_{j_1}\cdots\|\overline{u}\|_{j_b}\|u_x\|_m.\end{aligned} \tag{2.5}$$

Then by using the interpolation estimate in Lemma 5.1 in the Appendix, we can get

$$\|u\|_i \le C\|u\|_{p-1}^{\frac{i}{p-1}},\quad \|\overline{u}\|_j \le C\|\overline{u}\|_{p-1}^{\frac{j}{p-1}},\quad \|u_x\|_m \le C\|u_x\|_{p-1}^{\frac{m}{p-1}} \le C\|u\|_p^{\frac{m}{p-1}}. \tag{2.6}$$

Consequently we have $\big\|\frac{\partial^k F_1}{\partial\phi^k}(\xi,\phi)\big\|_{p-1} \le C(p)(\|u\|_p+\|\overline{u}\|_p)$. The estimate corresponding to $F_2$ can be obtained similarly, and we omit the details here. Hence we obtain

$$\Big\|\frac{\partial^k F_1}{\partial\phi^k}(\xi,\phi)\Big\|_{p-1} + \Big\|\frac{\partial^k F_2}{\partial\phi^k}(\xi,\phi)\Big\|_{p-1} \le C(p)(\|u\|_p+\|\overline{u}\|_p). \tag{2.7}$$

Then the conclusion that $\frac{\partial^k F}{\partial\phi^k}$ defines a map from $\mathcal{U}$ into $\mathbf{H}^{p-1}$ for any $\phi\in\mathbb{T}^n$ follows.

Now we investigate the first order Frèchet derivative of $\frac{\partial^k F}{\partial\phi^k}$ with respect to $\xi$. When $|k|=\ell$, then nothing remains to be done. Hence in the following we assume $|k|<\ell$. For any $\eta=(v,\overline{v})\in\mathbf{H}^p$, we get

$$\begin{aligned}\frac{\partial^{k+1}F_1}{\partial\xi\partial\phi^k}(\xi,\phi)(v)(x) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial^k F_1}{\partial\phi^k}(\xi+s\eta,\phi)\Big|_{s=0}\\ &= \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial^k}{\partial\phi^k}f(u+sv,\overline{u}+s\overline{v},\phi)(u_x+sv_x)\Big)\Big|_{s=0}\end{aligned}$$

$$= \mathbf{i}\Big(\frac{\partial^{k+1} f}{\partial\phi^k \partial z_1} u_x v + \frac{\partial^{k+1} f}{\partial\phi^k \partial z_2} u_x \overline{v} + \frac{\partial^k f}{\partial\phi^k} v_x\Big). \tag{2.8}$$

Note that $\frac{\partial^{k+1} f}{\partial\phi^k\partial z_1}, \frac{\partial^{k+1} f}{\partial\phi^k\partial z_2}, \frac{\partial^k f}{\partial\phi^k} \in C^{\overline{p}-\ell}(\mathbb{C}\times\mathbb{C}\times\mathbb{T}^n;\ \mathbb{C})$ and $\overline{p}-\ell > p$, we have

$$\Big\|\frac{\partial^{k+1} f}{\partial\phi^k\partial z_1} u_x\Big\|_{p-1}, \Big\|\frac{\partial^{k+1} f}{\partial\phi^k\partial z_2} u_x\Big\|_{p-1}, \Big\|\frac{\partial^k f}{\partial\phi^k}\Big\|_p \le C(p)(\|u\|_p + \|\overline{u}\|_p) \tag{2.9}$$

with a different constant $C$ depending on $p$. Thus, $\big\|\frac{\partial^{k+1} F_1}{\partial\xi\partial\phi^k}(\xi,\phi)(v)\big\|_{p-1} \le C(p)(\|u\|_p + \|\overline{u}\|_p)(\|v\|_p + \|\overline{v}\|_p)$, and the same estimate can be obtained for $F_2$. Therefore, for any $(\xi,\phi) \in \mathcal{U}\times\mathbb{T}^n$, the first Frèchet derivative $\frac{\partial^{k+1} F}{\partial\xi\partial\phi^k}(\xi,\phi)$ defines a bounded linear operator from $\mathbf{H}^p$ into $\mathbf{H}^{p-1}$. Denote $\mathcal{B}(\mathbf{H}^p;\mathbf{H}^{p-1})$ as the set of bounded linear operators from $\mathbf{H}^p$ into $\mathbf{H}^{p-1}$, then we can obtain $\frac{\partial^{k+1} F}{\partial\xi\partial\phi^k} : \mathcal{U}\subset\mathbf{H}^p \to \mathcal{B}(\mathbf{H}^p;\mathbf{H}^{p-1})$ for all $\phi\in\mathbb{T}^n$. For any other derivative of high order, we can handle in the same way. By now we have finished the proof.

We set some notations and definitions for the sake of convenience. Set $\mathcal{P}^p = \ell^p\times\ell^p$, and for $\zeta = (q,\widetilde{q})^{\mathrm{T}} \in \mathcal{P}^p$, define the norm $\|\zeta\|_p := \|q\|_p + \|\widetilde{q}\|_p$. For $\alpha\ge 1$, we use $X_\alpha$ and $\|\cdot\|^{X_\alpha}$ to represent the set of bounded linear operators from $\prod\limits_{\alpha}\mathcal{P}^p$ into $\mathcal{P}^{p-1}$ and the corresponding norm respectively. Here we just focus our attention on $\alpha\in\mathbb{N}$ for simplicity, and hence in Lemma 2.1 we choose $\ell, |k| \in \mathbb{N}$ also. Set $X_0 := \mathcal{P}^{p-1}$. For a given vector $R(\zeta,\phi) = (f(\zeta,\phi), g(\zeta,\phi))^{\mathrm{T}}$, define

$$\|R\|_{C^\ell} := \sup_{0\le|k|\le\ell}\ \sup_{0\le\alpha\le\ell-|k|}\ \sup_{(\zeta,\phi)\in\mathcal{O}\times\mathbb{T}^n} \Big\|\frac{\partial^{k+\alpha} R}{\partial\phi^k\partial q^\alpha}\Big\|^{X_\alpha}.$$

For the original system (2.2), set $F(\zeta,\phi) = (\epsilon\widetilde{f}(\zeta,\phi), \epsilon\widetilde{g}(\zeta,\phi))^{\mathrm{T}}$. Then based on Lemma 2.1, we conclude that for any $0\le|k|\le\ell$, $0\le\alpha\le\ell-|k|$ and $(\zeta,\phi)\in\mathcal{O}\times\mathbb{T}^n$, $\frac{\partial^{k+\alpha}\widetilde{F}}{\partial\phi^k\partial\zeta^\alpha}(\zeta,\phi)\in X_\alpha$, the estimate

$$\|F(\zeta,\phi)\|_{C^\ell} \le \epsilon \tag{2.10}$$

holds true. Then we have the following theorem.

**Theorem 2.1** *Consider the system*

$$\mathbf{i}\dot{\zeta} - \Lambda\zeta + F(\zeta,\omega t) = 0, \quad \zeta = (q,\widetilde{q}) \in \mathcal{P}^p,\ \omega\in\Pi, \tag{2.11}$$

*which fulfills the following hypotheses:*

(A1)

$$\Lambda = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \tag{2.12}$$

*where*

$$A = \mathrm{diag}(\lambda_i : i\ge 1) \tag{2.13}$$

*with* $\lambda_i = i^2$.

(A2) *The perturbation* $F(\zeta,\phi) = (F_1(\zeta,\phi), F_2(\zeta,\phi))^{\mathrm{T}} : \mathcal{O}\times\mathbb{T}^n \to \mathcal{P}^{p-1}$ *is finitely differentiable with respect to* $\zeta$ *and* $\phi$ *with*

$$\|F(\zeta,\phi)\|_{C^\ell} \le \epsilon. \tag{2.14}$$

*In addition,* $F_2(\zeta,\phi) = -\overline{F_1(\zeta,\phi)}$ *for* $\widetilde{q} = \overline{q}$ *and* $\phi \in \mathbb{T}^n$.

(A3) *The first order Frèchet derivative* $D_\zeta F(\zeta,\phi) = \left(\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}\right)$, *where* $P_i$, $1 \le i \le 4$ *is a function from* $\mathcal{O} \times \mathbb{T}^n$ *into the space of bounded operators from* $\ell^p$ *into* $\ell^{p-1}$. *Moreover,*

$$\overline{P}_1 = P_1^{\mathrm{T}} = -P_4, \quad P_2 = P_2^{\mathrm{T}}, \quad \overline{P}_2 = -P_3 \tag{2.15}$$

*for* $\widetilde{q} = \overline{q}$ *and* $\phi \in \mathbb{T}^n$.

*Then for any*

$$\ell > 3 + 100(1+\rho)(3n + 2\tau + 1), \tag{2.16}$$

*where* $\rho$ *and* $\tau$ *are two fixed positive constants with* $0 < \rho < \frac{1}{4}$ *and* $\tau > n+3$, *there exists* $\epsilon^* > 0$ *and a Cantor subset* $\Pi^\epsilon \subset \Pi$ *with* $\mathrm{Meas}(\Pi \setminus \Pi^\epsilon) \to 0$ *as* $\epsilon \to 0$, *such that for* $\epsilon < \epsilon^*$ *and* $\omega \in \Pi^\epsilon$, (2.11) *has a quasi-periodic solution with frequency* $\omega$.

From (1.3) in Section 1, we can find a Hamiltonian perturbation $P(q,\overline{q},\phi)$ such that $\widetilde{f}(q,\overline{q},\phi) = \partial_{\overline{q}} P$ and $\widetilde{g}(q,\overline{q},\phi) = -\partial_q P$. Then taking the "Reality" condition of $P$ into account, we can easily check the assumption (A3). Obviously, we can directly get the conclusion in Theorem 1.1 from Theorem 2.1, thus we will discuss the proof of Theorem 2.1 in the following. In the remaining part of the present section we list a well known and fundamental approximation result. Starting from the following lemma, we can set up a sequence of analytic functions which approximate to the original finitely differential one.

**Lemma 2.2** (Jackson, Moser, Zehnder) *Let* $X$ *be a Banach space and* $f \in C^\ell(\mathbb{R}^n; X)$ *for some* $\ell > 0$ *with finite* $C^\ell$ *norm over* $\mathbb{R}^n$. *Let* $\phi$ *be a radial-symmetric,* $C^\infty$ *function with support being the closure of the unit ball centered at the origin, where* $\phi$ *is completely flat and takes value* 1, *and let* $K = \widehat{\phi}$ *be its Fourier transform. For all* $\sigma > 0$, *define* $f_\sigma(x) := K_\sigma * f = \frac{1}{\sigma^n}\int_{\mathbb{R}^n} K\left(\frac{x-y}{\sigma}\right) f(y) \mathrm{d}y$. *Then there exists a constant* $C \ge 1$ *depending only on* $\ell$ *and* $n$ *such that the following holds: For any* $\sigma > 0$, *the function* $f_\sigma(x)$ *is a real-analytic function from* $\mathbb{C}^n$ *to* $X$ *such that if* $\triangle_\sigma^n$ *denotes the* $n$*-dimensional complex strip of width* $\sigma$, $\triangle_\sigma^n := \{x \in \mathbb{C}^n \mid |\mathrm{Im} x_j| \le \sigma,\ 1 \le j \le n\}$, *then for all* $\alpha \in \mathbb{N}^n$ *such that* $|\alpha| \le \ell$, *one has*

$$\sup_{x \in \triangle_\sigma^n} \Big\| \partial^\alpha f_\sigma(x) - \sum_{|\beta| \le \ell - |\alpha|} \frac{\partial^{\beta+\alpha} f(\mathrm{Re}\, x)}{\beta!} (\mathbf{i}\mathrm{Im} x)^\beta \Big\|^{X_\alpha} \le C \|f\|_{C^\ell} \sigma^{\ell - |\alpha|} \tag{2.17}$$

*and for all* $0 \le s \le \sigma$,

$$\sup_{x \in \triangle_s^n} \|\partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x)\|^{X_\alpha} \le C \|f\|_{C^\ell} \sigma^{\ell - |\alpha|}. \tag{2.18}$$

*Here* $X_\alpha$ *is the Banach space of bounded operators* $T : \prod_{|\alpha|} (\mathbb{R}^n) \to X$ *with the norm*

$$\|T\|_{X_\alpha} = \sup\{\|T(u_1, u_2, \cdots u_{|\alpha|})\| : \ \|u_i\| = 1,\ 1 \le i \le |\alpha|\}.$$

*The function* $f_\sigma$ *preserves periodicity (i.e., if* $f$ *is* $T$*-periodic in any of its variable* $x_j$, *so is* $f_\sigma$*). Finally, if* $f$ *depends on some parameter* $\xi \in \Pi \subset \mathbb{R}^n$ *and if the Lipschitz-norm of* $f$ *and its* $x$*-derivatives are uniformly bounded by* $\|f\|_{C^\ell}^{\mathcal{L}}$, *then all the above estimates hold with* $\|\cdot\|$ *replaced by* $\|\cdot\|^{\mathcal{L}}$.

This lemma is similar to the approximation theory obtained by Jackson, Moser and Zehnder, and the only difference is that we extend the applied range from $C^\ell(\mathbb{R}^n;\mathbb{C}^n)$ to $C^\ell(\mathbb{R}^n;X)$. The proof of this lemma consists in a direct check which is based on standard tools from calculus and complex analysis, for details see [27–28] and the references therein.

Fix a sequence of fast decreasing numbers $s_\nu \downarrow 0$, $\nu \ge 0$, and $s_0 \le \frac{1}{2}$, for $F(\phi) \in C^\ell(\mathbb{T}^n;X)$ we can construct a sequence of analytic and quasi-periodic functions $F^{(\nu)}(\phi)$ such that the following conclusions holds:

(1) $F^{(\nu)}(\phi)$ is analytic on the complex strip $\mathbb{T}^n_{s_\nu}$ of the width $s_\nu$ around $\mathbb{T}^n$.

