数 学 年 刊 A 辑
2026, 47A(1):11–32
DOI: 10.16205/j.cnki.cama.2026.0002

Littlewood-Paley 算子的混合弱型 Fefferman-Stein 不等式及其应用*

陈嵩箐[1] 方权清[2] 潘思慧[3]

提要 建立了 Littlewood-Paley 算子的混合弱型 Fefferman-Stein 不等式, 该结论即使在单权情形下也是一个新结论. 作为应用, 还建立了 Littlewood-Paley 算子的双权弱型不等式.

关键词 Littlewood-Paley 算子, 混合弱型 Fefferman-Stein 不等式, 双权弱型不等式
MR (2020) 主题分类 42B20, 42B25
中图法分类 O174.2
文献标志码 A
文章编号 1000-8314(2026)01-0011-22

1 引 言

设 $\psi(x,y)$ 为定义在远离对角线 $x\neq y$ 上的局部可积函数, 其中 $x,y\in\mathbb{R}^n$. 假设 $\psi(x,y)$ 满足如下条件: 存在正数 δ,A,γ_0,

(1) 尺寸条件:

$$|\psi(x,y)|\leqslant\frac{A}{(1+|x-y|)^{n+\delta}}.\tag{1.1}$$

(2) 光滑条件: 当 $|x-x'|<\frac{|x-y|}{2}$ 时,

$$|\psi(x,y)-\psi(x',y)|\leqslant\frac{A|x-x'|^{\gamma_0}}{(1+|x-y|)^{n+\delta+\gamma_0}},\tag{1.2}$$

当 $|y-y'|<\frac{|x-y|}{2}$ 时,

$$|\psi(x,y)-\psi(x,y')|\leqslant\frac{A|y-y'|^{\gamma_0}}{(1+|x-y|)^{n+\delta+\gamma_0}}.\tag{1.3}$$

考虑如下定义的平方函数 S_α 及 g_λ^*:

$$S_\alpha(f)(x):=\Big(\iint_{\Gamma_\alpha(x)}|\psi_t(f)(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{1/2},\quad\alpha>0,$$

$$g_\lambda^*(f)(x):=\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}|\psi_t(f)(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\Big)^{1/2},\quad\lambda>2,$$

本文 2026 年 1 月 3 日收到, 2026 年 3 月 5 日收到修改稿.
[1]湖北经济学院数字金融创新湖北省重点实验室, 武汉 430205; 湖北经济学院新财经交叉学科研究院, 武汉 430205. E-mail: chensongqing@hbue.edu.cn
[2]通信作者. 福建省金融信息处理重点实验室 (莆田学院), 福建 莆田 351100; 应用数学福建省高校重点实验室 (莆田学院), 福建 莆田 351100. E-mail: quanqingfang@163.com
[3]闽南师范大学数学与统计学院, 福建 漳州 363000. E-mail: a18922689812@163.com
*本文受到国家自然科学基金 (No. 12426654, No. 12426645), 福建省自然科学基金项目 (No. 2025J01361) 和莆田学院科研启动项目 (No. 2020002) 的资助.

其中 $\Gamma_\alpha(x)=\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<\alpha t\}$, $f\in\mathcal{S}(\mathbb{R}^n)$,

$$\psi_t(f)(x):=\frac{1}{t^n}\int_{\mathbb{R}^n}\psi\Big(\frac{x}{t},\frac{y}{t}\Big)f(y)\mathrm{d}y,\quad x\notin\operatorname{supp}f.$$

今后, 文中都事先假设 g_λ^* 对某个 $p_0>1$ 是 $L^{p_0}(\mathbb{R}^n)$ 有界的, 在此假设条件下, g_λ^* 是 $L^p(\mathbb{R}^n)$ 有界的, 也是弱 $(1,1)$ 的 (见 [1]). 由于 S_α 可被 g_λ^* 逐点控制, S_α 自然也有相应的有界性结果. 上述两类平方函数通常称为 Littlewood-Paley 算子. 一维情形下的 Littlewood-Paley 算子最初是 Littlewood 和 Paley 为研究傅里叶级数的二进分解而引入的 (见 [2–4]). 后来, Stein [5–7] 引进了高维情形的 Littlewood-Paley 算子, 并用以刻画 $L^p(\mathbb{R}^n)$ 空间. Littlewood-Paley 算子在 PDE 和其他数学领域也有很重要的应用, 如 Fabes, Jerison, Kenig [8] 通过对散度形式的椭圆算子平方根的多线性 Littlewood-Paley 型估计, 研究了非散度形式抛物方程柯西问题的解. 更多关于 Littlewood-Paley 算子的理论及其应用, 可参考文 [9–14].

另一方面, Fefferman 和 Stein [15] 的一个经典结果是, 对于任意的权 ω,

$$\int_{\mathbb{R}^n}Mf(x)^p\omega(x)\mathrm{d}x\lesssim\int_{\mathbb{R}^n}|f(x)|^pM\omega(x)\mathrm{d}x,\quad 1<p<\infty,\tag{1.4}$$

$$\omega(\{x\in\mathbb{R}^n:Mf(x)>t\})\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M\omega(x)\mathrm{d}x,\quad t>0,$$

其中 M 为 Hardy-Littlewood 极大算子. 上述两个不等式分别称为强型 Fefferman-Stein 不等式和弱型 Fefferman-Stein 不等式, 它在调和分析中有很重要的应用, 读者可参考文 [16] 的第 5 章和第 6 章, 其中有非常精彩的阐述. 然而 (1.4) 对于奇异积分算子 T 并不成立, Córdoba 和 Fefferman [17] 证明了当 $M\omega$ 换为 $M_r\omega=M(\omega^r)^{1/r}$, $1<r<\infty$ 时, (1.4) 成立. Wilson [18] 在 $1<p<2$ 情形下将文 [17] 中的结果改进到了 $M^2\sim M_{L\log L}$. Pérez [19] 将文 [18] 中的结果推广到了 $p>2$ 以及端点情形, 也可见文 [20]:

$$\int_{\mathbb{R}^n}Tf(x)^p\omega(x)\mathrm{d}x\lesssim\int_{\mathbb{R}^n}|f(x)|^pM_A\omega(x)\mathrm{d}x,\quad 1<p<\infty,\ A\in B_p,$$

$$\omega(\{x\in\mathbb{R}^n:|Tf(x)|>t\})\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_A\omega(x)\mathrm{d}x,\quad t>0,\ A\in\bigcup_{p>1}B_p,\tag{1.5}$$

这里 $A:[0,\infty)\to[0,\infty)$ 为 Young 函数, 即严格单调递增的连续凸函数且满足 $A(0)=0$, $\lim\limits_{t\to\infty}\frac{A(t)}{t}=\infty$,

$$M_A\omega(x):=\sup_{Q\ni x}\|\omega\|_{A,Q},$$

其中

$$\|\omega\|_{A,Q}:=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_QA\Big(\frac{|\omega(x)|}{\lambda}\Big)\mathrm{d}x\leqslant1\Big\}.$$

称 $A\in B_p$, 若存在正数 c, 使得

$$\int_c^\infty\frac{A(t)}{t^p}\frac{\mathrm{d}t}{t}<\infty,\quad 1<p<\infty.$$

Chang, Wilson 和 Wolff [21] 在 $p=2$ 时, 证明了卷积型的 S_α 也满足强型 Fefferman-Stein 不等式, Chanillo 和 Wheeden [22] 指出当 $p>2$ 时, S_α 不满足强型 Fefferman-Stein 不等式, 并将文 [21] 中的结果推广到 $1<p\leqslant 2$ 时:

$$\int_{\mathbb{R}^n} S_\alpha(f)(x)^p\omega(x)\mathrm{d}x \lesssim \int_{\mathbb{R}^n} |f(x)|^p M\omega(x)\mathrm{d}x.$$

最近, Cao 等人 [23] 将上述结果推广到了多线性 S_α 及 g_λ^*. 从已有结果来看, 自然会问下面的问题.

问题 1　在端点情形下, Littlewood-Paley 算子的 Fefferman-Stein 类型不等式如何建立?

Berra, Carena, Pradolini [24] 最近将 (1.5) 推广到混合弱型 Fefferman-Stein 估计: 给定 $\delta>0$, $q>1$, 设 $\varphi(z)=z(1+\log^+ z)^\delta$, $0\leqslant\mu\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\nu\in RH_\infty\cap A_q$, 则对每个 $t>0$ 和 $p>\max\{q,1+\frac{1}{\delta}\}$,

$$\mu\nu\Big(\Big\{x\in\mathbb{R}^n:\frac{|T(f\nu)(x)|}{\nu(x)}>t\Big\}\Big)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_{\varphi,\nu^{1-q'}}\mu(x)M(\Psi(\nu))(x)\mathrm{d}x,$$

其中 $f\in L_c^\infty(\mathbb{R}^n)$, $\Psi(z)=z^{p'+1-q'}\chi_{[0,1]}(z)+z^{p'}\chi_{[1,\infty)}(z)$,

$$M_{\varphi,\nu^{1-q'}}\mu(x)=\sup_{Q\ni x}\|\mu\|_{\varphi,Q,\nu^{1-q'}},$$
$$\|\mu\|_{\varphi,Q,\nu^{1-q'}}=\inf\Big\{\lambda>0:\frac{1}{\nu^{1-q'}(Q)}\int_Q\varphi\Big(\frac{|\mu(x)|}{\lambda}\Big)\nu^{1-q'}(x)\mathrm{d}x\leqslant 1\Big\}.$$

当 $\nu=1$ 时, 上式即为 Calderón-Zygmund 算子 T 的弱型 Fefferman-Stein 估计. 加权混合弱型估计肇端于 Sawyer [25] 的著名工作, Sawyer 为了研究 Hardy-Littlewood 极大算子加权有界性的新证明, 将 Hardy-Littlewood 极大算子的加权弱 (1,1) 不等式推广到加权混合弱型不等式, 但因权函数在分布集中会导致在证明该类型不等式时无法使用 Vitali 覆盖引理, 因此该推广是非平凡的. 在后续研究中也发现加权混合弱型估计可应用于加权外插理论、多线性算子理论以及遍历理论, 读者可参看文 [26–28]. 更多关于加权混合弱型的相关研究, 可参考文 [23, 29–33], 特别地, 文 [23] 中研究了 Littlewood-Paley 算子的加权混合弱型估计, 那么有下面的问题.

