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# 一个涉及变上限积分函数的 Hilbert 类 $n$ 重积分不等式的构造定理*

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**提要** 利用齐次核 Hilbert 型 $n$ 重积分不等式的构造定理和 Gamma 函数的性质, 讨论一个涉及变上限积分函数的 Hilbert 类 $n$ 重积分不等式成立的条件和最佳常数因子问题, 得到构造此类不等式的充要条件和最佳常数因子的计算公式. 最后作为应用给出若干特例.

**关键词** Hilbert 类 $n$ 重积分不等式, 变上限积分函数, 构造定理, 最佳常数因子
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## §1 引 言

设 $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, 有如下的 Hilbert 不等式 [1]:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y\leqslant\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|f\|_p\|g\|_q,$$

其中 $f\in L_p(0,+\infty)$, $g\in L_q(0,+\infty)$, 常数因子 $\frac{\pi}{\sin(\frac{\pi}{p})}$ 是最佳值. 由于 Hilbert 不等式在分析类学科和算子理论的研究中有重要应用, 因而受到广泛关注. 为了进一步推广研究, 人们首先将 Lebesgue 空间 $L_p(0,+\infty)$ 推广为加权形式:

$$L_p^{\varphi(x)}(0,+\infty)=\Big\{f(x):\|f\|_{p,\varphi(x)}=\Big(\int_0^{+\infty}\varphi(x)|f(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}}<+\infty\Big\},$$

其中权函数 $\varphi(x)>0$. 在此基础上, 引入多个独立参数, 得到了许多推广的 Hilbert 型不等式 [2−8]. 2016 年, 文 [9] 讨论了 Hilbert 型不等式的最佳搭配参数问题, 随后构建起了超齐次核 Hilbert 型不等式的最佳搭配参数理论, 丰富了 Hilbert 型不等式的基础理 论 [9−11], 文 [12] 进一步讨论了 Hilbert 型不等式的的构建问题, 得到了齐次核 Hilbert 型不等式成立的充要条件, 并获得了最佳常数因子的一般表达式, 至此, Hilbert 型不等式的基础理论框架得以初步建立 [13−19]. 2019 年, 文 [20] 考虑了齐次核的涉及部分和的 Hilbert 型级数不等式, 文 [21] 则考虑了涉及变上限积分函数的 Hilbert 型积分不等式, 文 [22] 将相关结

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果推广到了高维情形, 得到了具有最佳常数因子的不等式:

$$\int_0^{+\infty}\cdots\int_0^{+\infty}\frac{1}{\Big(\sum\limits_{i=1}^n x_i\Big)^\lambda}\prod_{i=1}^n f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$\leqslant\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n\frac{\lambda}{r_i}\Gamma\Big(\frac{\lambda}{r_i}\Big)\Big(\int_0^{+\infty}x_i^{-p_i\frac{\lambda}{r_i}-1}F_i^{p_i}(x_i)\mathrm{d}x_i\Big)^{\frac{1}{p_i}},\tag{1.1}$$

其中 $\sum\limits_{i=1}^n\frac{1}{p_i}=1\,(p_i>1)$, $\sum\limits_{i=1}^n\frac{1}{r_i}=1$, $F_i(x_i)=\int_0^{x_i}f_i(t)\mathrm{d}t$ (注: 文 [22] 中误将 $\frac{1}{\Gamma(\lambda)}$ 写为 $\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$). 不等式 (1.1) 的形式优美, 证明过程较为复杂, 但存在几点不足: (1) 参数结构繁琐; (2) 不等式中包含对未知函数 $f_i$ 的限制条件 $f_i(x)=o(\mathrm{e}^{tx})\,(x\to+\infty)$; (3) 没有得到此类不等式成立的参数条件. 针对这些不足, 本文利用关于齐次核 Hilbert 型 $n$ 重积分不等式的构造定理 [19], 考虑更一般的核函数

$$\frac{1}{\Big(\sum\limits_{i=1}^n\varphi_i(x_i)\Big)^\lambda}\quad(\varphi_i(x_i)>0,\ i=1,2,\cdots,n),$$

得到涉及变上限积分函数的 Hilbert 类 $n$ 重积分不等式的构造定理, 改进和推广文 [22] 的相关结论.

