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2 维可压缩 Navier-Stokes 方程带真空平坦稀疏波粘性消失极限

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提要　主要研究 2 维等熵可压缩 Navier-Stokes 方程带真空平坦稀疏波的粘性消失极限. 得到了关于粘性系数的一致收敛率. 主要创新和难点在于如何处理由高维稀疏波真空状态所引起的退化. 在真空附近利用适当截断和尺度伸缩技巧来处理真空引起的奇性问题. 最终, 通过引入截断参数和处理 2 维可压缩 Navier-Stokes 方程的粘性误差项, 得到了稀疏波真空附近的无粘性衰减率. 证明过程基于基本能量法.

关键词　可压缩 Navier-Stokes 方程, 平坦稀疏波, 真空, 粘性消失极限
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1　引　言

本文研究 2 维等熵可压缩 Navier-Stokes (NS) 方程

$$\begin{cases}\rho_t+\operatorname{div}(\rho\boldsymbol{u})=0,\\(\rho\boldsymbol{u})_t+\operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p(\rho)=\mu_1\Delta\boldsymbol{u}+(\mu_1+\lambda_1)\nabla\operatorname{div}\boldsymbol{u},\end{cases}\tag{1.1}$$

其中 $x:=(x_1,x_2)\in\Omega\subset\mathbb{R}^2$ 是空间变量, $t\geqslant 0$ 是时间变量. 方程中 $\rho=\rho(t,x)\geqslant 0$, $\boldsymbol{u}=\boldsymbol{u}(t,x)=(u_1,u_2,u_3)(t,x)$, $p=p(t,x)$ 分别表示流体密度、速度和压力. 压力项 $p=p(\rho)$ 满足 γ-定律

$$p(\rho)=\frac{\rho^\gamma}{\gamma},$$

其中 $\gamma>1$ 是绝热系数. 切粘滞系数 μ_1 和体粘滞系数 λ_1 满足以下条件:

$$\mu_1>0,\quad \mu_1+\lambda_1\geqslant 0.$$

可取定

$$\mu_1=\mu\varepsilon,\quad \lambda_1=\lambda\varepsilon,$$

其中 $\varepsilon>0$ 是粘性消失参数, μ 和 λ 是与 ε 无关的给定常数.

考虑 NS 方程 (1.1) 在空间区域为 $\Omega=\mathbb{R}\times\mathbb{T}$ ($\mathbb{T}=\mathbb{R}/\mathbb{Z}$ 表示环面), 给定如下初值

$$(\rho,\boldsymbol{u})(t=0,x)=(\rho_0,\boldsymbol{u}_0)(x)=(\rho_0,u_{10},u_{20})(x),\tag{1.2}$$

并且解在 x_1 方向无穷远处满足:

$$(\rho,u_1,u_2)\to(\rho_\pm,u_{1\pm},0),\quad x_1\to\pm\infty,\tag{1.3}$$

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其中 $\rho_\pm \geqslant 0$, $u_{1+} \neq u_{1-}$ 是给定的常数. 本文主要研究带真空的情形: $\rho_- = 0$, $\rho_+ > 0$, 并且 (ρ_+, u_{1+}) 通过 2-稀疏波与真空 $\rho_- = 0$ 相连接.

形式上来说, 当 $\varepsilon \to 0$ 时, 2 维可压缩 NS 方程 (1.1)–(1.3) 的解收敛到 2 维可压缩 Euler 方程

$$\begin{cases}\rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0,\\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = 0\end{cases} \tag{1.4}$$

在平坦 Riemann 初值

$$(\rho_0^r, \boldsymbol{u}_0^r)(x) = (\rho_0^r, \boldsymbol{u}_0^r)(x_1) = \begin{cases}(\rho_-, u_{1-}, 0), & x_1 < 0,\\ (\rho_+, u_{1+}, 0), & x_1 > 0\end{cases} \tag{1.5}$$

下的解. 但是 2 维可压缩 Euler 方程 (1.4)–(1.5) 的解会产生奇性, 例如激波和真空, 因此 NS 方程 (1.1)–(1.3) 的粘性消失问题通常是一个奇异极限问题. 值得注意的是, 2 维 Euler 方程的 Riemann 问题 (1.4)–(1.5) 与 1 维 Euler 方程

$$\begin{cases}\rho_t + (\rho u_1)_{x_1} = 0, \quad x_1 \in \mathbb{R},\ t \geqslant 0,\\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_{x_1} = 0\end{cases} \tag{1.6}$$

的 Riemann 初值

$$(\rho_0^r, u_{10}^r)(0, x_1) = \begin{cases}(\rho_-, u_{1-}), & x_1 < 0,\\ (\rho_+, u_{1+}), & x_1 > 0\end{cases} \tag{1.7}$$

问题息息相关 (见 [1–2]). 需要指出的是, 1 维可压缩 Euler 方程 Riemann 问题 (1.6)–(1.7) 包含 3 类基本波: 真正非线性特征域上的激波和稀疏波, 线性退化特征域上的接触间断波, 其中激波和稀疏波是非线性波. 并且 Liu-Smoller 在文 [3] 中指出非线性波中仅有稀疏波可以连接到真空 $\rho_\pm = 0$.

