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超空间中次正则函数的 Cauchy 积分公式*

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提要 在本文中, 首先给出了超空间中次正则函数 (sandwich 方程 $D_x f D_x = 0$ 的解) 的一些性质, 然后证明了超空间中的 Cauchy-Pompeiu 公式, 最后得到了超空间中的 Cauchy 积分公式和 Cauchy 积分定理.

关键词 超空间, 次正则函数, Cauchy-Pompeiu 公式, Cauchy 积分公式, Cauchy 积分定理
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§1 引 言

超空间创立于上世纪中叶, 是一种来源于量子力学而衍生的代数空间, 它既包含可交换变量又包含反交换变量 (Grassmann 代数的生成元). Berezin[1−2] 利用层理论研究超空间的方法广泛应用于现代代数几何理论.

许多学者对超空间进行了深入的研究. 从 2001 年起, Sommen 和 De Bie[3−6] 从函数论的角度研究了超空间, 将超空间与 Clifford 代数相结合, 形成一个新的代数系统, 并且他们将调和分析和 Clifford 分析的一些理论推广到超空间中. 2009 年 Sommen 等[7−8]给出了超空间中的一些基本理论, 例如 Stokes 定理和 Cauchy-Pompeiu 公式等. 2013 年袁洪芬等[9]在超空间中得到了超空间中 k-正则函数的分解定理.

近些年, Clifford 分析中次正则函数相关理论的研究已经相对成熟. 2017 年 García 等[10]给出了 Clifford 分析中的 Cauchy 积分公式. 2020 年 Blaya 等[11]给出了 Clifford 分析中次多正则函数的 Cauchy 积分公式, García 等[12]研究了次正则函数的分解以及在弹性理论中的应用. 2021 年 Santiesteban 等 [13] 介绍了在 Clifford 分析中的 (φ,ψ)-次正则函数.

本文的目标是在超空间中研究次正则函数的相关理论. 在上述工作的基础上, 我们首先研究了超空间中次正则函数的一些性质, 然后证明了超空间的 Cauchy-Pompeiu 公式, 最后得到了超空间中次正则函数的 Cauchy 积分公式和 Cauchy 积分定理.

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§2 预备知识

在超空间

$$\mathbf{R}^{m|2n}=\{(\underline{x},\underline{\grave{x}})\mid \underline{x}=(x_1,\cdots,x_m)\in\mathbf{R}^m,\ \underline{\grave{x}}=(\grave{x}_1,\cdots,\grave{x}_{2n})\in\Lambda_{2n}\}$$

中, 引入实代数

$$Alg\{x_i;\grave{x}_j\}\otimes Alg\{e_i;\grave{e}_j\}=Alg\{x_i,e_i;\grave{x}_j,\grave{e}_j\},$$

其中 $i=1,\cdots,m;\ j=1,\cdots,2n$, Λ_{2n} 是 Grassmann 代数, 标量代数 $Alg\{x_i;\grave{x}_j\}$ 记为 $\mathcal{P}$, 基底部分 $Alg\{e_i;\grave{e}_j\}$ 记为 $\mathcal{C}$, $\mathcal{P}$ 和 $\mathcal{C}$ 中元素是可交换的, 并且满足

$$\begin{cases}x_ix_j=x_jx_i, & i,j=1,\cdots,m,\\ \grave{x}_i\grave{x}_j=-\grave{x}_j\grave{x}_i, & i,j=1,\cdots,2n,\\ x_i\grave{x}_j=\grave{x}_jx_i, & i=1,\cdots,m;j=1,\cdots,2n\end{cases}$$

和

$$\begin{cases}e_je_k+e_ke_j=-2\delta_{jk}, & j,k=1,\cdots,m,\\ \grave{e}_{2j}\grave{e}_{2k}-\grave{e}_{2k}\grave{e}_{2j}=0, & j,k=1,\cdots,n,\\ \grave{e}_{2j-1}\grave{e}_{2k-1}-\grave{e}_{2k-1}\grave{e}_{2j-1}=0, & j,k=1,\cdots,n,\\ \grave{e}_{2j-1}\grave{e}_{2k}-\grave{e}_{2k}\grave{e}_{2j-1}=\delta_{jk}, & j,k=1,\cdots,n,\\ e_j\grave{e}_k+\grave{e}_ke_j=0, & j=1,\cdots,m;k=1,\cdots,2n.\end{cases}$$

当 $n=0$ 时, $\mathcal{P}\otimes\mathcal{C}\cong Cl_{0,m}(\mathbf{R})$.

定义 $\mathcal{P}\otimes\mathcal{C}$ 中向量为 $x=\underline{x}+\underline{\grave{x}}$, 其中

$$\underline{x}=\sum_{i=1}^m x_ie_i,\quad \underline{\grave{x}}=\sum_{j=1}^{2n}\grave{x}_j\grave{e}_j.$$

由文 [8], 有

$$x^2=\sum_{j=1}^n\grave{x}_{2j-1}\grave{x}_{2j}-\sum_{i=1}^m x_i^2=\underline{\grave{x}}^2+\underline{x}^2,$$

其中 $\underline{x}^2=-\sum\limits_{i=1}^m x_i^2$.

在本文中, 记 $\Omega\subset\mathbf{R}^m\backslash\{0\}$ 为一个非空连通开集, 其边界 $\partial\Omega$ 为可微、可定向和紧致的 Liapunov 曲面. 记 $Sc(\underline{x}e_i):=\underline{x}_i$ 为 $\underline{x}e_i$ 的标量部分, 其中 $i=1,\cdots,m$, $\underline{x}\in\mathbf{R}^m$.

