# Growth and distortion theorems for a subclass of holomorphic mappings *

Qinghua Xu[1 †], Taishun Liu[2] and Huiming Xu[3]

*E–mail address*: xuqh@mail.ustc.edu.cn, lts@ustc.edu.cn.

*(1.College of Mathematics and Information Science, JiangXi Normal University, NanChang 330022, China ;*

*2.Department of Mathematics, Huzhou Teacher's College, Huzhou 313000, China)*;

*3.College of Mathematics, Physics and Information Engineering, ZheJiang Normal University, JinHua 321004, China)*

**Abstract** Let $X$ be a complex Banach space with norm $\|\cdot\|$, $B$ be the unit ball in $X$. In this paper, we introduce a class of holomorphic mappings $\mathcal{M}_g$ on $B$. Let $f(x)$ be a normalized locally biholomorphic mappings on $B$ such that $(Df(x))^{-1}f(x) \in \mathcal{M}_g$ and $x = 0$ is the zero of order $k+1$ of $f(x) - x$. We investigate the growth theorem for $f(x)$. As applications, the distortion theorems for the Jacobian matrix $J_f(z)$ are obtained, where $f(z)$ belongs to the subclasses of starlike mappings defined on the unit polydisc $D^n$ in $\mathbb{C}^n$. These results unify and generalize many known results.

**Keywords**: Growth theorem, Distortion theorem, Subclasses of starlike mappings

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## 1. Introduction

In the case of one complex variable, the following growth, distortion theorem is well-known [1].

**Theorem A**. *Let $f$ be a normalized univalent holomorphic function on the unit disc $D$ in $\mathbb{C}$. Then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}, \quad z \in D, \tag{1}$$

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \quad z \in D.$$

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†Corresponding author

However, in the case of several complex variables, Cartan [2] pointed out that the above theorem does not hold.

Barnard et al.[3] and Chuaqui [4] extended the growth theorem (1) to normalized starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. After that, many mathematicians investigate the growth theorems for subclasses of starlike mappings ([5], [6], [7], [8]).

As for the distortion theorems for subclasses of normalized biholomorphic mappings, Pfaltzgraff and Suffridge [9] obtained a distortion result for a subclass of starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. Xu and Liu ([10], [11]) obtained a sharp distortion theorem for a subclass of biholomorphic mappings. Recently, Liu et al.([12], [13], [14]) obtained a distortion theorem for quasi-convex mappings, starlike mappings and a subclass of quasi-convex mappings on the unit polydisc $D^n$ in $\mathbb{C}^n$, respectively.

In this paper, we shall obtain growth theorem for a class of biholomorphic mappings. From it, the distortion theorems for subclasses of starlike mappings are obtained. These results generalize the related works of several authors.

Let $X$ be a complex Banach space with norm $\|\cdot\|$, $X^*$ be the dual space of $X$, $B$ be the unit ball in $X$, $D$ be the open unit disc in $\mathbb{C}$, $D^n$ represent the open unit polydisk in $\mathbb{C}^n$. Let $(\partial D)^n(0,r)$ be the distinguished boundary of the polydisc of radius $r$ with the center 0. For each $x \in X \setminus \{0\}$, we define $T(x) = \{l_x \in X^* : \|l_x\| \leq 1, l_x(x) = \|x\|\}$. According to the Hahn-Banach theorem, $T(x)$ is nonempty. Let $H(B)$ be the set of all holomorphic mappings from $B$ into $X$. Notice that for fixed $x \in X$, $\forall\alpha(\neq 0) \in \mathbb{C}$, when $l_x$ is chosen and fixed, then $\|\frac{|\alpha|}{\alpha}l_x\| = \|l_x\| \leq 1$, and $\frac{|\alpha|}{\alpha}l_x(\alpha x) = \frac{|\alpha|}{\alpha}\alpha l_x(x) = |\alpha|\|x\| = \|\alpha x\|$, so we can set $l_{\alpha x} = \frac{|\alpha|}{\alpha}l_x$. A holomorphic mapping $f : B \to X$ is said to be biholomorphic if the inverse $f^{-1}$ exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fre̒chet derivative $Df(x)$(When $X = \mathbb{C}^n$, $Df(x)$ is denoted by $J_f(z)$, where $J_f(z)$ is the Jacobian matrix of $f$ at point $z$) has a bounded inverse for each $x \in B$. We say that $f$ is normalized if $f(0) = 0$ and $Df(0) = I$, where $I$ represents the identity operator from $X$ into $X$. Let $S(B)$ be the set of all normalized biholomorphic mappings. We say that $f$ is starlike if $f$ is biholomorphic on $B$ and $f(B)$ is starlike with respect to the origin. Let $S^*(B)$ be the set of normalized starlike mappings on $B$.

