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Weighted Compact Commutator of Bilinear Fourier Multiplier Operator*

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Abstract Let T_σ be the bilinear Fourier multiplier operator with associated multiplier σ satisfying the Sobolev regularity that $\sup\limits_{\kappa\in\mathbb{Z}}\|\sigma_\kappa\|_{W^s(\mathbb{R}^{2n})}<\infty$ for some $s\in(n,2n]$. In this paper, it is proved that the commutator generated by T_σ and $\mathrm{CMO}(\mathbb{R}^n)$ functions is a compact operator from $L^{p_1}(\mathbb{R}^n,w_1)\times L^{p_2}(\mathbb{R}^n,w_2)$ to $L^p(\mathbb{R}^n,\nu_{\vec{w}})$ for appropriate indices $p_1,p_2,p\in(1,\infty)$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and weights w_1,w_2 such that $\vec{w}=(w_1,w_2)\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$.

Keywords Bilinear Fourier multiplier, Commutator, Bi(sub)linear maximal operator, Compact operator
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1 Introduction

As it is well known, the study of bilinear Fourier multiplier operator was origined by Coifman and Meyer. Let $\sigma\in L^\infty(\mathbb{R}^{2n})$. Define the bilinear Fourier multiplier operator T_σ by

$$T_\sigma(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\exp(2\pi\mathrm{i}x(\xi_1+\xi_2))\sigma(\xi_1,\xi_2)\mathcal{F}f_1(\xi_1)\mathcal{F}f_2(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{1.1}$$

for $f_1,f_2\in\mathscr{S}(\mathbb{R}^n)$, where and in the following, $\mathcal{F}f$ denotes the Fourier transform of f. Coifman and Meyer [6] proved that if $\sigma\in C^s(\mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\sigma(\xi_1,\xi_2)|\le C_{\alpha_1,\alpha_2}(|\xi_1|+|\xi_2|)^{-(|\alpha_1|+|\alpha_2|)} \tag{1.2}$$

for all $|\alpha_1|+|\alpha_2|\le s$ with $s\ge 4n+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1<p_1,p_2,p<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. For the case of $s\ge 2n+1$, Kenig-Stein [18] and Grafakos-Torres [12] improved Coifman and Meyer's multiplier theorem to the indices $\frac{1}{2}\le p\le 1$ by the multilinear Calderón-Zygmund operator theory. In the last several years, considerable attention has been paid to the behavior on function spaces for T_σ when the multiplier satisfies certain Sobolev regularity condition. A significant progress in this area was obtained by Tomita. Let $\Phi\in\mathscr{S}(\mathbb{R}^{2n})$ satisfy

$$\begin{cases}\operatorname{supp}\Phi\subset\Big\{(\xi_1,\xi_2):\dfrac{1}{2}\le|\xi_1|+|\xi_2|\le 2\Big\};\\ \displaystyle\sum_{\kappa\in\mathbb{Z}}\Phi(2^{-\kappa}\xi_1,2^{-\kappa}\xi_2)=1\quad\text{for all }(\xi_1,\xi_2)\in\mathbb{R}^{2n}\backslash\{0\}.\end{cases} \tag{1.3}$$

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For $\kappa \in \mathbb{Z}$, set

$$\sigma_\kappa(\xi_1,\xi_2)=\Phi(\xi_1,\xi_2)\sigma(2^\kappa\xi_1,2^\kappa\xi_2) \tag{1.4}$$

and

$$\|\sigma_\kappa\|_{W^s(\mathbb{R}^{2n})}=\Big(\int_{\mathbb{R}^{2n}}(1+|\xi_1|^2+|\xi_2|^2)^s|\mathcal{F}\sigma_\kappa(\xi_1,\xi_2)|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\Big)^{\frac{1}{2}}.$$

Tomita [21] proved that if σ satisfies the Sobolev regularity that

$$\sup_{\kappa\in\mathbb{Z}}\|\sigma_\kappa\|_{W^s(\mathbb{R}^{2n})}<\infty \tag{1.5}$$

for some $s\in(n,2n]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1,p_2\in(1,\infty)$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Grafakos and Si [11] considered the mapping properties from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when σ satisfies (1.5) and $p_1,p_2\in\big(\frac{2n}{s},\infty\big)$, then T is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Miyachi and Tomita [20] considered the problem to find minimal smoothness condition for bilinear Fourier multiplier. Let

$$\|\sigma_\kappa\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}=\Big(\int_{\mathbb{R}^{2n}}\langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2}|\mathcal{F}\sigma_\kappa(\xi_1,\xi_2)|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\Big)^{\frac{1}{2}},$$

where $\langle\xi_k\rangle:=(1+|\xi_k|^2)^{\frac{1}{2}}$. Miyachi and Tomita [20] proved that if

$$\sup_{\kappa\in\mathbb{Z}}\|\sigma_\kappa\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}<\infty \tag{1.6}$$

for some $s_1,s_2\in\big(\frac{n}{2},n\big]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $p_1,p_2\in(1,\infty)$ and $p\ge\frac{2}{3}$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Moreover, they also gave minimal smoothness condition for which T_σ is bounded from $H^{p_1}(\mathbb{R}^n)\times H^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The weighted estimates for the operator T_σ are also of great interest. As it is well known, when σ satisfies (1.2) for some $s\ge 2n+1$, then T_σ is a standard bilinear Calderón-Zygmund operator, and then by the weighted estimates with multiple weights for bilinear Calderón-Zygmund operators, which was established by Lerner et al. [19], we know that for any $p_1,p_2\in[1,\infty)$ and $p\in(0,\infty)$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, and weights w_1,w_2 such that $\vec{w}=(w_1,w_2)\in A_{\vec{p}}(\mathbb{R}^{2n})$ (for the definition of $A_{\vec{p}}(\mathbb{R}^{2n})$, see Definition 1.1 below),

$$\|T_\sigma(f_1,f_2)\|_{L^{p,\infty}(\mathbb{R}^n,\nu_{\vec{w}})}\lesssim\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)},$$

where and in the following, for indices p_1,p_2, we set $\vec{p}=(p_1,p_2)$ and $p\in(0,\infty)$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. By developing the ideas used in [19], Bui and Duong [4] established the weighted estimates with multiple weights for T_σ when σ satisfies (1.2) for some $s\in(n,2n]$. To consider the weighted estimates for T_σ when σ satisfies (1.5), Jiao [17] introduced the following class of multiple weights.

Definition 1.1 *Let $m\ge1$ be an integer, $w_1,\cdots,w_m$ be weights, $p_1,\cdots,p_m,p\in(0,\infty)$ with $\frac{1}{p}=\sum\limits_{k=1}^{m}\frac{1}{p_k}$, $r_k\in(0,p_k]$ $(1\le k\le m)$ and $\vec{r}=(r_1,\cdots,r_m)$. Set $\vec{w}=(w_1,\cdots,w_m)$,*

$\vec{p}=(p_1,\cdots,p_m)$ and $\nu_{\vec{w}}=\prod_{k=1}^m w_k^{\frac{p}{p_k}}$. We say that $\vec{w}\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ if

$$\sup_{B\subset\mathbb{R}^n}\Big(\frac{1}{|B|}\int_B\nu_{\vec{w}}(x)\,\mathrm{d}x\Big)^{\frac{1}{p}}\prod_{k=1}^m\Big(\frac{1}{|B|}\int_B w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,\mathrm{d}x\Big)^{\frac{1}{r_k}-\frac{1}{p_k}}<\infty,$$

where and in the following, when $p_k=r_k$, $\big(\frac{1}{|B|}\int_B w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(x)\,\mathrm{d}x\big)^{\frac{1}{r_k}-\frac{1}{p_k}}$ is understood as $\big(\inf_{x\in B}w_k\big)^{-\frac{1}{p_k}}$.

When $r_1=\cdots=r_m=1$, $A_{\vec{p}/\vec{r}}(\mathbb{R}^{mn})$ is just the weight class $A_{\vec{P}}(\mathbb{R}^{mn})$ introduced by Lerner et al. [19]. By some kernel estimates of the operator T_σ, Jiao proved that for $t_1,t_2\in[1,2)$ such that $\frac{1}{t_1}+\frac{1}{t_2}=\frac{s}{n}$, $p_k\in(t_k,\infty)$ with $k=1,2$, and w_1,w_2 such that $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n,w_1)\times L^{p_2}(\mathbb{R}^n,w_2)$ to $L^p(\mathbb{R}^n,\nu_{\vec{w}})$. For the weighted estimates with A_p weights when σ satisfies the regularity (1.6) (see [8, 15]), here and in the following, for $p\in[1,\infty)$, $A_p(\mathbb{R}^n)$ denotes the weight function class Muckenhoupt, and $A_\infty(\mathbb{R}^n)=\bigcup_{p\geq1}A_p(\mathbb{R}^n)$.