## §2 预备引理

为了简便起见, 本文中设 $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, $\|x\|_{\rho,n}=\Big(\sum\limits_{i=1}^n x_i^\rho\Big)^{\frac{1}{\rho}}$, 且变上限积分函数

$$F_i(x_i)=\int_0^{x_i}f_i(t)\mathrm{d}t,\quad i=1,2,\cdots,n.$$

**引理 2.1**[23] 设 $p_i>0$, $a_i>0$, $\alpha_i\geqslant 0$ $(i=1,2,\cdots,n)$, $\psi(u)$ 可测, 则

$$\int_{\sum\limits_{i=1}^n(\frac{x_i}{a_i})^{\alpha_i}\leqslant 1,x_i>0}\psi\Big(\sum_{i=1}^n\Big(\frac{x_i}{a_i}\Big)^{\alpha_i}\Big)\prod_{i=1}^n x_i^{p_i-1}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$=\frac{\prod\limits_{i=1}^n a_i^{p_i}\prod\limits_{i=1}^n\Big(\frac{p_i}{\alpha_i}\Big)}{\prod\limits_{i=1}^n\alpha_i\Gamma\Big(\sum\limits_{i=1}^n\frac{p_i}{\alpha_i}\Big)}\int_0^1\psi(u)u^{\sum\limits_{i=1}^n\frac{p_i}{\alpha_i}-1}\mathrm{d}u.$$

**引理 2.2**[19] 设 $n\geqslant 2$, $\sum\limits_{i=1}^n\frac{1}{p_i}=1\,(p_i>1)$, $\alpha_i\in\mathbb{R}\,(i=1,2,\cdots,n)$, $K(x_1,x_2,\cdots,x_n)$ 是 $\sigma$ 阶齐次非负可测函数, 且

$$W=\int_{\mathbb{R}_+^{n-1}}K(1,u_2,\cdots,u_n)\prod_{i=2}^n u^{-\frac{\alpha_i+1}{p_i}}\mathrm{d}u_2\cdots\mathrm{d}u_n<+\infty,$$

那么

(i) 当且仅当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=\sigma+n-1$ 时, 存在常数 $M>0$, 有 Hilbert 型 $n$ 重积分不等式

$$\int_{\mathbb{R}_+^n}K(x_1,x_2,\cdots,x_n)\prod_{i=1}^{n}f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n\leqslant M\prod_{i=1}^{n}\|f_i\|_{p_i,x_i^{\alpha_i}},\tag{2.1}$$

其中 $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)\,(i=1,2,\cdots,n)$.

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=\sigma+n-1$, 即 (2.1) 成立时, 其最佳常数因子为 $W$.

**引理 2.3** 设 $n\geqslant 2$, $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1(p_i>1)$, $\lambda>0$, $\alpha_i<p_i-1$ $(1,2,\cdots,n)$, $\lambda>\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)$, 那么

(i) 当且仅当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=n-1-\lambda$ 时, 存在常数 $M>0$, 有

$$\int_{\mathbb{R}_+^n}\frac{1}{\left(\sum\limits_{i=1}^{n}x_i\right)^{\lambda}}\prod_{i=1}^{n}f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n\leqslant M\prod_{i=1}^{n}\|f_i\|_{p_i,x_i^{\alpha_i}},\tag{2.2}$$

其中 $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)\,(i=1,2,\cdots,n)$.

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=n-1-\lambda$, 即 (2.2) 成立时, 其最佳常数因子为

$$M_0=\inf\{M\}=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).$$

**证** 记 $K(x_1,x_2,\cdots,x_n)=\frac{1}{\left(\sum\limits_{i=1}^{n}x_i\right)^{\lambda}}\,(x_i>0)$, 则 $K(x_1,x_2,\cdots,x_n)$ 是 $-\lambda$ 阶齐次非负函数, 于是引理 2.2 中的条件 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=\sigma+n-1$ 化为引理 2.3 的条件 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=n-1-\lambda$.