特别地, 1 维 Euler 方程 (1.6)–(1.7) 的 Riemann 解 (激波、稀疏波及二者的线性组合) 且取 $u_2 = 0$ 就是 2 维 Euler 方程 (1.4)–(1.5) 的解, 即为平坦 Riemann 解. 然而, 对于平坦激波初值 (令 (1.6)–(1.7) 的解为激波的 Riemann 初值 (1.5)), 除了 1 维 Euler 方程的激波解外, Chiodaroli 等人 [4–5] 证明了满足熵条件的高维 Euler 方程平坦 Riemann 问题 (1.4)–(1.5) 有无穷多个有界允许弱解, 其证明主要依赖于 De Lellis 和 Szekelyhidi [6] 提出的凸积分方法. 随后文 [7–8] 将文 [4–5] 的结论推广到组合波 (平面激波与接触间断波, 平面激波与稀疏波) 的 2 维 Riemann 初值问题上. 另一方面, 关于一致有界的允许弱解的唯一性，目前仅在高维平坦稀疏波的情形下得到了证明 (见 [9–11]), 即使初值中包含真空仍然成立. 这与高维平坦激波的情形形成鲜明对比. 这一唯一性结果表明, 即使包含真空, 平坦稀疏波的解也比平坦激波更为稳定. 基于此, 我们进一步研究了 2 维等熵可压缩 NS 方程中带真空平坦稀疏波的粘性消失极限问题.

准确地说, 当 $\rho_- = 0$ 时, 带真空的稀疏波是 1 维 Euler 方程 Riemann 问题 (1.6)–(1.7) 的唯一熵解, 即 1 维 Euler 方程 (1.6) 在给定单侧真空的 Riemann 初值

$$(\rho_0^r, u_{10}^r)(x_1) = \begin{cases}\rho_0^r \equiv 0, & x_1 < 0,\\ (\rho_+, u_{1+}), & x_1 > 0\end{cases} \tag{1.8}$$

下的唯一解. 同时, 这也是 2 维 Euler 方程平坦 Riemann 问题 (1.4)–(1.5) 在 $\rho_- = 0$ 情

况下的唯一解, 即在平坦 Riemann 初值

$$(\rho_0^r, \boldsymbol{u}_0^r)(x) = \begin{cases} \rho_0^r \equiv 0, & x_1 < 0, \\ (\rho_+, u_{1+}, 0), & x_1 > 0 \end{cases} \tag{1.9}$$

下的唯一解. 需要指出的是, 1 维基本波的粘性消失极限已被广泛研究, 其中文 [12–27] 研究了 1 维可压缩 NS 方程和 1 维带人工粘性的双曲守恒律方程, 文 [28–33] 证明了其他相关的结论. 尽管 1 维结果非常丰富, 然而由于高维粘性消失极限问题研究中存在诸多困难, 导致相关研究结果较为有限, 目前有文 [34–42].

值得一提的是, Huang 等人在文 [17] 中证明了一维等熵可压缩 NS 方程中连接到真空的稀疏波的粘性消失极限并得到了关于粘性一致的衰减率. 最近, Li 等人在文 [34–35] 中证明了 2 维等熵可压缩 NS 方程和 3 维完全 Navier-Stokes-Fourier 方程远离真空的平坦稀疏波的粘性消失极限. 随后, 文 [37–38] 进一步证明了 3 维等熵可压缩 NS 方程带真空平坦稀疏波的粘性消失极限. 最近, 文 [41–42] 将平坦稀疏波的粘性消失极限问题推广到其他方程. 受到文 [34, 37] 的启发, 本文研究 2 维等熵可压缩 NS 方程带真空平坦稀疏波的粘性消失极限以及获得关于粘性一致的收敛率.

2 维可压缩 Euler 方程 (1.4) 在初值 (1.9) 下的带真空平坦稀疏波, 也是 1 维可压缩 Euler 方程 (1.6) 在初值 (1.8) 下的带真空稀疏波解. 当 $\rho \geqslant 0$ 时, 1 维 Euler 方程 (1.6) 是双曲的, 有两个实特征值

$$\lambda_1(\rho, u_1) = u_1 - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)},$$

分别对应于右特征向量 $r_1(\rho, u_1), r_2(\rho, u_1)$. 若 $\rho > 0$, 则两个特征域是真正非线性的, 即

$$\nabla_{(\rho,u_1)}\lambda_i(\rho, u_1) \cdot r_i(\rho, u_1) \neq 0, \quad i = 1, 2,$$

对应的 i-Riemann 不变量 ($i = 1, 2$) 可定义为

$$\sum_i(\rho, u_1) = u_1 + (-1)^{i+1}\int^\rho \frac{\sqrt{p'(s)}}{s}\mathrm{d}s. \tag{1.10}$$

不失一般性, 这里仅考虑带真空的 2-稀疏波, 因为 1-稀疏波和 2 个稀疏波叠加的情形可用类似方法处理. 由于 2-Riemann 不变量 $\sum_2(\rho, u_1)$ 沿着 2-稀疏波曲线是常值, 则真空 $\rho_- = 0$ 附近的速度左状态 u_{1-} 需满足 $u_{1-} = \sum_2(\rho_+, u_{1+})$. 连接真空 $\rho_- = 0$ 和非真空 (ρ_+, u_{1+}) 的 2-稀疏波是 1 维 Euler 方程 (1.6) 的自相似解 $(\rho^{r_2}, u_1^{r_2})(\xi)\,(\xi = \frac{x_1}{t})$, 满足

$$\lambda_2(\rho^{r_2}, u_1^{r_2}) = \begin{cases} \rho^{r_2}(\xi) \equiv 0, & \xi < \lambda_2(0, u_{1-}) = u_{1-}, \\ \xi, & u_{1-} \leqslant \xi \leqslant \lambda_2(\rho_+, u_{1+}), \\ \lambda_2(\rho_+, u_{1+}), & \xi > \lambda_2(\rho_+, u_{1+}) \end{cases} \tag{1.11}$$