The commutator of the multiplier operator T_σ has been considered by many authors. Let T_σ be the multiplier operator defined by (1.1), $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$ and $\vec{b}=(b_1,b_2)$. Define the commutator of $\vec{b}$ and T_σ by

$$T_{\sigma,\vec{b}}(f_1,f_2)(x)=\sum_{k=1}^2[b_k,T_\sigma]_k(f_1,f_2)(x) \tag{1.7}$$

with

$$[b_1,T_\sigma]_1(f_1,f_2)(x)=b_1(x)T_\sigma(f_1,f_2)(x)-T_\sigma(b_1f_1,f_2)(x) \tag{1.8}$$

and

$$[b_2,T_\sigma]_2(f_1,f_2)(x)=b_2(x)T_\sigma(f_1,f_2)(x)-T_\sigma(f_1,b_2f_2)(x). \tag{1.9}$$

Bui and Duong [4] established the weighted estimates with multiple weights for $T_{\sigma,\vec{b}}$ when σ satisfies (1.2) for $s\in(n,2n]$. Hu and Yi [16] considered the behavior on $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ for $T_{\sigma,\vec{b}}$ when σ satisfies (1.6) for $s_1,s_2\in\big(\frac{n}{2},n\big]$, and showed that $T_{\sigma,\vec{b}}$ enjoys the same $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ mapping properties as that of the operator T_σ. Fairly recently, Hu [14] considered the compactness of $T_{\sigma,\vec{b}}$, and proved that if $b_1,b_2\in\mathrm{CMO}(\mathbb{R}^n)$, σ satisfies (1.6) for some $s_1,s_2\in\big(\frac{n}{2},n\big]$, then for $p_k\in(n/s_k,\infty)$ $(k=1,2)$ and $p\in[1,\infty)$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $T_{\sigma,\vec{b}}$ is a compact operator from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where and in the following, $\mathrm{CMO}(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincide with the space of functions of vanishing mean oscillation (see [3, 7] for details). Zhou and Li [22] considered the weighted compactness with A_p weights for $T_{\sigma,\vec{b}}$. By combining the ideas used in [2, 14], Zhou and Li showed that if $b_1,b_2\in\mathrm{CMO}(\mathbb{R}^n)$ and σ satisfies (1.6) for some $s_1,s_2\in\big(\frac{n}{2},n\big]$, then for $p_k\in(n/s_k,\infty)$ $(k=1,2)$, $p\in[1,\infty)$ with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, and $w_k\in A_{p_ks_k/n}(\mathbb{R}^n)$, $T_{\sigma,\vec{b}}$ is a compact operator from $L^{p_1}(\mathbb{R}^n,w_1)\times L^{p_2}(\mathbb{R}^n,w_2)$ to $L^p(\mathbb{R}^n,\nu_{\vec{w}})$.

The main purpose of this paper is to consider the weighted compactness of $T_{\sigma,\vec{b}}$ with multiple weights. We will show that if σ satisfies (1.5) and $b_1,b_2\in\mathrm{CMO}(\mathbb{R}^n)$, then for appropriate indices $p_1, p_2, p \in (1,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and weights w_1, w_2 such that $\vec{w} = (w_1, w_2) \in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$, $T_{\sigma,\vec{b}}$ is compact from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$. Our main result in this paper can be stated as follows.

Theorem 1.1 *Let σ be a multiplier satisfying* (1.5) *for some $s \in (n, 2n]$ and T_σ be the operator defined by* (1.1). *Let $t_1, t_2 \in [1,2)$ such that $\frac{1}{t_1} + \frac{1}{t_2} = \frac{s}{n}$, $b_1, b_2 \in \mathrm{CMO}(\mathbb{R}^n)$. Then for $p_k \in (t_k, \infty)$ with $k = 1, 2$, $p \in (1,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and weights w_1, w_2 such that $\vec{w} = (w_1, w_2) \in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$ and $\nu_{\vec{w}} \in A_p(\mathbb{R}^n)$, the commutator $T_{\sigma,\vec{b}}$ is a compact operators from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.*

Remark 1.1 It is well known that, the class $A_{\vec{P}}(\mathbb{R}^{2n})$ with $\vec{P} = (p_1, p_2)$ is really large than the weight class $\prod_{k=1}^{2} A_{p_k}(\mathbb{R}^n)$, and the weighted estimates with multiple weights $A_{\vec{P}}(\mathbb{R}^{2n})$ are more interesting and more refined than the weighted estimates with $A_{p_1}(\mathbb{R}^n) \times A_{p_2}(\mathbb{R}^n)$ for the bilinear Calderón-Zygmund operators (see [19]). To prove Theorem 1.1, we will employ the idea used in [2, 14]. However, the idea that controlling $T_{\sigma,\vec{b}}(f_1, f_2)$ by $\prod_{k=1}^{2} M_{n/s_k} f$ which was used in [14, 22] (even if the function $\mathcal{M}_{\vec{r}}(f_1, f_2)$ with $\vec{r} = \big(\frac{n}{s_1}, \frac{n}{s_2}\big)$ introduced by [17]) does not work. To overcome this difficulty, we establish some new estimates for the kernel of T_σ, and introduce a new subtle bi(sub)linear maximal operator to control $T_{\sigma,\vec{b}}$.

Throughout the article, C always denotes a positive constant that may vary from line to line but remains independent of the main variables. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. We use $B(x, R)$ to denote a ball centered at x with radius R and $C(x, R) = B(x, R) \backslash B\big(x, \frac{R}{2}\big)$. For a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we use λB to denote the ball concentric with B whose radius is λ times of B's. For any $\gamma \in [1,\infty]$, we use γ' to denote the dual exponent of γ, namely, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. For a locally integrable function f, Mf denotes the Hardy-Littlewood maximal function of f, and for $\tau \in (0,\infty)$,

$$M_\tau f(x) = (M(|f|^\tau)(x))^{\frac{1}{\tau}}.$$

Let $M^\sharp$ be the Fefferman-Stein sharp maximal operator. For $\epsilon > 0$, $M^\sharp_\epsilon$ denotes the operator defined by

$$M^\sharp_\epsilon f(x) = (M^\sharp(|f|^\epsilon)(x))^{\frac{1}{\epsilon}}.$$

2 A New Maximal Operator

To control the multilinear Calderón-Zygmund operators via the Fefferman-Stein sharp maximal operator, Lerner et al. [19] introduced the bi(sub)linear maximal operator $\mathcal{M}$ by

$$\mathcal{M}(f_1, f_2)(x) = \sup_{B \ni x} \prod_{k=1}^{2} \frac{1}{|B|} \int_B |f_k(y_k)| \mathrm{d}y_k.$$

For $r_1, r_2 \in (0,\infty)$, Jiao [17] generalized the operator $\mathcal{M}$, defined the maximal operator $\mathcal{M}_{\vec{r}}$ by

$$\mathcal{M}_{\vec{r}}(f_1,f_2)(x)=\sup_{B\ni x}\prod_{k=1}^{2}\Big(\frac{1}{|B|}\int_B |f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}},$$

and established the weighted norm inequalities with multiple weights $A_{\vec{p}/\vec{r}}(\mathbb{R}^{2n})$ for $\mathcal{M}_{\vec{r}}$. Let $\delta\in\mathbb{R}$ and $r_1,r_2\in[1,\infty)$. Define the bi(sub)linear maximal operators $\mathcal{M}^{(1)}_{\vec{r},\delta}$ and $\mathcal{M}^{(2)}_{\vec{r},\delta}$ by

$$\begin{aligned}\mathcal{M}^{(1)}_{\vec{r},\delta}(f_1,f_2)(x)=\sup_{B\ni x}&\sum_{j=1}^{\infty}2^{j\delta}2^{-\frac{jn}{r_1}}\Big(\frac{1}{|B|}\int_B|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}}\\ &\times\Big(\frac{1}{|2^jB|}\int_{2^jB}|f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}}\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^{(2)}_{\vec{r},\delta}(f_1,f_2)(x)=\sup_{B\ni x}&\sum_{j=1}^{\infty}2^{j\delta}2^{-\frac{jn}{r_2}}\Big(\frac{1}{|B|}\int_B|f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}}\\ &\times\Big(\frac{1}{|2^jB|}\int_{2^jB}|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}},\end{aligned}$$

respectively. It is obvious that for any $\delta<0$, $x\in\mathbb{R}^n$ and $k=1,2$,

$$\mathcal{M}^{(k)}_{\vec{r},\delta}(f_1,f_2)(x)\lesssim\mathcal{M}_{\vec{r}}(f_1,f_2)(x).$$

For the case of $\delta=0$ and $r_1=r_2=1$, these operators were introduced by Grafakos et al. in [9]. Although we do not know if the operator $\mathcal{M}_{\vec{r}}$ can be applied to prove Theorem 1.1, as the operator $\mathcal{M}$ do in the proof of the weighted compactness of the commutator of multilinear Calderón-Zygmund operators (see [2]), we will see that the operator $\mathcal{M}^{(k)}_{\vec{r},\delta}$ $(k=1,2)$ are suitable replacement of $\mathcal{M}_{\vec{r}}$ in our argument.

As it is well known, for a weight $w\in A_\infty(\mathbb{R}^n)$, there exists a positive constant θ, such that for any ball $B\subset\mathbb{R}^n$ and any measurable set $E\subset B$,

$$\frac{w(E)}{w(B)}\lesssim\Big(\frac{|E|}{|B|}\Big)^{\theta}. \tag{2.1}$$

For a fixed $\theta\in(0,1)$, set

$$R_\theta=\{w\in A_\infty(\mathbb{R}^n):\, w\,\text{satisfies (2.1)}\}.$$

Our result concerning the operators $\mathcal{M}^{(k)}_{\vec{r},\delta}$ can be stated as follows.