因为 $\alpha_i<p_i-1$, 故 $1-\frac{\alpha_i+1}{p_i}>0$. 又因为 $\lambda>\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)$, 故 $\lambda-\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)>0$, 于是根据引理 2.1, 有

$$\begin{aligned}W&=\int_{\mathbb{R}_+^{n-1}}K(1,u_2,\cdots,u_n)\prod_{i=2}^{n}u^{-\frac{\alpha_i+1}{p_i}}\mathrm{d}u_2\cdots\mathrm{d}u_n\\&=\int_{\mathbb{R}_+^{n-1}}\frac{1}{\left(1+\sum\limits_{i=2}^{n}u_i\right)^{\lambda}}\prod_{i=2}^{n}u^{-\frac{\alpha_i+1}{p_i}}\mathrm{d}u_2\cdots\mathrm{d}u_n\\&=\lim_{r\to+\infty}\int_{\sum\limits_{i=2}^{n}u_i\leqslant r,u_i>0}\frac{1}{\left(1+\sum\limits_{i=2}^{n}u_i\right)^{\lambda}}\prod_{i=2}^{n}u^{-\frac{\alpha_i+1}{p_i}}\mathrm{d}u_2\cdots\mathrm{d}u_n\\&=\lim_{r\to+\infty}\int_{\sum\limits_{i=2}^{n}\frac{u_i}{r}\leqslant 1,u_i>0}\frac{1}{\left(1+r\sum\limits_{i=2}^{n}\frac{u_i}{r}\right)^{\lambda}}\prod_{i=2}^{n}u^{-\frac{\alpha_i+1}{p_i}}\mathrm{d}u_2\cdots\mathrm{d}u_n\end{aligned}$$

$$
\begin{aligned}
&= \lim_{r\to+\infty} \frac{r^{\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)} \prod\limits_{i=2}^{n}\Gamma\left(1-\frac{\alpha_i+1}{p_i}\right)}{\Gamma\left(\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)\right)} \int_0^1 \frac{1}{(1+ru)^{\lambda}} u^{\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)-1}\mathrm{d}u \\
&= \lim_{r\to+\infty} \frac{\prod\limits_{i=2}^{n}\Gamma\left(1-\frac{\alpha_i+1}{p_i}\right)}{\Gamma\left(\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)\right)} \int_0^r \frac{1}{(1+t)^{\lambda}} t^{\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)-1}\mathrm{d}t \\
&= \frac{\prod\limits_{i=2}^{n}\Gamma\left(1-\frac{\alpha_i+1}{p_i}\right)}{\Gamma\left(\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)\right)} \int_0^{+\infty} \frac{1}{(1+t)^{\lambda}} t^{\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)-1}\mathrm{d}t \\
&= \frac{\prod\limits_{i=2}^{n}\Gamma\left(1-\frac{\alpha_i+1}{p_i}\right)}{\Gamma\left(\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)\right)} B\Big(\sum_{i=2}^{n}\Big(1-\frac{\alpha_i+1}{p_i}\Big), \lambda-\sum_{i=2}^{n}\Big(1-\frac{\alpha_i+1}{p_i}\Big)\Big) \\
&= \frac{1}{\Gamma(\lambda)}\prod_{i=2}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big)\Gamma\Big(\lambda-\sum_{i=2}^{n}\Big(1-\frac{\alpha_i+1}{p_i}\Big)\Big) < +\infty,
\end{aligned}
$$

其中 $B(u,v)$ 是 Beta 函数. 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=n-1-\lambda$ 时, 有 $\lambda-\sum\limits_{i=2}^{n}\left(1-\frac{\alpha_i+1}{p_i}\right)=1-\frac{\alpha_1+1}{p_1}$, 于是

$$
W=\frac{1}{\Gamma(\lambda)}\prod_{i=2}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big)\Gamma\Big(1-\frac{\alpha_1+1}{p_1}\Big)=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).
$$

综上并根据引理 2.2, 知引理 2.3 成立.