**Definition 1**. Let $g \in H(D)$ be a biholomorphic function such that $g(0) = 1$, $g(\bar{\xi}) = \overline{g(\xi)}$, for $\xi \in D$, $\Re eg(\xi) > 0$ on $\xi \in D$, and assume $g$ satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min\limits_{|\xi|=r} |g(\xi)| = \min\limits_{|\xi|=r} \Re eg(\xi) = g(-r) \\ \max\limits_{|\xi|=r} |g(\xi)| = \max\limits_{|\xi|=r} \Re eg(\xi) = g(r). \end{cases} \tag{2}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g = \left\{ p \in H(B) : p(0) = 0, Dp(0) = I, \frac{\|x\|}{l_x(p(x))} \in g(D), x \in B\backslash\{0\}, l_x \in T(x) \right\}.$$

In a slightly different manner, Definition 1 was considered by Kohr [15] on $B^n$ and by Graham et al.[16] on the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$. The set $\mathcal{M}_g$ has been important in the study of certain problems related to Löwner chains on the unit ball in $\mathbb{C}^n$ (see [17]).

Let $S_g^*(B)$ denote the subset of $S^*(B)$ consisting of those normalized locally biholomorphic mappings $f$ such that $[Df(x)]^{-1}f(x) \in \mathcal{M}_g$.

**Definition 2**. Let $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f \in H(B)$ is said to be starlike of order $\alpha$ if

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1+(1-2\alpha)\zeta}{1-\zeta}$, $\zeta \in D$.

We denote by $S_\alpha^*(B)$ the set of all starlike mappings of order $\alpha$ on $B$.

The following definition due to Roper and Suffridge [18].

**Definition 3**. Suppose $f : B \to X$ is a normalized locally biholomorphic mapping, denote

$$G_f(\alpha,\ \beta) = \frac{2\alpha}{l_u[(Df(\alpha u)^{-1}(f(\alpha u)) - f(\beta u))]} - \frac{\alpha+\beta}{\alpha-\beta}.$$

If

$$\Re e G_f(\alpha,\ \beta) \geq 0,\ \forall u \in \partial B,\ \alpha,\ \beta \in D,$$

then $f$ is said to be a quasi-convex mapping of type A on $B$.

Let $Q_A(B)$ denote the class of quasi-convex mappings of type A on $B$.

**Definition 4.** (see [19].) A normalized locally biholomorphic mapping $f : B \longrightarrow X$ is said to be quasi-convex if

$$\Re e l_x[(Df(x))^{-1}(f(x) - f(\xi x))] \geq 0, \quad \forall x \in B, \quad \xi \in \overline{D}, \quad l_x \in T(x).$$

Let $Q(B)$ denote the class of quasi-convex mappings on $B$.

**Remark 1.** In [19], it is proved that $Q(B) = Q_A(B)$. Roper and Suffridge [18] also proved that $Q_A(B) \subset S_{\frac{1}{2}}^*(B)$, and therefore $f$ satisfies the following relation

$$\left| \frac{l_x[(Df(x))^{-1}f(x)]}{\|x\|} - 1 \right| \leq 1. \tag{3}$$

From (3), we obtain

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1}{1-\zeta}$, $\zeta \in D$. Hence, we have

$$Q(B) = Q_A(B) \subset S_g^*(B) \ \text{ with } \quad g(\zeta) = \tfrac{1}{1-\zeta}, \quad \zeta \in D.$$

In 2006, Liu and Xu [20] defined quasi-convex mappings of order $\alpha$, which is a proper subset of the quasi-convex mappings.

**Definition 5**. Suppose $\alpha \in [0,1)$. A normalized locally biholomorphic mapping $f : B \longrightarrow X$ is said to be a quasi-convex mapping of order $\alpha$ if

$$\Re e l_x[(Df(x))^{-1}(f(x) - f(\xi x))] \geq \alpha(1 - \Re e\xi)\|x\|, \quad \forall x \in B, \quad \xi \in \overline{D}, \;\; l_x \in T(x).$$

Let $Q_\alpha(B)$ denote the class of quasi-convex mappings on $B$.