问题 2　是否可建立 Littlewood-Paley 算子的混合弱型 Fefferman-Stein 估计呢?

本文解决了上述两个问题, 主要结果如下.

定理 1.1　设 $\lambda>2$, $\alpha\geqslant 1$, ψ 满足 (1.1)–(1.3) 且 $0<\gamma_0<\min\{\frac{(\lambda-2)n}{2},\delta\}$. 设 $q>1$, μ 为权, $\nu\in RH_\infty\cap A_q$, $\Psi(z)=z^{p'+1-q'}\chi_{[0,1]}(z)+z^{p'}\chi_{[1,\infty)}(z)$. 则对于任意的 $f\in L_c^\infty(\mathbb{R}^n)$, $p>\max\{q,2\}$,

$$\mu\nu\Big(\Big\{x\in\mathbb{R}^n:\frac{g_\lambda^*(f\nu)(x)}{\nu(x)}>t\Big\}\Big)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_{\nu^{1-p'}}\mu(x)M(\Psi(\nu))(x)\mathrm{d}x,\quad t>0,$$

$$\mu\nu\Big(\Big\{x\in\mathbb{R}^n:\frac{S_\alpha(f\nu)(x)}{\nu(x)}>t\Big\}\Big)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_{\nu^{1-p'}}\mu(x)M(\Psi(\nu))(x)\mathrm{d}x,\quad t>0,$$

其中

$$M_\omega(f)(x):=\sup_{Q\ni x}\frac{1}{\omega(Q)}\int_Q|f(x)|\omega(x)\mathrm{d}x.$$

当 $\nu=1$ 时, 定理 1.1 的结论即为如下 Littlewood-Paley 算子的弱型 Fefferman-Stein 不等式.

推论 1.1 设 $\lambda>2$, $\alpha\geqslant 1$, ψ 满足 (1.1)–(1.3) 且 $0<\gamma_0<\min\big\{\frac{(\lambda-2)n}{2},\delta\big\}$. 设 μ 为权, 则对于任意的 $f\in L_c^\infty(\mathbb{R}^n)$,

$$\mu\Big(\Big\{x\in\mathbb{R}^n:g_\lambda^*(f)(x)>t\Big\}\Big)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M\mu(x)\mathrm{d}x,\quad t>0,$$
$$\mu\Big(\Big\{x\in\mathbb{R}^n:S_\alpha(f)(x)>t\Big\}\Big)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M\mu(x)\mathrm{d}x,\quad t>0.$$

作为应用, 本文还建立了 Littlewood-Paley 算子的双权弱型估计.

定理 1.2 设 $\lambda>2$, $\alpha\geqslant 1$, ψ 满足 (1.1)–(1.3) 且 $0<\gamma_0<\min\big\{\frac{(\lambda-2)n}{2},\delta\big\}$. 设 Young 函数 Φ 的补函数 $\overline{\Phi}\in B_{p'}$, $1<p<\infty$. 若 μ,ν 为一对权满足

$$\sup_Q\|\nu^{-1/p}\|_{p',Q}\|\mu^{1/p}\|_{\Phi,Q}<\infty.$$

则

$$t\mu(\{x\in\mathbb{R}^n:g_\lambda^*(f)(x)>t\})^{1/p}\lesssim\|f\|_{L^p(\nu)},\quad t>0,$$
$$t\mu(\{x\in\mathbb{R}^n:S_\alpha(f)(x)>t\})^{1/p}\lesssim\|f\|_{L^p(\nu)},\quad t>0,$$

其中 $\overline{\Phi}:[0,\infty)\to[0,\infty)$,

$$\overline{\Phi}(t)=\sup_{s>0}\{st-\Phi(s)\}.$$

本文结构安排如下: 第 2 部分将介绍一些需要用到的预备知识, 第 3 部分给出定理 1.1 的证明, 第 4 部分给出定理 1.2 的证明.

为了方便起见, 本文引入以下记号: 记 C 为常数, 但在不同的地方可能不同. 记 $B(x,r)$ 为中心在 $x\in\mathbb{R}^n$, 半径为 r 的开球. 对于任意可测集 $E\subset\mathbb{R}^n$, 记 $|E|$ 为 E 的 Lebesgue 测度, χ_E 为 E 的特征函数. 若 $f\leqslant Cg$, 则记 $f\lesssim g$; 若 $f\lesssim g\lesssim f$, 则记 $f\sim g$. 对于 $1\leqslant p\leqslant\infty$, 当 $\frac{1}{p}+\frac{1}{p'}=1$, 记 p' 是 p 的共轭指标.

2 预 备 知 识

本小节回顾一些需要用到的权的知识.

定义 2.1 权是定义在 $\mathbb{R}^n$ 上的几乎处处大于 0 的局部可积函数. 称权 $\omega\in A_p\,(1<$

$p < \infty$), 若存在常数 $C > 0$, 使得

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(y)\mathrm{d}y\Big)\Big(\frac{1}{|Q|}\int_Q \omega(y)^{1-p'}\mathrm{d}y\Big)^{p-1} < C, \tag{2.1}$$

其中上确界取遍所有的方体 $Q \subset \mathbb{R}^n$. 称权 $w \in A_1$, 若存在常数 $C > 0$, 使得

$$\frac{1}{|Q|}\int_Q w(y)\mathrm{d}y \leqslant C \inf_{y\in Q} w(y). \tag{2.2}$$

把满足(2.1)–(2.2)的最小常数 C 分别记为 $[\omega]_{A_p}$ 和 $[\omega]_{A_1}$.

A_p 权的一个重要性质是满足双倍条件, 即对每个方体 Q, 存在常数 $C > 1$, 使得 $\omega(2Q) \leqslant C\omega(Q)$.

定义 2.2　设 ω 为权, 称 $\omega \in RH_s\,(1 < s < \infty)$, 若存在常数 $C > 0$, 使得对每个方体 $Q \subset \mathbb{R}^n$,

$$\Big(\frac{1}{|Q|}\int_Q \omega(x)^s\mathrm{d}x\Big)^{1/s} \leqslant \frac{C}{|Q|}\int_Q \omega(x)\mathrm{d}x. \tag{2.3}$$

称 $\omega \in RH_\infty$, 若存在常数 $C > 0$, 使得对每个方体 $Q \subset \mathbb{R}^n$,

$$\sup_Q \omega \leqslant \frac{C}{|Q|}\int_Q \omega(x)\mathrm{d}x. \tag{2.4}$$

把满足(2.3)–(2.4)的最小常数 C 分别记为 $[\omega]_{RH_s}$ 和 $[\omega]_{RH_\infty}$.

以下引理是证明中需要用到的权性质.

引理 2.1 [34] 设 ω 为权, $1 < p < \infty$.

(1) 若 $\omega \in RH_\infty \cap A_p$, 则 $\omega^{1-p'} \in A_1$;

(2) 若 $\omega \in RH_\infty$, 则对于每个 $r > 0$, 有 $\omega^r \in RH_\infty$.

3 Littlewood-Paley 算子的混合弱型 Fefferman-Stein 不等式

本节将给出定理 1.1 的证明. 由于 S_α 可被 g_λ^* 逐点控制, 故今后只对 g_λ^* 证明相关结论.

