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Schwarzschild 时空中带记忆项的波动方程耦合方程组解的奇性*

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提要 本文研究 Schwarzschild 时空中非线性波动方程耦合方程组的 Cauchy 问题解的破裂性态. 问题的非线性项包含混合型记忆项、组合和幂次型记忆项、组合和导数型记忆项以及组合型记忆项. 当非线性项的指数满足一定假设时, 利用迭代方法建立解的生命跨度的上界估计. 创新之处是在 Schwarzschild 度量下分析非线性记忆项对解的生命跨度估计的影响. 据已有文献所知, 定理 1.1–1.4 中的结果是新的.

关键词 耦合方程组, 记忆项, 迭代方法, 破裂, 生命跨度估计

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§1 引言及主要结论

本文研究带记忆项的波动方程耦合方程组的 Cauchy 问题

$$\begin{cases}\Box_{g_S}u=f(v,v_t), \quad (t,x)\in\mathbb{R}^+\times\Omega,\\ \Box_{g_S}v=f(u,u_t), \quad (t,x)\in\mathbb{R}^+\times\Omega,\\ (u,u_t,v,v_t)(0,x)=\varepsilon(u_0,u_1,v_0,v_1)(x), \quad x\in\Omega,\end{cases} \tag{1.1}$$

其中 $\Box_{g_S}=\frac{1}{F(r)}(\partial_t^2-\frac{F(r)}{r^2}\partial_r(r^2F(r))\partial_r-\frac{F(r)}{r^2}\Delta_{\mathbb{S}^2})$ 是 Schwarzschild 度量 g_S 下的波算子, $F(r)=1-\frac{2M}{r}$. $\Omega=\{(r,w)\big|2M<r<\infty,\ w\in\mathbb{S}^2\}$ 表示黑洞的外部. 非线性项包含混合型记忆项 $f(v,v_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|v(\tau,x)|^q\mathrm{d}\tau$, $f(u,u_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|u_t(\tau,x)|^p\mathrm{d}\tau$, 组合和幂次型记忆项 $f(v,v_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(|v_t(\tau,x)|^{p_1}+|v(\tau,x)|^{q_1})\mathrm{d}\tau$, $f(u,u_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|u(\tau,x)|^{q_2}\mathrm{d}\tau$, 组合和导数型记忆项 $f(v,v_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(|v_t(\tau,x)|^{p_1}+|v(\tau,x)|^{q_1})\mathrm{d}\tau$, $f(u,u_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}|u_t(\tau,x)|^{p_2}\mathrm{d}\tau$ 以及组合型记忆项 $f(v,v_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(|v_t(\tau,x)|^{p_1}+|v(\tau,x)|^{q_1})\mathrm{d}\tau$, $f(u,u_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(|u_t(\tau,x)|^{p_2}+|u(\tau,x)|^{q_2})\mathrm{d}\tau$. 非线性项的指数满足 $0<\alpha<1$, $1<p,p_1,p_2,q,q_1,q_2<\infty$. $\Gamma(\alpha)=\int_0^{+\infty}t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t$ 为 Gamma 函数. 假设初值 $u_0(x)$, $u_1(x)$, $v_0(x)$, $v_1(x)$ 是非负函数并且不恒为 0. 正常数 $R_1<R_2$, 并且满足

$$\mathrm{supp}\,(u_0(x),u_1(x),v_0(x),v_1(x))\subset\{2M+R_1\leqslant r\leqslant 2M+R_2\}\times\mathbb{S}^2.$$

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ε 是刻画初值大小且充分小的正常数.

近来, 关于非线性波动方程已有许多研究结果. 同时 Minkowski 时空中非线性波动方程的 Cauchy 问题

$$\begin{cases} \Box u = f(u, u_t), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ (u, u_t)(0,x) = \varepsilon(u_0, u_1)(x), \quad x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

解的破裂及其生命跨度估计受到广泛关注 (见 [1–8]), 其中 $\Box = \partial_t^2 - \Delta$ 表示 Minkowski 度量下的波算子. 当非线性项为 $f(u, u_t) = |u|^p$ 时, 问题 (1.2) 具有 Strauss 临界指数 $p_S(n)$. 当 $n \geqslant 2$ 时, $p_S(n)$ 为二次方程

$$-(n-1)p^2 + (n+1)p + 2 = 0$$

的正根. 当 $n = 1$ 时, $p_S(1) = \infty$. 当 $f(u, u_t) = |u|^p$, $p > 2$ 时, 文 [7] 证明了四维情形问题 (1.2) 的整体解的存在性. 当 $f(u, u_t) = |u|^p$, $1 < p < p_S(n)\,(n \geqslant 3)$ 时, 文 [8] 利用 Kato 引理研究次临界情形变系数波动方程初边值问题的解的破裂及其生命跨度的上界估计. 文 [5] 证明了次临界情形外区域上二维波动方程不存在整体解. 文 [1, 3] 利用检验函数方法给出临界情形问题的解会在有限时间破裂. 当非线性项为 $f(u, u_t) = |u_t|^p$ 时, 问题 (1.2) 具有 Glassey 指数 $p_G(n) = 1 + \frac{2}{n-1}$, 其是小初值问题的解会破裂和存在整体解的分界点. 通过导出常微分不等式, 文 [8] 在次临界和临界情形 $(1 < p \leqslant p_G(n))$ 建立问题 (1.2) 解的生命跨度的上界估计.

Minkowski 时空中带幂次记忆项的波动方程的 Cauchy 问题

$$\begin{cases} \Box u = N_{\alpha,p}(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ (u, u_t)(0,x) = \varepsilon(u_0, u_1)(x), \quad x \in \mathbb{R}^n \end{cases} \tag{1.3}$$

引起许多关注, 其中 $N_{\alpha,p}(u) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)|u(\tau,x)|^p \mathrm{d}s$ (见 [9–13]). 注意到问题 (1.3) 具有临界指数 $p_S(n,\alpha)$. 当 $n = 1$ 时, $p_S(n,\alpha) = \infty$. 当 $n \geqslant 2$ 时, $p_S(n,\alpha)$ 是下列二次方程的正根

$$-(n-1)p^2 + (n + 2\alpha + 1)p + 2 = 0.$$

文 [9] 利用迭代方法得到问题 (1.3) 的解的破裂结果和生命跨度的上界估计. 注意到当 $\alpha \to 0$ 时, 临界指数 $p_S(n,\alpha)$ 趋近于 Strauss 指数 $p_S(n)$. 通过构造新的泛函并利用 Kato 引理, 文 [12] 证明带幂次记忆项的一维波动方程的初边值问题解的破裂. 文 [11] 研究了带阻尼项和幂次记忆项的波动方程的小初值问题

$$\begin{cases} \Box u + h(u_t) = N_{\alpha,p}(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ (u, u_t)(0,x) = \varepsilon(u_0, u_1)(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

当 $h(u_t) = u_t$ 时, 利用检验函数方法得到 n 维空间中问题 (1.4) 的解的破裂. 另外, 低维空间 $1 \leqslant n \leqslant 3$ 时, 利用加权能量法得到问题存在整体解. 当阻尼项为结构阻尼项 $h(u_t) = \mu(-\Delta)^{\frac{\sigma}{2}} u_t$ $(\mu > 0,\ 0 < \sigma < 2)$, $n \geqslant 1$ 时, 文 [10] 证明了问题 (1.4) 不存在整体解, 并利用检验函数方法得到解会在有限时间破裂. 更多结果参见文 [14–19].

Minkowski 时空中非线性波动方程耦合方程组的 Cauchy 问题

$$\begin{cases} u_{tt} - \Delta u = f(v, v_t), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ v_{tt} - \Delta v = f(u, u_t), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n \end{cases} \tag{1.5}$$

已被许多学者研究 (见 [20–25]). 文 [23] 利用检验函数方法研究 Moore-Gibson-Thompson 方程耦合方程组解的生命跨度的上界估计. 其中非线性项包含幂次非线性项 $f(v, v_t) = |v|^p$, $f(u, u_t) = |u|^q$, 导数非线性项 $f(v, v_t) = |v_t|^p$, $f(u, u_t) = |u_t|^q$ 以及组合非线性项 $f(v, v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f(u, u_t) = |u_t|^{p_2} + |u|^{q_2}$. 文 [20] 利用迭代方法得到带一般非线性记忆项 $f(v, v_t) = g_1 * |v|^p$, $f(u, u_t) = g_2 * |u|^q$ 的问题 (1.5) 的解的局部存在性以及解的破裂性态. 同时考虑 $g_1(t)$ 与 $g_2(t)$ 对解的破裂性态的相互影响. 文 [21] 证明了次临界和临界情形带混合记忆项 $f(v, v_t) = |v|^q$, $f(u, u_t) = |u_t|^p$ 的问题 (1.5) 解的生命跨度的上界估计. 文 [25] 研究带散射阻尼项和混合记忆项 $f(v, v_t) = |v|^q$, $f(u, u_t) = |u_t|^p$ 的问题 (1.5). 利用迭代方法和切片方法建立次临界和临界时解的生命跨度的上界估计. 利用检验函数方法和迭代方法, 文 [22] 在次临界和临界情形给出带阻尼项和组合记忆项 $f(v, v_t) = |v_t|^{p_1} + |v|^{q_1}$, $f(u, u_t) = |u_t|^{p_2} + |u|^{q_2}$ 的波动方程耦合方程组解的生命跨度的上界估计.

近来, 一些学者研究 Schwarzschild 时空中带小初值的非线性波动方程的 Cauchy 问题

$$\begin{cases} \Box_{g_S} u = f(u, u_t), \quad (t, x) \in \mathbb{R}^+ \times \Omega, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), \quad x \in \Omega \end{cases} \tag{1.6}$$

解的生命跨度估计. 文 [26] 研究 Schwarzschild 时空中的线性波动方程, 即问题 (1.6) 中非线性项为 $f(u, u_t) = 0$) 解的行为. 文 [27] 在初值满足一定假设时, 利用 Regge-Wheeler 坐标变换得到 $f(u, u_t) = |u|^p$ $(1 < p < 1+\sqrt{2})$ 时问题 (1.6) 解的破裂. 注意到假设条件中要求小初值的支集远离黑洞. 当 $p > 1+\sqrt{2}$ 时, 文 [28] 在 Kerr 时空中通过假设初值具有紧支集, 得到波动方程解的整体存在性. 当小初值不具有紧支集时, 文 [29] 在三维和四维 Schwarzschild 以及 Kerr 时空中超临界情形给出波动方程的整体解的存在性. 通过引入两个适当的检验函数并且利用 Kato 引理, 文 [30] 证明带小初值且 $f(u, u_t) = N_{\alpha,p}(u)$ $(2 \leqslant p < p_S(3, \alpha))$ 时问题 (1.6) 的整体解的不存在性, 但未给出解的生命跨度估计. 当 $2 \leqslant p \leqslant 1 + \sqrt{2}$ 时, 文 [31] 在 Schwarzschild 以及 Kerr 时空中给出 $f(u, u_t) = |u|^p$ 时 Cauchy 问题 (1.6) 的解的生命跨度的下界估计. 文 [32] 通过导出常微分不等式得到 $f(u, u_t) = |u|^p$ $(\frac{3}{2} \leqslant p \leqslant 2)$ 时问题 (1.6) 解的生命跨度的上界估计, 其中初值的支集可以靠近视界. 当 $1 < p \leqslant 2$ 时, 文 [33] 得到 $f(u, u_t) = |u_t|^p$ 时问题 (1.6) 解的破裂及其生命跨度的上界估计. 其中已去掉初值的支集远离黑洞的限制条件. 更多相关结果见文 [34–36].