(2) The sequence of functions $F^{(\nu)}(\phi)$ satisfies the bounds:

$$\sup_{\phi\in\mathbb{T}^n} \|F^{(\nu)}(\phi) - F(\phi)\| \le C\|F\|_{C^\ell} s_\nu^\ell, \tag{2.19}$$

$$\sup_{\phi\in\mathbb{T}^n_{s_{\nu+1}}} \|F^{(\nu+1)}(\phi) - F^{(\nu)}(\phi)\| \le C\|F\|_{C^\ell} s_\nu^\ell, \tag{2.20}$$

where $C$ denotes a constant depending only on $n$ and $\ell$.

(3) The first approximate $F^{(0)}$ is "small" with the perturbation $F$. Precisely speaking, for arbitrary $\phi \in \mathbb{T}^n_{s_0}$, we have

$$\begin{aligned}\|F^{(0)}(\phi)\| &\le \|F^{(0)}(\phi) - \sum_{|\alpha|\le\ell} \frac{\partial^\alpha F(\mathrm{Re}\,\phi)}{\alpha!}(\mathbf{i}\mathrm{Im}\,\phi)^\alpha\| + \Big\|\sum_{|\alpha|\le\ell} \frac{\partial^\alpha F(\mathrm{Re}\,\phi)}{\alpha!}(\mathbf{i}\mathrm{Im}\,\phi)^\alpha\Big\| \\ &\le C\|F\|_{C^\ell} s_0^\ell + \sum_{m=0}^{\ell} \|F\|_{C^m} s_0^m \le C\|F\|_{C^\ell}\sum_{m=0}^{\ell} s_0^m \le C\|F\|_{C^\ell},\end{aligned}$$

where constant $C$ is independent of $s_0$, and the last inequality holds true due to $s_0 \le \frac{1}{2}$.

(4) From (2.19), we have the equality below. For arbitrary $\phi \in \mathbb{T}^n$,

$$F(\phi) = F^{(0)}(\phi) + \sum_{\upsilon=0}^{+\infty} (F^{(\nu+1)}(\phi) - F^{(\nu)}(\phi)). \tag{2.21}$$

## 3 Reduction Lemma

In this section, we will give an iterative lemma, which is the key part of our proof. Let $m \ge 0$ be the $m$-th step, we introduce some recursive parameters.

(1) $\epsilon_0 = C\epsilon$, $C$ denotes a positive constant depending only on $n$ and $\overline{\rho}$,

(2) $\epsilon_m = \epsilon_0^{(1+\overline{\rho})^m}$, where $\overline{\rho}$ is a small constant satisfying $0 < \overline{\rho} < \frac{1}{4}$,

(3) $s_m = \epsilon_m^{\frac{1+\rho}{\ell-3}}$ with $0 < \overline{\rho} < \rho < \frac{1}{4}$, which dominates the width of the angle variable $\phi$,

(4) $\sigma_m = \frac{s_m}{18}$, which acts as a bridge from $s_m$ to $s_{m+1}$,

(5) $K_m = \frac{|\ln \epsilon_{m-1}^{\frac{2}{3}-\rho}|}{\sigma_m}$ and $L_m = \frac{|\ln \epsilon_m|}{\sigma_m}$, which denote the length of the truncation of Fourier series,

(6) $\alpha_m = \frac{\alpha_0}{2^m}$, which dominates the measure parameters excluded in the $m$-th iteration step,

(7) $C_{\lambda,m} = \frac{1}{2}\big(1 + \frac{1}{2^m}\big)$, hence $\frac{1}{2} \le C_{\lambda,m} \le 1$,

(8) $C_{\omega,m} = \big(2 - \frac{1}{2^m}\big)\epsilon_0^{\frac{2}{3}-\rho}$, hence $\epsilon_0^{\frac{2}{3}-\rho} \le C_{\omega,m} \le 2\epsilon_0^{\frac{2}{3}-\rho}$,

(9) $C_{\mu,m} = C_1\big(2 - \frac{1}{2^m}\big)\epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}$, hence $C_1\epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)} \le C_{\mu,m} \le 2C_1\epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}$, where $C_1$ is a positive constant depending only on $n$ and $\tau$, and $\tau$ is a fixed real number greater than $n+3$.

Before we give the iterative lemma, we list some notations.

(1) Suppose that $\mathcal{B}^{q,\overline{q}}$ is the set of bounded linear operators from $\ell^q$ into $\ell^{\overline{q}}$, and we define the operator norm of its element by $\|\cdot\|_{q,\overline{q}}$. Accordingly, we denote the set of bounded linear operators from $\mathcal{P}^q$ into $\mathcal{P}^{\overline{q}}$ by $\mathfrak{B}^{q,\overline{q}}$, and the operator norm of its element by $|||\cdot|||_{q,\overline{q}}$. Hence from the above section, we can conclude that $|||\cdot|||_{p,p-1} = \|\cdot\|^{X_1}$.

(2) Let $\mathbb{T}^n_{\mathbb{C}}$ be the complexification of $\mathbb{T}^n$, and define $\mathbb{T}^n_s = \{\phi \in \mathbb{T}^n_{\mathbb{C}} : |\mathrm{Im}\phi| = \max\limits_{1\le i\le n} |\mathrm{Im}\phi_i| < s\}$. Then for an analytic function $f : \mathbb{T}^n_s \to \mathcal{B}$ (here $\mathcal{B}$, a Banach space with norm $\|\cdot\|^{\mathcal{B}}$, may be $\mathbb{C}, \mathbb{C}^n, X_\alpha$ or $\mathfrak{B}^{q,\overline{q}}$) is analytic, define

$$\|f\|^{\mathcal{B}}_s := \sup_{\phi\in\mathbb{T}^n_s} \|f(\phi)\|^{\mathcal{B}}.$$

Furthermore, if $f$ has an additional (Lipschitz-continuous) dependence on $\omega \in \Pi$, we define the Lipschitz norm

$$\|f\|^{\mathcal{B},\mathcal{L}}_s := \|f\|^{\mathcal{B}}_s + \sup_{\substack{\phi\in\mathbb{T}^n_s\\ \omega\neq\omega'\in\Pi}} \frac{\|f(\phi,\omega) - f(\phi,\omega')\|^{\mathcal{B}}}{|\omega-\omega'|}.$$

(3) Choose $\epsilon$ and $\alpha_0$ such that

$$0 < \epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)} \ll \alpha_0 \ll 1. \tag{3.1}$$

(4) In what follows we use the notations $a \lessdot b$ to represent that there exists a constant $C$ independent of $m, \epsilon$ and $\alpha_0$ but may depending on $n, \tau$ and $\ell$ such that $a < Cb$ holds.

(5) Let

$$\Pi' = \Pi \setminus \bigcup_{k\in\mathbb{Z}^n\setminus\{0\}} \Big\{\omega \in \Pi :\ |k\cdot\omega| \le \frac{\alpha_0}{|k|^\tau}\Big\}.$$

Apparently we know that $\mathrm{Meas}(\Pi\setminus\Pi') \lessdot \alpha_0$.

**Lemma 3.1** *Assume that at the m-th iteration step, we have a system as follows*:

$$\mathbf{i}\dot{\chi} = (\Lambda_m + P_m(\omega t))\chi \tag{3.2}$$

*with* $\chi = (\psi, \widetilde{\psi})^{\mathrm{T}} \in \mathcal{P}^p$, $\omega \in \Pi_m$, $m \ge 1$ *which satisfies the following hypotheses:*

(H1)

$$\Lambda_m = \begin{pmatrix} A_m & 0 \\ 0 & -A_m \end{pmatrix},$$

*where*

$$A_m = \mathrm{diag}(\lambda_{i,m}(\omega) + \mu_{i,m}(\omega t;\omega) :\ i \ge 1)$$

*with*

$$0 < \lambda_{1,m} < \lambda_{2,m} < \cdots < \lambda_{i,m} < \cdots, \quad |\lambda_{i,m} - \lambda_{j,m}| \ge C_{\lambda,m}|i^2 - j^2|.$$

*Moreover,* $\lambda_{i,m}$ *is Lipschitz-continuous in* $\omega$ *and fulfills the estimate*

$$\sup_{\omega\neq\omega'\in\Pi_m} \frac{|\lambda_{i,m}(\omega) - \lambda_{i,m}(\omega')|}{|\omega-\omega'|} \le C_{\omega,m} i.$$

$\mu_{i,m}(\phi,\omega) : \mathbb{T}^n_{s_m} \times \Pi_m \to \mathbb{C}$ *is real analytic in* $\phi$, *Lipschitz-continuous in* $\omega$ *and of zero average, i.e.,* $\int_{\mathbb{T}^n} \mu_{i,m}(\phi)\mathrm{d}\phi = 0$. *It also fulfills the following estimates in* $\mathbb{T}^n_{s_m} \times \Pi_m$*:*

$$|\mu_{i,m}|_{s_m,\tau+1} \le C_{\mu,m} i, \quad |\mu_{i,m}|^{\mathcal{L}}_{s_m} \le C_{\omega,m} i, \tag{3.3}$$

*where* $|\mu|_{s,\tau+1} = \sum_{k\in\mathbb{Z}^n} |\widetilde{\mu}_k| \mathrm{e}^{|k|s}|k|^{\tau+1}$ *and* $\widetilde{\mu}_k = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \mu(\phi)\mathrm{e}^{-\mathbf{i}k\cdot\phi}\mathrm{d}\phi$ *denotes the k-th Fourier coefficient of* $\mu$.

(H2) $P_m(\phi,\omega) = \left(\begin{smallmatrix} P_{1,m} & P_{2,m} \\ P_{3,m} & P_{4,m}\end{smallmatrix}\right)\colon \mathbb{T}^n_{s_{m-1}-5\sigma_{m-1}} \times \Pi_m \to X_1$ *is real analytic with respect to* $\phi$, *Lipschitz-continuous in* $\omega$ *and satisfies the estimate:*

$$\|P_m\|^{X_1,\mathcal{L}}_{s_{m-1}-5\sigma_{m_1}} \le \epsilon_{m-1}^{\frac{2}{3}-\rho}. \tag{3.4}$$

*Furthermore, for* $\phi \in \mathbb{T}_n$ *and* $\omega \in \Pi_m$,

$$\overline{P}_{1,m} = P^{\mathrm{T}}_{1,m} = -P_{4,m}, \quad P^{\mathrm{T}}_{2,m} = P_{2,m}, \quad \overline{P}_{2,m} = -P_{3,m}. \tag{3.5}$$

(H3) *For any* $\omega \in \Pi_m$,

$$|k\cdot\omega + \lambda_{j,m}(\omega)| \ge \frac{\alpha_m j^2}{1+|k|^\tau}, \quad k \in \mathbb{Z}^n,\ j \ge 1, \tag{3.6}$$

$$|k\cdot\omega + \lambda_{i,m}(\omega) - \lambda_{j,m}(\omega)| \ge \frac{\alpha_m |i^2-j^2|}{1+|k|^\tau}, \quad k \in \mathbb{Z}^n,\ i,j\ge 1,\ i \ne j. \tag{3.7}$$

*Then there exist* $\Pi_{m+1} \subset \Pi_m$ *with* $\mathrm{Meas}(\Pi_m \setminus \Pi_{m+1}) \lessdot \alpha_{m+1}$ *and* $B_m(\phi,\omega)\colon \mathbb{T}^n_{s_m-4\sigma_m} \times \Pi_m \to \mathfrak{B}^{p,p} \cap \mathfrak{B}^{p-1,p-1}$ *which is real analytic with respect to* $\phi \in \mathbb{T}^n_{s_m-4\sigma_m}$, *Lipschitz-continuous in* $\omega \in \Pi_m$ *and satisfies*

$$|||B_m|||^{\mathcal{L}}_{p,p,s_m-4\sigma_m}, \quad |||B_m|||^{\mathcal{L}}_{p-1,p-1,s_m-4\sigma_m} \lessdot \epsilon_{m-1}^{\frac{5}{6}(\frac{2}{3}-\rho)} \tag{3.8}$$

*such that for any* $\omega \in \Pi_m$, *by the transformation* $\chi = \mathrm{e}^{B_m(\omega t)}\varphi$, (3.2) *can be changed into*

$$\mathbf{i}\dot{\varphi} = (\Lambda_{m+1} + P_{m+1}(\omega t))\varphi, \quad \varphi = (y,\widetilde{y})^{\mathrm{T}} \in \mathcal{P}^p,\ \omega \in \Pi_{m+1}. \tag{3.9}$$

*Moreover,* $\Lambda_{m+1}$, $P_{m+1}$ *fulfill* (H1)–(H3) *with* $m$ *replaced by* $m+1$.

The above result is similar to the iteration Lemma 3.2 in [17], and the key part of the proof is to find a suitable estimate for the solutions of the homological equations with large variable coefficients. Hence the process is parallel except the following two points: (1) The width of the angle variable $s_m$ relies on $\epsilon_m$ , hence the system (3.2) in our paper has weaker regularity, (2) $\|P_m\|^{X_1,\mathcal{L}}_{s_m}$ is controlled by $\epsilon_{m-1}^{\frac{2}{3}-\rho}$ instead of $\epsilon_m$ in [17]. Consequently, we concentrate our attention on these two aspects in the following.