The following proposition is a well-known result in one complex variable.

**Proposition 1**. If $g : D \longrightarrow D$ is a holomorphic function, $z = 1$ is not a singular point of g and $g(0) = 0, \; g(1) = 1$. Then $g'(1) \geq 1$.

**Remark 2**. Let $x \in B \setminus \{0\}$, and denote that

$$h_x(\xi) = l_x[(Df(x))^{-1}(f(x) - f(\xi x))] - \alpha(1 - \xi)\|x\|, \quad \xi \in \overline{D}.$$

By Definition 5, we know that as a function of $\xi$, $\Re e h_x(\xi) \geq 0$ and harmonic in $D$, so by the minimal value principle we have $\Re e h_x(0) > 0$. From this we obtain

$$| h_x(0) - h_x(\xi) | \leq | \overline{h_x(0)} + h_x(\xi) | .$$

Let

$$g(\xi) = \frac{\overline{h_x(0)}}{h_x(0)} \left( \frac{h_x(0) - h_x(\xi)}{\overline{h_x(0)} + h_x(\xi)} \right)$$

for $\xi \in \overline{D}$, then $g : D \longrightarrow D$ is a holomorphic function of $\xi$ with $g(0) = 0$, and $\xi = 1$ is not a singularity of $g(\xi)$, $g(1) = 1$. According to Proposition 1, we have

$$1 \leq g'(1) = \frac{\overline{h_x(0)}}{h_x(_0)} \cdot \frac{(1-\alpha)\|x\|\overline{h_x(0)} + (1-\alpha)\|x\|h_x(0)}{\overline{h_x(0)^2}} = \frac{2(1-\alpha)\|x\|\Re e h_x(0)}{| h_x(0) |^2}.$$

That is ,

$$| h_x(0) - (1-\alpha)\|x\| | \leq (1-\alpha)\|x\|.$$

More explicitly, it is

$$| l_x(Df(x))^{-1}f(x) - \|x\| | \leq (1-\alpha)\|x\|.$$

That is the same as

$$\left| \frac{l_x[(Df(x))^{-1}f(x)]}{(1-\alpha)\|x\|} - \frac{1}{1-\alpha} \right| \leq 1. \tag{4}$$

From (4), it follows that

$$[Df(x)]^{-1}f(x) \in \mathcal{M}_g,$$

where $g(\zeta) = \frac{1}{1-(1-\alpha)\zeta}$, $\zeta \in D$. Hence, we obtain that

$$Q_\alpha(B) \subset S_g^*(B) \;\text{ with }\; g(\zeta) = \tfrac{1}{1-(1-\alpha)\zeta}, \;\; \zeta \in D.$$

Let $f \in H(B)$ and let $k$ be a positive integer. We say that $z = 0$ is a zero of order $k$ of $f(z)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$ and $D^k f(0) \neq 0$.

Also, we denote by $S^*_{k+1}(B)$(respectively $S^*_{g,\ k+1}(B)$, $S^*_{\alpha,\ k+1}(B)$, $Q_{k+1}(B)$, $Q_{\alpha,\ k+1}(B)$ ), the subset of $S^*(B)$(respectively $S^*_g(B)$, $S^*_\alpha(B)$, $Q(B)$, $Q_\alpha(B)$ ) of mappings $f$ such that $x=0$ is a zero of order $k+1$ of $f(x)-x$.

## 2. Growth theorem

**Lemma 1.** (See [21].) *Suppose $x(t):[0,1]\to X$ is differentiable at the point $s$ which belongs to $[0,1]$, and $\|x(t)\|$ is differentiable at the point $s$ with respect to $t$. Then*

$$\Re e\left[l_{x(t)}\left(\frac{dx(t)}{dt}\right)\right]\Big|_{t=s}=\frac{d(\|x(t)\|)}{dt}\Big|_{t=s}.$$