受文 [20–23, 25, 30] 的启发, 本文拟证明带记忆项的非线性波动方程耦合方程组的 Cauchy 问题解的生命跨度的上界估计. 由于非线性记忆项的出现, 文 [32] 运用的 Kato 引理以及文 [21] 利用的检验函数方法均不适用于本文的问题. 根据不同的非线性项以及非线性项指数的不同取值范围, 本文利用文 [20, 22, 25] 中采用的迭代方法建立解的生命跨度估计. 注意到 Minkowski 时空是 Schwarzschild 时空的特殊情形. 本文将文 [20] 在平坦 Minkowski 时空中研究的问题推广为 Schwarzschild 时空中的问题. 当 $1 < p < p_S(3, \alpha)$ 并

且初值的支集远离黑洞时, 文 [30] 在 Schwarzschild 时空中研究带幂次记忆项的单个波动方程解的破裂. 注意到文 [20, 30] 中尚未得到解的生命跨度的估计. 本文将文 [21, 25] 中考虑的非线性项推广为 Schwarzschild 度量下的非线性记忆项 (见定理 1.1). 注意到本文给出 Schwarzschild 时空中其它非线性项对问题 (1.1) 解的生命跨度估计的影响. 并且非线性项包含组合和幂次型记忆项、组合和导数型记忆项以及组合型记忆项 (见定理 1.2–1.4). 据已有文献所知, 定理 1.1–1.4 中的结果是新的.

基于 Regge-Wheeler 坐标变换

$$s(r)=r+2M\ln(r-2M),$$

问题 (1.1) 可化为

$$\begin{cases} u_{tt}-u_{ss}-\dfrac{2F(s)}{r(s)}u_s-\dfrac{F(s)}{r^2(s)}\Delta_{\mathbb{S}^2}u=F(s)f(v,v_t), & (t,s)\in\mathbb{R}^+\times\mathbb{R},\\ v_{tt}-v_{ss}-\dfrac{2F(s)}{r(s)}v_s-\dfrac{F(s)}{r^2(s)}\Delta_{\mathbb{S}^2}v=F(s)f(u,u_t), & (t,s)\in\mathbb{R}^+\times\mathbb{R},\\ (u,u_t,v,v_t)(0,s)=\varepsilon(u_0,u_1,v_0,v_1)(s), & s\in\mathbb{R}. \end{cases}$$

为简化, 考虑径向解 $(u(t,s),v(t,s))$. 令 $\bar{u}(t,s)=r(s)u(t,s)$, $\bar{v}(t,s)=r(s)v(t,s)$, 则有

$$\begin{cases} \bar{u}_{tt}-\bar{u}_{ss}+W(s)\bar{u}=f_1(s,\bar{v},\bar{v}_t), & (t,s)\in\mathbb{R}^+\times\mathbb{R},\\ \bar{v}_{tt}-\bar{v}_{ss}+W(s)\bar{v}=f_2(s,\bar{u},\bar{u}_t), & (t,s)\in\mathbb{R}^+\times\mathbb{R},\\ (\bar{u},\bar{u}_t,\bar{v},\bar{v}_t)(0,s)=\varepsilon(\bar{u}_0,\bar{u}_1,\bar{v}_0,\bar{v}_1)(s), & s\in\mathbb{R}, \end{cases}\tag{1.7}$$

其中 $W(s)=\frac{2MF(s)}{r^3(s)}$. 假设初值 $\bar{u}_0(s),\bar{v}_0(s)\in H^1(\mathbb{R})$, $\bar{u}_1(s),\bar{v}_1(s)\in L^2(\mathbb{R})$ 是非负光滑函数, 并且不恒为 0. 此外, 存在正常数 R 使得

$$\operatorname{supp}(\bar{u}_0(s),\bar{u}_1(s),\bar{v}_0(s),\bar{v}_1(s))\subset\{s\big|\,|s|\leqslant R\}.$$

根据波动方程具有有限传播速度的性质可知, 当初值具有紧支集时, 解在任意时刻具有紧支集. 因此, 本文假设解具有紧支集.

下面给出本文中一些记号.

$$\begin{aligned}&\Gamma_{SG,1}(p,q)\\ &=\max\Big\{-1+\frac{\alpha+1}{p}+\frac{\alpha+1+(\alpha+2)p^{-1}}{pq-1},-1+\frac{\alpha}{q}+\frac{\alpha+2+(\alpha+1)q^{-1}}{pq-1}\Big\},\\ &\Gamma_{SG,2}(p,q)\\ &=\max\Big\{-1+\frac{\alpha+2}{p}+\frac{\alpha+1+(\alpha+2)p^{-1}}{pq-1},-1+\frac{\alpha+1}{q}+\frac{\alpha+2+(\alpha+1)q^{-1}}{pq-1}\Big\},\\ &\Gamma_{SG,3}(p,q)\\ &=\max\Big\{-1+\frac{\alpha+3}{p}+\frac{\alpha+3+\alpha p^{-1}}{pq-1},-1+\frac{\alpha-1}{q}+\frac{\alpha+4+(\alpha-1)q^{-1}}{pq-1}\Big\},\end{aligned}$$

$$\begin{aligned}&\Gamma_{SG,4}(p,q)\\ &=\max\Big\{-1+\frac{\alpha}{p}+\frac{\alpha+3+\alpha p^{-1}}{pq-1},-1+\frac{\alpha+2}{q}+\frac{\alpha+(\alpha+3)q^{-1}}{pq-1}\Big\},\end{aligned}$$

$$\Gamma_{CS,1}(p_1,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+1}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+1}{q_2}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1}\Big\},$$

$$\Gamma_{CS,2}(p_1,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+2}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+2}{q_2}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1}\Big\},$$

$$\Gamma_{CS,3}(p_1,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+1}{q_2}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1}\Big\},$$

$$\Gamma_{CS,4}(p_1,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+3}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+2}{q_2}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1}\Big\},$$

$$\Gamma_{CG,1}(p_1,p_2,q_1)$$
$$=\max\Big\{-1+\frac{\alpha}{p_1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)},\ -1+\frac{\alpha+1}{p_2}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1}\Big\},$$

$$\Gamma_{CG,2}(p_1,p_2,q_1)$$
$$=\max\Big\{-1+\frac{\alpha+1}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1p_2-1)},\ -1+\frac{\alpha+2}{p_2}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1}\Big\},$$

$$\Gamma_{CG,3}(p_1,p_2,q_1)$$
$$=\max\Big\{-1+\frac{\alpha-1}{p_1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)},\ -1+\frac{\alpha+1}{p_2}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1}\Big\},$$

$$\Gamma_{CG,4}(p_1,p_2,q_1)$$
$$=\max\Big\{-1+\frac{\alpha+2}{p_1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)},\ -1+\frac{\alpha+2}{p_2}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1}\Big\},$$

$$\Gamma_{CC,1}(p_1,p_2,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha}{p_1}+\frac{(\alpha+4)q_1+\alpha}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+3}{p_2}+\frac{\alpha q_2+\alpha+4}{p_2(q_1q_2-1)}\Big\},$$

$$\Gamma_{CC,2}(p_1,p_2,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+3}{p_1}+\frac{\alpha q_1+\alpha+4}{p_1(q_1q_2-1)},\ -1+\frac{\alpha}{p_2}+\frac{(\alpha+4)q_2+\alpha}{p_2(q_1q_2-1)}\Big\},$$

$$\Gamma_{CC,3}(p_1,p_2,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+3}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+2}{p_2}+\frac{(\alpha+2)q_2+\alpha+2}{p_2(q_1q_2-1)}\Big\},$$

$$\Gamma_{CC,4}(p_1,p_2,q_1,q_2)$$
$$=\max\Big\{-1+\frac{\alpha+2}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+\frac{\alpha+3}{p_2}+\frac{(\alpha+2)q_2+\alpha+2}{p_2(q_1q_2-1)}\Big\},$$

$$\begin{aligned}&\Gamma_{CC,5}(p_1,p_2,q_1,q_2)\\&=\max\Big\{-1+\frac{\alpha}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},-1+\frac{\alpha+1}{p_2}+\frac{(\alpha+2)q_2+\alpha+2}{p_2(q_1q_2-1)}\Big\},\\&\Gamma_{CC,6}(p_1,p_2,q_1,q_2)\\&=\max\Big\{-1+\frac{\alpha+1}{p_1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},-1+\frac{\alpha}{p_2}+\frac{(\alpha+2)q_2+\alpha+2}{p_2(q_1q_2-1)}\Big\}.\end{aligned}$$

$X\sim Y$ 表示存在正常数 C, 使得 $C^{-1}Y\leqslant X\leqslant CY$.

下面给出问题 (1.7) 的温和解的局部存在性引理.

引理 1.1 [37] 假设 $\bar u_0(s),\bar v_0(s)\in H^1(\mathbb{R})$, $\bar u_1(s),\bar v_1(s)\in L^2(\mathbb{R})$ 均为具有紧支集的函数, 并且满足 $\operatorname{supp}(\bar u_0(s),\bar u_1(s),\bar v_0(s),\bar v_1(s))\subset\{s\,\big|\,|s|\leqslant R\}$. 则存在正常数 T, 使得问题 (1.7) 具有唯一局部 (关于时间) 温和解 $(\bar u,\bar v)\in(C([0,T),H^1(\mathbb{R}))\cap C^1([0,T),L^2(\mathbb{R})))^2$.

本文主要结果如下.

定理 1.1 令 $p>1$, $q>1$. 若非线性项为 $f_1(s,\bar v,\bar v_t)=h_2(s)N_{\alpha,q}(\bar v)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h_2(s)|\bar v|^q\mathrm{d}\tau$, $f_2(s,\bar u,\bar u_t)=h_1(s)N_{\alpha,p}(\bar u_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h_1(s)|\bar u_t|^p\mathrm{d}\tau$ 的问题 (1.7) 的解 $(\bar u,\bar v)$ 满足 $\operatorname{supp}(\bar u,\bar v)\subset\{(t,s)\in[0,T)\times\mathbb{R}\,\big|\,|s|\leqslant t+R\}$, 则 $(\bar u,\bar v)$ 会在有限时间内破裂, 并且解的生命跨度的上界的估计满足

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-\Gamma_{SG,1}^{-1}(p,q)}, & \Gamma_{SG,1}>0,\ p\geqslant2,\ q\geqslant2,\ \dfrac12\leqslant\alpha<1,\\ C\varepsilon^{-\Gamma_{SG,2}^{-1}(p,q)}, & \Gamma_{SG,2}>0,\ 1<p<2,\ 1<q<2,\ 0<\alpha<1,\\ C\varepsilon^{-\Gamma_{SG,3}^{-1}(p,q)}, & \Gamma_{SG,3}>0,\ p\geqslant2,\ 1<q<2,\ \dfrac12\leqslant\alpha<1,\\ C\varepsilon^{-\Gamma_{SG,4}^{-1}(p,q)}, & \Gamma_{SG,4}>0,\ 1<p<2,\ q\geqslant2,\ 0<\alpha<1,\end{cases}\tag{1.8}$$

其中 $h_1(s)=F(s)r^{1-p}(s)$, $h_2(s)=F(s)r^{1-q}(s)$, C 为与 ε 无关的正常数.

定理 1.2 令 $1<q_1<6$. 若非线性项为 $f_1(s,\bar v,\bar v_t)=h_3(s)N_{\alpha,p_1}(\bar v_t)+h_4(s)N_{\alpha,q_1}(\bar v)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(h_3(s)|\bar v_t|^{p_1}+h_4(s)|\bar v|^{q_1})\mathrm{d}\tau$, $f_2(s,\bar u,\bar u_t)=h_6(s)N_{\alpha,q_2}(\bar u)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h_6(s)|\bar u|^{q_2}\mathrm{d}\tau$ 的问题 (1.7) 的解 $(\bar u,\bar v)$ 满足 $\operatorname{supp}(\bar u,\bar v)\subset\{(t,s)\,\big|\,|s|\leqslant t+R\}$, 则 $(\bar u,\bar v)$ 会在有限时间内破裂, 并且解的生命跨度的上界的估计满足

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-\Gamma_{CS,1}^{-1}(p_1,q_1,q_2)}, & \Gamma_{CS,1}>0,\ p_1\geqslant2,\ q_1\geqslant2,\\ & q_2\geqslant2,\ \dfrac12\leqslant\alpha<1,\\ C\varepsilon^{-\Gamma_{CS,2}^{-1}(p_1,q_1,q_2)}, & \Gamma_{CS,2}>0,\ 1<p_1<2,\ 1<q_1<2,\\ & 1<q_2<2,\ 0<\alpha<1,\\ C\varepsilon^{-\Gamma_{CS,3}^{-1}(p_1,q_1,q_2)}, & \Gamma_{CS,3}>0,\ 1<p_1<2,\ q_1\geqslant2,\\ & q_2\geqslant2,\ 0<\alpha<1,\\ C\varepsilon^{-\Gamma_{CS,4}^{-1}(p_1,q_1,q_2)}, & \Gamma_{CS,4}>0,\ p_1\geqslant2,\ 1<q_1<2,\\ & 1<q_2<2,\ \dfrac12\leqslant\alpha<1,\end{cases}\tag{1.9}$$

其中 $h_3(s)=F(s)r^{1-p_1}(s)$, $h_4(s)=F(s)r^{1-q_1}(s)$, $h_6(s)=F(s)r^{1-q_2}(s)$, C 为与 ε 无关的正常数.