**Proof** Now we include our system into a more general framework. Abbreviate the notations $\Lambda_m$, $A_m$, $P_m$, $\lambda_{i,m}$, $\mu_{i,m}$, $B_m$, and $\Gamma_{K_m}$ by $\Lambda$, $A$, $P$, $\lambda_i$, $\mu_i$, $B$ and $\Gamma_K$, and $\Lambda_{m+1}$, $A_{m+1}$, $P_{m+1}$, $\lambda_{i,m+1}$ and $\mu_{i,m+1}$ by $\Lambda^+$, $A^+$, $P^+$, $\lambda_i^+$ and $\mu_i^+$ respectively. Following the procedure in the proof of Lemma 3.2 in [17], we set $\chi = \mathrm{e}^{B(\phi)}\varphi$, and pluging into (3.2) yields (3.9), where

$$\Lambda_+ = \Lambda + \mathrm{diag}(P), \tag{3.10}$$

$$\begin{aligned} P_+ = {} & \{[\Lambda, B] - \mathbf{i}\dot{B} + P - \mathrm{diag}(P) - R\} + R + (\mathrm{e}^{-B}\Lambda\mathrm{e}^{B} - \Lambda - [\Lambda, B]) \\ & + (\mathrm{e}^{-B}P\mathrm{e}^{B} - P) - \mathbf{i}\Big(\mathrm{e}^{-B}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{B} - \dot{B}\Big). \end{aligned} \tag{3.11}$$

Thus we need to solve the homological equation for the unknown $B$:

$$[\Lambda, B] - \mathbf{i}\dot{B} + (P - \mathrm{diag}P) - R = 0, \tag{3.12}$$

where $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, $\mathrm{diag}P = \begin{pmatrix} \mathrm{diag}(P_{1,ii}) & 0 \\ 0 & \mathrm{diag}(P_{4,ii}) \end{pmatrix}$, $R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$. To this end, $B_i,\ 1 \le i \le 4$ should satisfy the homological equations:

$$-\mathbf{i}\partial_\omega B_{1,ij} + (\lambda_i - \lambda_j)B_{1,ij} + (\mu_i - \mu_j)B_{1,ij} + P_{1,ij} = R_{1,ij}, \quad i,j \ge 1,\ i \ne j, \tag{3.13}$$

$$-\mathbf{i}\partial_\omega B_{2,ij} + (\lambda_i + \lambda_j)B_{2,ij} + (\mu_i + \mu_j)B_{2,ij} + P_{2,ij} = R_{2,ij}, \quad i,j \ge 1, \tag{3.14}$$

$$-\mathbf{i}\partial_\omega B_{3,ij} - (\lambda_i + \lambda_j)B_{3,ij} - (\mu_i + \mu_j)B_{3,ij} + P_{3,ij} = R_{3,ij}, \quad i,j \ge 1, \tag{3.15}$$

$$-\mathbf{i}\partial_\omega B_{4,ij} + (\lambda_j - \lambda_i)B_{4,ij} + (\mu_j - \mu_i)B_{4,ij} + P_{4,ij} = R_{4,ij}, \quad i,j \ge 1,\ i \ne j. \tag{3.16}$$

We only give the details of solving (3.13) in the following, and (3.14)–(3.16) can be handled in the same way. In view of the proof of Lemma 3.2 in [17], we can obtain the estimates of the elements of $B_1$. Precisely speaking, we have the following statements:

(1) $B_{1,ii} = 0$.

(2) For $(i,j)$ with $0 < |\lambda_i - \lambda_j| < 2K_m$,

$$\|B_{1,ij}\|^{\mathcal{L}}_{s_m - 4\sigma_m} \lessdot \frac{1}{\alpha_m^2 \sigma_m^{2n+2\tau+1}\gamma_{ij}} \mathrm{e}^{\frac{8C_{\mu,m}\gamma_{ij}(s_m - \sigma_m)}{\alpha_0}} \|P_{1,ij}\|^{\mathcal{L}}_{s_m - \sigma_m}. \tag{3.17}$$

(3) For $(i,j)$ with $|\lambda_i - \lambda_j| \ge 2K_m$,

$$\|B_{1,ij}\|^{\mathcal{L}}_{s_m - 4\sigma_m} \lessdot \frac{1}{\sigma_m^{2n+1}} \frac{1}{\gamma_{ij}} \|P_{1,ij}\|^{\mathcal{L}}_{s_m - \sigma_m}. \tag{3.18}$$

By the assumptions $C_{\mu,m} \le 2C_1\varepsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}$, $\gamma_{ij} = |i^2 - j^2| \le 2\lambda_{ij} \le 4K_m$, $s_m - \sigma_m = 15\sigma_m = 17\frac{|\ln \epsilon_{m-1}^{\frac{2}{3}-\rho}|}{K_m}$, we can obtain

$$\frac{8C_{\mu,m}\gamma_{ij}(s_m - \sigma_m)}{\alpha_0} \le \frac{1088C_1\varepsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}|\ln \epsilon_{m-1}^{\frac{2}{3}-\rho}|}{\alpha_0}. \tag{3.19}$$

It follows from (3.1) that $\frac{1088C_1\varepsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}}{\alpha_0} \le \frac{1}{10}$. Consequently, we have

$$\|B_{1,ij}\|^{\mathcal{L}}_{s_m - 4\sigma_m} \lessdot \frac{\epsilon_{m-1}^{-\frac{1}{10}(\frac{2}{3}-\rho)}}{\alpha_m^2 \sigma_m^{2n+2\tau+1}} \frac{\|P_{1,ij}\|^{\mathcal{L}}_{s_m - \sigma_m}}{\gamma_{ij}}. \tag{3.20}$$

Taking (3.18) and (3.20) into account and using Lemma 5.2 in the Appendix, we get the estimate of $B_1 = (B_{1,ij})_{i,j \ge 1}$:

$$\|B_1\|^{\mathcal{L}}_{p,p,s_m - 4\sigma_m}, \|B_1\|^{\mathcal{L}}_{p-1,p-1,s_m - 4\sigma_m} \lessdot \frac{\epsilon_{m-1}^{-\frac{1}{10}(\frac{2}{3}-\rho)}}{\alpha_m^2 \sigma_m^{3n+2\tau+1}} \|P_1\|^{\mathcal{L}}_{p,p-1,s_m} \lessdot \frac{\epsilon_{m-1}^{\frac{9}{10}(\frac{2}{3}-\rho)}}{\alpha_m^2 \sigma_m^{3n+2\tau+1}}. \tag{3.21}$$

In view of (3.1), we can set $\epsilon_0^{\frac{1}{200}} \le \alpha_0$. Then by the definition of $\alpha_m$ and $\epsilon_m$, we get

$$\alpha_m \ge \epsilon_m^{\frac{1}{200}} \quad \text{for any } m \ge 0. \tag{3.22}$$

Taking the definition of $\sigma_m$ into account, together with (3.21) and (3.22), we have the following estimate:

$$\|B_1\|^{\mathcal{L}}_{p,p,s_m - 4\sigma_m}, \|B_1\|^{\mathcal{L}}_{p-1,p-1,s_m - 4\sigma_m}$$

$$\lessdot \frac{\epsilon_{m-1}^{\frac{9}{10}(\frac{2}{3}-\rho)}}{\alpha_m^2\sigma_m^{3n+2\tau+1}} \lessdot \frac{\epsilon_{m-1}^{\frac{9}{10}(\frac{2}{3}-\rho)}}{\epsilon_m^{\frac{1}{100}}\epsilon_m^{\frac{(1+\rho)(3n+2\tau+1)}{\ell-3}}} \lessdot \epsilon_{m-1}^{\frac{9}{10}(\frac{2}{3}-\rho)-\frac{1}{100}(1+\overline{\rho})-\frac{3n+2\tau+1}{\ell-3}(1+\rho)(1+\overline{\rho})}. \tag{3.23}$$

From the assumption

$$\ell > 3+100(1+\rho)(3n+2\tau+1), \quad 0<\overline{\rho}<\rho<\frac{1}{4}, \tag{3.24}$$

we can obtain $\frac{9}{10}\big(\frac{2}{3}-\rho\big)-\frac{1}{100}(1+\overline{\rho})-\frac{3n+2\tau+1}{\ell-3}(1+\rho)(1+\overline{\rho}) > \frac{9}{10}\big(\frac{2}{3}-\rho\big)-\frac{1}{50}(1+\rho) > \frac{5}{6}\big(\frac{2}{3}-\rho\big)$. Consequently, we have

$$\|B_1\|^{\mathcal{L}}_{p,p,s_m-4\sigma_m}, \ \|B_1\|^{\mathcal{L}}_{p-1,p-1,s_m-4\sigma_m} \lessdot \epsilon_{m-1}^{\frac{5}{6}(\frac{2}{3}-\rho)}. \tag{3.25}$$

For the other terms of $B$, i.e., $B_2, B_3, B_4$, the same results can be obtained. Thus, we finally get the estimate for $B$:

$$|||B|||^{\mathcal{L}}_{p,p,s_m-4\sigma_m}, \ |||B|||^{\mathcal{L}}_{p-1,p-1,s_m-4\sigma_m} \lessdot \epsilon_{m-1}^{\frac{5}{6}(\frac{2}{3}-\rho)}. \tag{3.26}$$

The remaining estimates for $\lambda^+$, $\mu^+$ and the new perturbed term $P^+$ can be handled in the classical way, and we do not give the proof here. For the detail we can refer to [30], and we just need to verify (3.5) for $P^+$.

From (3.13)–(3.16), it is easy to see that for real $\phi$, i.e., $\phi\in\mathbb{T}^n$,

$$\overline{B}_1(\phi,\omega) = -B_1^{\mathrm{T}}(\phi,\omega) = B_4(\phi,\omega), \quad B_2(\phi,\omega)=B_2^{\mathrm{T}}(\phi,\omega), \quad \overline{B}_2(\phi,\omega)=B_3(\phi,\omega). \tag{3.27}$$

Then by a direct calculation, we can obtain that $R$ satisfies (3.5). Furthermore, from Lemma 5.3 in Appendix, together with the fact that $\Lambda$ satisfies (3.5) and

$$\begin{aligned}
\mathrm{e}^{-B}P\mathrm{e}^{B} &= P+[P,B]+\frac{1}{2!}[[P,B],B]+\cdots,\\
\mathrm{e}^{-B}\Lambda\mathrm{e}^{B} &= \Lambda+[\Lambda,B]+\frac{1}{2!}[[\Lambda,B],B]+\cdots,\\
\mathrm{e}^{-B}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{B}) &= \dot{B}+\frac{1}{2!}[\dot{B},B]+\frac{1}{3!}[[\dot{B},B],B]+\cdots,
\end{aligned}$$

we can conclude that $P^+$ satisfies (3.5). Till now we complete the proof of this lemma.

**Remark 3.1** In the following, for a block matrix $P(\phi)=\begin{pmatrix} P_1 & P_2\\ P_3 & P_4\end{pmatrix}$, if the components $P_i$, $1\le i\le 4$ satisfy (3.5) for real $\phi$, we call $P(\phi)$ satisfies (3.5). Similarly, for a given operator $B(\phi)=\begin{pmatrix} B_1 & B_2\\ B_3 & B_4\end{pmatrix}$, if (3.27) can be fulfilled for any real $\phi$, we call $B(\phi)$ satisfies (3.27).

We can obtain a more general reduction lemma below by applying Lemma 3.1.

**Lemma 3.2** *Suppose that at the m-th iteration step, we have a system as follows:*

$$\mathbf{i}\dot{\eta} = (\Lambda+P^m(\omega t))\eta, \quad \eta=(v,\widetilde{v})\in\mathcal{P}^p, \ \omega\in\Pi_m, \ m\ge 1 \tag{3.28}$$

*with* $\Lambda$ *in* (2.12), *and* $\Pi_m$ *described by* (H3) *in Lemma* 3.1. *Moreover,* $P^m(\phi,\omega):\ \mathbb{T}^n\times\Pi_m\to X_1$ *has the form* $P^m(\phi,\omega)=\sum_{1\le i\le m}P_i(\phi,\omega)$. *In addition,* $P_i(\phi,\omega)$ *can be written as*

$$P_i(\phi,\omega)=P_i^{(i)}(\phi,\omega)+\sum_{\nu\ge i}(P_i^{(\nu+1)}(\phi,\omega)-P_i^{(\nu)}(\phi,\omega)), \quad 1\le i\le m \tag{3.29}$$

*and for* $\nu \geq i$, $P_i^{(\nu)}(\phi,\omega): \mathbb{T}^n_{s_\nu} \times \Pi_m \to X_1$ *is real analytic in* $\phi \in \mathbb{T}^n_{s_\nu}$, *Lipschitz-continuous in* $\omega$ *and satisfies* (3.5). *Moreover, we have the following estimates:*

$$\|P_i^{(i)}\|_{s_i}^{X_1,\mathcal{L}} \lessdot \epsilon_{i-1}^{\frac{2}{3}-\rho}, \quad \|P_i^{(\nu+1)} - P_i^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq i. \tag{3.30}$$

*Then there exists* $U^m(\phi,\omega)$ *defined on* $\mathbb{T}_{s_m-4\sigma_m} \times \Pi_m$ *with* $U^m(\phi,\omega) = \mathrm{e}^{B_1(\phi,\omega)} \cdots \mathrm{e}^{B_m(\phi,\omega)}$, *where* $B_i(\phi,\omega)$ *satisfies* (3.27) *and the following estimates:*

$$|||B_i(\phi,\omega)|||_{p,p,s_i-4\sigma_i}^{\mathcal{L}}, \ |||B_i(\phi,\omega)|||_{p-1,p-1,s_i-4\sigma_i}^{\mathcal{L}} \lessdot \epsilon_{i-1}^{\frac{5}{6}(\frac{2}{3}-\rho)}, \quad 1 \leq i \leq m \tag{3.31}$$

*such that by the transformation* $\eta = U^m(\omega t,\omega)\varphi$, (3.28) *can be changed into*

$$\mathbf{i}\dot{\varphi} = (\Lambda_{m+1} + Q_{m+1}(\omega t))\varphi, \quad \varphi = (y,\widetilde{y}) \in \mathcal{P}^p, \ \omega \in \Pi_{m+1}, \tag{3.32}$$

*where* $\Lambda_{m+1}$ *fulfills* (H1) *in Lemma* 3.1 *with* $m$ *replaced by* $m+1$, *and* $Q_{m+1}(\phi,\omega): \mathbb{T}^n \times \Pi_{m+1} \to X_1$ *can be written as*

$$Q_{m+1}(\phi,\omega) = \widetilde{Q}_{m+1}^{(m+1)} + \sum_{\nu \geq m+1} (Q_{m+1}^{(\nu+1)}(\phi,\omega) - Q_{m+1}^{(\nu)}(\phi,\omega)) \tag{3.33}$$