记

$$\mathcal{F}(\Omega)_{m|2n}=\{f\mid f:\Omega\otimes\Lambda_{2n}\to\mathcal{P}\otimes\mathcal{C}\}.$$

$C^k(\Omega)$ 表示 Ω 中 k 次连续可微实值函数的集合, 其中 $k\in N^*$, N^* 是正整数集.

记

$$C^k(\Omega)_{m|2n}=C^k(\Omega\otimes\Lambda_{2n}).$$

对于 $f\in C^1(\Omega)_{m|2n}$, 分别定义左和右超 Dirac 算子如下[8]:

$$D_xf=D_{\underline{\grave{x}}}f-D_{\underline{x}}f=2\sum_{j=1}^n\Big(\grave{e}_{2j}\frac{\partial f}{\partial\grave{x}_{2j-1}}-\grave{e}_{2j-1}\frac{\partial f}{\partial\grave{x}_{2j}}\Big)-\sum_{i=1}^m e_i\frac{\partial f}{\partial x_i},$$

$$fD_x = -fD_{\underline{\grave{x}}} - fD_{\underline{x}} = -2\sum_{j=1}^{n}\Big(\frac{f\partial}{\partial \grave{x}_{2j-1}}\grave{e}_{2j} - \frac{f\partial}{\partial \grave{x}_{2j}}\grave{e}_{2j-1}\Big) - \sum_{i=1}^{m}\frac{\partial f}{\partial x_i}e_i.$$

对于 $f\in C^1(\Omega)_{m|2n}$, 超空间中的超 Euler 算子定义如下[8]:

$$\mathbf{E}f = \sum_{j=1}^{2n}\grave{x}_j\frac{\partial f}{\partial \grave{x}_j} + \sum_{i=1}^{m}x_i\frac{\partial f}{\partial x_i} = \mathbf{E}_{\underline{\grave{x}}}f + \mathbf{E}_{\underline{x}}f,$$

其中

$$\mathbf{E}_{\underline{\grave{x}}}f = \sum_{j=1}^{2n}\grave{x}_j\frac{\partial f}{\partial \grave{x}_j},\quad \mathbf{E}_{\underline{x}}f = \sum_{i=1}^{m}x_i\frac{\partial f}{\partial x_i}.$$

将 $\mathcal{P}$ 分解为齐次多项式空间

$$\mathcal{P} = \bigoplus_{k=0}^{2n}\mathcal{P}_k = \Big(\bigoplus_{k\ \text{奇}}\mathcal{P}_k\Big)\bigoplus\Big(\bigoplus_{k\ \text{偶}}\mathcal{P}_k\Big) = \mathcal{P}_o\bigoplus\mathcal{P}_e,$$

其中

$$\mathcal{P}_k = \{\omega\in\mathcal{P}\mid \mathbf{E}_{\underline{\grave{x}}}\omega = k\omega,\ k\in N^*\},\quad \mathcal{P}_o = \bigoplus_{k\ \text{奇}}\mathcal{P}_k,\quad \mathcal{P}_e = \bigoplus_{k\ \text{偶}}\mathcal{P}_k.$$

对于 $f_i\in\mathcal{P}_i,\ i=1,\cdots,m$, 我们有

$$\frac{\partial f_i}{\partial \grave{x}_j} = \begin{cases}\dfrac{f_i\partial}{\partial \grave{x}_j}, & \text{若 } i \text{ 为奇数},\ j=1,\cdots,2n,\\ -\dfrac{f_i\partial}{\partial \grave{x}_j}, & \text{若 } i \text{ 为偶数},\ j=1,\cdots,2n.\end{cases} \tag{2.1}$$

对于 $f\in C^2(\Omega)_{m|2n}$, 超 Dirac 算子的平方是超 Laplace 算子:

$$\Delta f = D_x^2 f = 4\sum_{j=1}^{n}\frac{\partial^2 f}{\partial \grave{x}_{2j-1}\partial \grave{x}_{2j}} - \sum_{i=1}^{m}\frac{\partial^2 f}{\partial x_i^2} = \Delta_{\underline{\grave{x}}}f + \Delta_{\underline{x}}f.$$

这个算子的玻色子部分是

$$\Delta_{\underline{x}}f = -\sum_{i=1}^{m}\frac{\partial^2 f}{\partial x_i^2},$$

它是经典的 Laplace 算子, 费米子部分是

$$\Delta_{\underline{\grave{x}}}f = 4\sum_{j=1}^{n}\frac{\partial^2 f}{\partial \grave{x}_{2j-1}\partial \grave{x}_{2j}}.$$

定义 2.1 若 $f\in C^1(\Omega)_{m|2n}$ 满足

$$D_x f(x) = 0\ (f(x)D_x = 0),$$

则称 f 是 Ω 中的左 (右) 正则函数.

引理 2.1 [4] 若 $f\in C^2(\Omega)_{m|2n}$, 则

$$(D_x f(x))D_x = D_x(f(x)D_x).$$

由引理 2.1, 在本文中我们把 $(D_x f(x))D_x$ 与 $D_x(f(x)D_x)$ 统一记为 $D_x f(x)D_x$.

定义 2.2 若 $f \in C^2(\Omega)_{m|2n}$ 满足 $D_x f(x) D_x = 0$, 则称 f 是 Ω 中的次正则函数.

定义 2.3 若 $f \in C^2(\Omega)_{m|2n}$ 满足 $\Delta f(x) = 0$, 则称 f 是 Ω 中的调和函数.