定理 1.3 令 $1<q_1<6$. 若非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_3(s)N_{\alpha,p_1}(\bar{v}_t)+h_4(s)N_{\alpha,q_1}(\bar{v})=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(h_3(s)|\bar{v}_t|^{p_1}+h_4(s)|\bar{v}|^{q_1})\mathrm{d}\tau$, $f_2(s,\bar{u},\bar{u}_t)=h_5(s)N_{\alpha,p_2}(\bar{u}_t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h_5(s)|\bar{u}_t|^{p_2}\mathrm{d}\tau$ 的问题 (1.7) 的解 $(\bar{u},\bar{v})$ 满足 $\operatorname{supp}(\bar{u},\bar{v})\subset\{(t,s)\,\big|\,|s|\leqslant t+R\}$, 则 $(\bar{u},\bar{v})$ 会在有限时间内破裂, 并且解的生命跨度的上界的估计满足

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-\Gamma_{CG,1}^{-1}(p_1,p_2,q_1)}, & \Gamma_{CG,1}>0,\ p_1\geqslant2,\ p_2\geqslant2,\\ & q_1\geqslant2,\ \dfrac{1}{2}\leqslant\alpha<1,\\ C\varepsilon^{-\Gamma_{CG,2}^{-1}(p_1,p_2,q_1)}, & \Gamma_{CG,2}>0,\ 1<p_1<2,\ 1<p_2<2,\\ & 1<q_1<2,\ 0<\alpha<1,\\ C\varepsilon^{-\Gamma_{CG,3}^{-1}(p_1,p_2,q_1)}, & \Gamma_{CG,3}>0,\ 1<p_1<2,\ p_2\geqslant2,\\ & q_1\geqslant2,\ \dfrac{1}{2}\leqslant\alpha<1,\\ C\varepsilon^{-\Gamma_{CG,4}^{-1}(p_1,p_2,q_1)}, & \Gamma_{CG,4}>0,\ p_1\geqslant2,\ 1<p_2<2,\\ & 1<q_1<2,\ \dfrac{1}{2}\leqslant\alpha<1,\end{cases}\tag{1.10}$$

其中 $h_5(s)=F(s)r^{1-p_2}(s)$, C 为与 ε 无关的正常数.

定理 1.4 假设 $1<q_1,\,q_2<6$, $\frac{1}{2}\leqslant\alpha<1$. 若非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_3(s)N_{\alpha,p_1}(\bar{v}_t)+h_4(s)N_{\alpha,q_1}(\bar{v})=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(h_3(s)|\bar{v}_t|^{p_1}+h_4(s)|\bar{v}|^{q_1})\mathrm{d}\tau$, $f_2(s,\bar{u},\bar{u}_t)=h_5(s)N_{\alpha,p_2}(\bar{u}_t)+h_6(s)N_{\alpha,q_2}(\bar{u})=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(h_5(s)|\bar{u}_t|^{p_2}+h_6(s)|\bar{u}|^{q_2})\mathrm{d}\tau$ 的问题 (1.7) 的解 $(\bar{u},\bar{v})$ 满足 $\operatorname{supp}(\bar{u},\bar{v})\subset\{(t,s)\,\big|\,|s|\leqslant t+R\}$, 则 $(\bar{u},\bar{v})$ 会在有限时间内破裂. 并且解的生命跨度的上界的估计满足

$$T(\varepsilon)\leqslant\begin{cases}C\varepsilon^{-\Gamma_{CC,1}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,1}>0,\ 1<p_1<2,\ p_2\geqslant2,\\ & 1<q_1<2,\ q_2\geqslant2,\\ C\varepsilon^{-\Gamma_{CC,2}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,2}>0,\ p_1\geqslant2,\ 1<p_2<2,\\ & q_1\geqslant2,\ 1<q_2<2,\\ C\varepsilon^{-\Gamma_{CC,3}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,3}>0,\ p_1\geqslant2,\ 1<p_2<2,\\ & 1<q_1<2,\ 1<q_2<2,\\ C\varepsilon^{-\Gamma_{CC,4}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,4}>0,\ 1<p_1<2,\ p_2\geqslant2,\\ & 1<q_1<2,\ 1<q_2<2,\\ C\varepsilon^{-\Gamma_{CC,5}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,5}>0,\ 1<p_1<2,\ p_2\geqslant2,\\ & q_1\geqslant2,\ q_2\geqslant2,\\ C\varepsilon^{-\Gamma_{CC,6}^{-1}(p_1,p_2,q_1,q_2)}, & \Gamma_{CC,6}>0,\ p_1\geqslant2,\ 1<p_2<2,\\ & q_1\geqslant2,\ q_2\geqslant2,\end{cases}\tag{1.11}$$

其中 C 为与 ε 无关的正常数.

注 1.1 当 $\alpha\to 0$ 时, (1.8) 中具有混合型记忆项的问题 (1.1) 解的第一个生命跨度估计等价于文 [25] 中次临界情形三维耦合波动方程解的生命跨度估计 (见定理 1.1). 当 $\alpha\to 0$ 时, 具有组合类型记忆项的问题 (1.1) 等价于文 [23] 中研究的具有组合非线性 $|v_t|^{p_1}+|v|^{q_1}$, $|u_t|^{p_2}+|u|^{q_2}$ 的问题. 然而, 由于非线性记忆项中的指数范围限制, (1.11) 中解的生命跨度估计与文 [23] 中定理 1.4 中解的生命估计不同 (见定理 1.4).

假设

$$\begin{aligned}U(t)&=\int_{\mathbb{R}}\bar{u}(t,s)\phi_0(s)\mathrm{d}s, & V(t)&=\int_{\mathbb{R}}\bar{v}(t,s)\phi_0(s)\mathrm{d}s,\\ U_1(t)&=\int_{\mathbb{R}}\bar{u}(t,s)\Phi(t,s)\mathrm{d}s, & V_1(t)&=\int_{\mathbb{R}}\bar{v}(t,s)\Phi(t,s)\mathrm{d}s,\\ U_2(t)&=\int_{\mathbb{R}}\bar{u}_t(t,s)\Phi(t,s)\mathrm{d}s, & V_2(t)&=\int_{\mathbb{R}}\bar{v}_t(t,s)\Phi(t,s)\mathrm{d}s.\end{aligned}$$

§2 定理 1.1 的证明

§2.1 相关引理以及弱解的定义

引理 2.1 [27,30] 存在正函数 $\phi_0(s)$ 满足

$$\partial_s^2\phi_0(s)-W(s)\phi_0(s)=0,$$

并且

$$\phi_0(s)\sim\begin{cases}s, & s\to+\infty,\\ \mathrm{e}^{\frac{s}{2M}}+D, & s\to-\infty,\end{cases}\tag{2.1}$$

其中 D 为正常数.

引理 2.2 [27,30] 设 A 是正常数. 正函数 $\varphi_1(s)\sim\mathrm{e}^{As}$ 是方程

$$\big(-\partial_s^2+W(s)+A^2\big)\varphi_1(s)=0,$$

的解.

在引理 2.2 中令 $A=\frac{1}{2M}$, 可知正函数 $\varphi_1(s)\sim\mathrm{e}^{\frac{s}{2M}}$ 是方程

$$\Big(-\partial_s^2+W(s)+\frac{1}{4M^2}\Big)\varphi_1(s)=0\tag{2.2}$$

的解. 令 $\Phi(t,s)=\mathrm{e}^{-\frac{t}{2M}}\varphi_1(s)$. 则有

$$\begin{aligned}\Phi_t(t,s)&=-\frac{1}{2M}\Phi(t,s),\\ \Phi_{tt}(t,s)&=\frac{1}{4M^2}\Phi(t,s),\\ \Phi_{ss}(t,s)-W(s)\Phi(t,s)&=\frac{1}{4M^2}\Phi(t,s).\end{aligned}\tag{2.3}$$

引理 2.3 [38] 令 $h(s)=F(s)r^{1-p}(s)$, 则有

$$\int_{\mathbb{R}}(h(s))^{-\frac{1}{p-1}}\phi_0(s)\mathrm{d}s\leqslant\begin{cases}C(t+R)^3, & p\geqslant 2,\\ C(t+R)\mathrm{e}^{\frac{2-p}{p-1}\cdot\frac{t}{2M}}, & 1<p<2.\end{cases}\tag{2.4}$$

引理 2.4 [38] 令 $h(s)=F(s)r^{1-p}(s)$, 则有

$$\int_{\mathbb{R}}(h(s)\phi_0(s))^{-\frac{1}{p-1}}(\Phi(t,s))^{p'}\mathrm{d}s\leqslant\begin{cases}C(t+R)^{\frac{p-2}{p-1}}, & p\geqslant 2,\\ C(t+R), & 1<p<2.\end{cases}\tag{2.5}$$

定义 2.1 假设 $\bar{u}_0(s),\bar{v}_0(s)\in H^1(\mathbb{R})$, $\bar{u}_1(s),\bar{v}_1(s)\in L^2(\mathbb{R})$, 并且

$$(\bar{u},\bar{v})\in(C([0,T),H^1(\mathbb{R}))\cap C^1([0,T),L^2(\mathbb{R})))^2.$$

当问题 (1.7) 的非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_2(s)N_{\alpha,q}(\bar{v})$, $f_2(s,\bar{u},\bar{u}_t)=h_1(s)N_{\alpha,p}(\bar{u}_t)$ 时, 满足 $N_{\alpha,q}(\bar{v})\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,p}(\bar{u}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$. 当问题 (1.7) 的非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_3(s)N_{\alpha,p_1}(\bar{v}_t)+h_4(s)N_{\alpha,q_1}(\bar{v})$, $f_2(s,\bar{u},\bar{u}_t)=h_6(s)N_{\alpha,q_2}(\bar{u})$ 时, 满足 $N_{\alpha,p_1}(\bar{v}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,q_1}(\bar{v})\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,q_2}(\bar{u})\in L^1_{loc}([0,T)\times\mathbb{R})$. 当问题 (1.7) 的非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_3(s)N_{\alpha,p_1}(\bar{v}_t)+h_4(s)N_{\alpha,q_1}(\bar{v})$, $f_2(s,\bar{u},\bar{u}_t)=h_5(s)N_{\alpha,p_2}(\bar{u}_t)$ 时, 满足 $N_{\alpha,p_1}(\bar{v}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,q_1}(\bar{v})\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,p_2}(\bar{u}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$. 当问题 (1.7) 的非线性项为 $f_1(s,\bar{v},\bar{v}_t)=h_3(s)N_{\alpha,p_1}(\bar{v}_t)+h_4(s)N_{\alpha,q_1}(\bar{v})$, $f_2(s,\bar{u},\bar{u}_t)=h_5(s)N_{\alpha,p_2}(\bar{u}_t)+h_6(s)N_{\alpha,q_2}(\bar{u})$ 时, 满足 $N_{\alpha,p_1}(\bar{v}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,q_1}(\bar{v})\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,p_2}(\bar{u}_t)\in L^1_{loc}([0,T)\times\mathbb{R})$, $N_{\alpha,q_2}(\bar{u})\in L^1_{loc}([0,T)\times\mathbb{R})$. 则有

$$\begin{aligned}&\int_{\mathbb{R}}\bar{u}_t(t,s)\phi(t,s)\mathrm{d}s-\varepsilon\int_{\mathbb{R}}\bar{u}_1(s)\phi(0,s)\mathrm{d}s-\int_0^t\int_{\mathbb{R}}\bar{u}_\tau(\tau,s)\phi_\tau(\tau,s)\mathrm{d}s\mathrm{d}\tau\\&-\int_0^t\int_{\mathbb{R}}\bar{u}(\tau,s)\phi_{ss}(\tau,s)\mathrm{d}s\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}W(s)\bar{u}(\tau,s)\phi(\tau,s)\mathrm{d}s\mathrm{d}\tau\\=&\int_0^t\int_{\mathbb{R}}f_1(s,\bar{v},\bar{v}_\tau)\phi(\tau,s)\mathrm{d}s\mathrm{d}\tau,\end{aligned}\tag{2.6}$$

$$\begin{aligned}&\int_{\mathbb{R}}\bar{v}_t(t,s)\psi(t,s)\mathrm{d}s-\varepsilon\int_{\mathbb{R}}\bar{v}_1(s)\psi(0,s)\mathrm{d}s-\int_0^t\int_{\mathbb{R}}\bar{v}_\tau(\tau,s)\psi_\tau(\tau,s)\mathrm{d}s\mathrm{d}\tau\\&-\int_0^t\int_{\mathbb{R}}\bar{v}(\tau,s)\psi_{ss}(\tau,s)\mathrm{d}s\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}W(s)\bar{v}(\tau,s)\psi(\tau,s)\mathrm{d}s\mathrm{d}\tau\\=&\int_0^t\int_{\mathbb{R}}f_2(s,\bar{u},\bar{u}_\tau)\psi(\tau,s)\mathrm{d}s\mathrm{d}\tau,\end{aligned}\tag{2.7}$$