Theorem 2.1 *Let $r_1,r_2\in(0,\infty)$ and $\delta\in\mathbb{R}$, $p_1\in[r_1,\infty)$ and $p_2\in[r_2,\infty)$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Let w_1,w_2 be weights such that $\vec{w}\in A_{\vec{p}/\vec{r}}(\mathbb{R}^{2n})$ and $\nu_{\vec{w}}\in R_\theta$ for some θ such that $\delta<n\theta\min\{\frac{1}{p_1},\frac{1}{p_2}\}$. Then both of the operators $\mathcal{M}^{(1)}_{\vec{r},\delta}$, $\mathcal{M}^{(2)}_{\vec{r},\delta}$ are bounded from $L^{p_1}(\mathbb{R}^n,w_1)\times L^{p_2}(\mathbb{R}^n,w_2)$ to $L^{p,\infty}(\mathbb{R}^n,\nu_{\vec{w}})$. Moreover, if $p_k\in(r_k,\infty)$ with $k=1,2$, then these operators are bounded from $L^{p_1}(\mathbb{R}^n,w_1)\times L^{p_2}(\mathbb{R}^n,w_2)$ to $L^{p}(\mathbb{R}^n,\nu_{\vec{w}})$.*

To prove Theorem 2.1, we need the following characterization of $A_{\vec{p}/\vec{r}}(\mathbb{R}^{2n})$, which was proved in [17].

Lemma 2.1 *Let w_1, w_2 be weights, $p_1, p_2, p \in (0,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $r_k \in (0, p_k]$ $(k = 1,2)$. Then the following conditions are equivalent:*

(i) $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{2n})$;

(ii) $\nu_{\vec{w}} \in A_{p/r}(\mathbb{R}^n)$, *and for* $k = 1,2$, $w_k^{-\frac{1}{\frac{p_k}{r_k}-1}} \in A_{p_k r_k/r(p_k-r_k)}(\mathbb{R}^n)$ *if* $r_k \neq p_k$ *or* $w_k^{\frac{r}{p_k}} \in A_1(\mathbb{R}^n)$ *if* $r_k = p_k$, *here* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$.

Proof of Theorem 2.1 We first consider the case of $p_k \in (r_k, \infty)$ with $k = 1,2$. Since the argument for $\mathcal{M}^{(1)}_{\vec{r},\delta}$ and $\mathcal{M}^{(2)}_{\vec{r},\delta}$ are very similar, we only consider the operator $\mathcal{M}^{(1)}_{\vec{r},\delta}$. We will employ the ideas used in [9]. Let $M^c_{\nu_{\vec{w}}}$ be the centered maximal operator defined by

$$M^c_{\nu_{\vec{w}}} f(x) = \sup_{B:\,\text{ball centered at}\, x} \frac{1}{\nu_{\vec{w}}(B)} \int_B |f(y)| \nu_{\vec{w}}(y)\,\mathrm{d}y.$$

As it was pointed out in [9], it suffices to prove that for some $q_1, q_2 \in (0,1)$,

$$\mathcal{M}^{(1)}_{\vec{r},\delta}(f_1,f_2)(x) \lesssim \prod_{k=1}^{2} \Big\{ M^c_{\nu_{\vec{w}}}\Big(\Big(\frac{|f_k|^{p_k} w_k}{\nu_{\vec{w}}}\Big)^{q_k}\Big)(x)\Big\}^{\frac{1}{q_k p_k}}. \tag{2.2}$$

For each fixed k, we know by Lemma 2.1 that $w_k^{-\frac{1}{\frac{p_k}{r_k}-1}} \in A_{p_k r_k/r(p_k-r_k)}(\mathbb{R}^n)$, and so there exists a positive constant $\sigma_k > 1$ such that for any ball B,

$$\Big(\frac{1}{|B|}\int_B w_k^{-\frac{\sigma_k}{\frac{p_k}{r_k}-1}}(y)\,\mathrm{d}y\Big)^{\frac{1}{\sigma_k}} \lesssim \frac{1}{|B|}\int_B w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(y)\,\mathrm{d}y. \tag{2.3}$$

For $k = 1,2$, let

$$q_k = \frac{p r_k}{p r_k + r(p_k - r_k)\big(1 - \frac{1}{\sigma_k}\big)}, \quad \gamma_k = \frac{r(p_k q_k - r_k)}{r_k (p - r)(1 - q_k)}.$$

It is obvious that $\frac{p_k q_k}{r_k} > 1$, $\gamma_k > 1$, and

$$\frac{q_k \gamma_k'}{p_k q_k - r_k} = \frac{q_k r}{r(p_k q_k - r_k) - r_k(p-r)(1-q_k)}, \tag{2.4}$$

$$\frac{q_k(p_k - r_k)}{(p_k q_k - r_k) - r_k\Big(\frac{p}{r} - 1\Big)(1 - q_k)} = \sigma_k. \tag{2.5}$$

An application of the Hölder inequality gives that

$$\begin{aligned}\Big(\int_B |f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} \lesssim{}& \Big(\int_B |f_1(y)|^{q_1 p_1} w_1^{q_1}(y) \nu_{\vec{w}}^{1-q_1}(y)\mathrm{d}y\Big)^{\frac{1}{q_1 p_1}} \\ &\times \Big(\int_B \Big(w_1^{q_1}(y)\nu_{\vec{w}}^{1-q_1}(y)\Big)^{-\frac{1}{\frac{p_1 q_1}{r_1}-1}}\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{q_1 p_1}}\end{aligned} \tag{2.6}$$

and

$$\int_B \Big(w_1^{q_1}(y)\nu_{\vec{w}}^{1-q_1}(y)\Big)^{-\frac{1}{\frac{p_1 q_1}{r_1}-1}}\mathrm{d}y \le \Big(\int_B w_1^{-\frac{q_1\gamma_1'}{\frac{p_1 q_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{\gamma_1'}} \Big(\int_B \nu_{\vec{w}}^{-\frac{1}{\frac{p}{r}-1}}(y)\,\mathrm{d}y\Big)^{\frac{1}{\gamma_1}}. \tag{2.7}$$

On the other hand, we have by the inequalities (2.3)–(2.5) that

$$\begin{aligned}\int_B w_1^{-\frac{q_1\gamma_1'}{\frac{p_1q_1}{r_1}-1}}(y)\mathrm{d}y &= \int_B w_1^{-\frac{1}{\frac{p_1}{r_1}-1}\frac{q_1(p_1-r_1)}{(p_1q_1-r_1)-r_1(\frac{p}{r}-1)(1-q_1)}}(y)\mathrm{d}y\\ &\lesssim |B|^{1-\frac{q_1\gamma_1'(p_1-r_1)}{p_1q_1-r_1}}\Big(\int_B w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\,\mathrm{d}y\Big)^{\frac{q_1\gamma_1'(p_1-r_1)}{p_1q_1-r_1}}. \end{aligned}\tag{2.8}$$

Note that

$$\frac{1}{\gamma_1}\Big(\frac{1}{r_1}-\frac{1}{q_1p_1}\Big)+\Big(1-\frac{q_1\gamma_1'(p_1-r_1)}{p_1q_1-r_1}\Big)\frac{1}{\gamma_1'}\Big(\frac{1}{r_1}-\frac{1}{p_1q_1}\Big)=\frac{1}{p_1}-\frac{1}{p_1q_1}.$$

Combining the inequalities (2.6)–(2.8) then yields

$$\begin{aligned}\Big(\int_B|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} &\lesssim \Big(\int_B|f_1(y)|^{p_1q_1}w_1^{q_1}(y)\nu_{\vec{w}}^{1-q_1}(y)\mathrm{d}y\Big)^{\frac{1}{p_1q_1}}\\ &\quad\times\Big(\frac{1}{|B|}\int_B w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}}\\ &\quad\times|B|^{\frac{1}{r_1}-\frac{1}{p_1q_1}}\Big(\frac{1}{|B|}\int_B\nu_{\vec{w}}^{-\frac{1}{\frac{p}{r}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{\gamma_1}(\frac{1}{r_1}-\frac{1}{q_1p_1})}.\end{aligned}$$

Recall that $\nu_{\vec{w}}\in A_{p/r}(\mathbb{R}^n)$. Thus for each ball B,

$$\begin{aligned}\Big(\int_B|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} &\lesssim \Big(\frac{1}{\nu_{\vec{w}}(B)}\int_B|f_1(y)|^{p_1q_1}w_1^{q_1}(y)\nu_{\vec{w}}^{1-q_1}(y)\mathrm{d}y\Big)^{\frac{1}{p_1q_1}}\\ &\quad\times\Big(\frac{\nu_{\vec{w}}(B)}{|B|}\Big)^{\frac{1}{p_1}}|B|^{\frac{1}{r_1}}\Big(\frac{1}{|B|}\int_B w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\Big(\int_{2^jB}|f_2(y)|^{r_2}\mathrm{d}y\Big)^{\frac{1}{r_2}} &\lesssim \Big(\frac{1}{\nu_{\vec{w}}(2^jB)}\int_{2^jB}|f_2(z)|^{p_2q_2}w_2^{q_2}(z)\nu_{\vec{w}}^{1-q_2}(z)\mathrm{d}z\Big)^{\frac{1}{p_2q_2}}\\ &\quad\times(\nu_{\vec{w}}(2^jB))^{\frac{1}{p_2}}\Big(\int_{2^jB}w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}.\end{aligned}$$