引理 3.1　设 ω 为双倍权, $M_\omega f < \infty$ a.e., Q 为固定的方体. 则对于任意的 $x, y \in Q$,

$$M_\omega(f\chi_{\mathbb{R}^n\backslash 4\sqrt{n}Q})(x) \sim M_\omega(f\chi_{\mathbb{R}^n\backslash 4\sqrt{n}Q})(y).$$

证　固定 $x, y \in Q$, 令方体 $Q' \ni x$. 不妨设 $Q' \cap (\mathbb{R}^n\backslash 4\sqrt{n}Q) \neq \emptyset$. 则 $Q \subset 4\sqrt{n}Q$. 事实上, 由 $B\big(x_{Q'}, \frac{l_{Q'}}{2}\big) \subset Q'$ 可得 $B\big(x_{Q'}, \frac{l_{Q'}}{2}\big) \cap (\mathbb{R}^n\backslash 4\sqrt{n}Q) \neq \emptyset$, 其中 $x_{Q'}$, $l_{Q'}$ 分别表示方体 Q' 的中心和边长. 故存在 $z \in B\big(x_{Q'}, \frac{l_{Q'}}{2}\big) \cap (\mathbb{R}^n\backslash 4\sqrt{n}Q)$, 一方面, $|z - x_Q| > 2\sqrt{n}l_Q$, 其中 x_Q, l_Q 分别表示方体 Q 的中心和边长. 另一方面,

$$|z - x_Q| \leqslant |z - x| + |x - x_Q|$$

$$\begin{aligned}&\leqslant |z-x_{Q'}|+|x-x_{Q'}|+|x-x_Q|\\&\leqslant \frac{l_{Q'}}{2}+\frac{\sqrt{n}l_{Q'}}{2}+\frac{\sqrt{n}l_Q}{2}\leqslant \sqrt{n}l_{Q'}+\frac{\sqrt{n}l_Q}{2}.\end{aligned}$$

故 $l_{Q'}>\frac{3l_Q}{2}$. 从而任意的 $\overline{z}\in Q$,

$$\begin{aligned}|\overline{z}-x_{Q'}|&\leqslant |\overline{z}-x|+|x-x_{Q'}|\\&\leqslant \sqrt{n}l_Q+\frac{\sqrt{n}l_{Q'}}{2}\\&<\frac{2\sqrt{n}l_{Q'}}{3}+\frac{\sqrt{n}l_{Q'}}{2}<2\sqrt{n}l_{Q'}.\end{aligned}$$

由此即得 $Q\subset 4\sqrt{n}Q'$. 于是对每个 $Q'\ni x$, 由 ω 为双倍权, 可得

$$\begin{aligned}&\frac{1}{\omega(Q')}\int_{Q'}|f(x)|\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q}(x)\omega(x)\mathrm{d}x\\&\leqslant \frac{\omega(4\sqrt{n}Q')}{\omega(Q')}\frac{1}{\omega(4\sqrt{n}Q')}\int_{4\sqrt{n}Q'}|f(x)|\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q}(x)\omega(x)\mathrm{d}x\\&\lesssim M_\omega(f\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q})(y).\end{aligned}$$

则

$$M_\omega(f\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q})(x)\lesssim M_\omega(f\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q})(y).$$

同理可证

$$M_\omega(f\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q})(x)\gtrsim M_\omega(f\chi_{\mathbb{R}^n\setminus 4\sqrt{n}Q})(y).$$

于是引理得证.

引理 3.2 设 $\omega\in A_1$. 则对于使得 $M_\omega f<\infty$ a.e. 的每个函数 f, 有 $Mf\leqslant [\omega]_{A_1}M_\omega f$.

证 固定 x 且令方体 $Q\ni x$. 利用 $\omega\in A_1$, 有

$$\begin{aligned}\frac{1}{|Q|}\int_Q|f(x)|\mathrm{d}x&=\frac{\omega(Q)}{|Q|}\frac{1}{\omega(Q)}\int_Q|f(x)|\omega(x)\omega(x)^{-1}\mathrm{d}x\\&\leqslant [\omega]_{A_1}\inf_Q\omega\Big(\sup_Q\omega^{-1}\Big)\frac{1}{\omega(Q)}\int_Q|f(x)|\omega(x)\mathrm{d}x\\&\leqslant [\omega]_{A_1}M_\omega f(x).\end{aligned}$$

对所有这些方体 $Q\ni x$ 取上确界即证.

下面给出定理 1.1 的证明.

定理 1.1 的证明 先考虑 μ 有界的情形. 固定 $t>0$, 在水平 t 上对 f 关于测度 $\mathrm{d}\mu(x)=\nu(x)\mathrm{d}x$ 作 Calderón-Zygmund 分解. 从 $\nu\in RH_\infty$ 容易看出 μ 是倍测度. 由 Calderón-Zygmund 分解可得到二进方体族 $\{Q_j\}_{j=1}^\infty$ 满足

$$t<\frac{1}{\nu(Q_j)}\int_{Q_j}f(y)\nu(y)\mathrm{d}y\leqslant Ct.$$

记 $\Omega=\bigcup_{j=1}^{\infty}Q_j$, 则 $f(x)\leqslant t$ a.e. $x\in\mathbb{R}^n\backslash\Omega$. 同样地, 根据 Calderón-Zygmund 分解, $f=g+h$, 其中

$$g(x)=\begin{cases}f(x), & x\in\mathbb{R}^n\backslash\Omega;\\ \dfrac{1}{\nu(Q_j)}\displaystyle\int_{Q_j}f(y)\nu(y)\mathrm{d}y, & x\in Q_j,\end{cases}$$

$h(x)=\sum_{j=0}^{\infty}h_j(x)$,

$$h_j(x)=\Big(f(x)-\frac{1}{\nu(Q_j)}\int_{Q_j}f(y)\nu(y)\mathrm{d}y\Big)\chi_{Q_j}(x).$$

从而 $g(x)\leqslant ct$ a.e. $x\in\mathbb{R}^n$, h_j 支在 Q_j 上且

$$\int_{Q_j}h_j(x)\nu(x)\mathrm{d}x=0.$$

记 $Q_j^*=4\sqrt{n}Q_j$, $\Omega^*=\bigcup_j Q_j^*$. 则

$$\begin{aligned}\mu\nu\Big(\Big\{x\in\mathbb{R}^n:\frac{g_\lambda^*(f\nu)(x)}{\nu(x)}>t\Big\}\Big)&\leqslant\mu\nu\Big(\Big\{x\in\mathbb{R}^n\backslash\Omega^*:\frac{g_\lambda^*(g\nu)(x)}{\nu(x)}>\frac{t}{2}\Big\}\Big)+\mu\nu(\Omega^*)\\&\quad+\mu\nu\Big(\Big\{x\in\mathbb{R}^n\backslash\Omega^*:\frac{g_\lambda^*(h\nu)(x)}{\nu(x)}>\frac{t}{2}\Big\}\Big)\\&=:\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}\tag{3.1}$$

先处理 I. 记 $\mu^*=\mu\chi_{\mathbb{R}^n\backslash\Omega^*}$, 对 $p'\in(1,2]$, 由 Chebyshev 不等式和 g_λ^* 的强型 Fefferman-Stein 不等式 (见 [23]), 可得

$$\begin{aligned}\mathrm{I}&\lesssim\frac{1}{t^{p'}}\int_{\mathbb{R}^n}g_\lambda^*(g\nu)(x)^{p'}\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x\\&\lesssim\frac{1}{t^{p'}}\int_{\mathbb{R}^n}|g(x)\nu(x)|^{p'}M(\mu^*\nu^{1-p'})(x)\mathrm{d}x.\end{aligned}\tag{3.2}$$

为此，需处理 $M(\mu^*\nu^{1-p'})$. 固定 x 且方体 $Q\ni x$, 则

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x\\&=\frac{1}{|Q|}\int_{Q\cap\{\nu(x)^{1-p'}\leqslant e\}}\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x+\frac{1}{|Q|}\int_{Q\cap\{\nu(x)^{1-p'}\geqslant e\}}\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x.\end{aligned}$$

根据引理 2.1 可得 $\nu^{1-q'}\in A_1$, 于是

$$\begin{aligned}\frac{1}{|Q|}\int_{Q\cap\{\nu(x)^{1-p'}\leqslant e\}}\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x&\leqslant\frac{e}{|Q|}\int_Q\mu^*(x)\mathrm{d}x\\&=\frac{e}{\nu^{1-q'}(Q)}\int_Q\mu^*(x)\nu(x)^{1-q'}\nu(x)^{q'-1}\mathrm{d}x\frac{\nu^{1-q'}(Q)}{|Q|}\\&\lesssim\frac{1}{\nu^{1-q'}(Q)}\int_Q\mu^*(x)\nu(x)^{1-q'}\mathrm{d}x\sup_Q\nu^{q'-1}\inf_Q\nu^{1-q'}\\&\leqslant M_{\nu^{1-q'}}\mu^*(x).\end{aligned}$$

注意到

$$\nu^{p'}\max\{1,\nu^{1-q'}\}=\Psi(\nu).$$

取 $\tau=\frac{q'-1}{p'-1}-1$, 对于 $x\in Q$, 有

$$\begin{aligned}\frac{1}{|Q|}\int_{Q\cap\{\nu(x)^{1-p'}\geqslant e\}}\mu^*(x)\nu(x)^{1-p'}\mathrm{d}x&\leqslant\frac{1}{|Q|}\int_{Q\cap\{\nu(x)^{1-p'}\geqslant e\}}\mu^*(x)\nu(x)^{(1-p')(1+\tau)}\mathrm{d}x\\&\leqslant\frac{1}{|Q|}\int_Q\mu^*(x)\nu(x)^{1-q'}\mathrm{d}x\\&=\frac{1}{\nu^{1-q'}(Q)}\int_Q\mu^*(x)\nu(x)^{1-q'}\mathrm{d}x\frac{\nu^{1-q'}(Q)}{|Q|}\\&\lesssim M_{\nu^{1-q'}}\mu^*(x)\max\{1,\nu(x)^{1-q'}\}\\&\leqslant M_{\nu^{1-q'}}\mu^*(x)\nu(x)^{-p'}\Psi(\nu)(x).\end{aligned}$$