和

$$\sum_2(\rho^{r_2}(\xi), u_1^{r_2}(\xi)) = \sum_2(0, u_{1-}) = \sum_2(\rho_+, u_{1+}). \tag{1.12}$$

定义 2-稀疏波的动量为

$$m_1^{r_2}(\xi) := \begin{cases} \rho^{r_2}(\xi)u_1^{r_2}(\xi), & \rho^{r_2} > 0, \\ 0, & \rho^{r_2} = 0. \end{cases} \tag{1.13}$$

需要指出的是, 速度在真空区域无法定义, 但动量却是唯一确定的且为 0. 由于 1 维可压缩 Euler 方程 (1.6) 和初值 (1.8) 的带真空 2-稀疏波定义为 $(\rho^{r_2}, m_1^{r_2})(\xi = \frac{x_1}{t})$, 因此相应的 2 维可压缩 Euler 方程 (1.4) 和初值 (1.9) 的带真空平坦 2-稀疏波可定义为 $(\rho^{r_2}, m_1^{r_2}, 0)(\xi = \frac{x_1}{t})$.

定理 1.1 设 $(\rho^{r_2}, m_1^{r_2}, 0)(\frac{x_1}{t})$ 是由 (1.11)–(1.13) 定义的 2 维可压缩 Euler 方程 (1.4) 的带真空平坦 2-稀疏波解. 令 $T > 0$ 为任意固定的时刻, 绝热指数 $1 < \gamma < 19$, 则存在足够小的常数 $\varepsilon_0 > 0$, 使得对任意 $\varepsilon \in (0, \varepsilon_0]$, 存在一族依赖于 T 和初值 (3.1) 的 2 维可压缩 NS 方程 (1.1) 光滑解 $(\rho^\varepsilon, m_1^\varepsilon = \rho^\varepsilon u_1^\varepsilon, m_2^\varepsilon = \rho^\varepsilon u_2^\varepsilon)$, 满足

$$\begin{cases}(\rho^\varepsilon - \rho^{r_2}, m_1^\varepsilon - m_1^r, m_2^\varepsilon) \in C^0(0, T; L^2(\mathbb{R} \times \mathbb{T})),\\ (\nabla \rho^\varepsilon, \nabla \boldsymbol{m}^\varepsilon) \in C^0(0, T; H^1(\mathbb{R} \times \mathbb{T})),\\ \nabla^3 \boldsymbol{m}^\varepsilon \in L^2(0, T; L^2(\mathbb{R} \times \mathbb{T})).\end{cases}$$

此外, 对任意小的正常数 h, 存在与 ε 无关的正常数 $C_{h,T}$, 使得

$$\sup_{h \leqslant t \leqslant T} \Big\|(\rho^\varepsilon, m_1^\varepsilon, m_2^\varepsilon)(t, x_1, x_2) - (\rho^r, m_1^r, 0)\Big(\frac{x_1}{t}\Big)\Big\|_{L^\infty(\mathbb{R} \times \mathbb{T})} \leqslant C_{h,T} \varepsilon^{\frac{1}{3(\gamma+2)}} |\ln \varepsilon|.$$

除 (0,0) 外, $(\rho^\varepsilon, \boldsymbol{m}^\varepsilon) = (\rho^\varepsilon, m_1^\varepsilon, m_2^\varepsilon)(t, x_1, x_2)$ 逐点收敛到平面 2-稀疏波 $(\rho^{r_2}, m_1^{r_2}, 0)(\frac{x_1}{t})$, 即

$$(\rho^\varepsilon, m_1^\varepsilon, m_2^\varepsilon)(t, x_1, x_2) \to (\rho^{r_2}, m_1^{r_2}, 0)\Big(\frac{x_1}{t}\Big), \quad \text{a.e. } \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}.$$

注 1.1 定理 1.1 证明了 2 维可压缩 NS 方程 (1.1)–(1.3) 带真空平坦稀疏波的粘性消失极限. 同时, 引言中阐述了平坦稀疏波在高维可压缩 Euler 方程的一致有界可允许弱解中是唯一的, 即使初值包含真空结论仍然成立 (见 [9–11]). 对比之下, 定理 1.1 将唯一性结论推广到粘性消失极限问题上, 肯定了即使在粘性消失问题上带真空平坦稀疏波仍然是唯一的, 这与高维平坦激波的非唯一性形成鲜明对比 (见 [4–5]). 这些结论更加肯定了平坦稀疏波的稳定性行为.

注 1.2 事实上, 若 $\gamma \geqslant 19$, 利用相同的方法也能得到带真空平坦稀疏波的粘性消失极限和衰减率. 原因是, 虽然引理 2.4 只证明了 $1 < \gamma < 19$ 的双曲波估计, 但只要将文 [37] 中引理 2.5 省略的高阶导数详细展开容易得到 $\gamma \geqslant 19$ 的双曲波估计 (2.11), 并且后续误差估计步骤和 $1 < \gamma < 19$ 一致, 只是后续关于双曲波的估计均用 (2.11) 替换, 但考虑到较为繁琐并且 $1 < \gamma < 19$ 已经包含了一般的物理情形 $(1 < \gamma \leqslant 3)$, 故而将 $\gamma \geqslant 19$ 的情况省略. 此外, $\gamma \geqslant 19$ 得到带真空平坦稀疏波关于粘性一致的衰减率可能比 $1 < \gamma < 19$ 小, 因为前者双曲波的衰减率 (2.11) 比后者 (2.10) 更小, 参照 (3.7) 后关于最终衰减率的讨论.