Therefore, for each fixed $x\in\mathbb{R}^n$ and ball B containing x,

$$\begin{aligned}&\sum_{j=1}^{\infty}2^{\delta j}2^{-\frac{jn}{r_1}}\Big(\frac{1}{|B|}\int_B|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}}\Big(\frac{1}{|2^jB|}\int_{2^jB}|f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}}\\ &\lesssim\prod_{k=1}^{2}\Big\{M_{\nu_{\vec{w}}}^c\Big(\Big(\frac{|f_k|^{p_k}w_k}{\nu_{\vec{w}}}\Big)^{q_k}\Big)(x)\Big\}^{\frac{1}{q_kp_k}}\Big(\frac{\nu_{\vec{w}}(B)}{|B|}\Big)^{\frac{1}{p_1}}\\ &\quad\times\sum_{j=1}^{\infty}2^{\delta j}2^{-\frac{jn}{p_1}}\Big(\frac{\nu_{\vec{w}}(2^jB)}{|2^jB|}\Big)^{\frac{1}{p_2}}\prod_{k=1}^{2}\Big(\frac{1}{|2^jB|}\int_{2^jB}w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(y_k)\mathrm{d}y_k\Big)^{\frac{1}{r_k}-\frac{1}{p_k}}.\end{aligned}$$

This, along with the fact that $\vec{w} \in A_{\vec{p}/\vec{r}}(\mathbb{R}^{2n})$ and the fact that $\frac{\nu_{\vec{w}}(B)}{\nu_{\vec{w}}(2^jB)} \lesssim 2^{-jn\theta}$, leads to that

$$\begin{aligned}&\sum_{j=1}^{\infty} 2^{\delta j} 2^{-\frac{jn}{r_1}} \Big(\frac{1}{|B|}\int_B |f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} \Big(\frac{1}{|2^jB|}\int_{2^jB} |f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}}\\ &\lesssim \prod_{k=1}^{2}\Big\{M^c_{\nu_{\vec{w}}}\Big(\Big(\frac{|f_k|^{p_k}w_k}{\nu_{\vec{w}}}\Big)^{q_k}\Big)(x)\Big\}^{\frac{1}{q_kp_k}} \Big(\frac{\nu_{\vec{w}}(B)}{|B|}\Big)^{\frac{1}{p_1}} \times \sum_{j=1}^{\infty} 2^{\delta j}2^{-\frac{jn}{p_1}}\Big(\frac{\nu_{\vec{w}}(2^jB)}{|2^jB|}\Big)^{-\frac{1}{p_1}}\\ &\lesssim \prod_{k=1}^{2}\Big\{M^c_{\nu_{\vec{w}}}\Big(\Big(\frac{|f_k|^{p_k}w_k}{\nu_{\vec{w}}}\Big)^{q_k}\Big)(x)\Big\}^{\frac{1}{q_kp_k}},\end{aligned}$$

since $\delta < \frac{n\theta}{p_1}$. This establishes (2.2).

For the case of $p_k = r_k$ with $k = 1, 2$, the proof is similar to the case of $p_k \in (r_k, \infty)$ and is more simple. In fact, for each $x \in \mathbb{R}^n$ and ball $B \subset \mathbb{R}^n$ containing x, as in the proof of (2.2), we can verify that for $k = 1, 2$,

$$\begin{aligned}\Big(\int_B |f_k(y)|^{r_k}\mathrm{d}y\Big)^{\frac{1}{r_k}} &\lesssim \Big(\frac{\nu_{\vec{w}}(B)}{|B|}\Big)^{\frac{1}{p_k}} |B|^{\frac{1}{r_k}} \Big\{M^c_{\nu_{\vec{w}}}\Big(\Big(\frac{|f_k|^{p_k}w_k}{\nu_{\vec{w}}}\Big)\Big)(x)\Big\}^{\frac{1}{p_k}}\\ &\quad\times \Big(\frac{1}{|B|}\int_B w_k^{-\frac{1}{\frac{p_k}{r_k}-1}}(y_k)\mathrm{d}y_k\Big)^{\frac{1}{r_k}-\frac{1}{p_k}},\end{aligned}$$

which implies that

$$\mathcal{M}^{(1)}_{\vec{r},\delta}(f_1, f_2)(x) \lesssim \prod_{k=1}^{2}\Big\{M^c_{\nu_{\vec{w}}}\Big(\Big(\frac{|f_k|^{p_k}w_k}{\nu_{\vec{w}}}\Big)\Big)(x)\Big\}^{\frac{1}{p_k}}.$$

and then shows that $\mathcal{M}^{(1)}_{\vec{r},\delta}(f_1, f_2)$ is bounded from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^{p,\infty}(\mathbb{R}^n, \nu_{\vec{w}})$.

3 Proof of Theorem 1.1

Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ and $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$ satisfy (1.3). For $\kappa \in \mathbb{Z}$, define

$$\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = \Phi(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2)\sigma(\xi_1, \xi_2).$$

Then $\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = \sigma_\kappa(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2)$ and

$$\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = 2^{2\kappa n}\mathcal{F}^{-1}\sigma_\kappa(2^\kappa\xi_1, 2^\kappa\xi_2),$$

where $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f. For a positive integer N, let

$$\sigma^N(\xi_1, \xi_2) = \sum_{|\kappa|\le N} \widetilde{\sigma}_\kappa(\xi_1, \xi_2), \quad K^N(x; y_1, y_2) = \mathcal{F}^{-1}\sigma^N(x - y_1, x - y_2).$$

For an integer k with $1 \le k \le m$ and $x, y_1, y_2, x' \in \mathbb{R}^n$, let

$$W^N(x, x'; y_1, y_2) = K^N(x; y_1, y_2) - K^N(x'; y_1, y_2).$$

Lemma 3.1 *Let $q_1, q_2 \in [2, \infty)$, and $s_1, s_2 \ge 0$. Then*

$$\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}\sigma_\kappa(\xi_1, \xi_2)|^{q_1}\langle\xi_1\rangle^{s_1}\mathrm{d}\xi_1\Big)^{\frac{q_2}{q_1}}\langle\xi_2\rangle^{s_2}\mathrm{d}\xi_2\Big)^{\frac{1}{q_2}} \lesssim \|\sigma_\kappa\|_{W^{\frac{s_1}{q_1},\frac{s_2}{q_2}}(\mathbb{R}^{2n})}.$$

For the proof of Lemma 3.1, see Appendix A in [8].

Lemma 3.2 *Let σ be a bilinear multiplier satisfying* (1.5) *for some $s \in [0,\infty)$, $r_1, r_2 \in (1,2]$ and $\gamma \in (0,s]$. Then for every $x \in \mathbb{R}^n$ and $R > 0$,*

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{R\le|x-y_1|<2R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim(2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)+\mathcal{M}_{\vec{r}}(f_1,f_2)(x))\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\int_{|x-y_2|\ge R}\int_{|x-y_1|<R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim(2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).\end{aligned}\tag{3.2}$$

Furthermore, if $\gamma \in (0,s]$ and $-\gamma+\frac{n}{r_1}+\frac{n}{r_2}+1>0$, then

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{|x-y_1|<R}|x-y_1||\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim(2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}R(\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)+\mathcal{M}_{\vec{r}}(f_1,f_2)(x)).\end{aligned}\tag{3.3}$$

Proof By the Hölder inequality and Lemma 3.1, we have that for each $l \in \mathbb{Z}$,

$$\begin{aligned}&\int_{|x-y_2|<2^{l-1}R}\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim(2^\kappa 2^lR)^{-\gamma}\Big(\int_{|x-y_2|<2^{l-1}R}\Big(\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)|^{r_1'}\\&\quad\times\langle 2^\kappa(x-y_1)\rangle^{\gamma r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\prod_{k=1}^2\Big(\int_{B(x,2^lR)}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}}\\&\lesssim 2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})}(2^lR)^{-\gamma}\prod_{k=1}^2\Big(\int_{B(x,2^lR)}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\int_{C(x,2^j2^{l-1}R)}\int_{B(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim(2^j2^\kappa2^lR)^{-\gamma}\Big(\int_{C(x,2^j2^{l-1}R)}\Big(\int_{B(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y,x-z)|^{r_1'}\mathrm{d}y\Big)^{\frac{r_2'}{r_1'}}\\&\quad\times\langle 2^\kappa(x-z)\rangle^{\gamma r_2'}\mathrm{d}z\Big)^{\frac{1}{r_2'}}\Big(\int_{B(x,2^lR)}|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}}\Big(\int_{B(x,2^l2^jR)}|f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}}\\&\lesssim(2^{j+\kappa+l}R)^{-\gamma}2^{\kappa(\frac{n}{r_1}+\frac{n}{r_2})}\Big(\int_{B(x,2^lR)}|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}}\\&\quad\times\Big(\int_{B(x,2^l2^jR)}|f_2(w)|^{r_2}\mathrm{d}w\Big)^{\frac{1}{r_2}}.\end{aligned}\tag{3.5}$$