所以

$$M(\mu^*\nu^{1-p'})\lesssim M_{\nu^{1-q'}}(\mu^*)(x)\nu^{-p'}(x)\Psi(\nu)(x).$$

将其代入 (3.2), 可得

$$\begin{aligned}\mathrm{I}&\lesssim\frac{1}{t^{p'}}\int_{\mathbb{R}^n}|g(x)|^{p'}M_{\nu^{1-q'}}(\mu^*)(x)\Psi(\nu)(x)\mathrm{d}x\\&=\frac{1}{t^{p'}}\int_{\mathbb{R}^n}|g(x)||g(x)|^{p'-1}M_{\nu^{1-q'}}(\mu^*)(x)\Psi(\nu)(x)\mathrm{d}x\\&\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|g(x)|M_{\nu^{1-q'}}(\mu^*)(x)\Psi(\nu)(x)\mathrm{d}x\\&\leqslant\frac{1}{t}\int_{\mathbb{R}^n\setminus\Omega}|f(x)|M_{\nu^{1-q'}}(\mu)(x)\Psi(\nu)(x)\mathrm{d}x\\&\quad+\frac{1}{t}\int_\Omega\Big|\frac{1}{\nu(Q_j)}\int_{Q_j}f(y)\nu(y)\mathrm{d}y\Big|M_{\nu^{1-q'}}(\mu^*)(x)\Psi(\nu)(x)\mathrm{d}x.\end{aligned}\tag{3.3}$$

记 $\mu_j^*=\mu\chi_{\mathbb{R}^n\setminus4\sqrt{n}Q_j}$. 注意到 μ 有界, 故由引理 3.1 和 $\nu\in RH_\infty$, 可得

$$\begin{aligned}&\frac{1}{t}\int_\Omega\Big|\frac{1}{\nu(Q_j)}\int_{Q_j}f(y)\nu(y)\mathrm{d}y\Big|M_{\nu^{1-q'}}(\mu^*)(x)\Psi(\nu)(x)\mathrm{d}x\\\lesssim&\frac{1}{t}\sum_j\Big(\inf_{Q_j}M_{\nu^{1-q'}}(\mu_j^*)\Big)\frac{(\Psi\circ\nu)(Q_j)}{\nu(Q_j)}\int_{Q_j}|f(x)|\nu(x)\mathrm{d}x\\\leqslant&\frac{1}{t}[\nu]_{RH_\infty}\sum_j\Big(\inf_{Q_j}M_{\nu^{1-q'}}(\mu_j^*)\Big)\frac{(\Psi\circ\nu)(Q_j)}{|Q_j|}\int_{Q_j}|f(x)|\mathrm{d}x\\\leqslant&\frac{1}{t}[\nu]_{RH_\infty}\sum_j\int_{Q_j}|f(x)|M_{\nu^{1-q'}}(\mu)(x)M(\Psi(\nu))(x)\mathrm{d}x\\\lesssim&\frac{1}{t}\int_\Omega|f(x)|M_{\nu^{1-q'}}(\mu)(x)M(\Psi(\nu))(x)\mathrm{d}x.\end{aligned}$$

结合 (3.3), 可得

$$\mathrm{I}\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_{\nu^{1-q'}}(\mu)(x)M(\Psi(\nu))(x)\mathrm{d}x. \tag{3.4}$$

再处理 II. 由引理 2.1 及 ν 的倍测度性质, 可得

$$\begin{aligned}\mu\nu(Q_j^*)&=\nu^{1-q'}(Q_j^*)\frac{1}{\nu^{1-q'}(Q_j^*)}\int_{Q_j^*}\mu(x)\nu(x)^{1-q'}\nu(x)^{q'}\mathrm{d}x\\&\leqslant\nu^{1-q'}(Q_j^*)\frac{1}{\nu^{1-q'}(Q_j^*)}\int_{Q_j^*}\mu(x)\nu(x)^{1-q'}\mathrm{d}x\sup_{Q_j^*}\nu(x)^{q'}\\&\leqslant[\nu^{q'}]_{RH_\infty}\frac{\nu^{1-q'}(Q_j^*)}{|Q_j^*|}\nu^{q'}(Q_j^*)\inf_{Q_j^*}M_{\nu^{1-q'}}(\mu)(x)\\&\leqslant[\nu^{q'}]_{RH_\infty}[\nu^{1-q'}]_{A_1}\nu^{q'}(Q_j^*)\inf_{Q_j^*}\nu(x)^{1-q'}\inf_{Q_j^*}M_{\nu^{1-q'}}(\mu)(x)\\&\lesssim\nu(Q_j)\inf_{Q_j}M_{\nu^{1-q'}}(\mu)(x)\\&\leqslant\frac{1}{t}\int_{Q_j}|f(x)|\nu(x)M_{\nu^{1-q'}}(\mu)(x)\mathrm{d}x\\&\leqslant\frac{1}{t}\int_{Q_j}|f(x)|M_{\nu^{1-q'}}(\mu)(x)M(\Psi(\nu))(x)\mathrm{d}x.\end{aligned}$$

故

$$\mathrm{II}\leqslant\sum_j\mu\nu(Q_j^*)\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(x)|M_{\nu^{1-q'}}(\mu)(x)M(\Psi(\nu))(x)\mathrm{d}x. \tag{3.5}$$

最后处理 III. 显然

$$\begin{aligned}\mathrm{III}&\leqslant\mu\nu\Big(\Big\{x\in\mathbb{R}^n\backslash\Omega^*:\sum_j\frac{g_\lambda^*(h_j\nu)(x)}{\nu(x)}>\frac{t}{2}\Big\}\Big)\\&\lesssim\frac{1}{t}\int_{\mathbb{R}^n\backslash\Omega^*}\sum_j g_\lambda^*(h_j\nu)(x)\mu(x)\mathrm{d}x.\end{aligned} \tag{3.6}$$

记 Q_j 的中心为 x_{Q_j}, 边长为 l_{Q_j}. 利用 h_j 的消失性和 Minkowski 不等式, 有

$$\begin{aligned}&g_\lambda^*(h_j\nu)(x)\\&=\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\Big|\int_{\mathbb{R}^n}h_j(y)\nu(y)t^{-n}\Big[\psi\Big(\frac{z}{t},\frac{y}{t}\Big)-\psi\Big(\frac{z}{t},\frac{x_{Q_j}}{t}\Big)\Big]\mathrm{d}y\Big|^2\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}\\&\leqslant\int_{\mathbb{R}^n}\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\Big|t^{-n}\psi\Big(\frac{z}{t},\frac{y}{t}\Big)-t^{-n}\psi\Big(\frac{z}{t},\frac{x_{Q_j}}{t}\Big)\Big|^2\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}\\&\quad\times|h_j(y)|\nu(y)\mathrm{d}y\\&\leqslant J_j(x)+J_j'(x)+K_j(x),\end{aligned} \tag{3.7}$$

其中

$$\begin{aligned}J_j(x):=\int_{\mathbb{R}^n}\Big[\int_0^\infty\int_{|z-x_{Q_j}|\leqslant2|x_{Q_j}-y|}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}t^{-2n}\Big|\psi\Big(\frac{z}{t},\frac{y}{t}\Big)\Big|^2\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big]^{1/2}\\\times|h_j(y)|\nu(y)\mathrm{d}y,\end{aligned}$$

$$J_j'(x) := \int_{\mathbb{R}^n} \Big[\int_0^\infty \int_{|z-x_{Q_j}|\leqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} t^{-2n} \Big|\psi\Big(\frac{z}{t}, \frac{x_{Q_j}}{t}\Big)\Big|^2 \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big]^{1/2}$$
$$\times |h_j(y)|\nu(y)\mathrm{d}y,$$
$$K_j(x) := \int_{\mathbb{R}^n} \Big[\int_0^\infty \int_{|z-x_{Q_j}|\geqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} t^{-2n} \Big|\psi\Big(\frac{z}{t}, \frac{y}{t}\Big)$$
$$- \psi\Big(\frac{z}{t}, \frac{x_{Q_j}}{t}\Big)\Big|^2 \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big]^{1/2} |h_j(y)|\nu(y)\mathrm{d}y.$$

先处理 J_j. 因 $\gamma_0 < \frac{(\lambda-2)n}{2}$, 可选取 $\gamma > 0$ 充分小, 使得 $\gamma < \min\{\delta, \ \frac{n}{2}\}$, $2\gamma_0 + 2n + 2\gamma < n\lambda$, 利用 (1.1) 和 $\gamma < \delta$, 可得

$$\int_0^\infty \int_{|z-x_{Q_j}|\leqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} t^{-2n} \Big|\psi\Big(\frac{z}{t}, \frac{y}{t}\Big)\Big|^2 \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}$$
$$\lesssim \int_0^\infty \int_{|z-x_{Q_j}|\leqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \Big(\frac{t}{t+|y-z|}\Big)^{2n+2\delta} \frac{\mathrm{d}z\mathrm{d}t}{t^{3n+1}}$$
$$\leqslant \int_0^\infty \int_{|z-x_{Q_j}|\leqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma} \Big(\frac{t}{t+|y-z|}\Big)^{2n+2\gamma} \frac{\mathrm{d}z\mathrm{d}t}{t^{3n+1}},$$

当 $|z-x_{Q_j}| \leqslant 2|x_{Q_j}-y|$, $x \in \mathbb{R}^n\backslash\Omega^*$, $y \in Q_j$ 时, 有

$$|x-z| \geqslant |x-x_{Q_j}| - |z-x_{Q_j}| \geqslant |x-x_{Q_j}| - 2|y-x_{Q_j}| \geqslant \frac{3|x-x_{Q_j}|}{4},$$
$$|y-z| \leqslant |z-x_{Q_j}| + |y-x_{Q_j}| \leqslant \frac{3\sqrt{n}l_{Q_j}}{2}.$$