其中 $\phi(t,s),\psi(t,s)\in C^2([0,T)\times\mathbb{R})$, $t\in[0,T)$. 则 $(\bar{u},\bar{v})$ 称为问题 (1.7) 的弱解.

§2.2　问题 1.1 的证明

当 (2.6) 式中 $f_1(s,\bar{v},\bar{v}_t)=h_2(s)N_{\alpha,q}(\bar{v})$ 时, 令 $\phi(t,x)=\phi_0(x)$, 得到

$$U'(t)-U'(0)=\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}}h_2(s)\phi_0(s)|\bar{v}(\sigma,s)|^q\mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau.\tag{2.8}$$

利用 Hölder 不等式和引理 2.3, 则有

$$\int_{\mathbb{R}} h_2(s)\phi_0(s)|\bar{v}(t,s)|^q \mathrm{d}s \geqslant \frac{|V(t)|^q}{(\int_{\mathbb{R}}(h_2(s))^{-\frac{1}{q-1}}\phi_0(s)\mathrm{d}s)^{q-1}} \geqslant \begin{cases} C(t+R)^{-3(q-1)}|V(t)|^q, & q \geqslant 2, \\ C(t+R)^{-(q-1)}|V(t)|^q, & 1<q<2. \end{cases}$$

从而可得

$$U'(t) \geqslant \begin{cases} K\int_0^t\int_0^\tau (\tau-\sigma)^{\alpha-1}(\sigma+R)^{-3(q-1)}|V(\sigma)|^q \mathrm{d}\sigma\mathrm{d}\tau, & q \geqslant 2, \\ K\int_0^t\int_0^\tau (\tau-\sigma)^{\alpha-1}(\sigma+R)^{-(q-1)}|V(\sigma)|^q \mathrm{d}\sigma\mathrm{d}\tau, & 1<q<2. \end{cases} \tag{2.9}$$

当 (2.7) 式中 $f_2(s,\bar{u},\bar{u}_t) = h_1(s)N_{\alpha,p}(\bar{u}_t)$ 时, 令 $\psi(t,s) = \phi_0(s)$, 得到

$$V'(t) - V'(0) = \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau (\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}} h_1(s)\phi_0(s)|\bar{u}_t(\sigma,s)|^p \mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau. \tag{2.10}$$

计算得到

$$V(t) \geqslant \begin{cases} C\int_0^t\int_0^\tau\int_0^\sigma (\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(p-1)}|U'(\lambda)|^p \mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau, \ p \geqslant 2, \\ C\int_0^t\int_0^\tau\int_0^\sigma (\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(p-1)}|U'(\lambda)|^q \mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau, \ 1<p<2. \end{cases} \tag{2.11}$$

将 (2.6) 式中 $\phi(t,s)$ 令为 $\Phi(t,s) = \mathrm{e}^{-\frac{t}{2M}}\varphi_1(s)$, 则有

$$\begin{aligned} &\int_{\mathbb{R}} \bar{u}_t(t,s)\Phi(t,s)\mathrm{d}s + \frac{1}{2M}\int_{\mathbb{R}} \bar{u}(t,s)\Phi(t,s)\mathrm{d}s - \varepsilon\int_{\mathbb{R}}\Big(\frac{1}{2M}\bar{u}_0(s)+\bar{u}_1(s)\Big)\varphi_1(s)\mathrm{d}s \\ &+\int_0^t\int_{\mathbb{R}} \bar{u}(\tau,s)\Big(-\Phi_{ss}(\tau,s)+W(s)\Phi(\tau,s)+\frac{1}{4M^2}\Phi(\tau,s)\Big)\mathrm{d}\tau ds \\ =&\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}} h_2(s)\Phi(\sigma,s)|\bar{v}(\sigma,s)|^q \mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau. \end{aligned}$$

结合 (2.2) 式得到

$$\begin{aligned} &U_1'(t) + \frac{1}{M}U_1(t) - \varepsilon\int_{\mathbb{R}}\Big(\frac{1}{2M}\bar{u}_0(s)+\bar{u}_1(s)\Big)\varphi_1(s)\mathrm{d}s \\ =&\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}} h_2(s)\Phi(\sigma,s)|\bar{v}(\sigma,s)|^q \mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau \geqslant 0. \end{aligned}$$

于是可得

$$U_1(t) \geqslant \mathrm{e}^{-\frac{t}{M}}U_1(0) + M\varepsilon\int_{\mathbb{R}}\Big(\frac{1}{2M}\bar{u}_0(s)+\bar{u}_1(s)\Big)\varphi_1(s)\mathrm{d}s(1-\mathrm{e}^{-\frac{t}{M}}) \geqslant C\varepsilon. \tag{2.12}$$

在 (2.7) 式中令 $\phi(t,s) = \psi(t,s) = \mathrm{e}^{-\frac{t}{2M}}\varphi_1(s)$, 得到

$$V_1(t) \geqslant \mathrm{e}^{-\frac{t}{M}}V_1(0) + M\varepsilon\int_{\mathbb{R}}\Big(\frac{1}{2M}\bar{v}_0(s)+\bar{v}_1(s)\Big)\varphi_1(s)\mathrm{d}s(1-\mathrm{e}^{-\frac{t}{M}}) \geqslant C\varepsilon. \tag{2.13}$$

在 (2.6) 式中令 $\phi(t,s) = \Phi(t,s)$, 并且结合 (2.6) 式, 则有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \bar{u}_t\Phi\mathrm{d}s + \frac{1}{2M}\int_{\mathbb{R}}\Big(\bar{u}_t+\frac{1}{2M}\bar{u}\Big)\Phi\mathrm{d}s = \int_{\mathbb{R}} h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s.$$

进而得到

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\bar{u}_t+\frac{1}{2M}\bar{u}\Big)\Phi\mathrm{d}s-\varepsilon\int_{\mathbb{R}}\Big(\bar{u}_1(s)+\frac{1}{2M}\bar{u}_0(s)\Big)\varphi_1(s)\mathrm{d}s\\&=\int_0^t\int_{\mathbb{R}}h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s\mathrm{d}t.\end{aligned}$$

计算可得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\bar{u}_t\Phi\mathrm{d}s+\frac{1}{M}\int_{\mathbb{R}}\bar{u}_t\Phi\mathrm{d}s-\frac{\varepsilon}{2M}\int_{\mathbb{R}}(\bar{u}_1(s)+\frac{1}{2M}\bar{u}_0(s))\varphi_1(s)\mathrm{d}s\\&=\int_{\mathbb{R}}h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s+\frac{1}{2M}\int_0^t\int_{\mathbb{R}}h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s\mathrm{d}t.\end{aligned}\tag{2.14}$$

记

$$G(t)=\int_{\mathbb{R}}\bar{u}_t\Phi\mathrm{d}s-\frac{\varepsilon}{2}\int_{\mathbb{R}}\bar{u}_1(s)\varphi_1(s)\mathrm{d}s-\frac{1}{2}\int_0^t\int_{\mathbb{R}}h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s\mathrm{d}t.$$

利用 (2.14) 式, 得到

$$G'(t)+\frac{1}{M}G(t)=\frac{1}{2}\int_{\mathbb{R}}h_2(s)N_{\alpha,q}(\bar{v})\Phi\mathrm{d}s+\frac{1}{4M^2}\int_{\mathbb{R}}\bar{u}_0(s)\varphi_1(s)\mathrm{d}s\geqslant 0.$$

计算可知

$$U_2(t)\geqslant\frac{\varepsilon}{2}\int_{\mathbb{R}}\bar{u}_1(s)\varphi_1(s)\mathrm{d}s.\tag{2.15}$$

类似得到

$$V_2(t)\geqslant\frac{\varepsilon}{2}\int_{\mathbb{R}}\bar{v}_1(s)\varphi_1(s)\mathrm{d}s.\tag{2.16}$$

利用引理 2.4 和 (2.13) 式, 可得

$$\begin{aligned}\int_{\mathbb{R}}h_2(s)\phi_0(s)|\bar{v}(t,s)|^q\mathrm{d}s&\geqslant\frac{|V_1(t)|^q}{(\int_{\mathbb{R}}(h_2(s)\phi_0(s))^{-\frac{1}{q-1}}(\Phi(t,s))^{q'}\mathrm{d}s)^{q-1}}\\&\geqslant\begin{cases}C\varepsilon^q(t+R)^{-(q-2)}, & q\geqslant 2,\\ C\varepsilon^q(t+R)^{-(q-1)}, & 1<q<2.\end{cases}\end{aligned}\tag{2.17}$$

将 (2.17) 式代入 (2.8) 式, 则有

$$U'(t)\geqslant\begin{cases}C\varepsilon^q(t+R)^{-(q-2)}t^{\alpha+1}, & q\geqslant 2,\\ C\varepsilon^q(t+R)^{-(q-1)}t^{\alpha+1}, & 1<q<2.\end{cases}\tag{2.18}$$

类似地, 结合 (2.10) 式和 (2.15) 式得到

$$V(t)\geqslant\begin{cases}K\varepsilon^p(t+R)^{-(p-2)}t^{\alpha+2}, & p\geqslant 2,\\ K\varepsilon^p(t+R)^{-(p-1)}t^{\alpha+2}, & 1<p<2.\end{cases}\tag{2.19}$$

情形 1　$p\geqslant 2$, $q\geqslant 2$.

假设

$$V(t)\geqslant C_j(t+R)^{-b_j}t^{a_j},\quad t\geqslant 0,\ j\in\mathbb{N},\tag{2.20}$$

$$U'(t)\geqslant K_j(t+R)^{-\beta_j}t^{\alpha_j},\quad t\geqslant 0,\ j\in\mathbb{N}.\tag{2.21}$$

根据 (2.18) 式和 (2.19) 式可知, (2.20) 式中 $j=0$ 时, $C_0=K\varepsilon^p$, $b_0=p-2$, $a_0=\alpha+2$, (2.21) 式中 $j=0$ 时, $K_0=C\varepsilon^q$, $\beta_0=q-2$, $\alpha_0=\alpha+1$. 将 (2.20) 式代入 (2.9) 式, 得到

$$\begin{aligned}U'(t)&\geqslant KC_j^q\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}(\sigma+R)^{-3(q-1)-b_jq}\sigma^{a_jq}\mathrm{d}\sigma\mathrm{d}\tau\\&\geqslant\frac{KC_j^q}{(a_jq+2)^2}(t+R)^{-3(q-1)-b_jq}t^{a_jq+\alpha+1}.\end{aligned}$$

结合 (2.11) 式, 可知

$$\begin{aligned}V(t)&\geqslant\frac{CK^pC_j^{pq}}{(a_jq+2)^{2p}}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(p-1)-3p(q-1)-b_jpq}\\&\quad\times\lambda^{a_jpq+(\alpha+1)p}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau\\&\geqslant C_{j+1}(t+R)^{-b_{j+1}}t^{a_{j+1}},\end{aligned}$$

其中 $C_{j+1}\geqslant\frac{CK^pC_j^{pq}}{(a_jq+2)^{2p}(a_jpq+(\alpha+1)p+2)^3}$, $b_{j+1}=3(p-1)+3p(q-1)+b_jpq$, $a_{j+1}=a_jpq+(\alpha+1)p+\alpha+2$.