*with* $\widetilde{Q}_{m+1}^{(m+1)}(\phi,\omega)$, $Q_{m+1}^{(\nu)}(\phi,\omega)$ *being real analytic in* $\phi \in \mathbb{T}^n_{s_\nu}$, *Lipschitz-continuous in* $\omega \in \Pi_{m+1}$ *and satisfying* (3.5) *as well as the estimates*:

$$\|\widetilde{Q}_{m+1}^{(m+1)}\|_{s_{m+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_m^{\frac{2}{3}-\rho}, \quad \|Q_{m+1}^{(\nu+1)} - Q_{m+1}^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m+1. \tag{3.34}$$

**Proof** We proof this lemma by induction. When $m=1$, we have

$$P^1 = P_1 = P_1^{(1)} + \sum_{\nu \geq 1} (P_1^{(\nu+1)} - P_1^{(\nu)}) \tag{3.35}$$

with

$$\|P_1^{(1)}\|_{s_1}^{X_1,\mathcal{L}} \lessdot \epsilon_0^{\frac{2}{3}-\rho}, \quad \|P_1^{(\nu+1)} - P_1^{(\nu)}\|_{s_\nu}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}. \tag{3.36}$$

Consider the system

$$\mathbf{i}\dot{\eta} = (\Lambda + P_1^{(1)}(\omega t))\eta, \quad \eta \in \mathcal{P}^p, \ \omega \in \Pi_1, \tag{3.37}$$

and it is easy to check the hypotheses (H1)–(H3) in Lemma 3.1 are satisfied. Applying Lemma 3.1 to (3.37), we can find a set $\Pi_2 \subset \Pi_1$ and a linear transformation $\eta = \mathrm{e}^{B_1}\varphi$ such that (3.37) is changed into

$$\mathbf{i}\dot{\varphi} = (\Lambda_2 + \widetilde{P}_2(\omega t))\varphi, \quad \varphi \in \mathcal{P}^p, \ \omega \in \Pi_2,$$

where $\Lambda_2$ and $\widetilde{P}_2$ satisfy (H1)–(H3) in Lemma 3.1 with $m=2$. Moreover, the operator $B_1(\phi,\omega)$ : $\mathbb{T}^n_{s_1-4\sigma_1} \times \Pi_1 \to \mathfrak{B}^{p,p} \cap \mathfrak{B}^{p-1,p-1}$ satisfies the estimate:

$$|||B_1|||_{p,p,s_1-4\sigma_1}^{\mathcal{L}}, \ |||B_1|||_{p-1,p-1,s_1-4\sigma_1}^{\mathcal{L}} \lessdot \epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}. \tag{3.38}$$

Hence by the same transformation, in view of the expansion (3.35), the original equation (3.28) for $m=1$ can be changed into

$$\mathbf{i}\dot{\varphi} = (\Lambda_2 + \widetilde{P}_2)\varphi + \sum_{\nu\geq 1} \mathrm{e}^{-B_1}(P_1^{(\nu+1)} - P_1^{(\nu)})\mathrm{e}^{B_1}\varphi := (\Lambda_2 + \widetilde{Q}_2^{(2)})\varphi + \sum_{\nu\geq 2}(Q_2^{(\nu+1)} - Q_2^{(\nu)})\varphi,$$

where

$$\widetilde{Q}_2^{(2)} = \widetilde{P}_2 + \mathrm{e}^{-B_1}(P_1^{(2)} - P_1^{(1)})\mathrm{e}^{B_1}, \quad Q_2^{(\nu)} = \mathrm{e}^{-B_1}P_1^{(\nu)}\mathrm{e}^{B_1}, \quad \nu \geq 2.$$

Let $Q_2 = \widetilde{Q}_2^{(2)} + \sum_{\nu\geq 2}(Q_2^{(\nu+1)} - Q_2^{(\nu)})$. Thus from (3.36) and (3.38), we obtain

$$\|\widetilde{Q}_2^{(2)}\|_{s_2}^{X_1,\mathcal{L}} \lessdot \epsilon_1^{\frac{2}{3}-\rho}, \quad \|Q_2^{(\nu+1)} - Q_2^{(\nu)}\|_{s_2}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq 2, \tag{3.39}$$

which means that the lemma is true for $m=1$.

Now suppose that the lemma is true for $m-1$. At the $m$-th step, rewrite the system (3.28) as

$$\mathbf{i}\dot{\eta} = (\Lambda + P^{m-1}(\omega t) + P_m(\omega t))\eta, \quad \omega \in \Pi_m.$$

By induction, we can find a coordinate transformation $\eta = U^{m-1}\chi$ with $U^{m-1} = \mathrm{e}^{B_1}\cdots\mathrm{e}^{B_{m-1}}$ and $\Pi_m \subset \Pi_{m-1}$ changing the equation

$$\mathbf{i}\dot{\eta} = (\Lambda + P^{m-1}(\omega t))\eta, \quad \omega \in \Pi_{m-1} \tag{3.40}$$

into

$$\mathbf{i}\dot{\chi} = (\Lambda_m + Q_m(\omega t))\chi, \quad \chi \in \mathcal{P}^p,\ \omega \in \Pi_m, \tag{3.41}$$

where

$$Q_m = \widetilde{Q}_m^{(m)} + \sum_{\nu\geq m}(Q_m^{(\nu+1)} - Q_m^{(\nu)}) \tag{3.42}$$

and

$$\|\widetilde{Q}_m^{(m)}\|_{s_m-6\sigma_m}^{X_1,\mathcal{L}} \lessdot \epsilon_{m-1}^{\frac{2}{3}-\rho}, \quad \|Q_m^{(\nu+1)} - Q_m^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m. \tag{3.43}$$

Differentiating the transformation $\eta = U^{m-1}\chi$ with respect to $t$, we have $\dot{\eta} = \dot{U}^{m-1}\chi + U^{m-1}\dot{\chi}$. Inserting it into (3.28) and in view of (3.29), we can obtain the new system:

$$\begin{aligned}
\mathbf{i}\dot{\chi} &= (\Lambda_m + Q_m + (U^{m-1})^{-1}P_m U^{m-1})\chi \\
&= \Lambda_m\chi + (\widetilde{Q}_m^{(m)} + (U^{m-1})^{-1}P_m^{(m)}U^{m-1})\chi \\
&\quad + \sum_{\nu\geq m}(Q_m^{(\nu+1)} - Q_m^{(\nu)} + (U^{m-1})^{-1}(P_m^{(\nu+1)} - P_m^{\nu})U^{m-1})\chi \\
&:= (\Lambda_m + \widehat{P}_m)\chi + \sum_{\nu\geq m}(\widehat{P}_m^{(\nu+1)} - \widehat{P}_m^{(\nu)})\chi,
\end{aligned} \tag{3.44}$$

where

$$\|\widehat{P}_m\|_{s_m}^{X_1,\mathcal{L}} \lessdot \epsilon_{m-1}^{\frac{2}{3}-\rho}, \quad \|\widehat{P}_m^{(\nu+1)} - \widehat{P}_m^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m. \tag{3.45}$$

Consequently based on Lemma 3.1, there exist $\Pi_{m+1} \subset \Pi_m$ with $\mathrm{Meas}(\Pi_m \setminus \Pi_{m+1}) \lessdot \alpha_{m+1}$, and $B_m(\phi,\omega)\colon \mathbb{T}^n_{s_m-4\sigma_m} \times \Pi_m \to \mathfrak{B}^{p,p} \cap \mathfrak{B}^{p-1,p-1}$ satisfying

$$|||B_m|||^{\mathcal{L}}_{p,p,s_m-4\sigma_m},\ |||B_m|||^{\mathcal{L}}_{p-1,p-1,s_m-4\sigma_m} \lessdot \epsilon_{m-1}^{\frac{5}{6}(\frac{2}{3}-\rho)}. \tag{3.46}$$

By the transformation $\chi = \mathrm{e}^{B_m(\omega t)}\varphi$, we can change (3.44) into

$$\mathbf{i}\dot{\varphi} = (\Lambda_{m+1} + \widetilde{P}_{m+1})\varphi + \sum_{\nu \geq m} \mathrm{e}^{-B_m}(\widehat{P}_m^{(\nu+1)} - \widehat{P}_m^{(\nu)})\mathrm{e}^{B_m}\varphi \tag{3.47}$$

with

$$\|\widetilde{P}_{m+1}\|^{X_1,\mathcal{L}}_{s_m-5\sigma_m} \lessdot \epsilon_m^{\frac{2}{3}-\rho}, \quad \|\mathrm{e}^{-B_m}(\widehat{P}_m^{(\nu+1)} - \widehat{P}_m^{(\nu)})\mathrm{e}^{B_m}\|^{X_1,\mathcal{L}}_{s_{\nu+1}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m. \tag{3.48}$$

Let $\widetilde{Q}^{(m+1)}_{m+1} = \widetilde{P}_{m+1} + \mathrm{e}^{-B_m}(\widehat{P}_m^{(m+1)} - \widehat{P}_m^{(m)})\mathrm{e}^{B_m}$, $Q^{(\nu)}_{m+1} = \mathrm{e}^{-B_m}\widehat{P}_m^{(\nu)}\mathrm{e}^{B_m}$, $\nu \geq m+1$. Then from the assumption $0 < \rho < \frac{1}{4}$ we obtain

$$\|\widetilde{Q}^{(m+1)}_{m+1}\|^{X_1,\mathcal{L}}_{s_{m+1}} \lessdot \epsilon_m^{\frac{2}{3}-\rho}, \quad \|Q^{(\nu+1)}_{m+1} - Q^{(\nu)}_{m+1}\|^{X_1,\mathcal{L}}_{s_{\nu+1}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m+1. \tag{3.49}$$

Set $U^m := U^{m-1} \cdot \mathrm{e}^{B_m}$, then the lemma is true for $m$, and we finish the proof.

## 4 Iteration Process and Proof of Theorem 2.1

**Lemma 4.1** *Let us consider the system*

$$\mathbf{i}\dot{\varphi} - \Lambda_{m+1}\varphi + \widehat{F}_m = 0, \quad \varphi = (y, \widetilde{y}) \in \mathcal{P}^p \tag{4.1}$$

*defined on $\mathbb{T}^n_{s_m-5\sigma_m} \times \Pi_{m+1}$, where $\Lambda_{m+1}$ and $\Pi_{m+1}$ fulfill the hypotheses* (H1) *and* (H3) *in Lemma* 3.1 *with $m$ replaced by $m+1$, and the vector field $\widehat{F}_m(\phi,\omega) = (\widehat{f}_m(\phi,\omega), \widehat{g}_m(\phi,\omega))^{\mathrm{T}}$ : $\mathbb{T}^n_{s_m-5\sigma_m} \times \Pi_{m+1} \to \mathcal{P}^{p-1}$ is real analytic in $\phi \in \mathbb{T}^n_{s_m-5\sigma_m}$, Lipschitz-continuous in $\Pi_{m+1}$ and satisfies the estimate*

$$\|\widehat{F}_m\|^{X_0,\mathcal{L}}_{s_m-5\sigma_m} \lessdot \epsilon_m. \tag{4.2}$$

*Furthermore, we assume $\widehat{g}(\phi,\omega) = -\overline{\widehat{f}}(\phi,\omega)$ for any $\phi \in \mathbb{T}^n$ and $\omega \in \Pi_{m+1}$. Then there exists a quasi-periodic solution $\varphi_{m+1}(\phi,\omega) = (y_{m+1}(\phi,\omega), \widetilde{y}_{m+1}(\phi,\omega))^{\mathrm{T}}$ which is real analytic in $\phi \in \mathbb{T}^n_{s_m-9\sigma_m}$ and Lipschitz-continuous in $\omega \in \Pi_{m+1}$ such that*

$$\mathbf{i}\dot{\varphi}_{m+1} - \Lambda_{m+1}\varphi_{m+1} + \widehat{F}_m = r_m \tag{4.3}$$

*with $r_m = (r_m^1, r_m^2)$ and*

$$\|\varphi_{m+1}\|^{\mathcal{L}}_{p,s_m-9\sigma_m} \lessdot \epsilon_m^{\frac{5}{6}}, \quad \|r_m\|^{X_0,\mathcal{L}}_{s_m-9\sigma_m} \lessdot \epsilon_m^{\frac{3}{2}}. \tag{4.4}$$

*In addition, for any $\phi \in \mathbb{T}^n$, we have*

$$\widetilde{y}_{m+1} = \overline{y}_{m+1}, \quad r_m^2 = -\overline{r}_m^1. \tag{4.5}$$

**Proof** Abbreviate the notations $\Lambda_{m+1}$, $A_{m+1}$, $\widehat{F}_m$, $\widehat{f}_m$, $\widehat{g}_m$, $r_m$, $\lambda_{i,m+1}$, $\mu_{i,m+1}$, $\varphi_{m+1}$, $y_{m+1}$, $\widetilde{y}_{m+1}$ and $\Gamma_{L_m}$ by $\Lambda$, $A$, $\widehat{F}$, $\widehat{f}$, $\widehat{g}$, $r$, $\lambda_i$, $\mu_i$, $\varphi$, $y$, $\widetilde{y}$ and $\Gamma_L$, respectively. Then (4.3) can be written as

$$\begin{cases}\mathbf{i}\dot{y} - Ay + \widehat{f} = r^1,\\ \mathbf{i}\dot{\widetilde{y}} + A\widetilde{y} + \widehat{g} = r^2.\end{cases} \tag{4.6}$$

In the following we just find the solution to the first equation, and the second one can be solved similarly. Set $r = (r^1, r^2)$ and let $r^1$ be an infinite vector with elements $r_j^1$:

$$r_j^1 = \begin{cases}0, & |\lambda_j| < 2L_m,\\ (1-\Gamma_L)(\widehat{f}_j - \mu_j y_j), & |\lambda_j| \geq 2L_m,\end{cases} \tag{4.7}$$

where $\Gamma_L$ is the truncation operator $\Gamma_L f = \sum\limits_{|k|\leq L} \widehat{f}_k \mathrm{e}^{\mathbf{i}k\cdot\phi}$. Then $y_j$ satisfies equations as follows:

(1) For $j$ with $0 < |\lambda_j| < 2L_m$,

$$-\mathbf{i}\partial_\omega y_j + \lambda_j y_j + \mu_j(\phi) y_j = \widehat{f}_j. \tag{4.8}$$