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim (2^\kappa 2^l R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)+\mathcal{M}_{\vec{r}}(f_1,f_2)(x)),
\end{aligned}\tag{3.6}
$$

which gives (3.1) directly. We can also obtain from (3.5) (with $l=0$) that

$$
\begin{aligned}
&\int_{|x-y_2|\geq R}\int_{|x-y_1|<R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim 2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})}R^{-\gamma}\sum_{j=0}^{\infty}2^{-j\gamma}\Big(\int_{B(x,R)}|f_1(z)|^{r_1}\mathrm{d}z\Big)^{\frac{1}{r_1}}\times\Big(\int_{B(x,2^jR)}|f_2(w)|^{r_2}\mathrm{d}w\Big)^{\frac{1}{r_2}}\\
&\lesssim (2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).
\end{aligned}
$$

Finally, (3.6) implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|x-y_1|<R}|x-y_1||\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\leq\sum_{l=-\infty}^{-1}2^lR\int_{\mathbb{R}^n}\int_{C(x,2^lR)}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim (2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}R(\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)+\mathcal{M}_{\vec{r}}(f_1,f_2)(x)),
\end{aligned}
$$

since $-\gamma+\frac{n}{r_1}+\frac{n}{r_2}+1>0$. This completes the proof of Lemma 3.2.

Remark 3.1 Let σ be a bilinear multiplier satisfying (1.5) for some $s\in[0,\infty)$, $r_1,r_2\in(1,2]$ and $\gamma\in(0,s]$. As in the proof of (3.2), we can verify that, for each $R>0$ and $x,y\in\mathbb{R}$ with $|x-y|<R$,

$$
\begin{aligned}
&\int_{|y-y_2|\geq R}\int_{|y-y_1|<R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(y-y_1,y-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim (2^\kappa R)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).
\end{aligned}\tag{3.7}
$$

Lemma 3.3 *Let σ be a bilinear multiplier satisfying* (1.5) *for some $s\in[0,\infty)$, $r_1,r_2\in(1,2]$ and $\gamma\in(0,s]$. For $R>0$ and $x\in\mathbb{R}^n$ with $|x|>4R$, set*

$$
V^R_{\kappa,0}(x)=\int_{|y_2|\leq|x|}\int_{|y_1|\leq R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2
$$

and

$$
V^R_{\kappa,l}(x)=\int_{2^{l-1}|x|<|y_2|\leq 2^l|x|}\int_{|y_1|\leq R}|\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2
$$

for positive integer l. Then for any weights w_1, w_2 and $p_k\in(r_k,\infty)$ with $k=1,2$,

$$
\begin{aligned}
V^R_{\kappa,0}(x)&\lesssim|x|^{-\gamma}2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})}\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}\\
&\quad\times\Big(\int_{B(0,R)}w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}}\Big(\int_{B(0,|x|)}w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}
\end{aligned}
$$

and

$$V_{\kappa,l}^R(x) \lesssim (2^l|x|)^{-\gamma} 2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}$$
$$\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,2^l|x|)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}.$$

Proof Note that when $|y_1| \le R$ and $|x| > 2R$, $|x-y_1| \ge \frac{|x|}{2}$. As in the proof of Lemma 3.2, we obtain by Lemma 3.1 and the Hölder inequality,

$$\begin{aligned} V_{\kappa,0}^R(x) &\lesssim \Big(\int_{|y_2|\le|x|} \Big(\int_{|x-y_1|\ge\frac{|x|}{2}} |\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}} \\ &\quad\times \Big(\int_{B(0,R)} |f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} \Big(\int_{B(0,|x|)} |f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}} \\ &\lesssim (2^\kappa|x|)^{-\gamma} 2^{\kappa(\frac{n}{r_1}+\frac{n}{r_2})} \Big(\int_{B(0,R)} |f_1(y_1)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} \times \Big(\int_{B(0,|x|)} |f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}} \\ &\lesssim |x|^{-\gamma} 2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)} \\ &\quad\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,|x|)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}. \end{aligned}$$

Similarly, for $l \ge 1$, we have that

$$\begin{aligned} V_{\kappa,l}^R(x) &\lesssim \Big(\int_{C(0,2^l|x|)} \Big(\int_{|y_1|\le R} |\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}} \\ &\quad\times \Big(\int_{B(0,R)} |f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}} \Big(\int_{B(0,|x|)} |f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}} \\ &\lesssim |2^l x|^{-\gamma} 2^{\kappa(-\gamma+\frac{n}{r_1}+\frac{n}{r_2})} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)} \\ &\quad\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,|2^l x|)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}. \end{aligned}$$

This completes the proof of Lemma 3.3.

Lemma 3.4 *Let σ be a multiplier which satisfies* (1.5), $r_1, r_2 \in (1,2]$ *such that* $s \in \big(\frac{n}{r_1}+\frac{n}{r_2}, \frac{n}{r_1}+\frac{n}{r_2}+1\big)$. *Then for each $R>0$, $x, x' \in \mathbb{R}^n$ with $|x-x'| < \frac{R}{4}$, nonnegative integers j_1, j_2 with $j^* = \max\{j_1, j_2\} \ge 2$,*

$$\Big(\int_{S_{j_2}(B(x,R))} \Big(\int_{S_{j_1}(B(x,R))} |W^N(x,x';y_1,y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}} \lesssim \frac{|x-x'|^{s-\frac{n}{r_1}-\frac{n}{r_2}}}{|2^{j^*}B|^{\frac{s}{n}}}.$$

Proof We employ some estimates in [17]. Without loss of generality, we may assume that $j^* = j_1$. For $l \in \mathbb{Z}$, set

$$W_l(x,x';y_1,y_2) = \mathcal{F}^{-1}\widetilde{\sigma}_l(x-y_1,x-y_2) - \mathcal{F}^{-1}\widetilde{\sigma}_l(x'-y_1,x'-y_2)$$

and

$$J_{l;j_1\,j_2}=\Big(\int_{S_{j_2}(B(x,R))}\Big(\int_{S_{j_1}(B(x,R))}|W_l(x,x';\,y_1,y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}.$$

It was pointed out in [17] that

$$J_{l;\,j_1,j_2}\lesssim(2^{j_1}R)^{-s}2^{-l(s-\frac{n}{r_1}-\frac{n}{r_2})}.$$

On the other hand, by the proof of the inequality (3.6) in [17], we know that

$$J_{l;j_1,j_2}\lesssim 2^l|x-x'|(2^{j_1}R)^{-s}2^{-l(s-\frac{n}{r_1}-\frac{n}{r_2})}.$$

Therefore,

$$\begin{aligned}&\Big(\int_{S_{j_2}(B(x,R))}\Big(\int_{S_{j_1}(B(x,R))}|W^N(x,x';\,y_1,y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\\&\lesssim\sum_{l:\,2^l|x-x'|<1}J_{l;\,j_1,j_2}+\sum_{l:\,2^l|x-x'|\geq 1}J_{l;\,j_1,j_2}\lesssim\frac{|x-x'|^{s-\frac{n}{r_1}-\frac{n}{r_2}}}{|2^{j^*}B|^{\frac{s}{n}}}.\end{aligned}$$

This completes the proof of Lemma 3.4.

Lemma 3.5 *Let σ be a multiplier which satisfies* (1.5) *for some $s\in(n,2n]$, $t_1,t_2\in[1,2)$ such that $\frac{1}{t_1}+\frac{1}{t_2}=\frac{s}{n}$. Let $p_k\in(t_k,\infty)$ for $k=1,2$ and w_1,w_2 be weights such that $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$. Then for $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$,*

$$\|T_{\sigma,\vec{b}}(f_1,f_2)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\lesssim\sum_{j=1}^2\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\prod_{k=1}^2\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}.$$

Proof The proof here is fairly standard (see [4, 17]). For each fixed positive integer N, let $T_{\sigma,N}$ be the bilinear operator with kernel K^N in the sense that

$$T_{\sigma,N}(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}K^N(x;\,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2. \tag{3.8}$$

Let $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, $[b_1,T_{\sigma,N}]_1$ and $[b_2,T_{\sigma,N}]_2$ be the commutator of $T_{\sigma,N}$ as in (1.8) and (1.9) respectively. As in the proof of Theorem 3.1 in [17], we can prove that if $r_1,r_2\in(1,2]$ such that $\frac{s}{n}>\frac{1}{r_1}+\frac{1}{r_2}$, then for $\epsilon\in(0,t)$ with $\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}$,

$$M_\epsilon^\sharp([b_k,T_{\sigma,N}]_k(f_1,f_2))(x)\lesssim\|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)}(\mathcal{M}_{\vec{r}}(f_1,f_2)(x)+M_t(T_{\sigma,N}(f_1,f_2))(x)).$$