利用 (2.9) 式、(2.11) 式和 (2.21) 式, 则有

$$U'(t)\geqslant K_{j+1}(t+R)^{-\beta_{j+1}}t^{\alpha_{j+1}},$$

其中 $K_{j+1}\geqslant\frac{KC^qK_j^{pq}}{(\alpha_jpq+(\alpha+2)q+2)^{3q+2}}$, $\beta_{j+1}=3(q-1)+3q(p-1)+\beta_jpq$, $\alpha_{j+1}=\alpha_jpq+(\alpha+2)q+\alpha+1$.

计算可得

$$a_j=\Big(\alpha+2+\frac{(\alpha+1)p+\alpha+2}{pq-1}\Big)(pq)^j-\frac{(\alpha+1)p+\alpha+2}{pq-1},\tag{2.22}$$

$$\alpha_j=\Big(\alpha+1+\frac{(\alpha+2)q+\alpha+1}{pq-1}\Big)(pq)^j-\frac{(\alpha+2)q+\alpha+1}{pq-1},\tag{2.23}$$

$$b_j=(p+1)(pq)^j-3,\quad\beta_j=(q+1)(pq)^j-3,\tag{2.24}$$

$$\begin{aligned}C_j&\geqslant\frac{CK^pC_{j-1}^{pq}}{(a_j-\alpha)^{2p+3}}\geqslant M(pq)^{-(2p+3)j}C_{j-1}^{pq}\\&\geqslant\exp\Big\{(pq)^j\Big(\log C_0+\frac{\log M}{pq-1}-\frac{(2p+3)\log(pq)}{(pq-1)^2}\Big)\Big\},\end{aligned}\tag{2.25}$$

$$\begin{aligned}K_j&\geqslant\widetilde{M}(pq)^{-(3q+2)j}K_{j-1}^{pq}\\&\geqslant\exp\Big\{(pq)^j\Big(\log K_0+\frac{\log\widetilde{M}}{pq-1}-\frac{(3q+2)\log(pq)}{(pq-1)^2}\Big)\Big\}.\end{aligned}\tag{2.26}$$

利用 (2.20) 式、(2.22) 式以及 (2.24)–(2.25) 式, 得到

$$\begin{aligned}V(t)\geqslant{}&\exp\{(pq)^j(\log(K\varepsilon^pt^{-p+\alpha+1+\frac{(\alpha+1)p+\alpha+2}{pq-1}})-(p+1)\log2\\&-S_{p,q}(\infty))\}\times(t+R)^3t^{-\frac{(\alpha+1)p+\alpha+2}{pq-1}}.\end{aligned}\tag{2.27}$$

在 (2.27) 式中令 $j\to\infty$, 当

$$t>C\varepsilon^{-(-1+(\alpha+1)p^{-1}+\frac{\alpha+1+(\alpha+2)p^{-1}}{pq-1})^{-1}}$$

时, 可知 $V(t)\to\infty$. 因此, 得到生命跨度估计 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)p^{-1}+\frac{\alpha+1+(\alpha+2)p^{-1}}{pq-1})^{-1}}$.

类似地, 根据 (2.21) 式可知

$$U'(t) \geqslant \exp\{(pq)^j(\log(K_0 t^{-(q+1)+\alpha+1+\frac{(\alpha+2)q+\alpha+1}{pq-1}}) - (q+1)\log 2 \\ -\widetilde{S}_{p,q}(\infty))\} \times (t+R)^3 t^{-\frac{(\alpha+2)q+\alpha+1}{pq-1}}, \tag{2.28}$$

进而得到生命跨度估计 $T(\varepsilon) \leqslant C\varepsilon^{-(-1+\alpha q^{-1}+\frac{\alpha+2+(\alpha+1)q^{-1}}{pq-1})^{-1}}$. 于是可得 (1.8) 式中第一个生命跨度估计.

情形 2　$1<p<2$, $1<q<2$.

利用 (2.19) 式和 (2.20) 式, 可知 $C_0 = K\varepsilon^p$, $b_0 = p-1$, $a_0 = \alpha+2$. 结合 (2.18) 式和 (2.21) 式, 则有 $K_0 = C\varepsilon^q$, $\beta_0 = q-1$, $\alpha_0 = \alpha+1$. 将 (2.20) 式代入 (2.9) 式, 得到

$$U'(t) \geqslant \frac{KC_j^q}{(a_j q+2)^2}(t+R)^{-(q-1)-b_j q} t^{a_j q+\alpha+1}. \tag{2.29}$$

利用 (2.11) 式和 (2.29) 式, 可得

$$V(t) \geqslant \frac{CK^p C_j^{pq}}{(a_j q+2)^{2p}(a_j pq+(\alpha+1)p+2)^3}(t+R)^{-(p-1)-p(q-1)-b_j pq} \\ \times t^{a_j pq+(\alpha+1)p+\alpha+2}.$$

结合 (2.9) 式、(2.11) 式以及 (2.21) 式, 得到

$$U'(t) \geqslant \frac{KC^q K_j^{pq}}{(\alpha_j p+2)^{3q}(\alpha_j pq+(\alpha+2)q+3)^2}(t+R)^{-(q-1)-q(p-1)-\beta_j pq} \\ \times t^{\alpha_j pq+(\alpha+2)q+\alpha+1}.$$

计算可知

$$b_{j+1} = p(pq)^j - 1, \quad \beta_{j+1} = q(pq)^j - 1. \tag{2.30}$$

根据 (2.20) 式、(2.22) 式、(2.25) 式和 (2.30) 式, 则有

$$V(t) \geqslant \exp\{(pq)^j(\log(K\varepsilon^p t^{-p+\alpha+2+\frac{(\alpha+1)p+\alpha+2}{pq-1}}) - p\log 2 \\ - S_{p,q}(\infty))\} \times (t+R)\, t^{-\frac{(\alpha+1)p+\alpha+2}{pq-1}}.$$

从而得到生命跨度估计 $T(\varepsilon) \leqslant C\varepsilon^{-(-1+(\alpha+2)p^{-1}+\frac{\alpha+1+(\alpha+2)p^{-1}}{pq-1})^{-1}}$.

类似地, 利用 (2.21) 式、(2.23) 式、(2.26) 式和 (2.30) 式, 得到生命跨度估计 $T(\varepsilon) \leqslant C\varepsilon^{-(-1+(\alpha+1)q^{-1}+\frac{\alpha+2+(\alpha+1)q^{-1}}{pq-1})^{-1}}$. 于是可得 (1.8) 式中第二个生命跨度估计.

情形 3　$p \geqslant 2$, $1<q<2$.

假设 (2.20) 式中 $C_0 = K\varepsilon^p$, $b_0 = p-2$, $a_0 = \alpha+2$, (2.21) 式中 $K_0 = C\varepsilon^q$, $\beta_0 = q-1$, $\alpha_0 = \alpha+1$. 类似于情形 1 和情形 2 中的推导, 则有

$$b_{j+1} = pq+2p-3+b_j pq = \Big(p-1+\frac{2p-2}{pq-1}\Big)(pq)^j - \frac{pq+2p-3}{pq-1},$$
$$\beta_{j+1} = 3pq-2q-1+\beta_j pq = \Big(q+2-\frac{2q-2}{pq-1}\Big)(pq)^j - \frac{3pq-2q-1}{pq-1}.$$

计算得到

$$T(\varepsilon) \leqslant C\varepsilon^{-\max\{-1+(\alpha+3)p^{-1}+\frac{\alpha+3+p^{-1}}{pq-1},\ -1+(\alpha-1)q^{-1}+\frac{\alpha+4+(\alpha-1)q^{-1}}{pq-1}\}^{-1}}.$$

从而得到 (1.8) 式中第三个生命跨度估计.

情形 4 $1 < p < 2$, $q \geqslant 2$.

根据 (2.19) 式知 $C_0 = K\varepsilon^p$, $b_0 = p-1$, $a_0 = \alpha+2$. 利用 (2.18) 式, 得到 $K_0 = C\varepsilon^q$, $\beta_0 = q-2$, $\alpha_0 = \alpha+1$. 类似于情形 1 以及情形 2 的推导过程, 可得 (2.22)–(2.23) 式、(2.25)–(2.26) 式以及

$$b_{j+1} = 3pq - 2p - 1 + b_j pq = \Big(p + 2 - \frac{2p-2}{pq-1}\Big)(pq)^j - \frac{3pq-2p-1}{pq-1},$$
$$\beta_{j+1} = pq + 2q - 3 + \beta_j pq = \Big(q - 1 + \frac{2q-2}{pq-1}\Big)(pq)^j - \frac{pq+2q-3}{pq-1}.$$

于是得到 (1.8) 式中第四个生命跨度估计. 定理 1.1 证毕.

§3 定理 1.2 的证明

当 (2.6) 式中 $f_1(s,\bar{v},\bar{v}_t) = N_{\alpha,p_1,q_1}(\bar{v},\bar{v}_t)$ 时, 令 $\phi(t,s) = \phi_0(s)$, 得到

$$U(t) \geqslant \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)(h_3(s)|\bar{v}_t|^{p_1} + h_4(s)|\bar{v}|^{q_1})\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau.$$

从而可得

$$U(t) \geqslant \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_3(s)|\bar{v}_t(\lambda,s)|^{p_1}\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau \tag{3.1}$$

和

$$U(t) \geqslant \frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_4(s)|\bar{v}(\lambda,s)|^{q_1}\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau. \tag{3.2}$$

利用 Hölder 不等式、(2.16) 式和引理 2.4, 则有

$$\begin{aligned}\int_{\mathbb{R}}\phi_0(s)h_3(s)|\bar{v}_t|^{p_1}\mathrm{d}s &\geqslant \frac{|V_2(t)|^{p_1}}{(\int_{\mathbb{R}}(\phi_0(s)h_3(s))^{-\frac{1}{p_1-1}}(\Phi(t,s))^{p_1'}\mathrm{d}s)^{p_1-1}}\\ &\geqslant \begin{cases} C\varepsilon^{p_1}(t+R)^{-(p_1-2)}, & p_1 \geqslant 2,\\ C\varepsilon^{p_1}(t+R)^{-(p_1-1)}, & 1 < p_1 < 2.\end{cases}\end{aligned} \tag{3.3}$$

当 $p_1 \geqslant 2$ 时, 结合 (3.1) 式和 (3.3) 式, 得到

$$\begin{aligned}U(t) &\geqslant K\varepsilon^{p_1}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(p_1-2)}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau\\ &\geqslant K\varepsilon^{p_1}(t+R)^{-(p_1-2)}t^{\alpha+2}.\end{aligned} \tag{3.4}$$

类似地, 当 $1 < p_1 < 2$ 时, 可知

$$U(t) \geqslant K\varepsilon^{p_1}(t+R)^{-(p_1-1)}t^{\alpha+2}. \tag{3.5}$$

根据引理 2.3 得到

$$\begin{aligned}\int_{\mathbb{R}}\phi_0(s)h_4(s)|\bar{v}|^{q_1}\mathrm{d}s &\geqslant \frac{|V(t)|^{q_1}}{(\int_{\mathbb{R}}\phi_0(s)(h_4(s))^{-\frac{1}{q_1-1}}\mathrm{d}s)^{q_1-1}}\\ &\geqslant \begin{cases} C(t+R)^{-3(q_1-1)}|V(t)|^{q_1}, & q_1 \geqslant 2,\\ C(t+R)^{-(q_1-1)}|V(t)|^{q_1}, & 1 < q_1 < 2.\end{cases}\end{aligned} \tag{3.6}$$