则有

$$\int_0^\infty \int_{|z-x_{Q_j}|\leqslant 2|x_{Q_j}-y|} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} t^{-2n} \Big|\psi\Big(\frac{z}{t}, \frac{y}{t}\Big)\Big|^2 \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}$$
$$\lesssim \int_{|y-z|\leqslant 3\sqrt{n}l_{Q_j}/2} \int_0^{l_{Q_j}} \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma} \Big(\frac{t}{|y-z|}\Big)^{n-\gamma} \frac{\mathrm{d}z\mathrm{d}t}{t^{3n+1}}$$
$$+ \int_{|y-z|\leqslant 3\sqrt{n}l_{Q_j}/2} \int_{l_{Q_j}}^\infty \Big(\frac{t}{|x-z|}\Big)^{2n+2\gamma} \frac{\mathrm{d}z\mathrm{d}t}{t^{3n+1}}$$
$$\lesssim \frac{1}{|x-x_{Q_j}|^{2n+2\gamma}} \Big[\int_{|y-z|\leqslant 3\sqrt{n}l_{Q_j}/2} \frac{\mathrm{d}z}{|y-z|^{n-\gamma}} \int_0^{l_{Q_j}} t^{\gamma-1}\mathrm{d}t$$
$$+ \int_{|y-z|\leqslant 3\sqrt{n}l_{Q_j}/2} \mathrm{d}z \int_{l_{Q_j}}^\infty t^{-n+2\gamma-1}\mathrm{d}t\Big]$$
$$\sim \frac{l_{Q_j}^{2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma}}.$$

将上式代入 J_i, 根据 μ 有界, $\nu^{1-q'} \in A_1$, $M_{\nu^{1-q'}}(\mu) < \infty$, 利用引理 3.2, 可得

$$\frac{1}{t}\int_{\mathbb{R}^n\backslash\Omega^*} \sum_j J_j(x)\mu(x)\mathrm{d}x$$
$$\lesssim \frac{1}{t}\int_{\mathbb{R}^n\backslash\Omega^*} \sum_j \int_{Q_j} \frac{l_{Q_j}^{\gamma}}{|x-x_{Q_j}|^{n+\gamma}} |h_j(y)|\nu(y)\mathrm{d}y\mu(x)\mathrm{d}x$$

$$
\begin{aligned}
&\leqslant \frac{1}{t}\sum_j \int_{Q_j} |h_j(y)|\nu(y)\mathrm{d}y \sum_{k=1}^{\infty} \int_{2^{k-1}2\sqrt{n}l_{Q_j}<|x-x_{Q_j}|\leqslant 2^k 2\sqrt{n}l_{Q_j}} \frac{l_{Q_j}^{\gamma}}{|x-x_{Q_j}|^{n+\gamma}}\mu_j^*(x)\mathrm{d}x\\
&\leqslant \frac{1}{t}\sum_j \int_{Q_j} |h_j(y)|\nu(y)\mathrm{d}y \sum_{k=1}^{\infty} \int_{B(x_{Q_j},2^k 2\sqrt{n}l_{Q_j})} \frac{l_{Q_j}^{\gamma}}{(2^{k-1}2\sqrt{n}l_{Q_j})^{n+\gamma}}\mu_j^*(x)\mathrm{d}x\\
&\lesssim \frac{1}{t}\sum_j \int_{Q_j} |h_j(y)|\nu(y)\mathrm{d}y \inf_{Q_j} M(\mu_j^*)\\
&\leqslant \frac{1}{t}\sum_j \int_{Q_j} |h_j(y)|\nu(y) M(\mu_j^*)(y)\mathrm{d}y\\
&\lesssim \frac{1}{t}\int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y)\mathrm{d}y. \qquad (3.8)
\end{aligned}
$$

类似于 (3.8) 的处理, 可得

$$
\frac{1}{t}\int_{\mathbb{R}^n\setminus\Omega^*} \sum_j J_j'(x)\mu(x)\mathrm{d}x \lesssim \frac{1}{t}\int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y)\mathrm{d}y. \qquad (3.9)
$$

再处理 K_j. 将 $S:=\{(z,t)\in\mathbb{R}_+^{n+1}: |z-x_{Q_j}|>2|x_{Q_j}-y|\}$ 分解成以下部分:

$$
\begin{aligned}
S_1 &:= S\cap\{(z,t)\in\mathbb{R}_+^{n+1}: \max\{|z-x|,|z-y|\}\leqslant t\},\\
S_2 &:= S\cap\{(z,t)\in\mathbb{R}_+^{n+1}: t<|z-y|\},\\
S_3 &:= S\cap\{(z,t)\in\mathbb{R}_+^{n+1}: t<|z-x|\}.
\end{aligned}
$$

则由 (1.3), 可得

$$
\begin{aligned}
K_j(x) &\lesssim \int_{\mathbb{R}^n}\Big(\int_{S_1}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}l_{Q_j}^{2\gamma_0}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^n}\Big(\int_{S_2}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}l_{Q_j}^{2\gamma_0}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^n}\Big(\int_{S_3}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}l_{Q_j}^{2\gamma_0}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&=: K_{j1}(x)+K_{j2}(x)+K_{j3}(x). \qquad (3.10)
\end{aligned}
$$

为估计 K_{j1}, 将 S_1 拆分成以下集合:

$$
\begin{aligned}
S_{11} &:= S_1\cap\{(z,t)\in\mathbb{R}_+^{n+1}: |z-x_{Q_j}|\leqslant\sqrt{n}l_{Q_j}\},\\
S_{12} &:= S_1\cap\{(z,t)\in\mathbb{R}_+^{n+1}: |z-x_{Q_j}|>\sqrt{n}l_{Q_j},\ t<|z-x_{Q_j}+2\sqrt{n}l_{Q_j}|\},\\
S_{13} &:= S_1\cap\{(z,t)\in\mathbb{R}_+^{n+1}: |z-x_{Q_j}|>\sqrt{n}l_{Q_j},\ t\geqslant|z-x_{Q_j}+2\sqrt{n}l_{Q_j}|\}.
\end{aligned}
$$

当 $(z,t)\in S_{11}$, $x\in\mathbb{R}^n\setminus\Omega^*$ 时,

$$
t\geqslant|x-z|\geqslant|x-x_{Q_j}|-|z-x_{Q_j}|\geqslant\frac{|x-x_{Q_j}|}{2}.
$$

故

$$
l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{11}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y
$$

$$
\begin{aligned}
&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|x-x_{Q_j}|/2}^{\infty}\int_{|z-y|\leqslant t}\frac{t^{2\delta}}{t^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y. \qquad (3.11)
\end{aligned}
$$

当 $(z,t)\in S_{12}$, $y\in Q_j$ 时,

$$
\begin{aligned}
&t<|z-x_{Q_j}|+2\sqrt{n}l_{Q_j}\leqslant|z-x_{Q_j}|+2|z-x_{Q_j}|=3|z-x_{Q_j}|,\\
&t\geqslant|y-z|\geqslant|z-x_{Q_j}|-|y-x_{Q_j}|\geqslant|z-x_{Q_j}|-\frac{\sqrt{n}l_{Q_j}}{2}\geqslant\frac{|z-x_{Q_j}|}{2}.
\end{aligned}
$$

由此推出

$$
|x-x_{Q_j}|\leqslant|x-z|+|z-x_{Q_j}|\leqslant t+|z-x_{Q_j}|\leqslant 4|z-x_{Q_j}|.
$$

因此

$$
\begin{aligned}
&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{12}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|/2}^{3|z-x_{Q_j}|}\int_{|z-x_{Q_j}|\geqslant|x-x_{Q_j}|/4}\frac{t^{2\delta}}{|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|\geqslant|x-x_{Q_j}|/4}|z-x_{Q_j}|^{-2n-2\gamma_0}\int_{|z-x_{Q_j}|/2}^{3|z-x_{Q_j}|}\frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y. \qquad (3.12)
\end{aligned}
$$

当 $(z,t)\in S_{13}$ 时,

$$
2t\geqslant|x-z|+|z-x_{Q_j}|\geqslant|x-x_{Q_j}|,
$$

则

$$
\begin{aligned}
&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{13}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|x-x_{Q_j}|/2}^{\infty}\int_{|z-x_{Q_j}|<t}t^{-2n-2\gamma_0}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y. \qquad (3.13)
\end{aligned}
$$

结合 (3.11)–(3.13), 类似于 (3.8) 的估计, 可得

$$
\begin{aligned}
&\frac{1}{t}\int_{\mathbb{R}^n\setminus\Omega^*}\sum_j K_{j1}(x)\mu(x)\mathrm{d}x\\
&\lesssim\frac{1}{t}\int_{\mathbb{R}^n\setminus\Omega^*}\sum_j\int_{Q_j}\frac{l_{Q_j}^{\gamma_0}}{|x-x_{Q_j}|^{n+\gamma_0}}|h_j(y)|\nu(y)\mathrm{d}y\mu(x)\mathrm{d}x\\
&\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(y)|M_{\nu^{1-q'}}(\mu)(y)M(\Psi(\nu))(y)\mathrm{d}y. \qquad (3.14)
\end{aligned}
$$