(2) For $j$ with $|\lambda_j| \geq 2L_m$,

$$-\mathbf{i}\partial_\omega y_j + \lambda_j y_j + \Gamma_L(\mu_j(\phi) y_j) = \Gamma_L \widehat{f}_j, \quad \Gamma_L y_j = y_j. \tag{4.9}$$

Now we solve the homological equations (4.8)–(4.9) with large variable coefficient. First let us consider (4.8). By (3.3) and (3.6), we have $|\mu_j|_{s_m-5\sigma_m,\tau+1} \leq C_{\mu,m+1} j \leq C_{\mu,m+1} j^2$. Then applying Theorem 1.4 in [17] to (4.8), we have

$$\|y_j\|_{s_m-7\sigma_m} \lessdot \frac{1}{\alpha_{m+1} j^2 \sigma_m^{n+\tau}} \mathrm{e}^{\frac{2C_{\mu,m+1} j^2 (s_m-6\sigma_m)}{\alpha_0}} \|\widehat{f}_j\|_{s_m-6\sigma_m}. \tag{4.10}$$

In view of $C_{\mu,m+1} \leq 2C_1 \epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}$, $j^2 \leq 2\lambda_{j,m+1} \leq 4L_m$, and $s_m - 6\sigma_m = 12\sigma_m = 12\frac{|\ln \epsilon_m|}{L_m}$, we have

$$\frac{2C_{\mu,m+1} j^2 (s_m - 6\sigma_m)}{\alpha_0} \leq \frac{192 C_1 \epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)} |\ln \epsilon_m|}{\alpha_0}. \tag{4.11}$$

Thus by (3.1) we can obtain

$$\frac{192 C_1 \epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}}{\alpha_0} \leq \frac{1}{20}. \tag{4.12}$$

Hence from (4.10)–(4.12), we conclude that

$$\|y_j\|_{s_m-7\sigma_m} \lessdot \frac{\epsilon_m^{-\frac{1}{20}}}{\alpha_{m+1}\sigma_m^{n+\tau}} \frac{\|\widehat{f}_j\|_{s_m-6\sigma_m}}{j^2}. \tag{4.13}$$

Next we consider (4.9). By (3.3), we have

$$\sum_{k\in\mathbb{Z}^n} |(\widehat{\mu}_j)_k| \mathrm{e}^{|k|(s_m-6\sigma_m)} \leq |\mu_j|_{s_m-6\sigma_m,\tau+1} \leq C_{\mu,m+1} j. \tag{4.14}$$

In view of $C_{\mu,m+1}\leq 2C_1\epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}$, $j\leq\frac{2|\lambda_j|}{j}$, we have $C_{\mu,m+1}j\leq 4C_1\epsilon_0^{\frac{5}{6}(\frac{2}{3}-\rho)}\frac{|\lambda_j|}{j}\leq\frac{|\lambda_j|}{4j}$. Applying Lemma 2.6 in [17] to (4.9), we have

$$\|y_j\|_{s_m-7\sigma_m}\lessdot\frac{1}{|\lambda_j|\sigma_m^n}\|\widehat{f}_j\|_{s_m-6\sigma_m}, \tag{4.15}$$

$$\|(1-\Gamma_L)\mu_j y_j\|_{s_m-7\sigma_m}\lessdot\frac{1}{j\sigma_m^n}\mathrm{e}^{-\frac{9L_m\sigma_m}{10}}\|\widehat{f}_j\|_{s_m-6\sigma_m}. \tag{4.16}$$

Taking (4.13) and (4.15) into account, using Lemma 5.2 in Appendix, we can get the estimate of the $\ell^p$-norm of $y=(y_j)_{j\geq 1}$:

$$\|y\|_{p,s_m-7\sigma_m}\lessdot\frac{\epsilon_m^{-\frac{1}{20}}}{\alpha_{m+1}\sigma_m^{2n+\tau}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m}. \tag{4.17}$$

For the estimate of the Lipschitz norm, we proceed as follows. Given a function $B$ of $\omega$, set $\triangle B=B(\omega)-B(\omega')$. Applying $\triangle$ to (4.8)–(4.9), we obtain the following assertions:

(1) For $0<|\lambda_j|<2L_m$,

$$\begin{aligned}&-\mathbf{i}\partial_\omega(\triangle y_j)+\lambda_j(\omega)(\triangle y_j)+\mu_j(\phi,\omega)(\triangle y_j)\\&=\mathbf{i}\partial_{\triangle\omega}y_j(\omega')-(\triangle\lambda_j+\triangle\mu_j)y_j(\omega')+\triangle\widehat{f}_j.\end{aligned} \tag{4.18}$$

(2) For $|\lambda_j|\geq 2L_m$,

$$\begin{aligned}&-\mathbf{i}\partial_\omega(\triangle y_j)+\lambda_j(\omega)(\triangle y_j)+\Gamma_L(\mu_j(\phi,\omega)(\triangle y_j))\\&=\mathbf{i}\partial_{\triangle\omega}y_j(\omega')-\Gamma_L((\triangle\lambda_j+\triangle\mu_j)y_j(\omega')-\triangle\widehat{f}_j).\end{aligned} \tag{4.19}$$

Applying Theorem 1.4 in [17] to (4.18), for $0<|\lambda_j|<2L_m$, we obtain

$$\|\triangle y_j\|_{s_m-9\sigma_m}\lessdot\frac{1}{\alpha_{m+1}^2\sigma_m^{2n+2\tau+1}}\frac{\epsilon_m^{-\frac{1}{10}}}{j^2}(\|\widehat{f}_j\|_{s_m-6\sigma_m}|\triangle\omega|+\|\triangle\widehat{f}_j\|_{s_m-6\sigma_m}). \tag{4.20}$$

Then applying Lemma 2.6 in [17] to (4.19), for $|\lambda_j|\geq 2L_m$, we have

$$\|\triangle y_j\|_{s_m-9\sigma_m}\lessdot\frac{1}{\sigma_m^{2n+1}j^2}(\|\widehat{f}_j\|_{s_m-6\sigma_m}|\triangle\omega|+\|\triangle\widehat{f}_j\|_{s_m-6\sigma_m}), \tag{4.21}$$

$$\|(1-\Gamma_L)(\mu_j\triangle y_j)\|_{s_m-9\sigma_m}\lessdot\frac{\epsilon_m^{\frac{9}{10}}}{j\sigma_m^{n+1}}(\|\widehat{f}_j\|_{s_m-6\sigma_m}|\triangle\omega|+\|\triangle\widehat{f}_j\|_{s_m-6\sigma_m}). \tag{4.22}$$

Then from (4.20)–(4.21), using Lemma 5.2 in Appendix, we get

$$\|\triangle y\|_{p,s_m-9\sigma_m}\lessdot\frac{\epsilon_m^{-\frac{1}{10}}}{\alpha_{m+1}^2\sigma_m^{3n+2\tau+1}}(\|\widehat{f}\|_{p-1,s_m-5\sigma_m}|\triangle\omega|+\|\triangle\widehat{f}\|_{p-1,s_m-5\sigma_m}). \tag{4.23}$$

Divided by $|\triangle\omega|$, together with (4.17), we obtain

$$\|y\|^{\mathcal{L}}_{p,m-9\sigma_m}\lessdot\frac{\epsilon_m^{-\frac{1}{10}}}{\alpha_{m+1}^2\sigma_m^{3n+2\tau+1}}\|\widehat{f}\|^{\mathcal{L}}_{p-1,s_m-5\sigma_m}. \tag{4.24}$$

The estimate for $\widetilde{y}$ is the same as $y$ in (4.24). Hence we have

$$\|\varphi\|^{\mathcal{L}}_{p,s_m-9\sigma_m} \lessdot \frac{\epsilon_m^{-\frac{1}{10}}}{\alpha^2_{m+1}\sigma_m^{3n+2\tau+1}}\|\widehat{F}\|^{X_0,\mathcal{L}}_{s_m-5\sigma_m} \lessdot \frac{\varepsilon_m^{\frac{9}{10}}}{\alpha^2_m\sigma_m^{3n+2\tau+1}}. \tag{4.25}$$

In view of the definition of $\sigma_m$ and (3.22), we have

$$\|\varphi\|^{\mathcal{L}}_{p,s_m-9\sigma_m} \lessdot \epsilon_m^{\frac{9}{10}-\frac{1}{100}-\frac{(1+\rho)(3n+2\tau+1)}{\ell-3}} \lessdot \epsilon_m^{\frac{5}{6}}, \tag{4.26}$$

where the last inequality follows from the assumption on $\ell$ in (2.16).

We handle with the estimate of the remaining part $r^1$ as in [17]. First, we divide $r^1$ into three parts, that is $r^1 = r^1_1 + r^1_2 + r^1_3$, where $r^1_1$ and $r^1_2$ have the vector elements as follows:

$$r^1_{1,j} = \begin{cases} 0, & 0 < |\lambda_j| < 2L_m, \\ (1-\Gamma_L)(-\mu_j y_j), & |\lambda_j| \geq 2L_m, \end{cases} \tag{4.27}$$

$$r^1_{2,j} = \begin{cases} -(1-\Gamma_L)\widetilde{f}_j, & 0 < |\lambda_j| < 2L_m, \\ 0, & |\lambda_j| \geq 2L_m, \end{cases} \tag{4.28}$$

and $r^1_3$ is the truncation of $\widehat{f}$, that is $r^1_3 = (1-\Gamma_L)\widehat{f}$. From (4.16), (4.22) and

$$\begin{aligned} \|(1-\Gamma_L)\triangle\mu_j y_j\|_{s_m-9\sigma_m} &\lessdot \frac{\mathrm{e}^{-L_m\sigma_m}}{\sigma_m^n}\|\triangle\mu_j\|_{s_m-7\sigma_m}\|y_j\|_{s_m-7\sigma_m} \\ &\lessdot \frac{\epsilon_m}{\sigma_m^{2n}}\frac{\|\widehat{f}_j\|_{s_m-6\sigma_m}}{j}|\triangle\omega|, \end{aligned} \tag{4.29}$$

we get

$$\|r^1_1\|^{\mathcal{L}}_{p-1,s_m-9\sigma_m} \lessdot \frac{\epsilon_m^{\frac{9}{10}}}{\sigma_m^{3n+1}}\|\widetilde{f}\|^{\mathcal{L}}_{p-1,s_m-5\sigma_m} \lessdot \frac{1}{3}\epsilon_m^{\frac{3}{2}} \tag{4.30}$$

by Lemma 5.2 in Appendix. Since

$$\begin{aligned} \|r^1_{2,j}\|_{s_m-7\sigma_m} &\leq \sum_{|k|>L_m}|\widehat{(\widehat{f}_j)_k}|\mathrm{e}^{|k|(s_m-7\sigma_m)} \leq \|\widehat{f}_j\|_{s_m-5\sigma_m}\sum_{|k|>L_m}\mathrm{e}^{-2|k|\sigma_m} \\ &\lessdot \frac{\mathrm{e}^{-L_m\sigma_m}}{\sigma_m^n}\|\widehat{f}_j\|_{s_m-5\sigma_m} \lessdot \frac{\epsilon_m}{\sigma_m^n}\|\widehat{f}_j\|_{s_m-5\sigma_m}, \end{aligned}$$

applying Lemma 5.2 in Appendix, we obtain

$$\begin{aligned} \|r^1_2\|_{p-1,s_m-7\sigma_m} &\leq \frac{\epsilon_m}{\sigma_m^{2n}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m}\max\{j : |\lambda_j| \leq 2L_m\} \\ &\lessdot \frac{\epsilon_m L_m}{\sigma_m^{2n}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m} \lessdot \frac{\epsilon_m|\ln\epsilon_m|}{\sigma_m^{2n}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m} \\ &\lessdot \frac{\epsilon_m^{\frac{9}{10}}}{\sigma_m^{2n}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m} \lessdot \frac{1}{3}\epsilon_m^{\frac{1}{2}}\|\widehat{f}\|_{p-1,s_m-5\sigma_m}. \end{aligned}$$

Applying again Lemma 5.2 in Appendix to $\triangle r^1_2 = r^1_2(\omega) - r^1_2(\omega')$, we get $\|r^1_2\|^{\mathcal{L}}_{p-1,s_m-9\sigma_m} \lessdot \frac{1}{3}\epsilon_m^{\frac{3}{2}}$. Foe $r^1_3$, we obtain

$$\|r^1_3\|^{\mathcal{L}}_{p-1,s_m-9\sigma_m} \leq \frac{\mathrm{e}^{-L_m\sigma_m}}{\sigma_m^n}\|\widehat{f}\|^{\mathcal{L}}_{p-1,s_m-5\sigma_m} \lessdot \frac{1}{3}\epsilon_m^{\frac{3}{2}}. \tag{4.31}$$

By the same method, we can obtain the estimate with respect to $r^2$, and consequently we have

$$\|r\|_{s_m-9\sigma_m}^{X_0,\mathcal{L}} \lessdot \epsilon_m^{\frac{3}{2}}. \tag{4.32}$$

In view of $\widetilde{g}(\phi,\omega) = -\overline{\widetilde{f}(\phi,\omega)}$, we can get (4.5) by a direct calculation. Now we finish the proof.

**Remark 4.1** (1) For a given vector $F(\phi) = (F_1(\phi), F_2(\phi))^{\mathrm{T}}$, if for any $\phi \in \mathbb{T}^n$, we have

$$F_2(\phi) = -\overline{F_1(\phi)}, \tag{4.33}$$

then we say that the vector $F$ satisfies (4.33).

(2) Given an operator $P(\phi)$ fulfilling (3.5), and $\zeta(\phi) = (y(\phi), \widetilde{y}(\phi))^{\mathrm{T}}$, we assume that $\widetilde{y} = \overline{y}$ for real $\phi$. Then it is easy to conclude that the vector $P(\phi)\zeta(\phi)$ satisfies (4.33).