**引理 2.4** 设 $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1\,(p_i>1)$, $\alpha_i<p_i-1$, $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)$, $\varphi_i(x_i)$ 是可测函数 $(i=1,2,\cdots,n)$, 且 $\lim\limits_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=0\,(t>0)$, 则

$$
\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}F_i(x_i)=0.
$$

**证** 记 $q_i=\Big(\sum\limits_{j\neq i}^{n}\frac{1}{p_j}\Big)^{-1}$, 则 $\frac{1}{p_i}+\frac{1}{q_i}=1\,(p_i>1)$. 由于 $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)$, 根据 Hölder 积分不等式, 有

$$
\begin{aligned}
\lim_{x_i\to+\infty}|\mathrm{e}^{-\varphi_i(x_i)t}F_i(x_i)| &\leqslant \lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}\int_0^{x_i}|f_i(t)|\mathrm{d}t \\
&= \lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}\int_0^{x_i}t^{\frac{\alpha_i}{p_i}}(t^{-\frac{\alpha_i}{p_i}}|f_i(t)|)\mathrm{d}t \\
&\leqslant \lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}\Big(\int_0^{x_i}t^{\alpha_i}|f_i(t)|^{p_i}\mathrm{d}t\Big)^{\frac{1}{p_i}}\Big(\int_0^{x_i}t^{-\frac{\alpha_i}{p_i}q_i}\mathrm{d}t\Big)^{\frac{1}{q_i}} \\
&\leqslant \lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}\Big(\int_0^{+\infty}t^{\alpha_i}|f_i(t)|^{p_i}\mathrm{d}t\Big)^{\frac{1}{p_i}}\Big(\int_0^{x_i}t^{-\frac{\alpha_i}{p_i}q_i}\mathrm{d}t\Big)^{\frac{1}{q_i}} \\
&= \|f_i\|_{p_i,x_i^{\alpha_i}}\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}\Big(\int_0^{x_i}t^{-\frac{\alpha_i}{p_i}q_i}\mathrm{d}t\Big)^{\frac{1}{q_i}}.
\end{aligned}
$$

因为 $\alpha_i < p_i - 1$, 故 $1 - \frac{\alpha_i}{p_i-1} > 0$, 于是

$$\begin{aligned}&\Big(\int_0^{x_i} t^{-\frac{\alpha_i}{p_i}q_i}\mathrm{d}t\Big)^{\frac{1}{q_i}}\\&=\Big(\int_0^{x_i} t^{-\frac{\alpha_i}{p_i-1}}\mathrm{d}t\Big)^{\frac{p_i-1}{p_i}}\\&=\Big[\Big(1-\frac{\alpha_i}{p_i-1}\Big)^{-1}t^{1-\frac{\alpha_i}{p_i-1}}\Big|_0^{x_i}\Big]^{\frac{p_i-1}{p_i}}\\&=\Big(1-\frac{\alpha_i}{p_i-1}\Big)^{\frac{1-p_i}{p_i}}x_i^{1-\frac{\alpha_i+1}{p_i}},\end{aligned}$$

从而

$$\begin{aligned}&\lim_{x_i\to+\infty}|\mathrm{e}^{-\varphi_i(x_i)t}F_i(x_i)|\\&\leqslant\Big(1-\frac{\alpha_i}{p_i-1}\Big)^{\frac{1-p_i}{p_i}}\|f_i\|_{p_i,x_i^{\alpha_i}}\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=0,\end{aligned}$$

由此可得 $\lim\limits_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}F_i(x_i)=0$.

## §3　主要结果

**定理 3.1**　设 $n\geqslant 2$, $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1\,(p_i>1)$, $\lambda>0$, $\alpha_i<p_i-1$, $\varphi_i(x_i)$ 在 $(0,+\infty)$ 上是严格递增的可微函数, $\lim\limits_{x_i\to0^+}\varphi_i(x_i)=0$, $\lim\limits_{x_i\to+\infty}\varphi_i(x_i)=+\infty$, 且

$$\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=0\quad(i=1,2,\cdots,n).$$

那么

(i) 当且仅当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, 存在常数 $M>0$, 有 Hilbert 类积分不等式

$$\int_{\mathbb{R}_+^n}\frac{1}{\Big(\sum\limits_{i=1}^{n}\varphi_i(x_i)\Big)^{\lambda}}\prod_{i=1}^{n}f_i(x_i)\mathbf{d}x_1\cdots\mathbf{d}x_n\leqslant M\prod_{i=1}^{n}\|F_i\|_{p_i,\theta_i(x_i)},\tag{3.1}$$

其中 $\theta_i(x_i)=\varphi_i^{\alpha_i}(x_i)\varphi_i'(x_i)$, $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)\,(i=1,2,\cdots,n)$.