引理 2.2[14] 若 $f, g \in C^1(\Omega)_{m|0}$, 则

$$D_{\underline{x}}(f(x)g(x)) = (D_{\underline{x}} f(x))g(x) + \sum_{i=1}^{m} e_i f(x) \frac{\partial g(x)}{\partial x_i},$$

$$(f(x)g(x))D_{\underline{x}} = \sum_{i=1}^{m} \frac{\partial f(x)}{\partial x_i} g(x) e_i + f(x)(g(x) D_{\underline{x}}).$$

对于任意的 $f \in C^2(\Omega)_{m|2n} \otimes \mathcal{C}$, 定义在 $\Omega \otimes \Lambda_{2n}$ 上的积分如下:

$$\int_{\Omega \otimes \Lambda_{2n}} f = \int_{\Omega} \mathrm{d}\underline{x} \int_B f = \int_B \int_{\Omega} f \mathrm{d}\underline{x},$$

其中

$$\mathrm{d}\underline{x} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m, \quad \int_B f = \pi^{-n} \frac{\partial^{2n} f}{\partial \grave{x}_{2n} \cdots \partial \grave{x}_1}.$$

引理 2.3[7] 设非空开集 $\Sigma \subset \Omega$, $\overline{\Sigma} \subset \Gamma$ 为一个 m 维带有光滑边界 $\partial\Sigma$ 的可微紧致定向的流形, $\beta \in \Lambda_{2n}$, 若 $f, g \in C^1(\overline{\Sigma})_{m|2n} \otimes \mathcal{C}$, 则

$$\int_{\Sigma} \int_B [(f \widehat{\beta} D_x) g + f \beta (D_x g)] \mathrm{d}\underline{x} = - \int_{\partial\Sigma} \int_B f \beta \mathrm{d}\sigma_{\underline{x}} g + \int_{\Sigma} \int_B f(\beta D_{\grave{\underline{x}}}) g \mathrm{d}\underline{x},$$

其中式子左边的 $f\widehat{\beta}D_x$ 表示超 Dirac 算子 D_x 从右侧只作用在 f 上, 作用的结果与 β 相乘, $\mathrm{d}\sigma_{\underline{x}} = \sum\limits_{i=1}^{m} (-1)^{i-1} e_i \mathrm{d}\widehat{x_i}$, $\mathrm{d}\widehat{x_i} = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_m, i = 1, 2, \cdots, m$.

命题 2.1 若 $f \in \mathcal{P}_e$ 取值于 $\mathbf{R} \otimes \Lambda_{2n}$ 中, 则

$$D_x f(x) = f(x) D_x.$$

证 根据等式 (2.1) 可得

$$D_x f = D_{\grave{\underline{x}}} f - D_{\underline{x}} f = 2 \sum_{j=1}^{n} \Big(\grave{e}_{2j} \frac{\partial f}{\partial \grave{x}_{2j-1}} - \grave{e}_{2j-1} \frac{\partial f}{\partial \grave{x}_{2j}} \Big) - \sum_{i=1}^{m} e_i \frac{\partial f}{\partial x_i}$$

$$= -2 \sum_{j=1}^{n} \Big(\frac{f \partial}{\partial \grave{x}_{2j-1}} \grave{e}_{2j} - \frac{f \partial}{\partial \grave{x}_{2j}} \grave{e}_{2j-1} \Big) - \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} e_i = f D_x.$$

命题 2.2 若 $f \in \mathcal{P}_e$ 是一个定义于 Ω 中取值在 $\mathbf{R} \otimes \Lambda_{2n}$ 上的次正则函数, 则 f 是 Ω 中的调和函数.

命题 2.3 左正则函数和右正则函数均是次正则函数.

命题 2.4 若 $f \in C^3(\Omega)_{m|2n}$ 是一个次正则函数, 则 f 在 Ω 中满足

$$(D_x)^3 f(x) = f(x)(D_x)^3 = 0.$$

换而言之, f 是 Ω 中的双边 3-正则函数.

证 因为 f 是一个次正则函数, 我们有

$$D_x f(x) D_x = 0,$$

因此

$$(D_x)^3 f(x) = D_x(\Delta f(x)) = D_x f(x)\Delta = (D_x f(x) D_x) D_x = 0.$$

类似地, 我们有

$$f(x)(D_x)^3 = 0.$$

命题 2.5　在 Ω 中的每一个次正则函数 $f \in C^4(\Omega)_{m|2n}$ 都是 Ω 中的双调和函数, 即它在 Ω 中满足方程

$$(\Delta_x)^2 f(x) = 0.$$

§3　次正则函数的 Cauchy 积分公式

定义 3.1　若 $f \in C^1(\Omega)_{m|0}$ 满足

$$D_{\underline{x}} f(\underline{x}) = \delta(\underline{x}),$$

其中 $\delta(\underline{x})$ 是 Dirac 函数, 则称 f 是 Dirac 算子 $D_{\underline{x}}$ 的基本解.