Now let $p_k\in(t_k,\infty)$, w_1,w_2 be weights such that $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$. We can choose $\delta\in(0,1)$ which is close to 1, such that $\vec{w}\in A_{\delta\vec{p}/\vec{t}}(\mathbb{R}^{2n})$ and $r_k=\frac{t_k}{\delta}<p_k$ for $k=1,2$. Recall that by Lemma 2.2, $\vec{w}\in A_{\vec{p}/\vec{t}}(\mathbb{R}^{2n})$ implies that $\nu_{\vec{w}}\in A_{p/t}(\mathbb{R}^n)$. It then follows that for $k=1,2$,

$$\begin{aligned}\|[b_k,T_{\sigma,N}]_k(f_1,f_2)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}&\lesssim\|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)}(\|M_t(T_{\sigma,N}(f_1,f_2))\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})}\\&\quad+\|\mathcal{M}_{\vec{r}}(f_1,f_2)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})})\\&\lesssim\|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)}\prod_{k=1}^2\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)},\end{aligned}$$

if $b_1, b_2 \in L^\infty(\mathbb{R}^n)$. Note that for $b_1, b_2 \in L^\infty(\mathbb{R}^n)$ and $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$,

$$\lim_{N\to\infty} [b_k, T_{\sigma,N}]_k(f_1, f_2)(x) = [b_k, T_\sigma]_k(f_1, f_2)(x)$$

holds for almost everywhere $x \in \mathbb{R}^n$. Thus, by the Fatou lemma, for $k = 1, 2,\ b_1, b_2 \in L^\infty(\mathbb{R}^n)$ and $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$,

$$\|[b_k, T_\sigma]_k(f_1, f_2)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})} \lesssim \|b_k\|_{\mathrm{BMO}(\mathbb{R}^n)} \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}.$$

This, via a standard argument leads to our desired conclusion.

For a positive integer N, let $\mathscr{T}_{\sigma,N}$ be the operator defined by

$$\mathscr{T}_{\sigma,N}(f_1, f_2)(x) = \sup_{\epsilon>0} \Big| \int_{\max\limits_{1\le k\le 2} |x-y_k|>\epsilon} K^N(x;\, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2 \Big|.$$

Lemma 3.6 *Let σ be a multiplier which satisfies* (1.5) *for some $s \in (n, 2n]$, $r_1, r_2 \in (1, 2]$ such that $s \in \big(\frac{n}{r_1} + \frac{n}{r_2}, \frac{n}{r_1} + \frac{n}{r_2} + 1\big)$. Then for any $\gamma < \frac{n}{r_1} + \frac{n}{r_2}$, $\tau \in (0, \min\{1, r\})$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, and $x \in \mathbb{R}^n$,*

$$\mathscr{T}_{\sigma,N}(f_1, f_2)(x) \lesssim M_\tau(T_{\sigma,N}(f_1, f_2))(x) + \sum_{k=1}^{2} \mathcal{M}^{(k)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1, f_2)(x).$$

Proof We employ the ideas used in [9, 13]. For each fixed $\epsilon > 0$, let

$$T_{\sigma,N;\,\epsilon}(f_1, f_2)(x) = \int_{\max\limits_{1\le k\le 2} |x-y_k|>\epsilon} K^N(x;\, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2$$

and

$$\widetilde{T}_{\sigma,N;\,\epsilon}(f_1, f_2)(y, x) = \int_{\min\limits_{1\le k\le 2} |x-y_k|>\epsilon} K^N(y;\, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}y_1 \mathrm{d}y_2.$$

For functions f_1 and f_2, let

$$f_k^1(y_k) = f_k(y_k)\chi_{B(x,\epsilon)}(y_k), \quad f_k^2(y_k) = f_k(y_k)\chi_{\mathbb{R}^n\setminus B(x,\epsilon)}(y_k), \quad k = 1, 2.$$

A trivial computation shows that for $y \in B\big(x, \frac{\epsilon}{2}\big)$,

$$\begin{aligned} |\widetilde{T}_{\sigma,N;\epsilon}(f_1, f_2)(x, x)| &\le \big|\widetilde{T}_{\sigma,N;\epsilon}(f_1, f_2)(x, x) - \widetilde{T}_{\sigma,N;\epsilon}(f_1, f_2)(y, x)\big| \\ &\quad + |\widetilde{T}_{\sigma,N;\,\epsilon}(f_1, f_2)(y, x)| \\ &\lesssim \int_{\min\limits_{1\le k\le 2} |x-y_k|>\epsilon} \big|W^N(x, y; y_1, y_2) f_1(y_1) f_2(y_2)| \mathrm{d}y_1 \mathrm{d}y_2 \\ &\quad + \big|T_{\sigma,N}(f_1, f_2)(y) - T_{\sigma,N}(f_1^1, f_2^1)(y)\big| \\ &\quad + \big|T_{\sigma,N}(f_1^1, f_2^2)(y)\big| + \big|T_{\sigma,N}(f_1^2, f_2^1)(y)\big|. \end{aligned} \tag{3.9}$$

We obtain from Lemma 3.5 that

$$\begin{aligned}
&\int_{\min\limits_{1\le k\le 2}|x-y_k|>\epsilon}|W^N(x,y;y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim \sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\Big(\int_{S_{j_2}(B(x,\epsilon))}\Big(\int_{S_{j_1}(B(x,\epsilon))}|W^N(x,y;y_1,y_2)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}\\
&\quad\times\prod_{k=1}^{2}\Big(\int_{S_{j_k}(B(x,\epsilon))}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}}\\
&\lesssim \mathcal{M}_{\vec r}(f_1,f_2)(x). \qquad (3.10)
\end{aligned}$$

On the other hand, it follows from (3.7) that for $y\in B\big(x,\frac{\epsilon}{2}\big)$,

$$\begin{aligned}
&|T_{\sigma,N}(f_1^1,f_2^2)(y)|+|T_{\sigma,N}(f_1^2,f_2^1)(y)|\\
&\lesssim \sum_{|\kappa|\le N:\,2^\kappa\epsilon>1}(2^\kappa\epsilon)^{-s+\frac{n}{r_1}+\frac{n}{r_2}}\sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-s+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)\\
&\quad+\sum_{|\kappa|\le N:\,2^\kappa\epsilon\le 1}(2^\kappa\epsilon)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}\sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)\\
&\lesssim \sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x), \qquad (3.11)
\end{aligned}$$

where in the last inequality, we have invoked the estimate

$$\mathcal{M}^{(k)}_{\vec r,-s+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)\lesssim \mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x),$$

since $-s+\frac{n}{r_1}+\frac{n}{r_2}<-\gamma+\frac{n}{r_1}+\frac{n}{r_2}$. Similarly, we have that

$$\begin{aligned}
&|T_{\sigma,N;\,\epsilon}(f_1,f_2)(x)-\widetilde{T}_{\sigma,N;\epsilon}(f_1,f_2)(x,x)|\\
&\lesssim \int_{\substack{\max\limits_{1\le k\le 2}|x-y_k|>\epsilon\\ \min\limits_{1\le k\le 2}|x-y_k|<\epsilon}}|K^N(x;\,y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim \sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x). \qquad (3.12)
\end{aligned}$$

Combining the estimates (3.9)–(3.12) then leads to that for $y\in B\big(x,\frac{\epsilon}{2}\big)$,

$$\begin{aligned}
|T_{\sigma,N;\epsilon}(f_1,f_2)(x)|&\lesssim|\widetilde{T}_{\sigma,N;\,\epsilon}(f_1,f_2)(x,x)|+\sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)\\
&\lesssim|T_{\sigma,N}(f_1,f_2)(y)|+|T_{\sigma,N}(f_1^1,f_2^1)(y)|+\sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec r,-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).
\end{aligned}$$

Recall that $T_{\sigma,N}$ is bounded from $L^{r_1}(\mathbb{R}^n)\times L^{r_2}(\mathbb{R}^n)$ to $L^{r,\infty}(\mathbb{R}^n)$ (see [8, 17]). Applying the argument in the proof of the Kolmogorov inequality (see also [9, 13]), tells us that for

$\tau \in (0, \min\{1, r\})$,

$$\Big(\frac{1}{|B(x,\frac{\epsilon}{2})|}\int_{B(x,\frac{\epsilon}{2})}|T_{\sigma,N}(f_1^1,f_2^1)(y)|^{\tau}\mathrm{d}y\Big)^{\frac{1}{\tau}} \lesssim \prod_{k=1}^{2}\Big(\frac{1}{|B(x,\epsilon)|}\int_{B(x,\epsilon)}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}}.$$

Therefore, for each $x \in \mathbb{R}^n$ and $\epsilon > 0$,

$$\begin{aligned}|T_{\sigma,N;\,\epsilon}(f_1,f_2)(x)| &\lesssim \Big(\frac{1}{|B(x,\frac{\epsilon}{2})|}\int_{B(x,\frac{\epsilon}{2})}|T_{\sigma,N}(f_1,f_2)(y)|^{\tau}\mathrm{d}y\Big)^{\frac{1}{\tau}} \\ &\quad + \Big(\frac{1}{|B(x,\frac{\epsilon}{2})|}\int_{B(x,\frac{\epsilon}{2})}|T_{\sigma,N}(f_1^1,f_2^1)(y)|^{\tau}\mathrm{d}y\Big)^{\frac{1}{\tau}} \\ &\quad + \sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x) \\ &\lesssim M_{\tau}(T_{\sigma,N}(f_1,f_2))(x) + \sum_{k=1}^{2}\mathcal{M}^{(k)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x),\end{aligned}$$

which gives us the desired conclusion.