下面简要描述定理 1.1 的证明思路. 首先, 左状态 ρ_- 连接到真空会导致特征域退化, 即 1 维可压缩 Euler 方程不是严格双曲的, 因此需要在真空附近做截断处理. 取定截断参数 $\nu := \varepsilon^a |\ln \varepsilon| > 0$ (为何取这种形式见注 3.4), 定义连接 $(\nu, u_{1\nu})$ 和 (ρ_+, u_{1+}) 的截断 2-稀疏波, 其中 $u_{1\nu}$ 由 2-Riemann 不变量 (1.10) 唯一确定. 再者, 由于截断 2-稀疏波仅是 Lipschitz 连续的, 但后续分析中需要对其求导并进行误差估计, 因此需要将其光滑化得到磨光的截断 2-稀疏波. 紧接着, 由于稀疏波的无粘性在高维会导致相应的粘性误差项衰减率不足, 故引入双曲波以恢复稀疏波的粘性项. 最后, 通过时空尺度变换可以获得更快的收敛率. 此外, 收敛率主要由截断参数 ν 和 δ 决定 (见注 3.4 及 (3.9)).

本文组织如下: 在第 2 节, 首先将密度的左状态 ρ_- 在真空附近截断并光滑化, 得到磨光的截断 2-稀疏波, 然后引入双曲波构造逼近解. 第 3 节主要把原始问题转化成逼近解附近的扰动方程问题, 并基于标准的但被省略的局部存在性以及关于粘性一致的先验

估计证明定理 1.1. 最后一节利用基本能量法封闭解的 H^2 先验估计. 本文用到以下记号. 记 $H^l(\Omega)$ ($l \geqslant 0$, $l \in \mathbb{Z}$) 为一般的 Sobolev 空间且范数为 $\|\cdot\|_l$, 记 $L^2(\Omega) := H^0(\Omega)$ 为一般的平方可积空间且范数 $\|\cdot\| := \|\cdot\|_0$, 记 $L^\infty(\Omega)$ 为本质有界空间且范数 $\|\cdot\|_\infty$. 令 C 是与 $\varepsilon, \nu, \delta, T$ 无关的通用正常数, C_T 是与 ε, ν, δ 无关但与时间 T 有关的正常数. 空间区域 $\Omega = \mathbb{R} \times \mathbb{T}$, $\Omega_\varepsilon = \mathbb{R} \times (\mathbb{T}/\varepsilon)$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$ 表示环面).

2　带真空的平坦稀疏波

下面考虑 2 维可压缩 Euler 方程 (1.4) 和初值 (1.9) 的带真空平坦稀疏波, 即 1 维可压缩 Euler 方程 (1.6) 和初值 (1.8) 的带真空稀疏波.

2.1　1 维稀疏波

首先考虑 1 维 Euler 方程 (1.6) 不含真空的稀疏波解. 当 $\rho > 0$ 时, 1 维 Euler 方程 (1.6) 是严格双曲的, 有两个不同的特征值

$$\lambda_1(\rho, u_1) = u_1 - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)},$$

对应的两个右特征向量为 $r_1(\rho, u_1), r_2(\rho, u_1)$, 且特征值是真正非线性的, 即满足

$$\nabla_{\rho, u_1} \lambda_i(\rho, u_1) \cdot r_i(\rho, u_1) \neq 0, \quad i = 1, 2.$$

1 维 Euler 方程 (1.6) 对应的 i-Riemann 不变量

$$\sum_i(\rho, u_1) = u_1 + (-1)^{i+1} \int^\rho \frac{\sqrt{p'(s)}}{s} \mathrm{d}s$$

满足 $\nabla_{(\rho, u_1)} \sum\limits_i(\rho, u_1) \cdot r_i(\rho, u_1) \equiv 0$ ($i = 1, 2$), 因此 i-Riemann 不变量沿着 i-稀疏波曲线是常值. 本文只考虑 2-稀疏波, 因为 1-稀疏波和 2-稀疏波叠加的情形方法类似.

2.2　截断 2-稀疏波

给定含真空的 Riemann 初值

$$\begin{cases} \rho(0, x_1) = 0, & x_1 < 0, \\ (\rho, u_1)(0, x_1) = (\rho_+, u_{1+}), & x_1 > 0, \end{cases}$$

其中 ρ 左状态是真空, ρ 右状态 $\rho_+ > 0$, u_{1+} 是给定的常数, 1 维可压缩 Euler 方程 (1.6) 在上述初值下用 2-稀疏波连接. 沿着 2-稀疏波曲线, 2-Riemann 不变量 $\sum\limits_2(\rho, u_1)$ 是常值, 因此可确定 $u_{1-} = \sum\limits_2(\rho_+, u_{1+})$. 2-稀疏波连接真空 $(\rho = 0, u_{1-})$ 和 (ρ_+, u_{1+}) 的自相似解 $(\rho^{r_2}, u_1^{r_2})(\xi)$ $\left(\xi = \frac{x_1}{t}\right)$ 满足

$$\lambda_2(\rho^{r_2}, u_1^{r_2}) = \begin{cases} \rho^{r_2}(\xi) \equiv 0, & \xi < \lambda_2(0, u_{1-}) = u_{1-}, \\ \xi, & u_{1-} \leqslant \xi \leqslant \lambda_2(\rho_+, u_{1+}), \\ \lambda_2(\rho_+, u_{1+}), & \xi > \lambda_2(\rho_+, u_{1+}) \end{cases} \tag{2.1}$$