引理 3.1 [15]　Laplace 算子玻色子部分的次幂 $(\Delta_{\underline{x}})^l$ 的基本解是 $v_{2l}^{m|0}(\underline{x})$, 且满足

$$(\Delta_{\underline{x}})^j v_{2l}^{m|0}(\underline{x}) = v_{2l-2j}^{m|0}(\underline{x}), \quad j < l,\ j, l \in N^*,$$
$$(\Delta_{\underline{x}})^l v_{2l}^{m|0}(\underline{x}) = \delta(\underline{x}), \quad l \in N^*,$$

其中 $v_{2l}^{m|0}(\underline{x})$ 表示如下:

$$v_{2l}^{m|0}(\underline{x}) = \begin{cases} \dfrac{r^{2l-m}}{\gamma_{l-1}}, & m = 2k-1, \\ \dfrac{r^{2l-m}}{\acute{\gamma}_{l-1}}, & m = 2k,\ l < k, \\ \dfrac{-r^{2l-m}\log r}{\acute{\gamma}_{l-1}} + \frac{C_{l-k}}{\acute{\gamma}_{l-1}} r^{2l-m}, & m = 2k,\ l \geqslant k, \end{cases}$$

其中 $r = \sqrt{-\underline{x}^2}$, $C_0 = 0$, $C_l = \sum\limits_{j=1}^{l} \frac{1}{2j} + \sum\limits_{j=k}^{l+k-1} \frac{1}{2j}$,

$$\gamma_t = (-1)^{t+1}(2-m)4^t t! \frac{\Gamma\big(t+2-\frac{m}{2}\big)}{\Gamma\big(2-\frac{m}{2}\big)} \frac{2\pi^{\frac{m}{2}}}{\Gamma\big(\frac{m}{2}\big)}, \quad m = 2k-1,$$

$$\acute{\gamma}_t = \begin{cases} (-1)^t (2-m) 4^t t! \dfrac{(k-2)!}{(k-t-2)!} \dfrac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}, & m = 2k,\ k > 1,\ t < k-1, \\ (-1)^{t+1}(2-m)4^t (k-2)! t! (t+1-k)! \dfrac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}, & m = 2k, k > 1, t \geqslant k-1, \\ \acute{\gamma}_t = 4^t (t!)^2 \dfrac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}, & m = 2, \end{cases}$$

其中 $k \in N^*$, $t = l-1$.

注 3.1　(1) 在引理 3.1 中, $v_{2k+2}^{m|0}(\underline{x} - \underline{y})$ 是仅与 $r = \sqrt{-\underline{x}^2}$ 有关的实值函数.

(2) 令

$$v_{2k+1}^{m|0}(\underline{x}) = -D_{\underline{x}} v_{2k+2}^{m|0}(\underline{x}) = -v_{2k+2}^{m|0}(\underline{x}) D_{\underline{x}},$$

则 $v_{2k+1}^{m|0}(\underline{x}) \in \mathbf{R^m}$, 其中 $k \in N^*$.

(3) 由引理 3.1, 我们有

$$D_{\underline{x}} v_{2k+1}^{m|0}(\underline{x}) = v_{2k+1}^{m|0}(\underline{x}) D_{\underline{x}} = -v_{2k}^{m|0}(\underline{x}), \quad k \in N^*.$$

定义 3.2 若 $f \in C^1(\Omega)_{m|0}$ 满足

$$D_x f(x) = \delta(x),$$

其中

$$\delta(x) = \delta(\underline{x}) \frac{\pi^n}{n!} \underline{\grave{x}}^{2n},$$

则称 f 是超 Dirac 算子 D_x 的基本解.

引理 3.2 [8] 左 (右) 超 Dirac 算子 D_x 的基本解是

$$v_1^{m|2n}(x) = \sum_{k=0}^{n-1} 2 \frac{4^k k!}{(n-k-1)!} v_{2k+2}^{m|0}(\underline{x}) \underline{\grave{x}}^{2n-2k-1} + \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} v_{2k+1}^{m|0}(\underline{x}) \underline{\grave{x}}^{2n-2k}. \tag{3.1}$$

引理 3.3 [6] 超 Laplace 算子 Δ 的基本解是

$$v_2^{m|2n}(x) = \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} v_{2k+2}^{m|0}(\underline{x}) \underline{\grave{x}}^{2n-2k}, \quad k \in N^*. \tag{3.2}$$

引理 3.4 [8] 令 $f \in C^1(\Omega)_{m|0}$, $\underline{y} \in \Omega$ 且 $\overline{B}_{(\underline{y},R)}$ 是 Ω 中以 $\underline{y}$ 为心半径为 R 的球, 则有下列结论成立:

$$\lim_{R\to 0^+} \int_{B_{(\underline{y},R)}} v_k^{m|0}(\underline{x}-\underline{y}) f(\underline{x}) \mathrm{d}\underline{x} = 0, \quad \forall k \in N^*,$$

$$\lim_{R\to 0^+} \int_{\partial B_{(\underline{y},R)}} v_k^{m|0}(\underline{x}-\underline{y}) \mathrm{d}\sigma_{\underline{x}} f(\underline{x}) = \begin{cases} -f(\underline{y}), & k=1, \\ 0, & \forall k>1, \quad k \in N^*. \end{cases}$$

引理 3.5 [7] (Cauchy-Pompeiu 公式) 若 $f \in C^1(\Omega)_{m|2n}$, 则

$$\int_{\partial\Omega} \int_B v_1^{m|2n}(x-y) \mathrm{d}\sigma_{\underline{x}} f(x) + \int_\Omega \int_B v_1^{m|2n}(x-y)(D_x f(x)) \mathrm{d}\underline{x}$$
$$= \begin{cases} -f(y), & \underline{y} \in \Omega, \\ 0, & \underline{y} \in \mathbf{R}^n \backslash \overline{\Omega}. \end{cases}$$

与引理 3.5 的证明类似, 我们有如下结论.