Let φ be a non-negative function in $C_0^{\infty}(\mathbb{R}^{3n})$, which satisfies that $\operatorname{supp}\varphi \subset \{(x,y_1,y_2) : \max\{|x|,|y_1|,|y_2|\} < 1\}$, $\int_{\mathbb{R}^{3n}}\varphi(u)\mathrm{d}u = 1$. For $\beta > 0$, let $\chi^{\beta} = \chi^{\beta}(x,y_1,y_2)$ be the characteristic function of the set $\big\{(x,y_1,y_2) : \max\limits_{k=1,2}|x-y_k| \geq 3\frac{\beta}{2}\big\}$, and let

$$\psi^{\beta}(x;\,y_1,y_2) = \varphi_{\beta} * \chi^{\beta}(x;\,y_2,y_2),$$

where $\varphi_{\beta}(x,y_1,y_2) = \big(\frac{\beta}{4}\big)^{-3n}\varphi\big(\frac{4x}{\beta},\frac{4y_1}{\beta},\frac{4y_2}{\beta}\big)$. As it was pointed out in [2], $\psi^{\beta} \in C^{\infty}(\mathbb{R}^{3n})$, $\|\psi^{\beta}\|_{L^{\infty}} \leq 1$, $\operatorname{supp}\psi^{\beta} \subset \{(x;y_1,y_2) : \max\limits_{k=1,2}|x-y_k| \geq \beta\}$, and $\psi^{\beta}(x,y_1,y_2) = 1$ if $\max\limits_{k=1,2}|x-y_k| \geq 2\beta$. For a fixed $N \in \mathbb{N}$, let $T^{\beta}_{\sigma,N}$ be the bilinear operator defined by

$$T^{\beta}_{\sigma,N}(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}}\psi^{\beta}(x;y,z)K^N(x;y,z)f_1(y)f_2(z)\mathrm{d}y\mathrm{d}z. \tag{3.13}$$

As usual, for $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, let $[b_1, T^{\beta}_{\sigma,N}]_1$, $[b_2, T^{\beta}_{\sigma,N}]_2$ be the commutators of $T^{\beta}_{\sigma,N}$ as in (1.8)–(1.9).

Lemma 3.7 *Let σ be a multiplier satisfying* (1.5) *for some $s \in (n, 2n]$, $T_{\sigma,N}$ and $T^{\beta}_{\sigma,N}$ be the operators defined by* (3.8) *and* (3.13) *respectively. Let $r_1, r_2 \in (1,2]$ such that $s \in \big(\frac{n}{r_1}+\frac{n}{r_2}, \frac{n}{r_1}+\frac{n}{r_2}+1\big)$. Then for any $\gamma < \frac{n}{r_1}+\frac{n}{r_2}$,*

$$|[b_j,T_{\sigma,N}]_j(f_1,f_2)(x) - [b_j,T^{\beta}_{\sigma,N}]_j(f_1,f_2)(x)| \lesssim \beta\mathcal{M}^{(j)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x). \tag{3.14}$$

Proof Without loss of generality, we assume that $\|\nabla b_j\|_{L^{\infty}(\mathbb{R}^n)} = 1$. We deduce from

Lemma 3.2 that

$$\begin{aligned}
&|[b_j, T_{\sigma,N}]_j(f_1,f_2)(x) - [b_j, T^{\beta}_{\sigma,N}]_j(f_1,f_2)(x)| \\
&\lesssim \sum_{\kappa\in\mathbb{Z}} \int_{\max\limits_{k=1,2}|x-y_k|\le 2\beta} |x-y_j||\mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x-y_1,x-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \beta \sum_{\kappa\in\mathbb{Z}:2^\kappa\beta>1} (2^\kappa\beta)^{-s+\frac{n}{r_1}+\frac{n}{r_2}} (\mathcal{M}^{(j)}_{\vec{r},-s+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x) + \mathcal{M}_{\vec{r}}(f_1,f_2)(x)) \\
&\quad + \beta \sum_{\kappa\in\mathbb{Z}:2^\kappa\beta\le 1} (2^\kappa\beta)^{-\gamma+\frac{n}{r_1}+\frac{n}{r_2}} \big(\mathcal{M}^{(j)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x) + \mathcal{M}_{\vec{r}}(f_1,f_2)(x)\big) \\
&\lesssim \beta\mathcal{M}^{(j)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).
\end{aligned}$$

This completes the proof of Lemma 3.7.

Lemma 3.8 *Let $r\in(1,\infty)$, $w\in A_r(\mathbb{R}^n)$, $\mathcal{K}\subset L^r(\mathbb{R}^n,w)$. Suppose that*
(i) *$\mathcal{K}$ is bounded in $L^r(\mathbb{R}^n,w)$;*
(ii) *$\lim\limits_{A\to\infty}\int_{|x|>A}|f(x)|^r w(x)\,\mathrm{d}x = 0$, uniformly for $f\in\mathcal{K}$;*
(iii) *$\|f(\cdot)-f(\cdot+t)\|_{L^p(\mathbb{R}^n,w)}\to 0$ uniformly for $f\in\mathcal{K}$ as $|t|\to 0$.*
Then $\mathcal{K}$ is precompact in $L^r(\mathbb{R}^n,w)$.

This lemma was given in [5].

Proof of Theorem 1.1 We will employ some ideas from [2]. By Lemma 3.5, it suf5.690Td[())-1623(c

If we can prove this, it then follows from the Fatou lemma that both (3.15) and (3.16) are true with $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$ and $T^{\beta}_{\sigma,N}$ is replaced by T^{β}_{σ}, here T^{β}_{σ} is defined by

$$T^{\beta}_{\sigma}(f_1, f_2)(x) = \sum_{\kappa\in\mathbb{Z}} \int_{\mathbb{R}^n} \psi^{\beta}(x; y, z)\mathcal{F}^{-1}\widetilde{\sigma}_{\kappa}(x; y, z)f_1(y)f_2(z)\mathrm{d}y\mathrm{d}z.$$

Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^{p_k}(\mathbb{R}^n, w_k)$, we then know that (3.15) and (3.16) are true when $T^{\beta}_{\sigma,N}$ is replaced by T^{β}_{σ}. This, via Lemma 3.8, tells us that $[b_1, T^{\beta}_{\sigma}]_1$ is compact from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$. On the other hand, (3.14) together with the Fatou lemma and a familiar density argument, leads to that

$$\|[b_1, T_{\sigma}]_1(f_1, f_2) - [b_1, T^{\beta}_{\sigma}]_1(f_1, f_2)\|_{L^p(\mathbb{R}^n,\nu_{\vec{w}})} \lesssim \beta \prod_{k=1}^{2} \|f\|_{L^{p_k}(\mathbb{R}^n,w_k)}.$$

Therefore, $[b_1, T_{\sigma}]_1$ is compact from $L^{p_1}(\mathbb{R}^n, w_1) \times L^{p_2}(\mathbb{R}^n, w_2)$ to $L^p(\mathbb{R}^n, \nu_{\vec{w}})$.

We first prove the conclusion (a). Let $R > 0$ be large enough such that $\operatorname{supp} b_1 \subset B(0, R)$. For every fixed $x \in \mathbb{R}^n$ with $|x| > 2R$, set

$$\mathrm{U}^R_{N,0}(x) = \int_{|y_2|\le|x|} \int_{|y_1|\le R} |K^N(x - y_1, x - y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2$$

and

$$\mathrm{U}^R_{N,l}(x) = \int_{2^{l-1}|x|<|y_2|\le 2^l|x|} \int_{|y_1|\le R} |K^N(x - y_1, x - y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2.$$

We deduce from Lemma 3.3 that for integers $N > 0$ and $l \ge 0$,

$$\begin{aligned}
\mathrm{U}^R_{N,l}(x) &\lesssim \sum_{\kappa:2^{\kappa}R\ge 1} V^R_{\kappa,l}(x) + \sum_{\kappa:2^{\kappa}R\le 1} V^R_{\kappa,l}(x)\\
&\lesssim ((2^l|x|)^{-s}R^{s-\frac{n}{r_1}-\frac{n}{r_2}} + (2^l|x|)^{-\gamma}R^{\gamma-\frac{n}{r_1}-\frac{n}{r_2}}) \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}\\
&\quad\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,2^l|x|)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}},
\end{aligned}$$

if we choose $\gamma < \frac{n}{r_1} + \frac{n}{r_2}$. Let $A > 4R$. Recall that $p > 1$. It then follows directly that