利用 (3.2) 式和 (3.6) 式, 可得

$$U(t)\geqslant\begin{cases}K\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(q_1-1)}|V(\lambda)|^{q_1}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau,\\ \qquad q_1\geqslant 2,\\ K\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(q_1-1)}|V(\lambda)|^{q_1}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau,\\ \qquad 1<q_1<2.\end{cases}\tag{3.7}$$

当 (2.7) 式中 $f_2(s,\bar{u},\bar{u}_t)=N_{\alpha,q_2}(\bar{u})$ 时, 令 $\psi(t,s)=\phi_0(s)$, 得到

$$V(t)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_6(s)|\bar{u}(\lambda,s)|^{q_2}\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau.\tag{3.8}$$

利用引理 2.3, 则有

$$\begin{aligned}\int_{\mathbb{R}}\phi_0(s)h_6(s)|\bar{u}|^{q_2}\mathrm{d}s&\geqslant\frac{|U(t)|^{q_2}}{(\int_{\mathbb{R}}\phi_0(s)(h_6(s))^{-\frac{1}{q_2-1}}\mathrm{d}s)^{q_2-1}}\\&\geqslant\begin{cases}C(t+R)^{-3(q_2-1)}|U(t)|^{q_2},&q_2\geqslant 2,\\C(t+R)^{-(q_2-1)}|U(t)|^{q_2},&1<q_2<2.\end{cases}\end{aligned}\tag{3.9}$$

于是可得

$$V(t)\geqslant\begin{cases}C\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(q_2-1)}|U(\lambda)|^{q_2}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau,\\ \qquad q_2\geqslant 2,\\ C\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(q_2-1)}|U(\lambda)|^{q_2}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau,\\ \qquad 1<q_2<2.\end{cases}\tag{3.10}$$

另一方面, 利用 (2.12) 式和引理 2.4 得到

$$\begin{aligned}\int_{\mathbb{R}}\phi_0(s)h_6(s)|\bar{u}|^{q_2}\mathrm{d}s&\geqslant\frac{|U_1(t)|^{q_2}}{(\int_{\mathbb{R}}(\phi_0(s)h_6(s))^{-\frac{1}{q_2-1}}(\Phi(t,s))^{q_2'}\mathrm{d}s)^{q_2-1}}\\&\geqslant\begin{cases}C\varepsilon^{q_2}(t+R)^{-(q_2-2)},&q_2\geqslant 2,\\C\varepsilon^{q_2}(t+R)^{-(q_2-1)},&1<q_2<2,\end{cases}\end{aligned}\tag{3.11}$$

结合 (3.8) 式可知

$$V(t)\geqslant\begin{cases}C\varepsilon^{q_2}(t+R)^{-(q_2-2)}t^{\alpha+2},&q_2\geqslant 2,\\C\varepsilon^{q_2}(t+R)^{-(q_2-1)}t^{\alpha+2},&1<q_2<2.\end{cases}\tag{3.12}$$

情形 1 $p_1\geqslant 2$, $q_1\geqslant 2$, $q_2\geqslant 2$.

假设

$$U(t)\geqslant D_j(t+R)^{-a_j}t^{b_j},\quad t\geqslant 0,\ j\in\mathbb{N},\tag{3.13}$$

$$V(t)\geqslant \Delta_j(t+R)^{-\alpha_j}t^{\beta_j},\quad t\geqslant 0,\ j\in\mathbb{N}.\tag{3.14}$$

当 $p_1\geqslant 2$, $q_2\geqslant 2$ 时, 利用 (3.4) 式和 (3.12) 式, 可得 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-2$, $b_0=\alpha+2$ 以及 $\Delta_0=C\varepsilon^{q_2}$, $\alpha_0=q_2-2$, $\beta_0=\alpha+2$.

利用 (3.7) 式、(3.10) 式和 (3.13) 式, 则有

$$\begin{aligned}U(t) &\geqslant \frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_2+2)^{3q_1}}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(q_1q_2-1)-a_jq_1q_2}\\&\quad\times\lambda^{b_jq_1q_2+(\alpha+2)q_1}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau\\&\geqslant D_{j+1}(t+R)^{-a_{j+1}}t^{b_{j+1}},\end{aligned}$$

其中 $D_{j+1}\geqslant\frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_1q_2+(\alpha+2)q_1+2)^{3q_1+3}}$, $a_{j+1}=3(q_1q_2-1)+a_jq_1q_2$, $b_{j+1}=b_jq_1q_2+(\alpha+2)q_1+\alpha+2$.

类似得到

$$V(t)\geqslant\Delta_{j+1}(t+R)^{-\alpha_{j+1}}t^{\beta_{j+1}},$$

其中 $\Delta_{j+1}\geqslant\frac{CK^{q_2}\Delta_j^{q_1q_2}}{(\beta_jq_1q_2+(\alpha+2)q_2+2)^{3q_2+3}}$, $\alpha_{j+1}=3(q_1q_2-1)+\alpha_jq_1q_2$, $\beta_{j+1}=\beta_jq_1q_2+(\alpha+2)q_2+\alpha+2$.

计算可得

$$a_j=(p_1+1)(q_1q_2)^j-3,\quad \alpha_j=(q_2+1)(q_1q_2)^j-3, \tag{3.15}$$

$$b_j=\Big(\alpha+2+\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}\Big)(q_1q_2)^j-\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}, \tag{3.16}$$

$$\beta_j=\Big(\alpha+2+\frac{(\alpha+2)q_2+\alpha+2}{q_1q_2-1}\Big)(q_1q_2)^j-\frac{(\alpha+2)q_2+\alpha+2}{q_1q_2-1}. \tag{3.17}$$

当 $j>\max\{\frac{\log N}{(3q_1+3)\log(q_1q_2)}-\frac{q_1q_2}{q_1q_2-1},\ \frac{\log\widetilde{N}}{(3q_2+3)\log(q_1q_2)}-\frac{q_1q_2}{q_1q_2-1}\}$ 时, 则有

$$\begin{aligned}D_j&\geqslant\frac{KC^{q_1}D_{j-1}^{q_1q_2}}{(b_j-\alpha)^{3q_1+3}}\geqslant N(q_1q_2)^{-(3q_1+3)j}D_{j-1}^{q_1q_2}\\&\geqslant\exp\{(q_1q_2)^j(\log D_0-S_{q_1,q_2}(\infty))\},\end{aligned} \tag{3.18}$$

$$\Delta_j\geqslant\widetilde{N}(q_1q_2)^{-(3q_2+3)j}\Delta_{j-1}^{q_1q_2}\geqslant\exp\{(q_1,q_2)^j(\log\Delta_0-\widetilde{S}_{q_1q_2}(\infty))\}. \tag{3.19}$$

利用 (3.13) 式、(3.15)–(3.16) 式以及 (3.18) 式, 得到

$$\begin{aligned}U(t)\geqslant\exp\{(q_1q_2)^j(\log(D_0t^{-p_1+\alpha+1+\frac{(\alpha+1)q_1+\alpha+2}{q_1q_2-1}})-S_{q_1,q_2}(\infty)\\-(p_2+1)\log2\}(t+R)^3t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}},\end{aligned}$$

因此, 当 $j\to\infty$ 并且

$$t>C\varepsilon^{-(-1+(\alpha+1)p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)})^{-1}}$$

时, 可知 $U(t)\to\infty$.

类似地, 得到生命跨度估计 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)q_2^{-1}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1})^{-1}}$. 从而建立 (1.9) 式中第一个生命跨度估计.

情形 2 $1<p_1<2$, $1<q_1<2$, $1<q_2<2$.

对于 $1<p_1<2$, $1<q_2<2$, 当 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$ 以及 $\Delta_0=C\varepsilon^{q_2}$, $\alpha_0=q_2-1$, $\beta_0=\alpha+2$ 时, 假设 (3.13) 式和 (3.14) 式成立. 利用 (3.7) 式、(3.10) 式和 (3.13) 式, 可得

$$U(t)\geqslant\frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_2+2)^{3q_1}(b_jq_1q_2+(\alpha+2)q_1+2)^3}(t+R)^{-(q_1q_2-1)-a_jq_1q_2}$$

$$\times t^{b_j q_1 q_2+(\alpha+2)q_1+\alpha+2}.$$

于是得到 $D_{j+1} \geqslant \frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_1q_2+(\alpha+2)q_1+2)^{3q_1+3}}$, $a_{j+1}=a_jq_1q_2+q_1q_2-1$, $b_{j+1}=b_jq_1q_2+(\alpha+2)q_1+\alpha+2$.

类似可得 $\Delta_{j+1} \geqslant \frac{CK^{q_2}\Delta_j^{q_1q_2}}{(\beta_jq_1q_2+(\alpha+2)q_2+2)^{3q_2+3}}$, $\alpha_{j+1}=\alpha_jq_1q_2+q_1q_2-1$, $\beta_{j+1}=\beta_jq_1q_2+(\alpha+2)q_2+\alpha+2$.

计算得到

$$a_j=p_1(q_1q_2)^j-1, \quad \alpha_j=q_2(q_1q_2)^j-1. \tag{3.20}$$

结合 (3.13) 式、(3.16) 式、(3.18) 式和 (3.20) 式, 则有

$$\begin{aligned}U(t) \geqslant \exp\{&(q_1q_2)^j(\log(D_0t^{-p_1+\alpha+2+\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}})-S_{q_1,q_2}(\infty)\\&-p_1\log 2)\}(t+R)t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}}.\end{aligned}$$

于是得到 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+2)p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)})^{-1}}$.

根据 (3.14) 式、(3.17) 式和 (3.19)–(3.20) 式, 可得生命跨度估计

$$T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+2)q_2^{-1}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1})^{-1}}.$$

从而得到 (1.9) 式中第二个生命跨度估计.

情形 3　$1<p_1<2$, $q_1\geqslant 2$, $q_2\geqslant 2$.

当 $q_2\geqslant 2$ 时, 利用 (3.5) 式和 (3.12) 式可得 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$ 以及 $\Delta_0=C\varepsilon^{q_2}$, $\alpha_0=q_2-2$, $\beta_0=\alpha+2$.

类似于情形 1 和情形 2 的推导, 则有

$$\begin{aligned}a_{j+1}&=3(q_1-1)+3q_1(q_2-1)+a_jq_1q_2,\\ \alpha_{j+1}&=3(q_2-1)+3q_2(q_1-1)+\alpha_jq_1q_2.\end{aligned}$$

计算得到

$$a_j=(p_1+2)(q_1q_2)^j-3, \quad \alpha_j=(q_2+1)(q_1q_2)^j-3. \tag{3.21}$$

利用 (3.13) 式、(3.16) 式、(3.18) 式和 (3.21) 式, 可得

$$T(\varepsilon)\leqslant C\varepsilon^{-(-1+\alpha p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)})^{-1}}.$$

根据情形 1, 得到生命跨度估计 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)q_2^{-1}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1})^{-1}}$. 从而得到 (1.9) 式中第三个生命跨度估计.

情形 4　$p_1\geqslant 2$, $1<q_1<2$, $1<q_2<2$.

当 $1<q_2<2$ 时, 利用 (3.4) 式和 (3.12) 式, 得到 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$ 以及 $\Delta_0=C\varepsilon^{q_2}$, $\alpha_0=q_2-1$, $\beta_0=\alpha+2$.

类似于情形 2 的证明, 可得

$$\begin{aligned}a_{j+1}&=q_1-1+q_1(q_2-1)+a_jq_1q_2,\\ \alpha_{j+1}&=q_2-1+q_2(q_1-1)+\alpha_jq_1q_2,\end{aligned}$$

则有 $a_j=(p_1-1)(q_1q_2)^j-1$, $\alpha_j=q_2(q_1q_2)^j-1$.