定理 3.1 (Cauchy-Pompeiu 公式) 若 $f \in C^1(\Omega)_{m|2n}$, 则

$$\int_{\partial\Omega} \int_B f(x) \mathrm{d}\sigma_{\underline{x}} v_1^{m|2n}(x-y) + \int_\Omega \int_B (f(x) D_x) v_1^{m|2n}(x-y) \mathrm{d}\underline{x}$$
$$= \begin{cases} -f(y), & \underline{y} \in \Omega, \\ 0, & \underline{y} \in \mathbf{R}^n \backslash \overline{\Omega}. \end{cases}$$

定理 3.2　下列等式成立:

(1)

$$D_x((f(x)D_x)(\underline{x}-\underline{y}))=(D_xf(x)D_x)(\underline{x}-\underline{y})-\sum_{i=1}^{m}e_i(f(x)D_x)e_i.$$

(2)

$$\begin{aligned}&\grave{x}_{2n+2}D_x((f(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}}))-\grave{x}_{2n+2}(D_xf(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}})\\&=2\sum_{j=1}^{n}(\grave{e}_{2j}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j-1}-\grave{e}_{2j-1}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j}).\end{aligned}$$

证　根据引理 2.2, 我们有

(1)

$$\begin{aligned}D_x((f(x)D_x)(\underline{x}-\underline{y}))&=(D_xf(x)D_x)(\underline{x}-\underline{y})-\sum_{i=1}^{m}e_i(f(x)D_x)\frac{\partial(\underline{x}-\underline{y})}{\partial x_i}\\&=(D_xf(x)D_x)(\underline{x}-\underline{y})-\sum_{i=1}^{m}e_i(f(x)D_x)e_i.\end{aligned}$$

(2)

$$\begin{aligned}&\grave{x}_{2n+2}D_x((f(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}}))-\grave{x}_{2n+2}(D_xf(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}})\\&=2\sum_{j=1}^{m}\Big(e_{2j}(f(x)D_x)\grave{x}_{2n+2}\frac{\partial(\underline{\grave{x}}-\underline{\grave{y}})}{\partial\grave{x}_{2j}}-e_{2j}(f(x)D_x)\grave{x}_{2n+2}\frac{\partial(\underline{\grave{x}}-\underline{\grave{y}})}{\partial\grave{x}_{2j-1}}\Big)\\&=2\sum_{j=1}^{n}(\grave{e}_{2j}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j-1}-\grave{e}_{2j-1}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j}).\end{aligned}$$

定理 3.3　下列等式成立:

(1)

$$\begin{aligned}&v_1^{m|2n}(x-y)e_i+e_iv_1^{m|2n}(x-y)\\&=2\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}Sc(v_{2k+1}^{m|0}(\underline{x}-\underline{y})e_i)(\underline{\grave{x}}-\underline{\grave{y}})^{2n-2k},\quad i=1,\cdots,m.\end{aligned}$$

(2)

$$\begin{aligned}&v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j}-\grave{x}_{2n+1}\grave{e}_{2j-1}v_1^{m|2n}(x-y)\\&=2\sum_{k=0}^{n-1}\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\underline{\grave{x}}-\underline{\grave{y}})^{2n-2k-2}(\grave{x}_{2j-1}-\grave{y}_{2j-1})\grave{x}_{2n+1},\quad j=1,\cdots,n.\end{aligned}$$

(3)

$$\begin{aligned}&v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j-1}+\grave{x}_{2n+1}\grave{e}_{2j-1}v_1^{m|2n}(x-y)\\&=-2\sum_{k=0}^{n-1}\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\underline{\grave{x}}-\underline{\grave{y}})^{2n-2k-2}(\grave{x}_{2j}-\grave{y}_{2j})\grave{x}_{2n+1},\quad j=1,\cdots,n.\end{aligned}$$

证　根据注 3.1 和等式 (3.1), 我们有

(1)

$$\begin{aligned}&v_1^{m|2n}(x-y)e_i+e_iv_1^{m|2n}(x-y)\\&=\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}(v_{2k+1}^{m|0}(\underline{x}-\underline{y})e_i+e_iv_{2k+1}^{m|0}(\underline{x}-\underline{y}))(\grave{x}-\grave{y})^{2n-2k}\\&=2\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}Sc(v_{2k+1}^{m|0}(\underline{x}-\underline{y})e_i)(\grave{x}-\grave{y})^{2n-2k}.\end{aligned}$$

(2)

$$\begin{aligned}&v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j}+\grave{x}_{2n+1}\grave{e}_{2j}v_1^{m|2n}(x-y)\\&=\sum_{k=0}^{n-1}2\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{x}-\grave{y})^{2n-2k-1}\grave{e}_{2j}\grave{x}_{2n+1}\\&\quad+\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}v_{2k+1}^{m|0}(\underline{x}-\underline{y})\grave{x}_{2n+1}\grave{e}_{2j}(\grave{x}-\grave{y})^{2n-2k}\\&\quad-\sum_{k=0}^{n-1}2\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})\grave{e}_{2j}(\grave{x}-\grave{y})^{2n-2k-1}\grave{x}_{2n+1}\\&\quad-\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}v_{2k+1}^{m|0}(\underline{x}-\underline{y})\grave{x}_{2n+1}\grave{e}_{2j}(\grave{x}-\grave{y})^{2n-2k}\\&=\sum_{k=0}^{n-1}2\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{x}-\grave{y})^{2n-2k-2}(\grave{x}_{2j-1}-\grave{y}_{2j-1})(\grave{e}_{2j-1}\grave{e}_{2j}-\grave{e}_{2j}\grave{e}_{2j-1})\grave{x}_{2n+1}\\&=2\sum_{k=0}^{n-1}\frac{4^kk!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{x}-\grave{y})^{2n-2k-2}(\grave{x}_{2j-1}-\grave{y}_{2j-1})\grave{x}_{2n+1}.\end{aligned}$$

与 (2) 的证明类似, 直接可得 (3) 成立.