再估计 K_{j2}. 注意到 $S_2\subset S_2'\cup S_3$, 其中

$$
S_2':=S\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x|\leqslant t<|z-y|\}.
$$

则

$$\begin{aligned}K_{j2}(x) &\leqslant \int_{\mathbb{R}^n}\Big(\int_{S_2'}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}l_{Q_j}^{2\gamma_0}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&\quad+\int_{\mathbb{R}^n}\Big(\int_{S_3}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}l_{Q_j}^{2\gamma_0}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&=: K_{j2}'(x)+K_{j3}(x).\end{aligned}\tag{3.15}$$

将 S_2' 分解为

$$S_{21}:=S_2'\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|>2|x-x_{Q_j}|\},$$
$$S_{22}:=S_2'\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|\leqslant 2|x-x_{Q_j}|\}.$$

当 $(z,t)\in S_{21}$, $x\in\mathbb{R}^n\backslash\Omega^*$ 时,

$$t\geqslant|z-x|\geqslant|z-x_{Q_j}|-|x-x_{Q_j}|>\frac{|z-x_{Q_j}|}{2},$$
$$t<|z-y|\leqslant|z-x_{Q_j}|+|y-x_{Q_j}|\leqslant|z-x_{Q_j}|+|x-x_{Q_j}|<\frac{3|z-x_{Q_j}|}{2}.$$

于是

$$\begin{aligned}&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{21}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|>2|x-x_{Q_j}|}\int_{|z-x_{Q_j}|/2}^{3|z-x_{Q-j}|/2}t^{-2n-2\gamma_0}\frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.\end{aligned}\tag{3.16}$$

当 $(z,t)\in S_{22}$, $y\in Q_j$, $x\in\mathbb{R}^n\backslash\Omega^*$ 时, 有

$$t<|y-z|\leqslant|y-x_{Q_j}|+|z-x_{Q_j}|<|x-x_{Q_j}|+2|x-x_{Q_j}|=3|x-x_{Q_j}|,$$
$$t+|z-x_{Q_j}|\geqslant|z-x|+|z-x_{Q_j}|\geqslant|x-x_{Q_j}|.$$

故

$$\begin{aligned}&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{22}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_0^{3|x-x_{Q_j}|}\int_{|z-x|\leqslant t}\frac{t^{2\delta}}{|x-x_{Q_j}|^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\&\sim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.\end{aligned}\tag{3.17}$$

结合 (3.16)–(3.17), 类似于 (3.8) 的估计, 可得

$$\frac{1}{t}\int_{\mathbb{R}^n\backslash\Omega^*}\sum_j K_{j2}'(x)\mu(x)\mathrm{d}x\lesssim\frac{1}{t}\int_{\mathbb{R}^n}|f(y)|M_{\nu^{1-q'}}(\mu)(y)M(\Psi(\nu))(y)\mathrm{d}y.\tag{3.18}$$

由 (3.15), 为进一步估计 K_{j2}, 接下来需要估计 K_{j3}. 将 S_3 分解为

$$S_3':=S_3\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|<\sqrt{n}l_{Q_j}\},$$

$$S_3'' := S_3 \cap \{(z,t) \in \mathbb{R}_+^{n+1} : |z - x_{Q_j}| \geqslant \sqrt{n} l_{Q_j}\}.$$

对于 S_3', 进一步将其分解为

$$S_{31}' := S_3' \cap \{(z,t) \in \mathbb{R}_+^{n+1} : t > |z - x_{Q_j}|\},$$
$$S_{32}' := S_3' \cap \{(z,t) \in \mathbb{R}_+^{n+1} : t \leqslant |z - x_{Q_j}|\}.$$

当 $(z,t) \in S_{31}'$, $x \in \mathbb{R}^n \backslash \Omega^*$ 时,

$$2|x-z| > |x-z| + t > |x-z| + |z - x_{Q_j}| \geqslant |x - x_{Q_j}|,$$
$$t < |x-z| \leqslant |z - x_{Q_j}| + |x - x_{Q_j}| < \frac{3|x - x_{Q_j}|}{2}.$$

则对于 (3.8) 中的 γ, 有

$$\begin{aligned}
&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \\
&\leqslant \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma_0+2\gamma} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \\
&\lesssim \frac{t^{2n+2\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}} \frac{t^{2\delta}}{t^{2n+2\delta+2\gamma_0}} \frac{t^{n-\gamma}}{|z-x_{Q_j}|^{n-\gamma}} \\
&\leqslant \frac{t^{n+\gamma}}{|x-x_{Q_j}|^{2n+2\delta+2\gamma_0} |z-x_{Q_j}|^{n-\gamma}}.
\end{aligned}$$

故

$$\begin{aligned}
&l_{Q_j}^{\gamma_0} \int_{\mathbb{R}^n} \Big(\int_{S_{31}'} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}} \Big)^{1/2} |h_j(y)| \nu(y) \mathrm{d}y \\
&\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \Big(\int_{|z-x_{Q_j}|<\sqrt{n} l_{Q_j}} \int_0^{3|x-x_{Q_j}|/2} \frac{t^{n+\gamma}}{|x-x_{Q_j}|^{2n+2\delta+2\gamma_0} |z-x_{Q_j}|^{n-\gamma}} \frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}} \Big)^{1/2} \\
&\quad \times |h_j(y)| \nu(y) \mathrm{d}y \\
&\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}} \nu(y) \mathrm{d}y.
\end{aligned} \tag{3.19}$$

当 $(z,t) \in S_{32}'$, $x \in \mathbb{R}^n \backslash \Omega^*$ 时,

$$t \leqslant |z - x_{Q_j}| < \sqrt{n} l_{Q_j} \leqslant \frac{|x - x_{Q_j}|}{2},$$
$$|x-z| \geqslant |x - x_{Q_j}| - |z - x_{Q_j}| > \frac{|x - x_{Q_j}|}{2}.$$

则

$$\begin{aligned}
&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \\
&\leqslant \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma_0+2\gamma} \frac{t^{2\delta}}{t^{n+\delta+\gamma_0} |z-x_{Q_j}|^{n+\delta+\gamma_0}} \\
&\lesssim \frac{t^{n+\delta+\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma} |z-x_{Q_j}|^{n+\delta+\gamma_0}}.
\end{aligned}$$

将上述估计代入下式, 可得

$$
\begin{aligned}
& l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{32}'}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y \\
\lesssim{}& l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_0^{|x-x_{Q_j}|/2}\int_{|z-x_{Q_j}|\geqslant t}\frac{t^{n+\delta+\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}|z-x_{Q_j}|^{n+\delta+\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2} \\
& \times|h_j(y)|\nu(y)\mathrm{d}y \\
\lesssim{}& l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.
\end{aligned}\tag{3.20}
$$

对于 S_3'', 进一步将其分解为

$$
\begin{aligned}
S_{31}'' &:= S_3''\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}| \\
&\qquad\leqslant t<|z-x_{Q_j}|+2\sqrt{n}l_{Q_j},\ 2|z-x_{Q_j}|>|x-x_{Q_j}|\},\\
S_{32}'' &:= S_3''\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}| \\
&\qquad\leqslant t<|z-x_{Q_j}|+2\sqrt{n}l_{Q_j},\ 2|z-x_{Q_j}|\leqslant|x-x_{Q_j}|\},\\
S_{33}'' &:= S_3''\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|+2\sqrt{n}l_{Q_j}\leqslant t,\ 2|z-x_{Q_j}|>|x-x_{Q_j}|\},\\
S_{34}'' &:= S_3''\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|+2\sqrt{n}l_{Q_j}\leqslant t,\ 2|z-x_{Q_j}|\leqslant|x-x_{Q_j}|\},\\
S_{35}'' &:= S_3''\cap\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|>t,\ |x-z|>2|x-x_{Q_j}|\},\\
S_{36}'' &:= S_3''\cap\Big\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|>t,\ \frac{|x-x_{Q_j}|}{2}<|x-z|\leqslant 2|x-x_{Q_j}|\Big\},\\
S_{37}'' &:= S_3''\cap\Big\{(z,t)\in\mathbb{R}_+^{n+1}:|z-x_{Q_j}|>t,\ |x-z|\leqslant\frac{|x-x_{Q_j}|}{2}\Big\}.
\end{aligned}
$$

对任意 $(z,t)\in S_{31}''$, $x\in\mathbb{R}^n\backslash\Omega^*$, 有

$$
|z-x_{Q_j}|\leqslant t<|z-x_{Q_j}|+2\sqrt{n}l_{Q_j}\leqslant|z-x_{Q_j}|+|x-x_{Q_j}|<3|z-x_{Q_j}|.
$$

又由

$$
\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\leqslant t^{-2n-2\gamma_0},
$$

所以

$$
\begin{aligned}
& l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{31}''}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y \\
\leqslant{}& l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|>|x-x_{Q_j}|/2}\int_{|z-x_{Q_j}|}^{3|z-x_{Q_j}|}t^{-2n-2\gamma_0}\frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y \\
\lesssim{}& l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.
\end{aligned}\tag{3.21}
$$