**Lemma 2.** (See [22].) *Suppose $f$ is a starlike mapping on $B$, $x\in B\backslash\{0\}$, $x(t)=f^{-1}(tf(x))(0\leq t\leq 1)$. Then*
*(a) $\|x(t)\|$ is strictly increasing on $[0,1]$ with respect to $t$;*
*(b) $\|f(x)\|=\lim\limits_{t\to 0}\frac{\|x(t)\|}{t}$, $\frac{dx(t)}{dt}=\frac{1}{t}[Df(x(t))]^{-1}f(x(t))\,, t\in(0,1)\,.$*
**Lemma 3.** (See [23].) *If $f\in H(D)$, $g$ is a biholomorphic function on $D$, $f(0)=g(0), f'(0)=\cdots=f^{(k-1)}(0)=0$, and $f\prec g$. Then*

$$f(rD)\subseteq g(r^kD),\quad r\in(0,1),\ rD=\{\xi\in\mathbb{C}:|\xi|<r\}.$$

Using Lemma 3, we can prove the following.
**Lemma 4.** *Let $g:D\to\mathbb{C}$ satisfy the conditions of Definition 1. If $h\in\mathcal{M}_g$ and $x=0$ is the zero of order $k+1$ of $h(x)-x$, then*

$$\frac{\|x\|}{g(\|x\|^k)}\leq\Re el_x((h(x))\leq|l_x(h(x))|\leq\frac{\|x\|}{g(-\|x\|^k)}\tag{5}$$

*for all $x\in B$.*

**Proof.** Fix $x\in B\backslash\{0\}$, and denote $x_0=\frac{x}{\|x\|}$. Let $p:D\longrightarrow\mathbb{C}$ be given by

$$p(\xi)=\begin{cases}\frac{\xi}{l_x(h(\xi x_0))}, & \xi\neq 0,\\ 1, & \xi=0.\end{cases}$$

Then $p\in H(D)$, $p(0)=g(0)=1$, and since $h\in\mathcal{M}_g$, we deduce that

$$p(\xi)=\frac{\xi}{l_x(h(\xi x_0))}=\frac{\xi}{l_{x_0}(h(\xi x_0))}=\frac{\|\xi x_0\|}{l_{\xi x_0}(h(\xi x_0))}\in g(D),\quad \xi\in D.$$

Let $\psi(\xi)=\frac{1}{p(\xi)}$. This implies that $\psi(\xi)\in\frac{1}{g}(D)$ for all $\xi\in D$. Since $\psi(0)=\frac{1}{g}(0)=1$, we have $\psi\prec\frac{1}{g}$.

According to hypothesis of Lemma 4, we deduce that

$$\psi(\xi)=1+\sum_{m=k+1}^{\infty}\frac{l_z(D^mh(0)(z_0^m))}{m!}\xi^{m-1}.$$

It is easy to see that the function $\psi(\xi)$ satisfies the conditions of Lemma 3, hence we obtain

$$\psi(rD) \subseteq \frac{1}{g}(r^k D), \quad r \in (0,1),\ rD = \{\xi \in \mathbb{C} : |\xi| < r\}.$$

On the other hand, combining the maximum and minimum principles for harmonic functions with (2), we deduce that

$$\frac{1}{g(|\xi|^k)} \leq \Re e\psi(\xi) \leq |\psi(\xi)| \leq \frac{1}{g(-|\xi|^k)}, \quad \xi \in D.$$

Setting $\xi = \|x\|$ in the above relation, we obtain (5), as desired. This completes the proof. $\square$

In [6], Hamada and Honda have recently obtained a sharp growth result for mappings in the family $S^*_{g,\ k+1}(B)$, and $g$ satisfies a slightly different assumption than that in Definition 1. Stimulated by [6], we are now able to obtain the following growth result for the set $f \in S^*_{g,\ k+1}(B)$. This result generalizes [16, Theorem 2.2], [15, Theorem 2.3] and [10, Theorem 2].