定理 3.4 (Cauchy-Pompeiu 公式) 若 $f\in C^2(\Omega)_{m|2n}$, 则

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(x-y)\mathrm{d}\underline{x}\\&+\frac{1}{2}\int_{\partial\Omega}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y))\\&-\int_{\partial\Omega}\int_B f(x)\mathrm{d}\sigma_{\underline{x}}v_1^{m|2n}(x-y)-\frac{1}{2}\sum_{i=1}^{m}e_i\Big(\int_{\Omega}\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)e_i\\&-\frac{1}{2}\sum_{i=1}^{m}e_i\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i\\&-\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega}\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\&-\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega}\int_{B}v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega}\int_{B}v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&=\begin{cases}f(y), & \underline{y}\in\Omega,\\ 0, & \underline{y}\in\mathbf{R}^n\backslash\overline{\Omega}.\end{cases}
\end{aligned}\tag{3.3}
$$

证　(i) 若 $\underline{y}\in\Omega$, 取 $\delta>0$, 使得

$$B_{(\underline{y},\delta)}:=\{\underline{x}\in\mathbf{R}^m:|\underline{x}-\underline{y}|<\delta\}\subset\Omega.$$

记 $\Omega^\delta=\Omega\backslash\overline{B}_{(\underline{y},\delta)}$,

$$
\begin{aligned}
\mathrm{I}=&\int_{\Omega^\delta}\int_B\sum_{i=1}^{m}v_1^{m|2n}(x-y)e_i(f(x)D_x)e_i\mathrm{d}\underline{x}\\
&+2\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j-1}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j}\frac{\partial^2}{\partial\grave{x}_{2n+2}\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
&-2\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j}(f(x)D_x)\grave{x}_{2n+2}\grave{e}_{2j-1}\frac{\partial^2}{\partial\grave{x}_{2n+2}\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}.
\end{aligned}\tag{3.4}
$$

由等式 (3.4), 定理 3.2 以及引理 2.3, 我们有

$$
\begin{aligned}
\mathrm{I}=&\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)((D_xf(x)D_x)(\underline{x}-\underline{y})-D_x((f(x)D_x)(\underline{x}-\underline{y})))\mathrm{d}\underline{x}\\
&+\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{x}_{2n+2}(D_xf(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}})\frac{\partial^2}{\partial\grave{x}_{2n+2}\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
&-\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{x}_{2n+2}(D_x((f(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}})))\frac{\partial^2}{\partial\grave{x}_{2n+2}\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
=&\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(x-y)\mathrm{d}\underline{x}\\
&-\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(D_x((f(x)D_x)(\underline{x}-\underline{y})))\mathrm{d}\underline{x}\\
=&\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(x-y)\mathrm{d}\underline{x}\\
&+\int_{\partial\Omega^\delta}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y)).
\end{aligned}\tag{3.5}
$$

根据等式 (3.4), 定理 3.3 以及在 (3.1) 中 $v_1^{m|2n}(x-y)$ 的表达式, 我们有

$$
\begin{aligned}
\mathrm{I}=&\int_{\Omega^\delta}\int_B\sum_{i=1}^{m}\Big(-e_iv_1^{m|2n}(x-y)(f(x)D_x)e_i\\
&+2\sum_{k=0}^{n}\frac{4^kk!}{(n-k)!}Sc(v_{2k+1}^{m|0}(\underline{x}-\underline{y})e_i)(\underline{\grave{x}}-\underline{\grave{y}})^{2n-2k}(f(x)D_x)e_i\Big)\mathrm{d}\underline{x}\\
&+2\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}\Big(-\grave{x}_{2n+1}\grave{e}_{2j-1}v_1^{m|2n}(x-y)
\end{aligned}
$$

$$
\begin{aligned}
&-2\sum_{k=0}^{n-1}\frac{4^k k!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2k-2}(\grave{x}_{2j}-\grave{y}_{2j})\grave{x}_{2n+1}\Big)(f(x)D_x)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
&-2\int_{\Omega^\delta}\int_B\sum_{j=1}^{n}\Big(-\grave{x}_{2n+1}\grave{e}_{2j}v_1^{m|2n}(x-y)\\
&+2\sum_{k=0}^{n-1}\frac{4^k k!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2k-2}(\grave{x}_{2j-1}-\grave{y}_{2j-1})\grave{x}_{2n+1}\Big)\\
&\cdot(f(x)D_x)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
=&-\sum_{i=1}^{m}e_i\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)e_i\\
&-2\int_{\Omega^\delta}\int_B(f(x)D_x)\sum_{k=0}^{n}\frac{4^k k!}{(n-k)!}v_{2k+1}^{m|0}(\underline{x}-\underline{y})(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2k}\mathrm{d}\underline{x}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-4\int_{\Omega^\delta}\int_B(f(x)D_x)\sum_{k=0}^{n-1}\frac{4^k k!}{(n-k-1)!}v_{2k+2}^{m|0}(\underline{x}-\underline{y})(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2k-1}\mathrm{d}\underline{x}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
=&-\sum_{i=1}^{m}e_i\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)e_i-2\int_{\Omega^\delta}\int_B(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}.
\end{aligned}\tag{3.6}
$$

根据等式 (3.6), 引理 2.3 以及

$$
v_2^{m|2n}(x-y)D_x=v_1^{m|2n}(x-y),
$$

可得

$$
\begin{aligned}
\mathrm{I}=&-2\int_{\Omega^\delta}\int_B(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}\\
&+\sum_{i=1}^{m}e_i\Big(\int_{\Omega^\delta}\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)e_i\\
&+\sum_{i=1}^{m}e_i\Big(\int_{\partial\Omega^\delta}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega^\delta}\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}
\end{aligned}
$$

$$\begin{aligned}
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)(D_{x}f(x)D_{x})\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}. \qquad (3.7)
\end{aligned}$$