于是可得 (1.9) 式中第四个生命跨度估计, 即

$$T(\varepsilon)\leqslant C\varepsilon^{-\max\{-1+(\alpha+3)p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+2}{p_1(q_1q_2-1)},\ -1+(\alpha+2)q_2^{-1}+\frac{\alpha+2+(\alpha+2)q_2^{-1}}{q_1q_2-1}\}^{-1}}.$$

定理 1.2 证毕.

§4 定理 1.3 的证明

当 (2.6) 式中 $f_1(s,\bar{v},\bar{v}_t)=N_{\alpha,p_1,q_1}(\bar{v},\bar{v}_t)$ 时, 令 $\phi(t,s)=\phi_0(s)$, 得到

$$U'(t)-U'(0)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_{\mathbb{R}}\phi_0(s)\int_0^\tau(\tau-\sigma)^{\alpha-1}(h_3(s)|\bar{v}_t|^{p_1}+h_4(s)|\bar{v}|^{q_1})\mathrm{d}\sigma\mathrm{d}s\mathrm{d}\tau,$$

从而可得

$$U'(t)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_3(s)|\bar{v}_t(\sigma,s)|^{p_1}\mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau,\tag{4.1}$$

$$U'(t)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_4(s)|\bar{v}(\sigma,s)|^{q_1}\mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau.\tag{4.2}$$

利用引理 2.4 和 (2.16) 式, 则有

$$\int_{\mathbb{R}}\phi_0(s)h_3(s)|\bar{v}_t(t,s)|^{p_1}\mathrm{d}s\geqslant\begin{cases}K\varepsilon^{p_1}(t+R)^{-(p_1-2)}, & p_1\geqslant 2,\\ K\varepsilon^{p_1}(t+R)^{-(p_1-1)}, & 1<p_1<2.\end{cases}$$

于是得到

$$U'(t)\geqslant\begin{cases}K\varepsilon^{p_1}(t+R)^{-(p_1-2)}t^{\alpha+1}, & p_1\geqslant 2,\\ K\varepsilon^{p_1}(t+R)^{-(p_1-1)}t^{\alpha+1}, & 1<p_1<2.\end{cases}\tag{4.3}$$

根据引理 2.3, 得到

$$\int_{\mathbb{R}}\phi_0(s)h_4(s)|\bar{v}(t,s)|^{q_1}\mathrm{d}s\geqslant\begin{cases}K(t+R)^{-3(q_1-1)}|V(t)|^{q_1}, & q_1\geqslant 2,\\ K(t+R)^{-(q_1-1)}|V(t)|^{q_1}, & 1<q_1<2.\end{cases}\tag{4.4}$$

将 (4.4) 式代入 (4.2) 式, 则有

$$U'(t)\geqslant\begin{cases}K\displaystyle\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}(\sigma+R)^{-3(q_1-1)}|V(\sigma)|^{q_1}\mathrm{d}\sigma\mathrm{d}\tau, & q_1\geqslant 2,\\ K\displaystyle\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}(\sigma+R)^{-(q_1-1)}|V(\sigma)|^{q_1}\mathrm{d}\sigma\mathrm{d}\tau, & 1<q_1<2.\end{cases}\tag{4.5}$$

当 (2.7) 式中 $f_2(s,\bar{u},\bar{u}_t)=N_{\alpha,p_2}(\bar{u})$ 时, 令 $\psi(t,s)=\phi_0(s)$, 得到

$$V'(t)-V'(0)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_5(s)|\bar{u}_t(\sigma,s)|^{p_2}\mathrm{d}s\mathrm{d}\sigma\mathrm{d}\tau.$$

从而可得

$$V(t)\geqslant C\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb{R}}\phi_0(s)h_5(s)|\bar{u}_t(\lambda,s)|^{p_2}\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau.\tag{4.6}$$

另一方面, 利用引理 2.3 得到

$$\int_{\mathbb{R}}\phi_0(s)h_5(s)|\bar{u}_t(t,s)|^{p_2}\mathrm{d}s\geqslant\begin{cases}C(t+R)^{-3(p_2-1)}|U'(t)|^{p_2}, & p_2\geqslant 2,\\ C(t+R)^{-(p_2-1)}|U'(t)|^{p_2}, & 1<p_2<2.\end{cases}$$

从而得到

$$V(t)\geqslant\begin{cases}C\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(p_2-1)}|U'(\lambda)|^{p_2}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau, \\ \qquad\qquad p_2\geqslant 2, \\ C\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(p_2-1)}|U'(\lambda)|^{p_2}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau, \\ \qquad\qquad 1<p_2<2.\end{cases}\tag{4.7}$$

另一方面, 利用引理 2.4、(2.15) 式和 (4.6) 式, 则有

$$V(t)\geqslant\begin{cases}C\varepsilon^{p_2}(t+R)^{-(p_2-2)}t^{\alpha+2}, & p_2\geqslant 2,\\ C\varepsilon^{p_2}(t+R)^{-(p_2-1)}t^{\alpha+2}, & 1<p_2<2.\end{cases}\tag{4.8}$$

情形 1　$p_1\geqslant 2$**,** $q_1\geqslant 2$**,** $p_2\geqslant 2$**.**

假设

$$U'(t)\geqslant D_j(t+R)^{-a_j}t^{b_j},\quad t\geqslant 0,\ j\in\mathbb{N},\tag{4.9}$$

$$V(t)\geqslant \Delta_j(t+R)^{-\alpha_j}t^{\beta_j},\quad t\geqslant 0,\ j\in\mathbb{N}.\tag{4.10}$$

当 $p_1\geqslant 2$, $p_2\geqslant 2$ 时, 得到 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-2$, $b_0=\alpha+1$ 以及 $\Delta_0=C\varepsilon^{p_2}$, $\alpha_0=p_2-2$, $\beta_0=\alpha+2$.

利用 (4.5) 式、(4.7) 式和 (4.9) 式, 可得

$$\begin{aligned}U'(t)&\geqslant\frac{KC^{q_1}D_j^{q_1p_2}}{(b_jp_2+2)^{3q_1}}\int_0^t\int_0^\tau(\tau-\sigma)^{\alpha-1}(\sigma+R)^{-3(q_1-1)-3q_1(p_2-1)-a_jq_1p_2}\\&\quad\times t^{b_jq_1p_2+(\alpha+2)q_1}\mathrm{d}\sigma\mathrm{d}\tau\\&\geqslant D_{j+1}(t+R)^{-a_{j+1}}t^{b_{j+1}},\end{aligned}$$

其中 $D_{j+1}\geqslant\frac{KC^{q_1}D_j^{q_1p_2}}{(b_jq_1p_2+(\alpha+2)q_2+2)^{3q_1+2}}$, $a_{j+1}=3(q_1p_2-1)+a_jq_1p_2$, $b_{j+1}=b_jq_1p_2+(\alpha+2)q_1+\alpha+1$. 计算可知

$$a_j=(p_1+1)(q_1p_2)^j-3,$$

$$b_j=\Big(\alpha+1+\frac{(\alpha+2)q_1+\alpha+1}{q_1p_2-1}\Big)(q_1p_2)^j-\frac{(\alpha+2)q_1+\alpha+2}{q_1p_2-1},\tag{4.11}$$

$$D_j\geqslant B(q_1p_2)^{-(3q_1+2)j}D_{j-1}^{q_1p_2}\geqslant\exp\{(q_1p_2)^j(\log D_0-S_{q_1,p_2}(\infty))\}.\tag{4.12}$$

类似可得 $\Delta_{j+1}\geqslant\frac{CK^{p_2}\Delta_j^{q_1p_2}}{(\beta_jq_1p_2+(\alpha+1)p_2+2)^{2p_2+3}}$, $\alpha_{j+1}=3(q_1p_2-1)+\alpha_jq_1p_2$, $\beta_{j+1}=\beta_jq_1p_2+(\alpha+1)p_2+\alpha+2$. 所以

$$\alpha_j=(p_2+1)(q_1p_2)^j-3,$$

$$\beta_j=\Big(\alpha+2+\frac{(\alpha+1)p_2+\alpha+21}{q_1p_2-1}\Big)(q_1p_2)^j-\frac{(\alpha+1)p_2+\alpha+2}{q_1p_2-1},$$

$$D_j\geqslant\widetilde{B}(q_1p_2)^{-(2p_2+3)j}\Delta_{j-1}^{q_1p_2}\geqslant\exp\{(q_1p_2)^j(\log\Delta_0-\widetilde{S}_{q_1,p_2}(\infty))\}.$$

根据 (4.9) 式, 得到

$$U'(t)\geqslant\exp\{(q_1p_2)^j(\log D_0-S_{q_1,p_2}(\infty))\}(t+R)^{-(p_1+1)(q_1p_2)^j+3}$$

$$\times t^{(\alpha+1+\frac{(\alpha+2)q_1+\alpha+1}{q_1p_2-1})(q_1p_2)^j-\frac{(\alpha+2)q_1+\alpha+2}{q_1p_2-1}}$$

$$\geqslant \exp\{(q_1p_2)^j(\log(D_0t^{-p_1+\alpha+\frac{(\alpha+2)q_1+\alpha+1}{q_1p_2-1}})-S_{q_1,p_2}(\infty)$$

$$-(p_1+1)\log 2)\}(t+R)^3t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1p_2-1}},$$

于是可得 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+\alpha p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)})^{-1}}$.

类似地, 得到生命跨度估计 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)p_2^{-1}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1})^{-1}}$. 从而得到 (1.10) 式中的第一个生命跨度估计.

情形 2 $1<p_1<2$, $1<q_1<2$, $1<p_2<2$.

假设当 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$, (4.9) 成立; 当 $\Delta_0=C\varepsilon^{p_2}$, $\alpha_0=p_2-1$, $\beta_0=\alpha+2$ 时 (4.10) 成立.

类似于情形 1 的证明过程, 则有

$$a_{j+1}=q_1p_2-1+a_jq_1p_2,\quad \alpha_{j+1}=q_1p_2-1+\alpha_jq_1p_2.$$

于是可得

$$a_j=p_1(q_1p_2)^j-1,\quad \alpha_j=p_2(q_1p_2)^j-1. \tag{4.13}$$

利用 (4.9) 式和 (4.11)–(4.13) 式, 得到

$$U'(t)\geqslant \exp\{(q_1p_2)^j(\log(D_0t^{-p_1+\alpha+1+\frac{(\alpha+2)q_1+\alpha+1}{q_1p_2-1}})-S_{q_1,p_2}(\infty)$$

$$-p_1\log 2)\}(t+R)\,t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1p_2-1}}.$$

计算可知 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)})^{-1}}$.

类似得到 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+2)p_2^{-1}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1})^{-1}}$. 从而得到 (1.10) 式中第二个生命跨度估计.

情形 3 $1<p_1<2$, $q_1\geqslant 2$, $p_2\geqslant 2$.

结合 (4.3) 式和 (4.8) 式, 可得

$$D_0=K\varepsilon^{p_1},\ a_0=p_1-1,\ b_0=\alpha+1,$$

$$\Delta_0=C\varepsilon^{p_2},\ \alpha_0=p_2-2,\ \beta_0=\alpha+2.$$

类似于情形 1 和情形 2 的推导过程, 得到 $a_{j+1}=3(q_1-1)+3q_1(p_2-1)+a_jq_1p_2$, $\alpha_{j+1}=3(p_2-1)+3p_2(q_1-1)+\alpha_jq_1p_2$. 所以

$$a_j=(p_1+2)(q_1p_2)^j-3,\quad \alpha_j=(p_2+1)(q_1p_2)^j-3.$$

计算可得

$$U'(t)\geqslant \exp\{(q_1p_2)^j(\log(D_0t^{-p_1+\alpha-1+\frac{(\alpha+2)q_1+\alpha+1}{q_1p_2-1}})-S_{q_1,p_2}(\infty)$$

$$-(p_1+2)\log 2)\}(t+R)^3t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1p_2-1}}.$$

从而, 当 $j\to\infty$ 并且

$$t>C\varepsilon^{-(-1+(\alpha-1)p_1^{-1}+\frac{(\alpha+2)q_1+\alpha+1}{p_1(q_1p_2-1)})^{-1}}$$

时, 则有 $U'(t)\to\infty$.