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, (3.1) 的最佳常数因子为

$$M_0=\inf\{M\}=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).$$

**证**　(i) 根据 Gamma 函数的性质, 有

$$\frac{1}{\Big(\sum\limits_{i=1}^{n}\varphi_i(x_i)\Big)^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^{+\infty}t^{\lambda-1}\mathrm{e}^{-t\sum\limits_{i=1}^{n}\varphi_i(x_i)}\mathrm{d}t,$$

于是

$$
\begin{aligned}
I := &\int_{\mathbb{R}_+^n} \frac{1}{\Big(\sum\limits_{i=1}^n \varphi_i(x_i)\Big)^\lambda} \prod_{i=1}^n f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n \\
= &\frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \prod_{i=1}^n f_i(x_i)\Big(\int_0^{+\infty} t^{\lambda-1}\mathrm{e}^{-t\sum\limits_{i=1}^n \varphi_i(x_i)}\mathrm{d}t\Big)\mathrm{d}x_1\cdots\mathrm{d}x_n \\
= &\frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1}\Big(\prod_{i=1}^n \int_0^{+\infty} \mathrm{e}^{-t\varphi_i(x_i)} f_i(x_i)\mathrm{d}x_i\Big)\mathrm{d}t,
\end{aligned}
$$

根据引理 2.4, 有

$$
\begin{aligned}
\int_0^{+\infty} \mathrm{e}^{-t\varphi_i(x_i)} f_i(x_i)\mathrm{d}x_i &= \int_0^{+\infty} \mathrm{e}^{-t\varphi_i(x_i)}\mathrm{d}F_i(x_i) \\
&= \lim_{x_i\to+\infty} \mathrm{e}^{-t\varphi_i(x_i)}F_i(x_i) + t\int_0^{+\infty} \varphi_i'(x_i)\mathrm{e}^{-t\varphi_i(x_i)}F_i(x_i)\mathrm{d}x_i \\
&= t\int_0^{+\infty} \mathrm{e}^{-t\varphi_i(x_i)}F_i(x_i)\mathrm{d}\varphi_i(x_i) = t\int_0^{+\infty} \mathrm{e}^{-tu_i}F_i(\varphi_i^{-1}(u_i))\mathrm{d}u_i,
\end{aligned}
$$

从而

$$
\begin{aligned}
I = &\frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1}\Big(\prod_{i=1}^n t\int_0^{+\infty} \mathrm{e}^{-tu_i}F_i(\varphi_i^{-1}(u_i))\mathrm{d}u_i\Big)\mathrm{d}t \\
= &\frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \prod_{i=1}^n F_i(\varphi_i^{-1}(u_i))\Big(\int_0^{+\infty} t^{\lambda+n-1}\mathrm{e}^{-t\sum\limits_{i=1}^n u_i}\mathrm{d}t\Big)\mathrm{d}u_1\cdots\mathrm{d}u_n \\
= &\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \frac{1}{\Big(\sum\limits_{i=1}^n u_i\Big)^{\lambda+n}} \prod_{i=1}^n F_i(\varphi_i^{-1}(u_i))\mathrm{d}u_1\cdots\mathrm{d}u_n.
\end{aligned}
$$