由等式 (3.5) 和 (3.7), 我们有

$$\begin{aligned}
&-2\int_{\Omega^{\delta}}\int_{B}(f(x)D_{x})v_{1}^{m|2n}(x-y)\mathrm{d}\underline{x}+\sum_{i=1}^{m}e_{i}\Big(\int_{\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)(D_{x}f(x)D_{x})\mathrm{d}\underline{x}\Big)e_{i}\\
&+\sum_{i=1}^{m}e_{i}\Big(\int_{\partial\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})\Big)e_{i}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)(D_{x}f(x)D_{x})\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)(D_{x}f(x)D_{x})\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega^{\delta}}\int_{B}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
=&\int_{\Omega^{\delta}}\int_{B}v_{1}^{m|2n}(x-y)(D_{x}f(x)D_{x})(x-y)\mathrm{d}\underline{x}\\
&+\int_{\partial\Omega^{\delta}}\int_{B}v_{1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_{x})(x-y)). \qquad (3.8)
\end{aligned}$$

且

$$\int_{\partial\Omega^{\delta}}\int_{B}=\int_{\partial\Omega}\int_{B}-\int_{\partial\overline{B}_{(\underline{y},\delta)}}\int_{B}.$$

利用引理 3.4, 我们有

$$\begin{aligned}
&\lim_{\delta\to0}\int_{\partial\overline{B}_{(\underline{y},\delta)}}\int_{B}v_{1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_{x})(x-y))\\
=&\lim_{\delta\to0}\int_{\partial\overline{B}_{(\underline{y},\delta)}}\int_{B}\frac{1}{n!}v_{1}^{m|0}(\underline{x}-\underline{y})(\underline{\grave{x}}-\underline{\grave{y}})^{2n}\mathrm{d}\sigma_{\underline{x}}((f(x)D_{x})(\underline{x}-\underline{y}))\\
&+\lim_{\delta\to0}\int_{\partial\overline{B}_{(\underline{y},\delta)}}\int_{B}\frac{1}{n!}v_{1}^{m|0}(\underline{x}-\underline{y})(\underline{\grave{x}}-\underline{\grave{y}})^{2n}\mathrm{d}\sigma_{\underline{x}}((f(x)D_{x})(\underline{\grave{x}}-\underline{\grave{y}}))=0.
\end{aligned}$$

类似地

$$\lim_{\delta\to0}\int_{\partial\overline{B}_{(\underline{y},\delta)}}v_{2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_{x})=0.$$

因此

$$\lim_{\delta\to 0}\int_{\partial\Omega^\delta}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y))$$
$$=\int_{\partial\Omega}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y)),$$
$$\lim_{\delta\to 0}\int_{\partial\Omega^\delta}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)=\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x).$$

又

$$\lim_{\delta\to 0}\int_{\Omega^\delta}\int_B=\int_\Omega\int_B.$$

在等式 (3.8) 中取 $\delta\to 0$, 我们有

$$\begin{aligned}
&-2\int_\Omega\int_B(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}+\sum_{i=1}^m e_i\Big(\int_\Omega\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)e_i\\
&+\sum_{i=1}^m e_i\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i\\
&+2\sum_{j=1}^n \grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_\Omega\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+2\sum_{j=1}^n \grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^n \grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_\Omega\int_B v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^n \grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
=&\int_{\Omega^\delta}\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(x-y)\mathrm{d}\underline{x}\\
&+\int_{\partial\Omega}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y)).
\end{aligned}\tag{3.9}$$

利用定理 3.1, 我们有

$$2\int_\Omega\int_B(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}=-2\int_{\partial\Omega}\int_B f(x)\mathrm{d}\sigma_{\underline{x}}v_1^{m|2n}(x-y)-2f(\underline{y}).$$

由等式 (3.9) 可以得到等式 (3.3).

(ii) 若 $\underline{y}\in\mathbf{R}^n\backslash\overline{\Omega}$, 根据定理 3.2, 有

$$\begin{aligned}
&\int_\Omega\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(x-y)\mathrm{d}\underline{x}\\
=&\int_\Omega\int_B v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{x}_{2n+2}(D_xf(x)D_x)(\grave{\underline{x}}-\grave{\underline{y}})\frac{\partial^2}{\partial\grave{x}_{2n+2}\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
&+\int_\Omega\int_B v_1^{m|2n}(x-y)(D_xf(x)D_x)(\underline{x}-\underline{y})\mathrm{d}\underline{x}\\
=&2\int_\Omega\int_B\sum_{j=1}^n v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j-1}(f(x)D_x)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}
\end{aligned}$$

$$
\begin{aligned}
&-2\int_{\Omega}\int_{B}\sum_{j=1}^{n}v_1^{m|2n}(x-y)\grave{x}_{2n+1}\grave{e}_{2j}(f(x)D_x)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\mathrm{d}\underline{x}\\
&+\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)(D_x((f(x)D_x)(\underline{\grave{x}}-\underline{\grave{y}})))\mathrm{d}\underline{x}\\
&+\int_{\Omega}\int_{B}\sum_{i=1}^{m}v_1^{m|2n}(x-y)e_i(f(x)D_x)e_i\mathrm{d}\underline{x}\\
&+\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)(D_x((f(x)D_x)(\underline{x}-\underline{y})))\mathrm{d}\underline{x}\\
=\ &\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4+\mathrm{J}_5.
\end{aligned}
\tag{3.10}
$$