和

$$\sum_2(\rho^{r_2}(\xi), u_1^{r_2}(\xi)) = \sum_2(0, u_{1-}) = \sum_2(\rho_+, u_{1+}).$$

定义 2-稀疏波的动量

$$m_1^{r_2}(\xi)=\begin{cases}\rho^{r_2}(\xi)u_1^{r_2}(\xi), & \rho^{r_2}>0,\\ 0, & \rho^{r_2}=0.\end{cases} \tag{2.2}$$

带真空的稀疏波是有奇性的, 即特征域线性退化, 需要在真空附近截断. 令 $\nu=\varepsilon^a|\ln\varepsilon|$, $a>0$ 是待定的参数 (ν 的选取参见注 3.4). 给定截断 2-稀疏波初值

$$(\rho,u_1)(0,x_1)=\begin{cases}(\nu,u_{1\nu}), & x_1<0,\\ (\rho_+,u_{1+}), & x_1>0,\end{cases}$$

2-Riemann 不变量沿着 2-稀疏波是常值, $\sum\limits_2(\rho_+,u_{1+})=\sum\limits_2(\nu,u_{1\nu})$ 可以确定 $u_{1\nu}$ 的表达式. 截断 2-稀疏波解 $(\rho_\nu^{r_2},u_{1\nu}^{r_2})(\xi)\ \big(\xi=\frac{x_1}{t}\big)$, 满足

$$\lambda_2(\rho_\nu^{r_2},u_{1\nu}^{r_2})=\begin{cases}\lambda_2(\nu,u_{1\nu}), & \xi<\lambda_2(\nu,u_{1\nu}),\\ \xi, & \lambda_2(\nu,u_{1\nu})\leqslant\xi\leqslant\lambda_2(\rho_+,u_{1+}),\\ \lambda_2(\rho_+,u_{1+}), & \xi>\lambda_2(\rho_+,u_{1+}),\end{cases}$$

截断 2-稀疏波的动量

$$m_{1\nu}^{r_2}=\rho_\nu^{r_2}(\xi)u_{1\nu}^{r_2}(\xi),$$

2-Riemann 不变量

$$\sum_2(\rho_\nu^{r_2}(\xi),u_{1\nu}^{r_2}(\xi))=\sum_2(\nu,u_{1\nu})=\sum_2(\rho_+,u_{1+}).$$

引理 2.1 (见 [3, 17]) 存在常数 $\nu_0\in(0,1)$, 使得对任意 $\nu\in(0,\nu_0]$, $t>0$, 有

$$\Big\|(\rho_\nu^{r_2},m_{1\nu}^{r_2})\Big(\frac{\cdot}{t}\Big)-(\rho^{r_2},m_1^{r_2})\Big(\frac{\cdot}{t}\Big)\Big\|_{L^\infty(\mathbb{R})}\leqslant C\nu. \tag{2.3}$$

2.3 磨光截断 2-稀疏波

截断 2-稀疏波仅仅是 Lipschitz 连续的, 但后续需要对 2-稀疏波求高阶导数并做估计, 因此我们将截断 2-稀疏波磨光. 考虑 1 维无粘 Burgers 方程

$$\begin{cases}\omega_t+\omega\omega_{x_1}=0,\\ \omega(0,x_1)=\begin{cases}\omega_-, & x_1<0,\\ \omega_+, & x_1>0,\end{cases}\end{cases}$$

若 $\omega_-<\omega_+$, 方程的稀疏波解

$$\omega^r(x_1,t)=\omega^r\Big(\frac{x_1}{t}\Big)=\begin{cases}\omega_-, & \dfrac{x_1}{t}<\omega_-,\\ \dfrac{x_1}{t}, & \omega_-<\dfrac{x_1}{t}<\omega_+,\\ \omega_+, & \dfrac{x_1}{t}>\omega_+.\end{cases}$$

磨光的稀疏波解

$$\begin{cases}\omega_t+\omega\omega_{x_1}=0,\\ \omega(t=0,x_1)=\omega_\delta(x_1)=\omega\Big(\dfrac{x_1}{\delta}\Big)=\dfrac{\omega_-+\omega_+}{2}+\dfrac{\omega_--\omega_+}{2}\cdot\tanh\Big(\dfrac{x_1}{\delta}\Big),\end{cases} \tag{2.4}$$

其中 $\delta=\varepsilon^a$ 是待定参数. 特别地, 截断参数 $\nu=\varepsilon^a|\ln\varepsilon|$ 与此处 $\delta=\varepsilon^a$ 选取相同的 a, 这是为了得到最佳衰减率 (见注 3.4). 由特征线方法有以下结论.