**Theorem 1**. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $f \in S^*_{g,\ k+1}(B)$, then*

$$\|x\| \exp \int_0^{\|x\|} \left[g(-y^k) - 1\right] \frac{dy}{y} \leq \|f(x)\| \leq \|x\| \exp \int_0^{\|x\|} \left[g(y^k) - 1\right] \frac{dy}{y}, \quad x \in B. \tag{6}$$

**Proof.** Since $f \in S^*_{g,\ k+1}(B)$, we deduce from Lemma 4 that

$$\frac{\|x\|}{g(\|x\|^k)} \leq \Re e l_x(Df(x))^{-1}f(x) \leq \frac{\|x\|}{g(-\|x\|^k)} \tag{7}$$

for all $x \in B$. Fix $x \in B \setminus \{0\}$, let $x(t) = f^{-1}(tf(x))(0 \leq t \leq 1)$. According to (a) of Lemma 2, we obtain that $\|x(t)\|$ is strictly increasing on [0,1]. Hence, $\|x(t)\|$ is differentiable on [0,1] a.e. . From Lemmas 1, 2(b) and (7), we deduce that for $t \in (0,1]$

$$\frac{\|x(t)\|}{g(\|x(t)\|^k)} \leq t\frac{d\|x(t)\|}{dt} \leq \frac{\|x(t)\|}{g(-\|x(t)\|^k)}, \tag{8}$$

and we may rewrite (8) as

$$\frac{g(-\|x(t)\|^k)}{\|x(t)\|}\frac{d\|x(t)\|}{dt} \leq \frac{1}{t} \leq \frac{g(\|x(t)\|^k)}{\|x(t)\|}\frac{d\|x(t)\|}{dt}.$$

Integrating both sides of the above inequalities with respect to $t$ and making a change of variable, we obtain

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(-y^k)\}dy}{y} = \int_\varepsilon^1 \frac{g(-\|x(t)\|^k))\}}{\|x(t)\|}\frac{d\|x(t)\|}{dt}dt \leq \int_\varepsilon^1 \frac{1}{t}dt,$$

and

$$\int_{\|x(\varepsilon)\|}^{\|x\|} \frac{g(y^k)dy}{y} = \int_\varepsilon^1 \frac{g(\|x(t)\|^k}{\|x(t)\|}\frac{d\|x(t)\|}{dt}dt \geq \int_\varepsilon^1 \frac{1}{t}dt,$$

where $0 < \varepsilon < 1$. It is elementary to verify that

$$\log \frac{\|x(\varepsilon)\|}{\varepsilon} \geq \int_{\|x(\varepsilon)\|}^{\|x\|} \Big[ g(-y^k) - 1 \Big] \frac{dy}{y} + \log \|x\|, \tag{9}$$

and

$$\log \frac{\|x(\varepsilon)\|}{\varepsilon} \leq \int_{\|x(\varepsilon)\|}^{\|x\|} \Big[ g(y^k) - 1 \Big] \frac{dy}{y} + \log \|x\|. \tag{10}$$

If we now let $\varepsilon \longrightarrow 0+$ in the above inequalities (9), (10) and use Lemma 2(b), we have

$$\|x\| \exp \int_0^{\|x\|} \Big[ g(-y^k) - 1 \Big] \frac{dy}{y} \leq \|f(x)\| \leq \|x\| \exp \int_0^{\|x\|} \Big[ g(y^k) - 1 \Big] \frac{dy}{y}, \quad x \in B,$$

as claimed. This completes the proof of Theorem 1. $\square$

## 3. Distortion theorem

In this section, we will give distortion theorems for subclasses of starlike mappings along a unit direction in $S^*_{g\ k+1}(D^n)$.

**Theorem 2**. *Let $g : D \to \mathbb{C}$ satisfy the conditions of Definition 1. If $f \in S^*_{g,\ k+1}(D^n)$, then for any $z \in D^n \setminus \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z) f(z)}{\|J_f^{-1}(z) f(z)\|}\right)$, such that*

$$g(-\|z\|^k) \exp \int_0^{\|x\|} \Big[ g(-y^k) - 1 \Big] \frac{dy}{y} \leq \|J_f(z)\xi(z)\| \leq g(\|z\|^k) \exp \int_0^{\|x\|} \Big[ g(y^k) - 1 \Big] \frac{dy}{y}.$$