根据等式 (3.10) 和引理 2.3, 我们有

$$
\begin{aligned}
\mathrm{J}_3+\mathrm{J}_5&=\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)\Big(D_x((f(x)D_x)(x-y))\Big)\mathrm{d}\underline{x}\\
&=-\int_{\partial\Omega}\int_{B}v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y)).
\end{aligned}
\tag{3.11}
$$

类似等式 (3.6), 我们有

$$
\begin{aligned}
\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_4=&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-\sum_{i=1}^{m}e_i\Big(\int_{\Omega}\int_{B}v_1^{m|2n}(x-y)(f(x)D_x)\mathrm{d}\underline{x}\Big)e_i\\
&-2\int_{\Omega}\int_{B}(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}.
\end{aligned}
\tag{3.12}
$$

由定理 3.1, 我们有

$$
\int_{\Omega}\int_{B}(f(x)D_x)v_1^{m|2n}(x-y)\mathrm{d}\underline{x}=-\int_{\partial\Omega}\int_{B}f(x)\mathrm{d}\sigma_{\underline{x}}v_1^{m|2n}(x-y).
\tag{3.13}
$$

根据等式 (3.12)–(3.13), 引理 2.3 和

$$
v_2^{m|2n}(x-y)D_x=v_1^{m|2n}(x-y),
$$

可得

$$
\begin{aligned}
\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_4=&\,2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\Omega}\int_{B}v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&+2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega}\int_{B}v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\Omega}\int_{B}v_2^{m|2n}(x-y)(D_xf(x)D_x)\mathrm{d}\underline{x}\Big)\grave{e}_{2j}\frac{\partial}{\partial\grave{x}_{2n+1}}\\
&-2\sum_{j=1}^{n}\grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega}\int_{B}v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j-1}\frac{\partial}{\partial\grave{x}_{2n+1}}
\end{aligned}
$$

$$
\begin{aligned}
&+2\int_{\partial\Omega}\int_B f(x)\mathrm{d}\sigma_{\underline{x}} v_1^{m|2n}(x-y)\\
&+\sum_{i=1}^{m} e_i\Big(\int_{\Omega}\int_B v_2^{m|2n}(x-y)(D_x f(x) D_x)\mathrm{d}\underline{x}\Big)e_i\\
&+\sum_{i=1}^{m} e_i\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i. \qquad (3.14)
\end{aligned}
$$

将等式 (3.11) 和 (3.14) 代入等式 (3.10), 我们得到如下结论.

定理 3.5 (Cauchy 积分公式) 若 $f \in C^2(\Omega)_{m|2n}$ 是 Ω 中的次正则函数, 则对于任意的 $\underline{y} \in \Omega$, 有

$$
\begin{aligned}
f(y)=&\frac{1}{2}\int_{\partial\Omega}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y))-\int_{\partial\Omega}\int_B f(x)\mathrm{d}\sigma_{\underline{x}} v_1^{m|2n}(x-y)\\
&-\frac{1}{2}\sum_{i=1}^{m} e_i\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i\\
&-\sum_{j=1}^{n} \grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j-1}\frac{\partial}{\partial \grave{x}_{2n+1}}\\
&+\sum_{j=1}^{n} \grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j}\frac{\partial}{\partial \grave{x}_{2n+1}}. \qquad (3.15)
\end{aligned}
$$

定理 3.6 (Cauchy 积分定理) 若 $f \in C^2(\Omega)_{m|2n}$ 是 Ω 中的次正则函数, 则对于任意的 $\underline{y} \in \mathbf{R}^n\backslash\overline{\Omega}$, 有

$$
\begin{aligned}
&\frac{1}{2}\int_{\partial\Omega}\int_B v_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}((f(x)D_x)(x-y))-\int_{\partial\Omega}\int_B f(x)\mathrm{d}\sigma_{\underline{x}} v_1^{m|2n}(x-y)\\
&-\frac{1}{2}\sum_{i=1}^{m} e_i\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)e_i\\
&-\sum_{j=1}^{n} \grave{x}_{2n+1}\grave{e}_{2j-1}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j-1}\frac{\partial}{\partial \grave{x}_{2n+1}}\\
&+\sum_{j=1}^{n} \grave{x}_{2n+1}\grave{e}_{2j}\Big(\int_{\partial\Omega}\int_B v_2^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}(f(x)D_x)\Big)\grave{e}_{2j}\frac{\partial}{\partial \grave{x}_{2n+1}}=0. \qquad (3.16)
\end{aligned}
$$

注 3.2 由于 $\grave{x}_{2n+1}\partial_{\grave{x}_{2n+1}}=1$, 对于等式中 $\grave{x}_{2n+1}$ 和 $\partial\grave{x}_{2n+1}$ 只参与交换位置来改变符号, 最终会抵消掉, 即 $f(y)$ 的积分表示不含 $\grave{x}_{2n+1}$ 和 $\partial\grave{x}_{2n+1}$. 换言之, (3.3) 与 (3.16) 是有意义的.

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A Cauchy Integral Formula for Inframonogenic Functions in Superspace

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Abstract In this paper, first some properties for inframonogenic functions (the solution of sandwich equation $D_x f D_x = 0$) in superspace are given. Then a Cauchy-Pompeiu formula in superspace is proved. Finally the Cauchy integral formula and the Cauchy integral theorem for inframonogenic functions in superspace are obtained.

Keywords Superspace, Inframonogenic functions, Cauchy-Pompeiu formula, Cauchy integral formula, Cauchy integral theorem

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