对任意 $(z,t)\in S_{32}''$, $x\in\mathbb{R}^n\backslash\Omega^*$,

$$
t<|z-x_{Q_j}|+2\sqrt{n}l_{Q_j}\leqslant|z-x_{Q_j}|+|x-x_{Q_j}|\leqslant\frac{3|x-x_{Q_j}|}{2},
$$

$$|x-z| \geqslant |x-x_{Q_j}|-|z-x_{Q_j}| \geqslant \frac{|x-x_{Q_j}|}{2}.$$

结合

$$\begin{aligned}&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\\ \leqslant& \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma_0+2\gamma}\frac{t^{2\delta}}{t^{2n+2\delta+2\gamma_0}}\\ \lesssim& \frac{t^{2n+2\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}}\frac{t^{2\delta}}{t^{2n+2\delta+2\gamma_0}},\end{aligned}$$

可得

$$\begin{aligned}&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S''_{32}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\ \lesssim& l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_0^{3|x-x_{Q_j}|/2}\int_{|z-x_{Q_j}|\leqslant t}\frac{t^{2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\ \lesssim& l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.\end{aligned}\tag{3.22}$$

对任意 $(z,t)\in S''_{33}$. 显然 $\min\{t,|z-x_{Q_j}|\}>\frac{|x-x_{Q_j}|}{2}$. 利用

$$\begin{aligned}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} &\leqslant \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2\delta}}\frac{1}{(t+|z-x_{Q_j}|)^{2n+2\gamma_0}}\\ &\leqslant \frac{1}{|z-x_{Q_j}|^{2n+2\gamma_0}},\end{aligned}$$

可得

$$\begin{aligned}&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S''_{33}}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\ \leqslant& l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|>|x-x_{Q_j}|/2}\int_{|x-x_{Q_j}|/2}^{\infty}|z-x_{Q_j}|^{-2n-2\gamma_0}\frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\ \lesssim& l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y.\end{aligned}\tag{3.23}$$

对任意 $(z,t)\in S''_{34}$,

$$t<|z-x|\leqslant|z-x_{Q_j}|+|x-x_{Q_j}|\leqslant\frac{3|x-x_{Q_j}|}{2}.$$

又

$$\begin{aligned}&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\\ \leqslant& \Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma_0+2\gamma}t^{-2n-2\gamma_0}\\ \lesssim& \frac{t^{2n+2\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}}t^{-2n-2\gamma_0}.\end{aligned}$$

故

$$\begin{aligned}
&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{34}''}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\lesssim l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_0^{3|x-x_{Q_j}|/2}\int_{|z-x_{Q_j}|<t}\frac{t^{2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\lesssim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y. \qquad (3.24)
\end{aligned}$$

对任意 $(z,t)\in S_{35}''$,

$$|z-x_{Q_j}|\geqslant|z-x|-|x-x_{Q_j}|>|x-x_{Q_j}|.$$

另外

$$\begin{aligned}
&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\\
&\leqslant\Big(\frac{t}{t+|x-z|}\Big)^{2n+2\gamma_0-2\gamma}\frac{t^{2\delta}}{|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}}\\
&\leqslant\frac{t^{2n+2\gamma_0-2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0-2\gamma}}\frac{t^{2\delta}}{|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}}.
\end{aligned}$$

于是

$$\begin{aligned}
&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{35}''}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\leqslant l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_{|z-x_{Q_j}|>|x-x_{Q_j}|}\int_0^{|z-x_{Q_j}|}\frac{t^{2n+2\gamma_0+2\delta-2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\delta-2\gamma}|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}t\mathrm{d}z}{t^{n+1}}\Big)^{1/2}\\
&\quad\times|h_j(y)|\nu(y)\mathrm{d}y\\
&\lesssim l_{Q_j}^{\gamma_0}\int_{Q_j}\frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}}\nu(y)\mathrm{d}y. \qquad (3.25)
\end{aligned}$$

对任意 $(z,t)\in S_{36}''$, 有

$$t<|z-x_{Q_j}|\leqslant|z-x|+|x-x_{Q_j}|\leqslant3|x-x_{Q_j}|,$$

且

$$\begin{aligned}
&\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\\
&\lesssim\frac{t^{2n+2\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}}\frac{t^{2\delta}}{|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}},
\end{aligned}$$

故

$$\begin{aligned}
&l_{Q_j}^{\gamma_0}\int_{\mathbb{R}^n}\Big(\int_{S_{36}''}\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda}\frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}|h_j(y)|\nu(y)\mathrm{d}y\\
&\lesssim l_{Q_j}^{\gamma_0}\int_{Q_j}\Big(\int_0^{3|x-x_{Q_j}|}\int_{|z-x_{Q_j}|>t}\frac{t^{2n+2\delta+2\gamma_0+2\gamma}}{|x-x_{Q_j}|^{2n+2\gamma_0+2\gamma}|z-x_{Q_j}|^{2n+2\delta+2\gamma_0}}\frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}}\Big)^{1/2}\\
&\quad\times|h_j(y)|\nu(y)\mathrm{d}y
\end{aligned}$$

$$\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}} \nu(y) \mathrm{d}y. \tag{3.26}$$

最后, 对任意 $(z,t) \in S_{37}''$, 有

$$t < |z - x_{Q_j}| \leqslant |z-x| + |x - x_{Q_j}| \leqslant \frac{3|x-x_{Q_j}|}{2},$$

利用 $\frac{|x-x_{Q_j}|}{2} \leqslant |z-x_{Q_j}| \leqslant \frac{3|x-x_{Q_j}|}{2}$,

$$\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \lesssim \Big(\frac{t}{|x-z|}\Big)^{2n+2\gamma_0} \frac{t^{2\delta}}{|x-x_{Q_j}|^{2n+2\delta+2\gamma_0}},$$

可得

$$\begin{aligned}
&l_{Q_j}^{\gamma_0} \int_{\mathbb{R}^n} \Big(\int_{S_{37}''} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}} \Big)^{1/2} |h_j(y)|\nu(y)\mathrm{d}y \\
&\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \Big(\int_0^{3|x-x_{Q_j}|/2} \int_{|z-x|>t} \Big(\frac{t}{|x-z|}\Big)^{2n+2\gamma_0} \frac{t^{2\delta}}{|x-x_{Q_j}|^{2n+2\delta+2\gamma_0}} \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}} \Big)^{1/2} \\
&\quad \times |h_j(y)|\nu(y)\mathrm{d}y \\
&\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}} \nu(y) \mathrm{d}y.
\end{aligned} \tag{3.27}$$

结合 (3.19)–(3.27), 可得

$$\begin{aligned}
K_{j3} &:= l_{Q_j}^{\gamma_0} \int_{\mathbb{R}^n} \Big(\int_{S_3} \Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} \frac{t^{2\delta}}{(t+|z-x_{Q_j}|)^{2n+2\delta+2\gamma_0}} \frac{\mathrm{d}z\mathrm{d}t}{t^{n+1}} \Big)^{1/2} |h_j(y)|\nu(y)\mathrm{d}y \\
&\lesssim l_{Q_j}^{\gamma_0} \int_{Q_j} \frac{|h_j(y)|}{|x-x_{Q_j}|^{n+\gamma_0}} \nu(y) \mathrm{d}y.
\end{aligned}$$

类似于 (3.8) 的估计, 可得

$$\frac{1}{t} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j K_{j3}(x) \mu(x) \mathrm{d}x \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y) \mathrm{d}y. \tag{3.28}$$

由 (3.18), (3.15) 及 (3.28), 可得

$$\frac{1}{t} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j K_{j2}(x) \mu(x) \mathrm{d}x \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y) \mathrm{d}y. \tag{3.29}$$

于是由 (3.10), (3.14) 和 (3.28)–(3.29), 可得

$$\frac{1}{t} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j K_j(x) \mu(x) \mathrm{d}x \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y) \mathrm{d}y.$$

结合 (3.6)–(3.9), 可得

$$\begin{aligned}
\mathrm{III} &\lesssim \frac{1}{t} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j (K_j(x) + J_j(x) + J_j'(x)) \mu(x) \mathrm{d}x \\
&\lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y) \mathrm{d}y.
\end{aligned}$$

结合 (3.4)–(3.5), 在 μ 有界时, 证明了

$$\mu\nu\Big(\Big\{x \in \mathbb{R}^n : \frac{g_\lambda^*(f\nu)(x)}{\nu(x)} > t\Big\}\Big) \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y) \mathrm{d}y.$$

对于更一般的 μ 及每个 $N \in \mathbb{N}$, 记 $\mu_N(x) = \min\{\mu(x), N\}$, 则

$$\begin{aligned}\mu_N\nu\Big(\Big\{x \in \mathbb{R}^n : \frac{g_\lambda^*(f\nu)(x)}{\nu(x)} > t\Big\}\Big) &\leqslant \frac{C}{t}\int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu_N)(y) M(\Psi(\nu))(y)\mathrm{d}y \\ &\leqslant \frac{C}{t}\int_{\mathbb{R}^n} |f(y)| M_{\nu^{1-q'}}(\mu)(y) M(\Psi(\nu))(y)\mathrm{d}y,\end{aligned}$$

其中 C 是与 N 无关的常数. 观察到 $\mu_N \nearrow \mu$, 对上式使用单调收敛定理即可证明定理 1.1 对于更一般的 μ 成立.

4 Littlewood-Paley 算子的双权弱型不等式

本节证明定理 1.2. 证明中需要用到如下 Hardy-Littlewood 极大算子 M 的双权强型有界性.

引理 4.1 (见 [35]) 设 $1 < p < \infty$, A 为 Young 函数, 满足 $\overline{A} \in B_p$, 若一对权 (μ, ν) 满足

$$\sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p}\|_{A,Q} < \infty,$$

则 $M : L^p(v) \to L^p(u)$.