(3) Given $B(\phi) = \begin{pmatrix} B_1(\phi) & B_2(\phi) \\ B_3(\phi) & B_4(\phi) \end{pmatrix}$ such that for any real $\phi$,

$$\overline{B_1(\phi)} = B_4(\phi), \quad \overline{B_2(\phi)} = B_3(\phi), \tag{4.34}$$

then we call that $B(\phi)$ satisfies (4.34). It is easy to check that the operator $\mathrm{e}^{B(\phi)}$ fulfills (4.34) if $B(\phi)$ satisfies (4.34).

(4) Given an operator $B(\phi)$ satisfying (4.34) and a vector $F(\phi)$ satisfying (4.33), we can obtain that the vector $B(\phi)F(\phi)$ satisfies (4.33).

Now with the above preparation work at hand, we begin to make the Newton iteration process clear. Setting $\mathcal{F}(\zeta) = \mathbf{i}\dot{\zeta} - \Lambda\zeta + F(\zeta,\phi)$, we have the following lemma.

**Lemma 4.2** (Iteration Lemma) *Assume that for $m \geq 1$, at the $m$-th iteration step, we have a solution $\zeta_m(\phi,\omega) = (q_m(\phi,\omega), \widetilde{q}_m(\phi,\omega))^{\mathrm{T}} : \mathbb{T}^n_{s_m} \times \Pi_m \to \mathcal{O}$ fulfilling the following hypotheses*:

(T1) *$\zeta_m(\phi,\omega)$ has an expansion of the form $\zeta_m(\phi,\omega) = \sum\limits_{1\leq i\leq m} \eta_i(\phi,\omega)$, where for any $1 \leq i \leq m$, $\eta_i(\phi,\omega) = (v_i(\phi,\omega), \widetilde{v}_i(\phi,\omega))^{\mathrm{T}}$ is real analytic in $\phi \in \mathbb{T}^n_{s_{i-1}-9\sigma_{i-1}}$, Lipschitz-continuous in $\omega \in \Pi_i$, and satisfies the estimate*

$$\|\eta_i\|_{p,s_{i-1}-9\sigma_{i-1}}^{\mathcal{L}} \lessdot \epsilon_{i-1}^{\frac{5}{6}}. \tag{4.35}$$

*Moreover, $\widetilde{v}_i = \overline{v}_i$ for any real $\phi$.*

(T2) *$\zeta_m$ is an $\epsilon_m$-approximate solution to the system (2.11), that is, $\|\mathcal{F}(\zeta_m)\|_{\mathbb{T}^n\times\Pi_m}^{X_0} \lessdot \epsilon_m$. In addition, we have*

$$\begin{aligned}\mathcal{F}(\zeta_m) = h_m + \check{P}_m\eta_m &+ \sum_{1\leq i\leq m-1}\sum_{\nu\geq m-1}(\check{P}_i^{(\nu+1)} - \check{P}_i^{(\nu)})\eta_i \\ &+ \sum_{0\leq i\leq m-1}\sum_{\nu\geq m-1}(G_i^{(\nu+1)} - G_i^{(\nu)}) + g_m,\end{aligned} \tag{4.36}$$

*which satisfies the following assumptions:*

(a) *$h_m(\phi,\omega) = (h_m^1(\phi,\omega), h_m^2(\phi,\omega))^{\mathrm{T}}$ is real analytic in $\phi \in \mathbb{T}^n_{s_m}$ and fulfills the estimate*

$$\|h_m\|_{s_m}^{X_0,\mathcal{L}} \lessdot \epsilon_{m-1}^{\frac{3}{2}}.$$

*In addition, $h_m(\phi,\omega)$ satisfies (4.33) for any $\omega \in \Pi_m$.*

(b) $\check{P}_m = \widetilde{\check{P}}_m^{(m)} + \sum_{\nu\geq m}(\check{P}_m^{(\nu+1)} - \check{P}_m^{(\nu)})$ *and* $\widetilde{\check{P}}_m^{(m)} : \mathbb{T}^n_{s_m}\times\Pi_m \to X_1$, $\check{P}_m^{(\nu)}(\phi,\omega) : \mathbb{T}^n_{s_\nu}\times\Pi_m \to X_1$, $\nu \geq m$ *are real analytic in* $\phi\in\mathbb{T}^n_{s_\nu}$, *Lipschitz-continuous in* $\omega$ *and satisfies* (3.5). *Moreover, we have the following estimates:*

$$\|\widetilde{\check{P}}_m^{(m)}\|_{s_m}^{X_1,\mathcal{L}} \lessdot \epsilon_{m-1}^{\frac{2}{3}-\rho}, \quad \|\check{P}_m^{(\nu+1)} - \check{P}_m^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu\geq m. \tag{4.37}$$

(c) *For* $1\leq i\leq m-1$ *and* $\nu\geq m-1$, $\check{P}_i^{(\nu)}(\phi,\omega) : \mathbb{T}^n_{s_m}\times\Pi_m \to X_1$ *is real analytic in* $\phi\in\mathbb{T}^n_{s_\nu}$, *Lipschitz-continuous in* $\omega$ *and fulfills* (3.5) *as well as the estimates*:

$$\|\check{P}_i^{(\nu+1)} - \check{P}_i^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}. \tag{4.38}$$

(d) *For* $0\leq i\leq m-1$ *and* $\nu\geq m-1$, $G_i^{(\nu)}(\phi,\omega) : \mathbb{T}^n_{s_m}\times\Pi_m \to X_0$ *is real analytic in* $\phi\in\mathbb{T}^n_{s_\nu}$, *Lipschitz-continuous in* $\omega$ *and fulfills* (4.33) *and the estimates:*

$$\|G_i^{(\nu+1)} - G_i^{(\nu)}\|_{s_{\nu+1}}^{X_0,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}. \tag{4.39}$$

(e) *For* $m\geq 1$,

$$g_m = \int_0^1 D_\zeta^2 f(\zeta_{m-1}+s\eta_m,\phi)\eta_m^2(1-s)\mathrm{d}s. \tag{4.40}$$

*Then there exists* $\Pi_{m+1}\subset\Pi_m$ *with* $\mathrm{Meas}(\Pi_m\setminus\Pi_{m+1})\lessdot\alpha_{m+1}$ *and a quasi-periodic solution* $\zeta_{m+1}(\phi,\omega)$ *defined on* $\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}$ *such that* (T1) *and* (T2) *hold true with* $m$ *replaced by* $m+1$.

**Proof** Set $\zeta_{m+1} = \zeta_m + \eta_{m+1}$, then

$$\begin{aligned}\mathcal{F}(\zeta_{m+1}) &= \mathcal{F}(\zeta_m) + D_\zeta\mathcal{F}(\zeta_m)\eta_{m+1} + \int_0^1 D_\zeta^2 F(\zeta_m + s\eta_{m+1})\eta_{m+1}^2(1-s)\mathrm{d}s\\ &= \mathcal{F}(\zeta_m) + \mathbf{i}\dot{\eta}_{m+1} - \Lambda\eta_{m+1} + D_\zeta F(\zeta_m)\eta_{m+1} + \int_0^1 D_\zeta^2 F(\zeta_m+s\eta_{m+1})\eta_{m+1}^2(1-s)\mathrm{d}s\\ &=: \mathcal{F}(\zeta_m) + \mathbf{i}\dot{\eta}_{m+1} - \Lambda\eta_{m+1} + P^m\eta_{m+1} + g_{m+1}.\end{aligned} \tag{4.41}$$

First let us investigate the higher order term $g_m$. For convenience, we set

$$D_\zeta^2 F(\zeta_{m-1}+s\eta_m,\phi) := A(\widetilde{\zeta}_m^s(\phi,\omega),\phi) := T_m(\phi,\omega,s)$$

with $D_\zeta^2 F(\zeta,\phi) = A(\zeta,\phi)$, $\widetilde{\zeta}_m^s(\phi,\omega) = \zeta_{m-1}(\phi,\omega) + s\eta_m(\phi,\omega)$. Then for a fixed constant $s$, we can obtain

$$\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}} = \sum_{j_1+\cdots+j_k+i=\ell-3}\frac{\partial^i}{\partial\phi^i}\frac{\partial^k A}{\partial\zeta^k}\frac{\partial^{j_1}\widetilde{\zeta}_m^s}{\partial\phi^{j_1}}\cdots\frac{\partial^{j_k}\widetilde{\zeta}_m^s}{\partial\phi^{j_k}}. \tag{4.42}$$

Observe that for $0\leq s\leq 1$,

$$\|\widetilde{\zeta}_m^s\|_{p,s_m}^{\mathcal{L}} = \|\zeta_{m-1}+s\eta_m\|_{p,s_m}^{\mathcal{L}} \lessdot \sum_{1\leq i\leq m}\epsilon_{i-1}^{\frac{5}{6}} \ll 1, \tag{4.43}$$

then in view of $i+k\le \ell-3$, we have

$$\Big\|\frac{\partial^i}{\partial\phi^i}\frac{\partial^k A}{\partial\zeta^k}\Big\|_{s_m}\le \|F(\zeta,\phi)\|_{C^\ell}. \tag{4.44}$$

Here the norm is defined as the operator norm from $\prod_k \mathcal{P}^p$ to $X_2$. Thus we have

$$\Big\|\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}\Big\|_{\mathbb{T}^n}^{X_2}\le \|F\|_{C^\ell}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big\|_{p,\mathbb{T}^n}\cdots\Big\|\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big\|_{p,\mathbb{T}^n}. \tag{4.45}$$

For the estimate of the Lipschitz norm, we proceed as follows: As what mentioned before, for a given function $T(\omega)$, set $\triangle T=T(\omega)-T(\omega')$. Applying $\triangle$ to (4.42) we can obtain

$$\begin{aligned}\triangle\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}=&\sum_{j_1+\cdots+j_k+i=\ell-3}\triangle\Big(\frac{\partial^i}{\partial\phi^i}\frac{\partial^k A}{\partial\zeta^k}\Big)\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\cdots\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\\&+\sum_{j_1+\cdots+j_k+i=\ell-3}\frac{\partial^i}{\partial\phi^i}\frac{\partial^k A}{\partial\zeta^k}\triangle\Big(\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big)\cdots\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\\&+\cdots+\sum_{j_1+\cdots+j_k+i=\ell-3}\frac{\partial^i}{\partial\phi^i}\frac{\partial^k A}{\partial\zeta^k}\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\cdots\triangle\Big(\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big).\end{aligned}$$

Observe that

$$\triangle\Big(\frac{\partial^i}{\partial\phi^i}\frac{\partial^k}{\partial\zeta^k}A(\widetilde{\zeta}^s_m(\phi,\omega),\phi)\Big)=\frac{\partial^i}{\partial\phi^i}\frac{\partial^{k+1}}{\partial\zeta^{k+1}}A(\theta\widetilde{\zeta}^s_m(\phi,\omega)+(1-\theta)\widetilde{\zeta}^s_m(\phi,\omega'),\phi)\triangle\widetilde{\zeta}^s_m, \tag{4.46}$$

then in view of $i+k+1\le\ell-2$, we have

$$\Big\|\frac{\partial^i}{\partial\phi^i}\frac{\partial^{k+1}}{\partial\zeta^{k+1}}A(\theta\widetilde{\zeta}^s_m(\phi,\omega)+(1-\theta)\widetilde{\zeta}^s_m(\phi,\omega')\Big\|_{\mathbb{T}^n}\le\|F(\zeta,\phi)\|_{C^\ell}, \tag{4.47}$$

where the norm is defined as the operator norm from $\prod_{k+1}\mathcal{P}^p$ to $X_2$. Thus

$$\begin{aligned}\Big\|\triangle\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}\Big\|_{\mathbb{T}^n}^{X_2}\le&\|F\|_{C^\ell}\|\triangle\widetilde{\zeta}^s_m\|_{p,\mathbb{T}^n}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big\|_{p,\mathbb{T}^n}\cdots\Big\|\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big\|_{p,\mathbb{T}^n}\\&+\|F\|_{C^\ell}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\triangle\Big(\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big)\Big\|_{p,\mathbb{T}^n}\cdots\Big\|\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big\|_{p,\mathbb{T}^n}\\&+\cdots+\|F\|_{C^\ell}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big\|_{p,\mathbb{T}^n}\cdots\Big\|\triangle\Big(\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big)\Big\|_{p,\mathbb{T}^n}.\end{aligned}$$

Divided by $|\triangle\omega|$, together with (4.43) and (4.45), we can obtain

$$\Big\|\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}\Big\|_{\mathbb{T}^n}^{X_2,\mathcal{L}}\le\|F\|_{C^\ell}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\frac{\partial^{j_1}\widetilde{\zeta}^s_m}{\partial\phi^{j_1}}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}}\cdots\Big\|\frac{\partial^{j_k}\widetilde{\zeta}^s_m}{\partial\phi^{j_k}}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}}. \tag{4.48}$$

Applying Cauchy estimate with respect to $\eta_i$ on $\mathbb{T}^n_{s_{i-1}-9\sigma_{i-1}}$, together with the definition of $s_m$, we get for $0\le j\le\ell-3$, $1\le i\le m$,

$$\Big\|\frac{\partial^j\eta_i}{\partial\phi^j}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}}\lessdot j!\epsilon_{i-1}^{\frac{5}{6}-\frac{j(1+\rho)}{\ell-3}}, \tag{4.49}$$

thus

$$\Big\|\frac{\partial^j\widetilde{\zeta}_m^s}{\partial\phi^j}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}} \lessdot \sum_{1\le i\le m}\epsilon_{i-1}^{\frac{5}{6}-\frac{j(1+\rho)}{\ell-3}} \lessdot \begin{cases}1, & \frac{5}{6}-\frac{j(1+\rho)}{\ell-3}\ge 0,\\ \epsilon_{m-1}^{\frac{5}{6}-\frac{j(1+\rho)}{\ell-3}}, & \frac{5}{6}-\frac{j(1+\rho)}{\ell-3}<0.\end{cases} \tag{4.50}$$