根据引理 2.3, 当且仅当 $\sum\limits_{i=1}^n \frac{\alpha_i}{p_i} = n-1-(\lambda+n) = -1-\lambda$ 时, 存在常数 $\overline{M}>0$, 使得

$$
\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{1}{\Big(\sum\limits_{i=1}^n u_i\Big)^{\lambda+n}} \prod_{i=1}^n F_i(\varphi_i^{-1}(u_i))\mathrm{d}u_1\cdots\mathrm{d}u_n \\
\leqslant &\overline{M}\prod_{i=1}^n \Big(\int_0^{+\infty} u_i^{\alpha_i}|F_i(\varphi_i^{-1}(u_i))|^{p_i}\mathrm{d}u_i\Big) \\
= &\overline{M}\prod_{i=1}^n \Big(\int_0^{+\infty} \varphi_i^{\alpha_i}(x_i)\varphi_i'(x_i)|F_i(x_i)|^{p_i}\mathrm{d}x_i\Big) \\
= &\overline{M}\prod_{i=1}^n \Big(\int_0^{+\infty} \theta_i(x_i)|F_i(x_i)|^{p_i}\mathrm{d}x_i\Big) = \overline{M}\prod_{i=1}^n \|F_i\|_{p_i,\theta_i(x_i)}.
\end{aligned}
\tag{3.2}
$$

记 $M = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\overline{M}$, 则当且仅当 $\sum\limits_{i=1}^n \frac{\alpha_i}{p_i} = -1-\lambda$ 时, 有

$$
I \leqslant M\prod_{i=1}^n \|F_i\|_{p_i,\theta_i(x_i)}.
$$

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$, 即 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=n-1-(\lambda+n)$ 时, 根据引理 2.3, (3.2) 的最佳常数因子是 $\frac{1}{\Gamma(\lambda+n)}\prod\limits_{i=1}^{n}\Gamma\big(1-\frac{\alpha_i+1}{p_i}\big)$, 故 (3.1) 的最佳常数因子为

$$M_0=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\frac{1}{\Gamma(\lambda+n)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big)=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).$$

**注 3.1** 若在定理 3.1 中取 $\varphi_i(x_i)=a_ix_i\,(a_i>0)$, 则当 $t>0$ 时, 必有

$$\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=\lim_{x_i\to+\infty}\frac{x_i^{1-\frac{\alpha_i+1}{p_i}}}{\mathrm{e}^{a_ix_it}}=0\quad(i=1,2,\cdots,n),$$

故在此种情形下, 定理 3.1 中的条件 $\lim\limits_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=0\,(t>0)$ 可以去掉.

**推论 3.1** 设 $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1\,(p_i>1)$, $\lambda>0$, $\alpha_i<p_i-1$, $a_i>0\,(i=1,2,\cdots,n)$.

(i) 当且仅当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, 存在常数 $M>0$, 使得

$$\int_{\mathbb{R}_+^n}\frac{1}{\Big(\sum\limits_{i=1}^{n}a_ix_i\Big)^{\lambda}}\prod_{i=1}^{n}f_i(x_i)\mathbf{d}x_1\cdots\mathbf{d}x_n\leqslant M\prod_{i=1}^{n}\|F_i\|_{p_i,x_i^{\alpha_i}},\tag{3.3}$$

其中 $f_i(x_i)\in L_{p_i}^{x_i^{\alpha_i}}(0,+\infty)\,(i=1,2,\cdots,n)$.

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, (3.3) 的最佳常数因子为

$$M_0=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}a_i^{\frac{\alpha_i+1}{p_i}}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).$$

**证** 取 $\varphi_i(x_i)=a_ix_i$, 则 $\theta_i(x_i)=\varphi_i^{\alpha_i}(x_i)\varphi_i'(x_i)=a_i^{\alpha_i+1}x_i^{\alpha_i}$, 于是

$$\begin{aligned}&\prod_{i=1}^{n}\|F_i\|_{p_i,\theta_i(x_i)}\\&=\prod_{i=1}^{n}\Big(\int_0^{+\infty}a_i^{\alpha_i+1}x_i^{\alpha_i}|F_i(x_i)|^{p_i}\mathrm{d}x_i\Big)^{\frac{1}{p_i}}\\&=\prod_{i=1}^{n}a_i^{\frac{\alpha_i+1}{p_i}}\prod_{i=1}^{n}\Big(\int_0^{+\infty}x_i^{\alpha_i}|F_i(x_i)|^{p_i}\mathrm{d}x_i\Big)^{\frac{1}{p_i}}\\&=\prod_{i=1}^{n}a_i^{\frac{\alpha_i+1}{p_i}}\prod_{i=1}^{n}\|F_i\|_{p_i,x_i^{\alpha_i}},\end{aligned}$$

由此并根据定理 3.1, 知推论 3.1 成立.