$$\begin{aligned}
&\Big(\int_{2^{j-1}A<|x|\le 2^jA} \Big|[b_1, T^{\beta}_{\sigma,N}]_1(f_1, f_2)(x)\Big|^p \nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\lesssim \sum_{l=0}^{\infty} \Big(\int_{2^{j-1}A<|x|\le 2^jA} |\mathrm{U}^R_{N,l}(x)|^p\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\lesssim \sum_{l=0}^{\infty} \Big(\int_{B(0,2^jA)} \nu_{\vec{w}}(y)\mathrm{d}y\Big)^{\frac{1}{p}} (2^{j+l}A)^{-s}R^{s-\frac{n}{r_1}-\frac{n}{r_2}} \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}\\
&\quad\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,2^{l+j}A)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}\\
&\quad+ \sum_{l=0}^{\infty} \Big(\int_{B(0,2^jA)} \nu_{\vec{w}}(y)\mathrm{d}y\Big)^{\frac{1}{p}} (2^{j+l}A)^{-\gamma}R^{\gamma-\frac{n}{r_1}-\frac{n}{r_2}} \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}\\
&\quad\times \Big(\int_{B(0,R)} w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}} \Big(\int_{B(0,2^{l+j}A)} w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}.
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}&\sum_{l=0}^{\infty}\Big(\int_{B(0,2^jA)}\nu_{\vec{w}}(y)\mathrm{d}y\Big)^{\frac{1}{p}}(2^{j+l}A)^{-s}R^{s-\frac{n}{r_1}-\frac{n}{r_2}}\\&\quad\times\Big(\int_{B(0,R)}w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}}\Big(\int_{B(0,2^{l+j}A)}w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}\\&\lesssim R^{s-\frac{n}{r_1}-\frac{n}{r_2}}\sum_{l=0}^{\infty}(2^{l+j}A)^{-s+\frac{n}{r_1}+\frac{n}{r_2}}\\&\lesssim\Big(\frac{R}{A}\Big)^{s-\frac{n}{r_1}-\frac{n}{r_2}}2^{j(-s+\frac{n}{r_1}+\frac{n}{r_2})}.\end{aligned}$$

On the other hand, noting that $w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}\in A_\infty(\mathbb{R}^n)$, there exists a constant $\zeta\in\big(0,\frac{1}{r_1}+\frac{1}{r_2}\big)$ such that

$$\int_{B(0,R)}w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y_1)\mathrm{d}y_1\lesssim(2^{-(j+l)}RA^{-1})^{n\zeta}\int_{B(0,2^{j+l}A)}w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y_1)\mathrm{d}y_1,$$

which, in turn, implies that

$$\begin{aligned}&\sum_{l=0}^{\infty}\Big(\int_{B(0,2^jA)}\nu_{\vec{w}}(y)\mathrm{d}y\Big)^{\frac{1}{p}}(2^{j+l}A)^{-\gamma}R^{\gamma-\frac{n}{r_1}-\frac{n}{r_2}}\\&\quad\times\Big(\int_{B(0,R)}w_1^{-\frac{1}{\frac{p_1}{r_1}-1}}(y)\mathrm{d}y\Big)^{\frac{1}{r_1}-\frac{1}{p_1}}\Big(\int_{B(0,2^{l+j}A)}w_2^{-\frac{1}{\frac{p_2}{r_2}-1}}(z)\mathrm{d}z\Big)^{\frac{1}{r_2}-\frac{1}{p_2}}\\&\lesssim\Big(\frac{R}{A}\Big)^{\gamma+n\zeta-\frac{n}{r_1}-\frac{n}{r_2}}2^{j(-\gamma-n\zeta+\frac{n}{r_1}+\frac{n}{r_2})},\end{aligned}$$

if we choose

$$\gamma\in\Big(-n\zeta+\frac{n}{r_1}+\frac{n}{r_2},\frac{n}{r_1}+\frac{n}{r_2}\Big).$$

Thus, for $b_1\in C_0^\infty(\mathbb{R}^n)$, we have that for some constant $\eta>0$,

$$\Big(\int_{|x|>A}\Big|[b_1,T_{\sigma,N}^{\beta}]_1(f_1,f_2)(x)\Big|^p\nu_{\vec{w}}(x)\mathrm{d}x\Big)^{\frac{1}{p}}\lesssim\Big(\frac{R}{A}\Big)^{\eta}\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n,w_k)}.$$

This leads to the conclusion (a).

We turn our attention to conclusion (b). Let

$$\gamma\in\Big(0,\frac{n}{r_1}+\frac{n}{r_2}\Big).$$

Set

$$W^{N,\beta}(x+u,x;\,y_1,y_2)=K^{N,\beta}(x+u;\,y_1,y_2)-K^{N,\beta}(x;\,y_1,y_2),$$

and set

$$\begin{aligned}\mathrm{J}_1^{\beta}(f_1,f_2)(x)&=(b_1(x)-b_1(x+u))\int_{\mathbb{R}^{2n}}K^{N,\beta}(x;y,z)f_1(y)f_2(z)\mathrm{d}y\mathrm{d}z,\\\mathrm{J}_2^{\beta}(f_1,f_2)(x)&=\int_{\mathbb{R}^{2n}}W^{N,\beta}(x+u,x;y,z)(b_1(y)-b_1(x+u))f_1(y)f_2(z)\mathrm{d}y\mathrm{d}z.\end{aligned}$$

As in the proof of Lemma 3.7, we obtain by Lemma 3.2 that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{2n}} K^{N,\beta}(x;y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad -\int_{\max\limits_{k=1,2}|x-y_k|\geq 2\beta} K^{N}(x;y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2\Big| \\
&\lesssim \int_{\beta\leq \max\limits_{k=1,2}|x-y_k|\leq 2\beta} |K^{N}(x;y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x).
\end{aligned}$$

Thus,

$$|\mathrm{J}_1^{\beta}(f_1,f_2)(x)| \lesssim |u|\big(\mathscr{T}_{\sigma,N}(f_1,f_2)(x)+\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)\big). \tag{3.17}$$

Note that $|\psi^{\beta}(x+u;y_1;y_2)-\psi^{\beta}(x;y_1,y_2)|\lesssim \frac{|u|}{\beta}$, and

$$\begin{aligned}
|W^{N,\beta}(x+u,x;y_1,y_2)| \leq{}& |W^{N}(x+u,x;\,y_1,y_2)||\psi^{\beta}(x+u;\,y_1;\,y_2)| \\
&+|K^{N}(x;y_1,y_2)||\psi^{\beta}(x+u;y_1;y_2)-\psi^{\beta}(x;y_1,y_2)|.
\end{aligned}$$

Let $|u|\leq\frac{\beta}{2}$. By Lemma 3.2 and Lemma 3.4 and the argument used in the proof of Lemma 3.7, we deduce that

$$\begin{aligned}
|\mathrm{J}_2^{\beta}(f_1,f_2)(x)| \lesssim{}& \int_{\max\limits_{k=1,2}|x-y_k|>\frac{\beta}{2}} |W^{N}(x+u,x;y_1,y_2)f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&+\frac{|u|}{\beta}\int_{\substack{\max\limits_{k=1,2}|x-y_k|\leq 3\beta \\ \max\limits_{k=1,2}|x-y_k|\geq\frac{\beta}{2}}} |K^{N}(x;y_1,y_2)f_1(y_2)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
\lesssim{}& \sum_{\substack{j_1,j_2\geq 0\\ \max\{j_1,j_2\}\geq 1}} \Big(\int_{S_{j_2}(B(x,\frac{\beta}{4}))}\Big(\int_{S_{j_1}(B(x,\frac{\beta}{4}))}|W^{N}(x,x+u;y,z)|^{r_1'}\mathrm{d}y\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}z\Big)^{\frac{1}{r_2'}} \\
&\times\Big(\int_{S_{j_1}(B(x,\frac{\beta}{4}))}|f_1(y)|^{r_1}\mathrm{d}y\Big)^{\frac{1}{r_1}}\Big(\int_{S_{j_2}(B(x,\frac{\beta}{4}))}|f_2(z)|^{r_2}\mathrm{d}z\Big)^{\frac{1}{r_2}} \\
&+\frac{|u|}{\beta}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x) \\
\lesssim{}& \sum_{\max\{j_1,j_2\}\geq 1}\frac{|u|^{s-\frac{n}{r_1}-\frac{n}{r_2}}}{|2^{j^*}B(x,\frac{\beta}{4})|^{\frac{s}{n}}}\prod_{k=1}^{2}\Big(\int_{S_{j_k}(B(x,\frac{\beta}{4}))}|f_k(y_k)|^{r_k}\mathrm{d}y_k\Big)^{\frac{1}{r_k}} \\
&+\frac{|u|}{\beta}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x) \\
\lesssim{}& \Big(\frac{|u|}{\beta}\Big)^{\varrho}\mathcal{M}^{(1)}_{\vec{r},-\gamma+\frac{n}{r_1}+\frac{n}{r_2}}(f_1,f_2)(x)
\end{aligned} \tag{3.18}$$

with $\varrho=\min\big\{1,s-\frac{n}{r_1}-\frac{n}{r_2}\big\}$. Note that

$$|[b_1,T^{\beta}_{\sigma,N}]_1(f_1,f_2)(x+u)-[b_1,T^{\beta}_{\sigma,N}]_1(f_1,f_2)(x)|\lesssim\sum_{k=1}^{2}\mathrm{J}_k^{\beta}(f_1,f_2)(x).$$

The conclusion (b) now follows from (3.17)–(3.18), Lemma 3.6 and Theorem 2.1, if we choose γ such that $0 < -\gamma + \frac{n}{r_1} + \frac{n}{r_2} < n\theta \min\{\frac{1}{p_1}, \frac{1}{p_2}\}$.

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