Set $\Omega=\big\{j\ge 0:\ \frac{5}{6}-\frac{j(1+\rho)}{\ell-3}<0\big\}=\big\{j\ge 0:\ j>\frac{5(\ell-3)}{6(1+\rho)}\big\}$. Consequently, from (4.50) we obtain

$$\begin{aligned}\Big\|\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}\Big\|_{\mathbb{T}^n}^{X_2,\mathcal{L}} &\le \|F\|_{C^\ell}\sum_{j_i+\cdots+j_k+i=\ell-3}\Big\|\frac{\partial^{j_1}\widetilde{\zeta}_m^s}{\partial\phi^{j_1}}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}}\cdots\Big\|\frac{\partial^{j_k}\widetilde{\zeta}_m^s}{\partial\phi^{j_k}}\Big\|_{p,\mathbb{T}^n}^{\mathcal{L}}\\ &\le \|F\|_{C^\ell}\sum_{\star}\epsilon_{m-1}^{\frac{5}{6}-\frac{j_{i_1}(1+\rho)}{\ell-3}}\cdots\epsilon_{m-1}^{\frac{5}{6}-\frac{j_{i_t}(1+\rho)}{\ell-3}}\\ &\le \|F\|_{C^\ell}\sum_{\star}\epsilon_{m-1}^{\frac{5}{6}t-\frac{(j_{i_1}+\cdots+j_{i_t})(1+\rho)}{\ell-3}},\end{aligned} \tag{4.51}$$

where $\star$ means the admissible index set with $t$ indices $j_{i_1},\cdots j_{i_t}$ lying in $\Omega$. Then $\frac{5}{6}t-\frac{(j_{i_1}+\cdots+j_{i_t})(1+\rho)}{\ell-3}\ge-\frac{1}{6}-\rho$, and thus

$$\Big\|\frac{\partial^{\ell-3}T_m(\phi,\omega,s)}{\partial\phi^{\ell-3}}\Big\|_{\mathbb{T}^n}^{X_2,\mathcal{L}} \lessdot \epsilon_{m-1}^{-\frac{1}{6}-\rho}.$$

By integrating with respect to $s$, we have

$$N_m(\phi,\omega):=\int_0^1 T_m(\phi,\omega,s)(1-s)\mathrm{d}s:\mathbb{T}^n\times\Pi_m\to X_2$$

with $\|N_m\|_{C^{\ell-3}}^{\mathcal{L}}\lessdot\epsilon_{m-1}^{-\frac{1}{6}-\rho}$. Now we apply Lemma 2.2 to $N_m$, then for $\{s_\nu\}_{\nu\ge m}$, in view of (2.21), we obtain $N_m(\phi,\omega)=N_m^{(m)}+\sum\limits_{\nu\ge m}(N_m^{(\nu+1)}-N_m^{(\nu)})$, where for $\nu\ge m$, $N_m^{(\nu)}(\phi,\omega)$ is analytic in $\phi\in\mathbb{T}^n_{s_\nu}$, Lipschitz-continuous in $\omega\in\Pi_m$ and satisfies

$$\|N_m^{(m)}\|_{s_m}^{X_2,\mathcal{L}}\lessdot\|N_m\|_{C^{\ell-3}}^{\mathcal{L}}\lessdot\epsilon_{m-1}^{-\frac{1}{6}-\rho},\quad \|N_m^{(\nu+1)}-N_m^{(\nu)}\|_{s_\nu}^{X_2,\mathcal{L}}\lessdot\|N_m\|_{C^{\ell-3}}^{\mathcal{L}}s_\nu^{\ell-3}\lessdot\epsilon_{m-1}^{-\frac{1}{6}-\rho}\epsilon_\nu^{1+\rho}.$$

Then from (4.40), together with (4.35), we get

$$g_m=N_m^{(m)}\eta_m^2+\sum_{\nu\ge m}(N_m^{(\nu+1)}-N_m^{(\nu)})\eta_m^2:=G_m^{(m)}+\sum_{\nu\ge m}(G_m^{(\nu+1)}-G_m^{(\nu)}), \tag{4.52}$$

and for $\nu\ge m$, in view of the assumption that $0<\rho<\frac{1}{4}$, we have

$$\|G_m^{(m)}\|_{s_m}^{X_0,\mathcal{L}}\lessdot\epsilon_{m-1}^{-\frac{1}{6}-\rho}\epsilon_{m-1}^{\frac{5}{3}}\lessdot\epsilon_{m-1}^{\frac{3}{2}-\rho}\lessdot\epsilon_{m-1}^{1+\rho},\quad \|G_m^{(\nu+1)}-G_m^{(\nu)}\|_{s_{\nu+1}}^{X_0,\mathcal{L}}\lessdot\epsilon_{m-1}^{-\frac{1}{6}-\rho}\epsilon_\nu^{1+\rho}\epsilon_{m-1}^{\frac{5}{3}}\lessdot\epsilon_\nu^{1+\rho}.$$

Set $F_m=h_m+\widetilde{\check{P}}_m^{(m)}\eta_m+\sum\limits_{1\le i\le m-1}(\check{P}_i^{(m)}-\check{P}_i^{(m-1)})\eta_i+\sum\limits_{0\le i\le m-1}(G_i^{(m)}-G_i^{(m-1)})+G_m^{(m)}$. Then $F_m(\phi,\omega)$ is real analytic in $\phi\in\mathbb{T}^n_{s_m}$, Lipschitz-continuous in $\omega\in\Pi_m$ and satisfies (4.33) and the estimate

$$\|F_m\|_{s_m,\mathcal{P}^{p-1}}^{\mathcal{L}}\lessdot\epsilon_{m-1}^{\frac{3}{2}}+\epsilon_{m-1}^{\frac{2}{3}-\rho}\epsilon_{m-1}^{\frac{5}{6}}+m\epsilon_{m-1}^{1+\rho}\lessdot\epsilon_{m-1}^{1+\bar{\rho}}=\epsilon_m. \tag{4.53}$$

In addition

$$\mathcal{F}(\zeta_m) - F_m = \sum_{1\le i\le m}\sum_{\nu\ge m}(\check{P}_i^{(\nu+1)} - \check{P}_i^{(\nu)})v_i + \sum_{0\le i\le m}\sum_{\nu\ge m}(G_i^{(\nu+1)} - G_i^{(\nu)}). \tag{4.54}$$

In the following we get down to find the solution $\eta_{m+1}$ of the homological equation

$$\mathbf{i}\eta - \Lambda\eta + P^m\eta + F_m = 0. \tag{4.55}$$

Now let us give some more analysis concerning the operator $P^m(\phi) := D_\zeta F(\zeta_m, \phi)$ defined in (4.36). Using Taylor's formula, we have

$$P^m = D_\zeta F(\zeta_m) = D_\zeta F(\zeta_{m-1}) + \int_0^1 D_\zeta^2 F(\zeta_{m-1} + s\eta_m)\eta_m \mathrm{d}s,$$

then by induction, we get

$$\begin{aligned} P^m &= D_\zeta F(\zeta_1) + \int_0^1 D_\zeta^2 F(\zeta_1 + s\eta_2)\eta_2 \mathrm{d}s + \cdots + \int_0^1 D_\zeta^2 F(\zeta_{m-1} + s\eta_m)\eta_m \mathrm{d}s \\ &:= \sum_{1\le i\le m} P_i. \end{aligned} \tag{4.56}$$

For $2 \le i \le m$, we set $T_i(\phi,\omega,s) := D_\zeta^2 F(\zeta_{i-1} + s\eta_i, \phi), M_i(\phi,\omega) := \int_0^1 T_i(\phi,\omega,s)\mathrm{d}s$. Thus by the same analysis as above, we have

$$\|T_i(\phi,\omega,s)\|_{C^{\ell-3}}^{\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}, \quad \|M_i(\phi,\omega)\|_{C^{\ell-3}}^{\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}, \quad 2 \le i \le m. \tag{4.57}$$

Hence, by applying Lemma 2.2, we can obtain $M_i = M_i^{(i)} + \sum\limits_{\nu\ge i}(M_i^{(\nu+1)} - M_i^{(\nu)})$ with

$$\|M_i^{(i)}\|_{s_i}^{X_2,\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}, \quad \|M_i^{(\nu+1)} - M_i^{(\nu)}\|_{s_{\nu+1}}^{X_2,\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}\epsilon_\nu^{1+\rho}. \tag{4.58}$$

Accordingly, for $2 \le i \le m$,

$$P_i = M_i^{(i)}\eta_i + \sum_{\nu\ge i}(M_i^{(\nu+1)} - M_i^{(\nu)})\eta_i := P_i^{(i)} + \sum_{\nu\ge i}(P_i^{(\nu+1)} - P_i^{(\nu)}). \tag{4.59}$$

Taking (4.35) into account, we have

$$\|P_i^{(i)}\|_{s_i}^{X_1,\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}\epsilon_{i-1}^{\frac{5}{6}} \lessdot \epsilon_{i-1}^{\frac{2}{3}-\rho}, \quad \|P_i^{(\nu+1)} - P_i^{(\nu)}\|_{s_{\nu+1}}^{X_2,\mathcal{L}} \lessdot \epsilon_{i-1}^{-\frac{1}{6}-\rho}\epsilon_\nu^{1+\rho}\epsilon_{i-1}^{\frac{5}{6}} \lessdot \epsilon_\nu^{1+\rho}.$$

For $i = 1$, we have

$$D_\zeta F(\zeta_1) = D_\zeta F(0,\phi) + \int_0^1 D_\zeta^2 F(s\eta_1)\eta_1 \mathrm{d}s, \tag{4.60}$$

where $D_\zeta F(0,\phi) \in C^{\ell-1}(\mathbb{T}^n, X_1)$ satisfies $\|D_\zeta F(0,\phi)\|_{C^{\ell-1}} \lessdot \epsilon_0$, and the second term can be handled similarly. Then by applying Lemma 2.2, we can obtain

$$D_\zeta F(0,\phi) = P_0^{(1)} + \sum_{\nu\ge 1}(P_0^{(\nu+1)} - P_0^{(\nu)}), \tag{4.61}$$

where

$$\|P_0^{(1)}\|_{s_1}^{X_1} \lessdot \epsilon_0, \quad \|P_0^{(\nu+1)} - P_0^{(\nu)}\|_{s_{\nu+1}}^{X_1} \lessdot \epsilon_0 s_v^{\ell-1} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq 1. \tag{4.62}$$

Hence we have $P_1 = P_1^{(1)} + \sum_{\nu\geq 1}(P_1^{(\nu+1)} - P_1^{(\nu)})$ with

$$\|P_1^{(1)}\|_{s_1}^{X_1} \lessdot \epsilon_0^{\frac{2}{3}-\rho}, \quad \|P_1^{(\nu+1)} - P_1^{(\nu)}\|_{s_{\nu+1}}^{X_1} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq 1. \tag{4.63}$$

It is easy to verify that $P^m$ fulfills the hypotheses in Lemma 3.2, then we can find an operator $U^m(\phi,\omega)$ defined on $\mathbb{T}^n_{s_m-4\sigma_m} \times \Pi_m$ fulfilling (4.34) and the corresponding coordinate transformation $\eta = U^m(\omega t,\omega)\varphi$. Differentiating it with respect to $t$, we have $\dot\eta = \dot U^m\varphi + U^m\dot\varphi$. Inserting it into (4.55), we obtain

$$\mathbf{i}\dot\varphi - \Lambda_{m+1}\varphi + Q_{m+1}\varphi + \widehat{F}_m = 0, \tag{4.64}$$

where $Q_{m+1}$ is defined by (3.33)–(3.34) and

$$\|\widehat{F}_m\|_{s_m-4\sigma_m}^{X_0,\mathcal{L}} = \|(U^m)^{-1}F_m\|_{s_m}^{X_0,\mathcal{L}} \lessdot \epsilon_m. \tag{4.65}$$

Moreover, we can conclude that $\widehat{F}_m$ satisfies (4.33) in view of Remark 4.1 in this section.

Consider the system $\mathbf{i}\dot\varphi - \Lambda_{m+1}\varphi + \widehat{F}_m = 0$ defined on $\mathbb{T}^n_{s_m-4\sigma_m} \times \Pi_{m+1}$, then applying Lemma 4.1, we can find a solution $\varphi_{m+1}(\phi,\omega)$ such that

$$\mathbf{i}\dot\varphi_{m+1} - \Lambda_{m+1}\varphi_{m+1} + \widehat{F}_m = r_m$$

with

$$\|\varphi_{m+1}\|_{p,s_m-9\sigma_m}^{\mathcal{L}} \lessdot \epsilon_m^{\frac{5}{6}}, \quad \|r_m\|_{s_m-9\sigma_m}^{X_0,\mathcal{L}} \lessdot \epsilon_m^{\frac{3}{2}}. \tag{4.66}$$

Hence (4.64) becomes

$$\mathbf{i}\dot\varphi_{m+1} - \Lambda_{m+1}\varphi_{m+1} + Q_{m+1}\varphi_{m+1} + \widehat{F}_m = Q_{m+1}\varphi_{m+1} + r_m. \tag{4.67}$$

Let $\eta_{m+1} = U^{m+1}\varphi_{m+1}$, from (3.31) and (4.66) we have

$$\|\eta_{m+1}\|_{p,s_m-9\sigma_m}^{\mathcal{L}} \lessdot \epsilon_m^{\frac{5}{6}}. \tag{4.68}$$