引理 2.2 (见 [17, 26])　对任意的 $\delta>0$, (2.4) 有唯一的全局光滑解 $\omega_\delta^r(t,x_1)$，且具备下面性质:

(1) 对任意的 $x_1\in\mathbb{R},\ t\geqslant 0,\ \delta>0$, 有

$$\omega_-<\omega_\delta^r(t,x_1)<\omega_+,\quad \partial_{x_1}\omega_\delta^r(t,x_1)>0.$$

(2) 对任意的 $t>0,\ \delta>0,\ p\in[1,\infty]$, 有

$$\begin{aligned}
&\|\partial_{x_1}\omega_\delta^r(t,.)\|_{L^p}\leqslant C(\omega_+-\omega_-)^{\frac{1}{p}}(\delta+t)^{-1+\frac{1}{p}},\\
&\|\partial_{x_1x_1}\omega_\delta^r(t,.)\|_{L^p}\leqslant C(\delta+t)^{-1}\delta^{-1+\frac{1}{p}},\\
&\|\partial_{x_1x_1x_1}\omega_\delta^r(t,.)\|_{L^p}\leqslant C(\delta+t)^{-1}\delta^{-2+\frac{1}{p}},\\
&|\partial_{x_1x_1}\omega_\delta^r(t,x_1)|\leqslant \frac{4}{\delta}\partial_{x_1}\omega_\delta^r(t,x_1).
\end{aligned}$$

(3) 存在常数 $\delta_0\in(0,1)$, 使得 $\delta\in(0,\delta_0],\ t>0$, 有

$$\left\|\omega_\delta^r(t,.)-\omega^r\left(\frac{\cdot}{t}\right)\right\|_{L^\infty(\mathbb{R})}\leqslant C\delta t^{-1}[\ln(1+t)+|\ln\delta|].$$

下面考虑将截断 2-稀疏波 $(\rho_\nu^{r_2},u_{1\nu}^{r_2})(\xi)$ $\left(\xi=\frac{x_1}{t}\right)$ 磨光，得到磨光截断 2-稀疏波 $(\overline{\rho}_{\nu,\delta},\overline{u}_{1\nu,\delta})(t,x_1)$, 有

$$\begin{cases}
\omega_+=\lambda_2(\rho_+,u_{1+}),\omega_-=\lambda_2(\nu,u_{1\nu}),\\
\omega_\delta^r(t,x_1)=\lambda_2(\overline{\rho}_{\nu,\delta},\overline{u}_{1\nu,\delta})(t,x_1),\\
\sum_2(\overline{\rho}_{\nu,\delta},\overline{u}_{1\nu,\delta})(t,x_1)=\sum_2(\nu,u_{1\nu})=\sum_2(\rho_{\rho_+},u_{1+}).
\end{cases}\tag{2.5}$$

记 $(\overline{\rho}_{\nu,\delta},\overline{u}_{1\nu,\delta})(t,x_1)$ 为 $(\overline{\rho},\overline{u}_1)(t,x_1)$, 其满足 1 维 Euler 方程

$$\begin{cases}
\overline{\rho}_t+(\overline{\rho}\overline{u}_1)_{x_1}=0,\\
(\overline{\rho}\overline{u}_1)_t+\left(\overline{\rho}\overline{u}_1^2+p(\overline{\rho})\right)_{x_1}=0,\\
(\overline{\rho},\overline{u}_1)(0,x_1):=(\overline{\rho}_0,\overline{u}_{10})(x_1),
\end{cases}\tag{2.6}$$

有以下结论.

引理 2.3 (见 [17, 26])　磨光的截断 2-稀疏波 $(\overline{\rho},\overline{u}_1)(t,x_1)$ 有以下性质:

(1) 对任意的 $x\in\mathbb{R},\ t\geqslant 0$, 有 $\nu<\overline{\rho}(t,x_1)<\rho_+,\ \overline{u}_{1x_1}(t,x_1)=\frac{2}{\gamma+1}(\omega_\delta^r)_{x_1}>0,\ \overline{\rho}_{x_1}=\overline{\rho}^{\frac{3-\gamma}{2}}\overline{u}_{1x_1},\ \overline{\rho}_{x_1x_1}=\overline{\rho}^{\frac{3-\gamma}{2}}\overline{u}_{x_1x_1}+\frac{3-\gamma}{2}\overline{\rho}^{2-\gamma}(\overline{u}_{1x_1})^2$.

(2) 对任意的 $t>0,\ \delta>0,\ p\in[1,\infty]$, 有以下估计:

$$\begin{aligned}
&\|\overline{u}_{1x_1}(t,.)\|_{L^p}\leqslant C(\omega_+-\omega_-)^{\frac{1}{p}}(\delta+t)^{-1+\frac{1}{p}},\\
&\|\overline{u}_{1x_1x_1}(t,.)\|_{L^p}\leqslant C(\delta+t)^{-1}\delta^{-1+\frac{1}{p}},\\
&\|\overline{u}_{1x_1x_1x_1}(t,.)\|_{L^p}\leqslant C(\delta+t)^{-1}\delta^{-2+\frac{1}{p}}.
\end{aligned}$$

(3) 存在常数 $\delta_0\in(0,1)$, 使得 $\delta\in(0,\delta_0],\ t>0$,

$$\|(\overline{\rho},\overline{u}_1)(t,.)-(\rho_\nu^{r_2},u_{1\nu}^{r_2})\left(\frac{\cdot}{t}\right)\|_{L^\infty(\mathbb{R})}\leqslant C\delta t^{-1}[\ln(1+t)+|\ln\delta|].\tag{2.7}$$