**注 3.2** 在推论 3.1 中取 $a_i=1$, $\alpha_i=-p_i\frac{\lambda}{r_i}-1$, 由于 $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1$, $\sum\limits_{i=1}^{n}\frac{1}{r_i}=1$, 故

$$\sum_{i=1}^{n}\frac{\alpha_i}{p_i}=\sum_{i=1}^{n}\Big(-\frac{\lambda}{r_i}-\frac{1}{p_i}\Big)=-\lambda\sum_{i=1}^{n}\frac{1}{r_i}-\sum_{i=1}^{n}\frac{1}{p_i}=-1-\lambda,$$

$$\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}a_i^{\frac{\alpha_i+1}{p_i}}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big)=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1+\frac{\lambda}{r_i}\Big)=\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\lambda}{r_i}\Gamma\Big(\frac{\lambda}{r_i}\Big),$$

由此便可得到文 [22] 中的 (1), 且其常数因子是最佳的.

**推论 3.2** 设 $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1\,(p_i>1)$, $\lambda>0$, $\alpha_i<p_i-1$, $\theta_i(x_i)=\mathrm{e}^{x_i}(\mathrm{e}^{x_i}-1)^{\alpha_i}$ $(i=1,2,\cdots,n)$, 则

(i) 当且仅当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, 存在常数 $M>0$, 使得

$$\int_{\mathbb{R}_+^n}\frac{1}{\Big(\sum\limits_{i=1}^{n}\mathrm{e}^{x_i}-n\Big)^{\lambda}}\prod_{i=1}^{n}f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n\leqslant M\prod_{i=1}^{n}\|F_i\|_{p_i,\theta_i(x_i)},\tag{3.4}$$

其中 $f_i(x_i)\in L_{p_i}^{\theta_i(x_i)}(0,+\infty)$ $(i=1,2,\cdots,n)$.

(ii) 当 $\sum\limits_{i=1}^{n}\frac{\alpha_i}{p_i}=-1-\lambda$ 时, (3.4) 的最佳常数因子为

$$\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\Big(1-\frac{\alpha_i+1}{p_i}\Big).$$

**证** 令 $\varphi_i(x_i)=\mathrm{e}^{x_i}-1$, 则 $\varphi_i(x_i)$ 是 $(0,+\infty)$ 上的严格递增函数, $\lim\limits_{x_i\to0^+}\varphi_i(x_i)=0$, $\lim\limits_{x_i\to+\infty}\varphi_i(x_i)=+\infty$, $\varphi_i^{\alpha_i}(x_i)\varphi_i'(x_i)=\mathrm{e}^{x_i}(\mathrm{e}^{x_i}-1)^{\alpha_i}$, 且当 $t>0$ 时, 有

$$0\leqslant\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=\lim_{x_i\to+\infty}\frac{x_i^{1-\frac{\alpha_i+1}{p_i}}}{\mathrm{e}^{(\mathrm{e}^{x_i}-1)t}}\leqslant\lim_{x_i\to+\infty}\frac{x_i^{1-\frac{\alpha_i+1}{p_i}}}{\mathrm{e}^{x_it}}=0,$$

故

$$\lim_{x_i\to+\infty}\mathrm{e}^{-\varphi_i(x_i)t}x_i^{1-\frac{\alpha_i+1}{p_i}}=0.$$

综上并根据定理 3.1, 知推论 3.2 成立.

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# A Construction Theorem for $n$-Multipe Hilbert-Type Integral Inequality Involving Variational Upper Limit Integral Functions

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**Abstract** Using the construction theorem of the homogeneous kernel $n$-multiple Hilbert-type integral inequality and the properties of the Gamma function, the authors discuss the conditions under which an $n$-multiple Hilbert-type integral inequality involving variational upper limit integral functions can be established and the optimal constant factor, obtain the necessary and sufficient condition for constructing this inequality and the formula for calculating the optimal constant factor, and finally, give several special cases.

**Keywords** $n$-Multiple Hilbert-type integral inequality, Variational upper limit integral function, Construction theorem, Optimal constant factor

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