**Proof.** For $z \in D^n \setminus \{0\}$ and any $\xi \in (\partial D)^n(0, \|z\|)$, we have

$$|\xi_1| = |\xi_2| = \cdots = |\xi_n| = \|z\|.$$

Note that

$$l_\xi = (0, \cdots, 0, \frac{\|\xi\|}{\xi_i}, 0, \cdots, 0).$$

Set $w(z) = J_f^{-1}(z) f(z)$. Then there exists an $i$ such that

$$\begin{aligned} \|w(z)\| = |w_i(z)| &\leq \max_{\xi \in (\partial D)^n(0, \|z\|)} |w_i(\xi)| \\ &= \max_{\xi \in (\partial D)^n(0, \|z\|)} \left| \frac{\|\xi\|}{\xi_i} w_i(\xi) \right| \\ &= \max_{\xi \in (\partial D)^n(0, \|z\|)} |l_\xi[w(\xi)]| \\ &= \max_{\xi \in (\partial D)^n(0, \|z\|)} |l_\xi[J_f^{-1}(\xi) f(\xi)]|. \end{aligned}$$

According to Lemma 4, we have

$$|l_\xi(J_f^{-1}(\xi) f(\xi)| \leq \frac{\|\xi\|}{g(-\|\xi\|^k)} = \frac{\|z\|}{g(-\|z\|^k)},$$

and thus

$$\|J_f^{-1}(z)f(z)\| = \|w(z)\| \leq \frac{\|z\|}{g(-\|z\|^k)}. \tag{11}$$

On the other hand, $\|l_z\| \leq 1$ and Lemma 4 show that

$$\|J_f^{-1}(z)f(z)\| \geq \|l_z[J_f^{-1}(z)f(z)]\| \geq \frac{\|z\|}{g(\|z\|^k)}. \tag{12}$$

By combining (11) and (12), we have

$$\frac{\|z\|}{g(\|z\|^k)} \leq \|J_f^{-1}(z)f(z)\| \leq \frac{\|z\|}{g(-\|z\|^k)}. \tag{13}$$

Set $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, where $z \in D^n \setminus \{0\}$. In view of Theorem 1, we have

$$\|z\| \exp \int_0^{\|z\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq \|f(z)\| \leq \|z\| \exp \int_0^{\|z\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}, \quad z \in D^n. \tag{14}$$

Consequently, from (13) and (14) we obtain

$$\begin{aligned} g(-\|z\|^k) \exp \int_0^{\|z\|} \Big[g(-y^k) - 1\Big] \frac{dy}{y} \leq & \|J_f(z)\xi(z)\| = \frac{\|f(z)\|}{\|J_f^{-1}(z)f(z)\|} \\ \leq & g(\|z\|^k) \exp \int_0^{\|z\|} \Big[g(y^k) - 1\Big] \frac{dy}{y}. \end{aligned}$$

as claimed. This completes the proof of Theorem 2. □

Now, we obtain the following corollaries from Theorem 2.

For $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in D$, we have the following distortion theorem along a unit direction of the polydisc $D^n$ for the set $S_{k+1}^*(D^n)$. When $k = 1$, Corollary 1 was obtained by Liu et al. [13].

**Corollary 1**. *If $f \in S_{k+1}^*(D^n)$, then for any $z \in D^n \backslash \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1 - \|z\|^k}{(1 + \|z\|^k)^{\frac{2}{k}+1}} \leq \|J_f(z)\xi(z)\| \leq \frac{1 + \|z\|^k}{(1 - \|z\|^k)^{\frac{2}{k}+1}}.$$

According to Remark 1, for $g(\zeta) = \frac{1}{1-\zeta}$, $\zeta \in D$, we have the following distortion theorem along a unit direction of the polydisc $D^n$ for the set $Q_{k+1}(D^n)$. When $k = 1$, Corollary 2 was obtained by Liu et al. [12].

**Corollary 2**. *If $f \in Q_{k+1}(D^n)$, then for any $z \in D^n \backslash \{0\}$, there exists a unit vector $\xi(z)$ $\left(\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

$$\frac{1}{(1 + \|z\|^k)^{\frac{1}{k}+1}} \leq \|J_f(z)\xi(z)\| \leq \frac{1}{(1 - \|z\|^k)^{\frac{1}{k}+1}}.$$

According to Remark 2, for $g(\zeta)=\frac{1}{1-(1-\alpha)\zeta}$, $\zeta\in D$, we have the following distortion theorem along a unit direction of the polydisc $D^n$ for the set $Q_{\alpha,\ k+1}(D^n)$ due to Wang et al. [14].

**Corollary 3**. *If $f\in Q_{\alpha,\ k+1}(D^n)$, then for any $z\in D^n\setminus\{0\}$, there exists a unit vector $\xi(z)\left(\xi(z)=\frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}\right)$, such that*

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