定理 1.2 的证明 由 (μ, ν) 条件及引理 4.1, 知

$$M : L^{p'}(\mu^{-p'/p}) \to L^{p'}(\nu^{-p'/p}).$$

固定 f, 不妨设 $f \in C_c^\infty(\mathbb{R}^n)$. 记 $\Omega_t = \{x \in \mathbb{R}^n : g_\lambda^*(f)(x) > t\}$, $t > 0$. 由对偶定理及推论 1.1, 存在非负函数 g, 满足 $\|g\|_{L^{p'}(\mathbb{R}^n)} = 1$, 使得

$$\begin{aligned}\mu(\Omega_t)^{1/p} = \|\mu^{1/p}\chi_{\Omega_t}\|_{L^p(\mathbb{R}^n)} &= \int_{\Omega_t} \mu(x)^{1/p} g(x)\mathrm{d}x \\ &\lesssim \frac{1}{t}\int_{\mathbb{R}^n} |f(x)| M(\mu^{1/p} g)(x)\mathrm{d}x \\ &= \frac{1}{t}\int_{\mathbb{R}^n} |f(x)| \nu(x)^{1/p}\nu(x)^{-1/p} M(\mu^{1/p} g)(x)\mathrm{d}x \\ &\leqslant \Big(\frac{1}{t^p}\int_{\mathbb{R}^n} |f(x)|^p \nu(x)\mathrm{d}x\Big)^{1/p}\Big(\int_{\mathbb{R}^n} M(\mu^{1/p} g)(x)^{p'}\nu(x)^{-p'/p}\mathrm{d}x\Big)^{1/p'} \\ &\lesssim \Big(\frac{1}{t^p}\int_{\mathbb{R}^n} |f(x)|^p \nu(x)\mathrm{d}x\Big)^{1/p}\|g\|_{L^{p'}(\mathbb{R}^n)} \\ &= \Big(\frac{1}{t^p}\int_{\mathbb{R}^n} |f(x)|^p \nu(x)\mathrm{d}x\Big)^{1/p}.\end{aligned}$$

致谢 感谢审稿专家的宝贵意见.

参 考 文 献

[1] Xue Q Y, Yan J Q. On multilinear square function and its application to multilinear Littlewood-Paley operators with non-convolution type kernels [J]. *J Math Anal Appl*, 2015, 422(2):1342–1362.

[2] Littlewood J E, Paley R E A C. Theorems on Fourier series and power series [J]. *J London Math Soc*, 1931, 6(3):230–233.

[3] Littlewood J E, Paley R E A C. Theorems on Fourier series and power series (II) [J]. *Proc London Math Soc*, 1936, 42(1):52–89.

[4] Littlewood J E, Paley R E A C. Theorems on Fourier series and power series (III) [J]. *Proc London Math Soc*, 1937, 43(2):105–126.

[5] Stein E M. On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz [J]. *Trans Amer Math Soc*, 1958, 88:430–466.

[6] Stein E M. On some functions of Littlewood-Paley and Zygmund [J]. *Bull Amer Math Soc*, 1961, 67:99–101.

[7] Stein E M. Singular integrals and differentiability properties of functions [M]. Princeton Mathematical Series, No. 30, Princeton, NJ: Princeton University Press, 1970.

[8] Fabes E B, Jerison D, Kenig C. Multilinear Littlewood-Paley estimates with applications [J]. *Proc Nat Acad Sci USA*, 1982, 79(18):5746–5750.

[9] Bui T A, Hormozi M. Weighted bounds for multilinear square functions [J]. *Potential Anal*, 2017, 46(1):135–148.

[10] Chen Y P, Ding Y. L^p boundedness for Littlewood-Paley operators [J]. *J Math Anal Appl*, 2011, 377(2):889–904.

[11] Fabes E B, Jerison D, Kenig C. Multilinear square functions and partial differential equations [J]. *Amer J Math*, 1985, 107(6):1325–1368.

[12] Lerner A K. Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals [J]. *Adv Math*, 2011, 226(5):3912–3926.

[13] Lerner A K. On sharp aperture-weighted estimates for square functions [J]. *J Fourier Anal Appl*, 2014, 20(4):784–800.

[14] Shi S G, Xue Q Y, Yabuta K. On the boundedness of Littlewood-Paley g_λ^* function [J]. *J Math Pures Appl*, 2014, 101(3):394–413.

[15] Fefferman C, Stein E M. Some maximal inequalities [J]. *Amer J Math*, 1971, 93:107–115.

[16] García-Cuerva J, Rubio de Francia J L. Weighted norm inequalities and related topics [M]. North-Holland Mathematics Studies, 116, Notas de Matemática, Mathematical Notes, 104, Amsterdam: North-Holland Publishing Co., 1985.

[17] Córdoba A, Fefferman C. A weighted norm inequality for singular integrals [J]. *Studia Math*, 1976, 57(1):97–101.

[18] Wilson J M. Weighted norm inequalities for the continuous square function [J]. *Trans Amer Math Soc*, 1989, 314(2):661–692.

[19] Pérez C. Weighted norm inequalities for singular integral operators [J]. *J London Math Soc*, 1994, 49(2):296–308.

[20] Cruz-Uribe D, Martell J M, Pérez C. Weights, extrapolation and the theory of Rubio de Francia [M]. Volume 215 of Operator Theory: Advances and Applications, Basel: Birkhäuser/Springer Basel AG, 2011.

[21] Chang S Y A, Wilson J M, Wolff T H. Some weighted norm inequalities concerning Schrödinger operators [J]. *Comment Math Helv*, 1985, 60(2):217–246.

[22] Chanillo S, Wheeden R. Some weighted norm inequalities for the area integral [J]. *Indiana Univ Math J*, 1987, 36(2):277–294.

[23] Cao M M, Hormozi M, Ibañez-Firnkorn G, et al. Weak and strong type estimates for multilinear Littlewood-Paley operators [J]. *J Fourier Anal Appl*, 2021, 27(4): Paper No 62, 42 pp.

[24] Berra F, Carena M, Pradolini G. Mixed inequalities of Fefferman-Stein type for singular integral operators [J]. *J Math Sci (NY)*, 2022, 266(3):461–475.

[25] Sawyer E. A weighted weak type inequality for the maximal function [J]. *Proc Amer Math Soc*, 1985, 93(4):610–614.

[26] Lerner A K, Ombrosi S, Pérez C, et al. New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory [J]. *Adv Math*, 2009, 220(4):1222–1264.

[27] Martín-Reyes F J. Mixed weak type inequalities one-sided operators and ergodic theorems [J]. *Rev Un Mat Argentina*, 2009, 50(2):51–61.

[28] Roure Perdices E. Restricted weak type extrapolation of multi-variable operators and related topics [D]. Universitat De Barcelona, Doctorat de Matemàtiques i Informàtica, 2019.

[29] Caldarelli M, Rivera-Ríos I P. A sparse approach to mixed weak type inequalities [J]. *Math Z*, 2020, 296(1–2):787–812.

[30] Cao M M, Ibañez-Firnkorn G, Rivera-Ríos I P, et al. A class of multilinear bounded oscillation operators on measurable spaces and applications [J]. *Math Ann*, 2024, 388(4):3627–3755.

[31] Li K W, Ombrosi S, Pérez C. Proof of an extension of E. Sawyer's conjecture about weighted mixed weak-type estimates [J]. *Math Ann*, 2019, 374(1–2):907–929.

[32] Wen Y M, Hou X M, Zhang J. A note on fractional type integrals associated to operators [J]. *Collect Math*, 2026, 77(1):133–148.

[33] Wen Y M, Wu H X. Weighted variational inequalities for heat semigroups associated with Schrödinger operators related to critical radius functions [J]. *J Fourier Anal Appl*, 2025, 31(3):Paper No 30, 39 pp.

[34] Cruz-Uribe D, Neugebauer C J. The structure of the reverse Hölder classes [J]. *Trans Amer Math Soc*, 1995, 347(8):2941–2960.

[35] Pérez C. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights [J]. *Proc London Math Soc*, 1995, 71(1):135–157.

Mixed Weak Type Inequalities of Fefferman-Stein Type for Littlewood-Paley Operators and Its Application

CHEN Songqing[1] FANG Quanqing[2] PAN Sihui[3]

[1]Hubei Key Laboratory of Digital Finance Innovation, Hubei University of Economics, Wuhan 430205, China; Interdisciplinary Research Institute in New Finance and Economics, Hubei University of Economics, Wuhan 430205, China. E-mail: chensongqing@hbue.edu.cn

[2]Corresponding author. Fujian Key Laboratory of Financial Information Processing, Putian University, Putian 351100, Fujian, China; Key Laboratory of Applied Mathematics, Putian University, Putian 351100, Fujian, China. E-mail: quanqingfang@163.com

[3]School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, Fujian, China. E-mail: a18922689812@163.com

Abstract In this paper, the authors establish mixed weak type inequalities of Fefferman-Stein type for Littlewood-Paley operators. The results are also new even in the case of one-weight. As applications, the authors also establish the two-weight weak type inequalities for Littlewood-Paley operators.

Keywords Littlewood-Paley operators, Mixed weak type inequalities of Fefferman-Stein type, Two-weight weak type inequalities

2020 MR Subject Classification 42B20, 42B25