Taking (3.31) and (3.33)–(3.34) into account, the homological equation (4.55) has the form:

$$\begin{aligned}
&\mathbf{i}\dot\eta_{m+1} - \Lambda\eta_{m+1} + P^m\eta_{m+1} + F_m \\
&= \mathbf{i}\dot U^m\varphi_{m+1} + \mathbf{i}U^m\dot\varphi_{m+1} - \Lambda U^m\varphi_{m+1} + P^mU^m\varphi_{m+1} + F_m \\
&= U^m(\mathbf{i}\dot\varphi_{m+1} - (U^m)^{-1}\Lambda U^m\varphi_{m+1} + (U^m)^{-1}P^mU^m\varphi_{m+1} + \mathbf{i}(U^m)^{-1}\dot U^m\varphi_{m+1} + (U^m)^{-1}F_m) \\
&= U^m(\mathbf{i}\dot\varphi_{m+1} - \Lambda_{m+1}\varphi_{m+1} + Q_{m+1}\varphi_{m+1} + \widehat{F}_m) \\
&= U^m(Q_{m+1}\varphi_{m+1} + r_m) = U^mQ_{m+1}(U^m)^{-1}\eta_{m+1} + U^mr_m \\
&:= h_{m+1} + \check P_{m+1}\eta_{m+1},
\end{aligned} \tag{4.69}$$

where $\check P_{m+1} = \widetilde{\check P}_{m+1}^{(m+1)} + \sum_{\nu\geq m+1} \check P_{m+1}^{(\nu+1)} - \check P_{m+1}^{(\nu)}$ satisfies

$$\|\widetilde{\check P}_{m+1}^{(m+1)}\|_{s_{m+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_m^{\frac{2}{3}-\rho}, \quad \|\check P_{m+1}^{(\nu+1)} - \check P_{m+1}^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}} \lessdot \epsilon_\nu^{1+\rho}, \quad \nu \geq m+1, \tag{4.70}$$

and $h_{m+1}(\phi,\omega)$ is real analytic in $\phi\in\mathbb{T}^n_{s_{m+1}}$ with

$$\|h_{m+1}\|^{X_0,\mathcal{L}}_{s_{m+1}}\lessdot\epsilon_m^{\frac{3}{2}}. \tag{4.71}$$

Consequently, from (4.41), (4.54) and (4.69), we can obtain

$$\begin{aligned}\mathcal{F}(\zeta_{m+1})&=h_{m+1}+\check{P}_{m+1}\eta_{m+1}+\sum_{1\le i\le m}\sum_{\nu\ge m}(\check{P}_i^{(\nu+1)}-\check{P}_i^{(\nu)})\eta_i\\&\quad+\sum_{0\le i\le m}\sum_{\nu\ge m}(G_i^{(\nu+1)}-G_i^{(\nu)})+g_{m+1},\end{aligned}$$

and from (4.38)–(4.39), (4.68), (4.70)–(4.71) and Remarks 4.1 in this section, we get that

$$\|\mathcal{F}(\zeta_{m+1})\|^{X_0}_{\mathbb{T}^n\times\Pi_m}\lessdot\epsilon_{m+1}$$

and all the assumptions in (T1) and (T2) hold with $m$ replaced by $m+1$. By now we finish the proof.

**Proof of Theorem 2.1** It remains to find the first approximate solution $\eta_1$ to (2.11) and then to show that (T1) and (T2) in Lemma 4.2 are true for $m=1$. By using Taylor's formula we have

$$\begin{aligned}\mathcal{F}(\zeta_1)&=\mathcal{F}(0)+D_\zeta\mathcal{F}(0)\eta_1+\int_0^1 D_\zeta^2\mathcal{F}(s\eta_1)\eta_1^2(1-s)\mathrm{d}s\\&=F(0,\phi)+\mathbf{i}\dot{\eta}_1-\Lambda\eta_1+D_\zeta F(0,\phi)\eta_1+\int_0^1 D_\zeta^2F(s\eta_1)\eta_1^2(1-s)\mathrm{d}s\\&:=F_0(\phi)+\mathbf{i}\dot{\eta}_1-\Lambda\eta_1+P_0(\phi)\eta_1+g_1,\end{aligned} \tag{4.72}$$

where $F_0(\phi)\in C^\ell(\mathbb{T}^n;X_0)$ and $P_0(\phi)\in C^{\ell-1}(\mathbb{T}^n;X_1)$ satisfy

$$\|F_0(\phi)\|_{C^\ell},\quad \|P_0(\phi)\|_{C^{\ell-1}}\le\|F(q,\phi)\|_{C^\ell}\lessdot\epsilon_0. \tag{4.73}$$

Then applying Lemma 2.2 to $F_0(\phi)$, we get $F_0=F_0^{(0)}+\sum\limits_{\nu\ge0}(F_0^{(\nu+1)}-F_0^{(\nu)})$ with

$$\|F_0^{(0)}\|^{X_0}_{s_0}\lessdot\epsilon_0,\quad \|F_0^{(\nu+1)}-F_0^{(\nu)}\|^{X_0}_{s_{\nu+1}}\lessdot\epsilon_0s_\nu^\ell\lessdot\epsilon_\nu^{1+\rho},\quad \nu\ge0. \tag{4.74}$$

From (4.61)–(4.62), we get $P_0=P_0^{(0)}+\sum\limits_{\nu\ge0}(P_0^{(\nu+1)}-P_0^{(\nu)})$ with the estimate

$$\|P_0^{(0)}\|^{X_1}_{s_0}\lessdot\epsilon_0,\quad \|P_0^{(\nu+1)}-P_0^{(\nu)}\|^{X_1}_{s_{\nu+1}}\lessdot\epsilon_\nu^{1+\rho},\quad \nu\ge0. \tag{4.75}$$

The homological equation we hope to investigate is

$$\mathbf{i}\dot{\eta}-\Lambda\eta+P_0\eta+F_0^{(0)}=0. \tag{4.76}$$

Fix $k,i$ and $j$ and set $\mathcal{R}_{kij}:=\big\{\omega\in\Pi'\colon |k\cdot\omega+i^2-j^2|\le\frac{\alpha_0|i^2-j^2|}{1+|k|^\tau}\big\}$, $i\ne j$, $\Pi_0:=\Pi'\setminus\big(\bigcup\limits_{k\in\mathbb{Z}^n,i\ne j}\mathcal{R}_{kij}\big)$. Then it is easy to check $\mathrm{Meas}(\Pi'\setminus\Pi_0)\lessdot\alpha_0$. Hence from Lemma 3.2 in [17] we know that for $\omega\in\Pi_0$, there exists an operator $B_0(\phi,\omega):\mathbb{T}^n_{s_0-4\sigma_0}\times\Pi_0\to\mathfrak{B}^{p,p}\cap\mathfrak{B}^{p-1,p-1}$ and

$$|||B_0|||^{\mathcal{L}}_{p,p,s_0-4\sigma_0},\ |||B_0|||^{\mathcal{L}}_{p-1,p-1,s_0-4\sigma_0}\lessdot\epsilon_0^{\frac{5}{6}} \tag{4.77}$$

such that by the coordinate transformation $\eta=\mathrm{e}^{B_0}\varphi$, we can change the system $\mathbf{i}\dot\eta=(\Lambda+P_0^{(0)})\eta$ into the new system $\mathbf{i}\dot\varphi=(\Lambda_1+\widetilde P_1)\varphi$, where $\Lambda_1$ apparently fulfills (H1) in Lemma 3.1 and $\|\widetilde P_1\|_{s_1}^{X_1,\mathcal{L}}\lessdot\epsilon_0^{\frac32}$. Then by the same transformation, (4.76) can be changed into

$$\mathbf{i}\dot\varphi-\Lambda_1\varphi+\widetilde P_1\varphi+\sum_{\nu\ge0}\mathrm{e}^{-B_0}(P_0^{(\nu+1)}-P_0^{(\nu)})\mathrm{e}^{B_0}\varphi+\mathrm{e}^{-B_0}F_0^{(0)}=\mathbf{i}\dot\varphi-\Lambda_1\varphi+Q_1\varphi+\widehat F_0=0$$

with $Q_1=\widetilde Q_1^{(1)}+\sum_{\nu\ge1}(Q_1^{(\nu+1)}-Q_1^{(\nu)})$ and

$$\widetilde Q_1^{(1)}:=\widetilde P_1+\mathrm{e}^{-B_0}(P_0^{(1)}-P_0^{(0)})\mathrm{e}^{B_0},\quad Q_1^{(\nu)}=\mathrm{e}^{-B_0}P_0^{(\nu)}\mathrm{e}^{B_0},\quad \nu\ge1.$$

In addition, from (4.75) and (4.77) we obtain

$$\|\widetilde Q_1^{(1)}\|_{s_1}^{X_1,\mathcal{L}}\lessdot\epsilon_0^{\frac32}+\epsilon_0^{1+\rho}\lessdot\epsilon_0^{1+\rho},\quad \|Q_1^{(\nu+1)}-Q_1^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}}\lessdot\epsilon_\nu^{1+\rho},\quad \nu\ge1.$$

Applying Lemma 4.1, we can find a real analytic solution $\varphi_1(\phi)$ such that

$$\mathbf{i}\dot\varphi_1-\Lambda_1\varphi_1+\widehat F_0=r_0$$

and $\|\varphi_1\|_{p,s_0-9\sigma_0}^{\mathcal{L}}\lessdot\epsilon_0^{\frac56}$, $\|r_0\|_{s_0-9\sigma_0}^{X_0,\mathcal{L}}\lessdot\epsilon_0^{\frac32}$. Setting $\eta_1=\mathrm{e}^{B_0}\varphi_1$, similarly to (4.69), we conclude

$$\mathbf{i}\dot\eta_1-\Lambda\eta_1+P_0\eta_1+F_0^{(0)}=\check P_1\eta_1+h_1,$$

where

$$\check P_1=\mathrm{e}^{B_0}Q_1\mathrm{e}^{-B_0}=\widetilde{\check P}_1^{(1)}+\sum_{\nu\ge1}(\check P_1^{(\nu+1)}-\check P_1^{(\nu)}),\quad h_1=\mathrm{e}^{B_0}r_0.$$

Hence

$$\|\widetilde{\check P}_i^{(1)}\|_{s_1}^{X_1,\mathcal{L}}\lessdot\epsilon_0^{1+\rho},\quad \|\check P_1^{(\nu+1)}-\check P_1^{(\nu)}\|_{s_{\nu+1}}^{X_1,\mathcal{L}}\lessdot\epsilon_\nu^{1+\rho},\quad \nu\ge1,\quad \|h_1\|_{s_1}^{X_0,\mathcal{L}}\lessdot\epsilon_0^{\frac32}. \tag{4.78}$$

Thus by setting $G_0^{(\nu)}=F_0^{(\nu)}$ for $\nu\ge0$, we obtain

$$\mathcal{F}(\eta_1)=h_1+\check P_1\eta_1+\sum_{\nu\ge0}(G_0^{(\nu+1)}-G_0^{(\nu)})+g_1,$$

which fulfills (T1)–(T2) in Lemma 4.2 with even better estimates.

Finally, letting $\Pi^\epsilon=\bigcap_{i\ge0}\Pi_i$, $\zeta^\infty(\omega t,\omega)=\lim_{m\to\infty}\zeta_m(\omega t,\omega)=(q^\infty(\omega t,\omega),\widetilde q^\infty(\omega t,\omega))$, then $\zeta^\infty(\omega t,\omega)$ is a quasi-periodic solution of frequency $\omega\in\Pi^\epsilon$ for (2.11) and $\widetilde q^\infty(\omega t,\omega)=\overline{q^\infty(\omega t,\omega)}$. In addition, $\mathrm{Meas}(\Pi\backslash\Pi^\epsilon)\lessdot\sum_{m\ge0}\alpha_m\lessdot\alpha_0(\epsilon)\to0$, as $\epsilon\to0$ and $\|\zeta^\infty(\phi,\omega)\|_{p,\mathbb{T}^n\times\Pi^\epsilon}\lessdot\sum_{m\ge1}\epsilon_{m-1}^{\frac56}\lessdot\epsilon^{\frac56}$. We finally finish the proof of Theorem 2.1, and the proof of Theorem 1.1 immediately follows.

## 5 Appendix

**Lemma 5.1** *Define the Sobolev space*

$$H^p([0,2\pi])=\Big\{u(x):\ u(0)=u(2\pi),\ \|u\|_p:=\Big(\int_0^{2\pi}\Big|\frac{\mathrm{d}^pu}{\mathrm{d}x^p}\Big|^2\mathrm{d}x\Big)^{\frac12}<\infty\Big\}.$$

*Then we have the interpolation estimate below*

$$\|u\|_r^{m-n}\le\|u\|_n^{m-r}\|u\|_m^{r-n}\quad for\ \ 0\le n<r<m\le p.$$

**Proof** The proof can be found in [22], we omit it here.

**Lemma 5.2** *Let* $F = (F_{ij})_{i,j\geq 1}$ *be a bounded operator on* $\ell^2$*. Assume that the matrix elements* $(F_{ij})$ *are analytic functions of* $\phi \in \mathbb{T}^n_s$*. Let* $R = (R_{ij})_{i,j\geq 1}$ *be another operator with matrix elements depending analytically on* $\phi \in \mathbb{T}^n_{s'}$ *and* $\|R_{ij}\|_{s'} \leq \frac{1}{|i-j|}\|F_{ij}\|_{s-\sigma}$*,* $i \neq j$*, and* $R_{ii} = 0$*. Then* $R$ *is a bounded operator on* $\ell^2$ *and* $\|R\|_{s'} \leq \frac{4^{n+1}}{\sigma^n}\|F\|_s$*. Moreover, let* $f = (f_j)_{j\geq 1}$ *be a vector in* $\ell^2$ *with the vector elements* $f_j$ *being analytic function of* $\phi \in \mathbb{T}^n_s$*. Assume that* $r = (r_j)_{j\geq 1}$ *is another vector of which the vector elements depend analytically on* $\phi \in \mathbb{T}^n_{s'}$ *and* $\|r_j\|_{s'} \leq \frac{1}{j}\|f_j\|_{s-\sigma}$*,* $j \geq 1$*. Then* $r \in \ell^2$ *and the* $\ell^2$*-norm* $\|r\|_{s'} \leq \frac{4^{n+1}}{\sigma^n}\|f\|_s$*.*

**Proof** The proof of this lemma can be found in [17].

**Lemma 5.3** *Let* $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$ *be an operator with the elements* $P_i$*,* $1 \leq i \leq 4$ *fulfilling* (3.5)*,* $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ *be another operator satisfying*

$$\overline{B}_1 = -B_1^{\mathrm{T}} = B_4, \quad B_2 = B_2^{\mathrm{T}}, \quad \overline{B}_2 = B_3. \tag{5.1}$$

*Then the operator* $[P, B] = PB - BP$ *satisfies* (3.5) *also.*

**Proof** The conclusion of this lemma can be obtained by a direct calculation.

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