2.4 双曲波

如果取 $(\overline{\rho},\overline{u}_1)(t,x_1)$ 作为粘性消失的解,但由于稀疏波的粘性误差项 $(2\mu+\lambda)\varepsilon\overline{u}_{1x_1x_1}$ 的存在导致衰减率不足, 这里需要引入双曲波取抵消粘性误差项的作用 (见 [34–35]). 令 $(d_1,d_2)(t,x_1)$ 是如下双曲方程的解

$$\begin{cases} d_{1t}+d_{2x_1}=0, \qquad\qquad x_1\in\mathbb{R},\ t\in(0,T], \\ d_{2t}+\Big(-\dfrac{\overline{m}_1^2}{\overline{\rho}^2}d_1+p'(\overline{\rho})d_1+\dfrac{2\overline{m}_1}{\overline{\rho}}d_2\Big)_{x_1} \\ \quad=(2\mu_1+\lambda_1)\overline{u}_{1x_1x_1}=(2\mu+\lambda)\varepsilon\overline{u}_{1x_1x_1}, \\ (d_1,d_2)(0,x_1)=(0,0), \end{cases} \tag{2.8}$$

其中 $\overline{m}_1:=\overline{\rho u}_1$ 表示磨光的截断 2-稀疏波的动量. 关于 (d_1,d_2) 的估计有以下结论.

引理 2.4 (见 [37]) 双曲波方程 (2.8) 的解 $(d_1,d_2)(t,x_1)$, 有以下估计

$$\Big\|\frac{\partial^k}{\partial {x_1}^k}(d_1,d_2)(t,.)\Big\|^2\leqslant\frac{C_T\cdot\varepsilon^2}{(\nu^\alpha\delta^2)^{k+1}},\quad k=0,1,2,3,$$

其中 $\alpha=\gamma-1\ (1<\gamma<19)$. 因此由 Sobolev 不等式, 有

$$\sup_{0\leqslant t\leqslant T}\Big\|\frac{\partial^k}{\partial x_1^k}(d_1,d_2)(t,.)\Big\|_{L^\infty(\mathbb{R})}\leqslant\frac{C_T\cdot\varepsilon}{(\nu^\alpha\delta^2)^{\frac{2k+3}{4}}},\quad k=0,1,2. \tag{2.9}$$

注 2.1 引理 2.4 表明双曲波的估计依赖于截断参数 ν, 即双曲波在真空附近也会退化. 下面简述引理 2.4 的证明思路, 详细证明见文 [37] 中引理 2.4–2.5. 首先将方程 (2.8) 对角化, 其中需要用到一个重要的解耦关系式, 文 [37] 中的 (2.11). 再对对角化后的双曲波进行加权能量估计, 权值为 $\overline{\rho}^\alpha$, 引入这一权值是为了保留恰当好项

$$\int_0^t\int_{\mathbb{R}}\overline{u}_{1x_1}\Big|\frac{\partial^k D_1}{\partial x_1^k}\Big|^2\mathrm{d}x_1\mathrm{d}t_1,\quad k=0,1,2,3,$$

进而封闭 D_2 的高阶导数估计. 最后将上述估计转化为原始双曲波 (2.8) 的估计, 为了后续计算方便, 这里取 $\alpha=\gamma-1\,(1<\gamma<19)$. 更准确地说, 若 $1<\gamma<19$, 有

$$\Big\|\frac{\partial^k}{\partial {x_1}^k}(d_1,d_2)(t,.)\Big\|^2\leqslant\frac{C_T\cdot\varepsilon^2}{(\nu^{\gamma-1}\delta^2)^{k+1}},\quad k=0,1,2,3, \tag{2.10}$$

而且这已经包含了一般的物理情形 $(1<\gamma<3)$. 若 $\gamma\geqslant 19$, 有

$$\Big\|\frac{\partial^k}{\partial {x_1}^k}(d_1,d_2)(t,.)\Big\|^2\leqslant\frac{C_T\cdot\varepsilon^2}{\nu^{k(\gamma-1)+\alpha}\delta^{2(k+1)}},\quad k=0,1,2,3, \tag{2.11}$$

其中 $\alpha>\frac{5\gamma-23}{4}$. 注意到文 [37] 中引理 2.5 没有 $\gamma\geqslant 19$ 下双曲波的估计, 但其证明过程若把高阶导数估计详细展开, 即可得这一结论.

需要指出的是, 利用 (2.11) 可得 $\gamma\geqslant 19$ 情形下的粘性消失极限和衰减率, 其步骤与 $1<\gamma<19$ 的情形一致, 只是后续关于双曲波的估计均用 (2.11) 替换, 但考虑到较为繁琐并且 $1<\gamma<19$ 已经包含了一般的物理情形 $(1<\gamma<3)$, 故而将 $\gamma\geqslant 19$ 的情况省略.

2.5 构造逼近解

结合本节 4 部分的叙述, 构造 2 维 NS 方程 (1.1) 和初值 (3.1) 的逼近解 $(\widetilde{\rho},\widetilde{m}_1)(t,x_1)$ 为

$$\widetilde{\rho}(t,x_1)=(\overline{\rho}+d_1)(t,x_1),\quad \widetilde{m}_1(t,x_1)=(\overline{m}_1+d_2)(t,x_1):=\widetilde{\rho}\widetilde{u}_1(t,x_1),$$

其满足如下方程组

$$\begin{cases}\widetilde{\rho}_t+(\widetilde{\rho}\widetilde{u}_1)_{x_1}=0,\\ (\widetilde{\rho}\widetilde{u}_1)_t+(\widetilde{\rho}\widetilde{u}_1^2+p(\widetilde{\rho}))_{x_1}=(2\mu+\lambda)\varepsilon\overline{u}_{1x_1x_1}+(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{x_1}\\ \qquad\qquad +(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{x_1},\\ (\widetilde{\rho},\widetilde{u}_1)(t=0,x_1)=(\overline{\rho}_0,\overline{u}_{10})(x_1).\end{cases}\tag{2.12}$$