类似可得 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+1)p_2^{-1}+\frac{\alpha+1+(\alpha+2)p_2^{-1}}{q_1p_2-1})^{-1}}$. 从而得到 (1.10) 式中第三个生命跨度估计.

情形 4　$p_1\geqslant 2$, $1<q_1<2$, $1<p_2<2$.

当 $p_1\geqslant 2$ 且 $1<p_2<2$ 时, 结合 (4.3) 式和 (4.8) 式, 得到 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-2$, $b_0=\alpha+1$ 并且 $\Delta_0=C\varepsilon^{p_2}$, $\alpha_0=p_2-1$, $\beta_0=\alpha+2$.

利用与情形 2 类似的证明过程, 得到

$$a_j=(p_1-1)(q_1p_2)^j-1,\quad \alpha_j=p_2(q_1p_2)^j-1.$$

于是得到 (1.10) 式中第四个生命跨度估计. 定理 1.3 证毕.

§5　定理 1.4 的证明

类似于定理 1.2 的证明过程, 当 (2.6) 式中 $f_1(s,\bar v,\bar v_t)=N_{\alpha,p_1,q_1}(\bar v,\bar v_t)$ 时, 令 $\phi(t,s)=\phi_0(s)$, 可知 (3.4)–(3.5) 式以及 (3.7) 式成立.

另一方面, 当 (2.7) 式中 $f_2(s,\bar u,\bar u_t)=N_{\alpha,p_2,q_2}(\bar u,\bar u_t)$ 时, 选取检验函数 $\psi(t,s)=\phi_0(s)$. 则有

$$V(t)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb R}\phi_0(s)h_5(s)|\bar u_t(\lambda,s)|^{p_2}\mathrm ds\mathrm d\lambda\mathrm d\sigma\mathrm d\tau,$$

$$V(t)\geqslant\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}\int_{\mathbb R}\phi_0(s)h_6(s)|\bar u(\lambda,s)|^{q_2}\mathrm ds\mathrm d\lambda\mathrm d\sigma\mathrm d\tau.$$

结合引理 2.4 和 (2.15) 式, 得到

$$\int_{\mathbb R}\phi_0(s)h_5(s)|\bar u_t(t,s)|^{p_2}\mathrm ds\geqslant\begin{cases}C\varepsilon^{p_2}(t+R)^{-(p_2-2)}, & p_2\geqslant 2,\\ C\varepsilon^{p_2}(t+R)^{-(p_2-1)}, & 1<p_2<2.\end{cases}$$

从而可得

$$V(t)\geqslant\begin{cases}C\varepsilon^{p_2}(t+R)^{-(p_2-2)}t^{\alpha+2}, & p_2\geqslant 2,\\ C\varepsilon^{p_2}(t+R)^{-(p_2-1)}t^{\alpha+2}, & 1<p_2<2.\end{cases}\tag{5.1}$$

利用引理 2.3, 可知

$$\int_{\mathbb R}\phi_0(s)h_6(s)|\bar u(t,s)|^{q_2}\mathrm ds\geqslant\begin{cases}C(t+R)^{-3(q_2-1)}|U(t)|^{q_2}, & q_2\geqslant 2,\\ C(t+R)^{-(q_2-1)}|U(t)|^{q_2}, & 1<q_2<2.\end{cases}$$

所以

$$V(t)\geqslant\begin{cases}C\displaystyle\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-3(q_2-1)}|U(\lambda)|^{q_2}\mathrm d\lambda\mathrm d\sigma\mathrm d\tau, & \\ \hfill q_2\geqslant 2, & \\ C\displaystyle\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(q_2-1)}|U(\lambda)|^{q_2}\mathrm d\lambda\mathrm d\sigma\mathrm d\tau, & \\ \hfill 1<q_2<2. & \end{cases}\tag{5.2}$$

下面讨论 p_1, p_2, q_1, q_2 不同范围时解的生命跨度估计.

情形 1　$1<p_1<2$, $p_2\geqslant 2$, $1<q_1<2$, $q_2\geqslant 2$.

假设

$$U(t)\geqslant D_j(t+R)^{-a_j}t^{b_j},\quad t\geqslant 0,\ j\in\mathbb{N},\tag{5.3}$$

且 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$ 以及

$$V(t)\geqslant \Delta_j(t+R)^{-\alpha_j}t^{\beta_j},\quad t\geqslant 0,\ j\in\mathbb{N},\tag{5.4}$$

且 $\Delta_0=K\varepsilon^{p_2}$, $\alpha_0=p_2-2$, $\beta_0=\alpha+2$.

将 (5.3) 式代入 (5.2) 式, 可得

$$V(t)\geqslant \frac{CD_j^{q_2}}{(b_jq_2+2)^3}(t+R)^{-3(q_2-1)-a_jq_2}t^{b_jq_2+\alpha+2}.$$

结合 (3.7) 式得到

$$\begin{aligned}U(t)&\geqslant \frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_2+2)^{3q_1}}\int_0^t\int_0^\tau\int_0^\sigma(\sigma-\lambda)^{\alpha-1}(\lambda+R)^{-(q_1-1)-3q_1(q_2-1)-a_jq_1q_2}\\&\quad\times\lambda^{b_jq_1q_2+(\alpha+2)q_1}\mathrm{d}\lambda\mathrm{d}\sigma\mathrm{d}\tau\\&\geqslant D_{j+1}(t+R)^{a_{j+1}}t^{b_{j+1}},\end{aligned}$$

其中 $D_{j+1}\geqslant\frac{KC^{q_1}D_j^{q_1q_2}}{(b_jq_1q_2+(\alpha+2)q_1+2)^{3q_1+3}}$, $a_{j+1}=3q_1q_2-2q_1-1+a_jq_1q_2$, $b_{j+1}=b_jq_1q_2+(\alpha+2)q_1+\alpha+2$.

计算可知

$$\begin{aligned}a_j&=\Big(p_1+2-\frac{2q_1-2}{q_1q_2-1}\Big)(q_1q_2)^j-\frac{3q_1q_2-2q_1-1}{q_1q_2-1},\\b_j&=\Big(\alpha+2+\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}\Big)(q_1q_2)^j-\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1},\\D_j&\geqslant\frac{KC^{q_1}D_j^{q_1q_2}}{(b_j-\alpha)^{3q_1+3}}\geqslant\exp\{(q_1q_2)^j(\log D_0-S_{q_1,q_2}(\infty))\}.\end{aligned}$$

利用 (3.7) 式、(5.2) 式和 (5.4) 式, 则有

$$\begin{aligned}&\Delta_{j+1}\geqslant\frac{CK^{q_2}\Delta_j^{q_1q_2}}{(\beta_jq_1q_2+(\alpha+2)q_2+2)^{3q_2+3}}\quad\alpha_{j+1}=q_1q_2+2q_2-3+\alpha_jq_1q_2,\\&\beta_{j+1}=\beta_jq_1q_2+(\alpha+2)q_2+\alpha+2.\end{aligned}$$

于是得到

$$\begin{aligned}\alpha_j&=\Big(p_2-1+\frac{2q_2-2}{q_1q_2-1}\Big)(q_1q_2)^j-\frac{q_1q_2+2q_2-3}{q_1q_2-1},\\\beta_j&=\Big(\alpha+2+\frac{(\alpha+2)q_2+\alpha+2}{q_1q_2-1}\Big)(q_1q_2)^j-\mathrm{frac}(\alpha+2)q_2+\alpha+2q_1q_2-1,\\\Delta_j&\geqslant\frac{CK^{q_2\Delta_j^{q_1q_2}}}{(\beta_j-\alpha)^{3q_2+3}}\geqslant\exp\{(q_1q_2)^j(\log\Delta_0-\widetilde{S}_{q_1,q_2}(\infty))\}.\end{aligned}$$

利用 (5.3) 式得到

$$\begin{aligned}U(t)\geqslant\exp\Big\{&(q_1q_2)^j\Big(\log(D_0t^{-p_1+\alpha+\frac{2q_1-2}{q_1q_2-1}+\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}})-S_{q_1,q_2}(\infty)\\&-\Big(p_1+2-\frac{2q_1-2}{q_1q_2-1}\Big)\log 2\Big)\Big\}(t+R)^{\frac{3q_1q_2-2q_1-1}{q_1q_2-1}}t^{-\frac{(\alpha+2)q_1+\alpha+2}{q_1q_2-1}}.\end{aligned}$$

从而 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+\alpha p_1^{-1}+\frac{(\alpha+4)q_1+\alpha}{p_1(q_1q_2-1)})^{-1}}$.

类似地, 得到生命跨度估计 $T(\varepsilon)\leqslant C\varepsilon^{-(-1+(\alpha+3)p_2^{-1}+\frac{\alpha q_2+\alpha+4}{p_2(q_1q_2-1)})^{-1}}$. 从而可得 (1.11) 式中第一个生命跨度估计.

情形 2　$p_1\geqslant 2,\ 1<p_2<2,\ q_1\geqslant 2,\ 1<q_2<2$.

类似于情形 1 的推导, 可得 (1.11) 式中第二个生命跨度估计.

情形 3　$p_1\geqslant 2,\ 1<p_2<2,\ 1<q_1<2,\ 1<q_2<2$.

当 $1<p_2<2$ 时, 根据 (3.4) 式和 (5.1) 式, 可知 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-2$, $b_0=\alpha+2$ 并且 $\Delta_0=C\varepsilon^{p_2}$, $\alpha_0=p_2-1$, $\beta_0=\alpha+2$. 计算得到

$$a_{j+1}=q_1q_2-1+a_jq_1q_2,\quad \alpha_{j+1}=q_1q_2-1+\alpha_jq_1q_2.$$

于是

$$a_j=(p_1-1)(q_1q_2)^j-1,\quad \alpha_j=p_2(q_1q_2)^j-1.$$

类似于情形 1 的推导, 可得 (1.11) 式中第三个生命跨度估计.

情形 4　$1<p_1<2,\ p_2\geqslant 2,\ 1<q_1<2,\ 1<q_2<2$.

此情形的证明类似于情形 3. 于是得到 (1.11) 式中第四个生命跨度估计.

情形 5　$1<p_1<2,\ p_2\geqslant 2,\ q_1\geqslant 2,\ q_2\geqslant 2$.

当 $j=0$ 时, (5.3) 式中 $D_0=K\varepsilon^{p_1}$, $a_0=p_1-1$, $b_0=\alpha+2$, 并且 (5.4) 式中 $\Delta_0=C\varepsilon^{p_2}$, $\alpha_0=p_2-2$, $\beta_0=\alpha+2$. 计算得到

$$a_{j+1}=3(q_1q_2-1)+a_jq_1q_2,\quad \alpha_{j+1}=3(q_1q_2-1)+\alpha_jq_1q_2.$$

从而可得

$$a_j=(p_1+2)(q_1q_2)^j-3,\quad \alpha_j=(p_2+1)(q_1q_2)^j-3.$$

于是得到 (1.11) 式中第五个生命跨度估计.

情形 6　$p_1\geqslant 2,\ 1<p_2<2,\ q_1\geqslant 2,\ q_2\geqslant 2$.

此情形的证明过程类似于本节中情形 5. 因此, 得到 (1.11) 式中第六个生命跨度估计. 定理 1.4 证毕.

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Formation of Singularities of Solutions to the Coupled System of Wave Equations with Memory Terms in Schwarzschild Spacetime

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Abstract The main purpose of this work is to consider blow-up dynamics of solutions to the Cauchy problem for coupled system of nonlinear wave equations in Schwarzschild spacetime. The nonlinear terms in the problem include mixed type memory terms, combined and power type memory terms, combined and derivative type memory terms as well as combined type memory terms. Furthermore, upper bound lifespan estimates of solutions are established by imposing certain assumptions on the exponents in the nonlinear terms and making use of the iteration method. The main novelty is that that authors analyze the effects of nonlinear memory terms on lifespan estimates of solutions under the Schwarzschild metric. To the best of the authors' knowledge, the results in Theorems 1.1–1.4 are new.

Keywords Coupled system, Memory terms, Iteration method, Blow-up, Lifespan estimates

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