根据引理 2.4 可知

$$\|(d_1,d_2)\|_{L^\infty(\mathbb{R})}\leqslant C_T\frac{\varepsilon}{\nu^{\frac{3}{4}\alpha}\delta^{\frac{3}{2}}},$$

同时由引理 2.3 中 (1) 知 $\nu<\overline{\rho}<\rho_+$ 以及 ν,δ 的取值 (3.9) 可推出

$$\Big\|\frac{(d_1,d_2)}{\overline{\rho}}\Big\|_{L^\infty(\mathbb{R})}\leqslant C_T\frac{\varepsilon}{\nu^{\frac{3}{4}\alpha+1}\delta^{\frac{3}{2}}}\leqslant C_T\varepsilon^{\beta'},$$

其中 β' 是任意小的正常数. 因此

$$\widetilde{\rho}\sim\overline{\rho},\quad \frac{\nu}{2}<\widetilde{\rho}<\rho_+.\tag{2.13}$$

3 问 题 的 转 化

下面主要证明定理 1.1. 需利用 2 维 NS 方程 (1.1) 的解 $(\rho^\varepsilon,u_1^\varepsilon,u_2^\varepsilon)$ 和渐近方程 (2.12) 的解 $(\widetilde{\rho},\widetilde{u}_1,0)$, 定义扰动项 (ϕ,Ψ):

$$\phi:=\rho^\varepsilon(t,x_1,x_2)-\widetilde{\rho}(t,x_1),$$

$$\Psi:=(\psi_1,\psi_2)^{\mathrm{T}}(t,x_1,x_2)=(u_1^\varepsilon,u_2^\varepsilon)^{\mathrm{T}}(t,x_1,x_2)-(\widetilde{u}_1,0)^{\mathrm{T}}(t,x_1),$$

这里 $(\rho^\varepsilon,u_1^\varepsilon,u_2^\varepsilon)$ 是 2 维 NS 方程 (1.1) 在给定的初值

$$(\rho^\varepsilon,u_1^\varepsilon,u_2^\varepsilon)(t=0,x_1,x_2):=(\overline{\rho}_0,\overline{u}_{10},0)(x_1)+(\phi_0,\psi_{10},\psi_{20})(x_1,x_2)\tag{3.1}$$

下的解. 为了得到更高的衰减率 (命题 3.1), 对时空变量做尺度伸缩变换

$$\tau=\frac{t}{\varepsilon},\quad y_1=\frac{x_1}{\varepsilon},\quad y_2=\frac{x_2}{\varepsilon},$$

并且仍用 $(\phi,\Psi)(\tau,y_1,y_2)$ 表示扰动项, 利用 2 维 NS 方程 (1.1) 减去逼近方程 (2.12) 得

到扰动方程:

$$\begin{cases}\phi_\tau+\rho\operatorname{div}\Psi+\phi\widetilde{u}_{1y_1}+u\nabla\phi+\psi_1\widetilde{\rho}_{y_1}=0,\\ \rho\Psi_\tau+\rho u_1\Psi_{y_1}+\rho u_2\Psi_{y_2}+(\rho\psi_1\widetilde{u}_{1y_1},0)^{\mathrm{T}}+p'(\rho)\nabla\phi\\ +\Big((p'(\rho)-\frac{\rho}{\widetilde{\rho}}p'(\widetilde{\rho}))\widetilde{\rho}_{y_1},0\Big)^{\mathrm{T}}+\Big((2\mu+\lambda)\frac{\overline{u}_{1y_1y_1}\phi}{\widetilde{\rho}},0\Big)^{\mathrm{T}}\\ +\Big(\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\phi}{\widetilde{\rho}},0\Big)^{\mathrm{T}}\\ +\Big(\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\phi}{\widetilde{\rho}},0\Big)^{\mathrm{T}}\\ =\mu\Delta\Psi+(\mu+\lambda)\nabla\operatorname{div}\Psi+\Big((2\mu+\lambda)\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1},0\Big)^{\mathrm{T}}\\ -((\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2-2\overline{u}_1d_2)_{y_1},0)^{\mathrm{T}}-((p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1},0)^{\mathrm{T}},\end{cases}\tag{3.2}$$

由 (3.1) 给定的扰动方程初值为

$$(\phi,\Psi)(\tau=0,y_1,y_2):=(\phi_0,\psi_{10},\psi_{20})(\varepsilon y_1,\varepsilon y_2),\tag{3.3}$$

并满足

$$E_0^2:=\frac{1}{\nu^2}\|\phi_0\|^2_{H^2(\Omega_\varepsilon)}+\|\Psi_0\|^2_{H^2(\Omega_\varepsilon)}=O\Big(\frac{\varepsilon}{\nu^{2\alpha+2}\delta^4}\Big).$$

$\forall\,\tau_1\in\big(0,\frac{T}{\varepsilon}\big)$, 扰动方程 (3.2)–(3.3) 选取的解空间为

$$\begin{aligned}&\Pi(0,\tau_1)\\ =&\left\{(\phi,\Psi)\,\middle|\,\begin{array}{l}(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac12}\Psi)\in C^0(0,\tau_1;L^2(\Omega_\varepsilon)),(\nabla\phi,\nabla\Psi)\in C^0(0,\tau_1;H^1(\Omega_\varepsilon)),\\ \nabla\Psi\in L^2(0,\tau_1;H^2(\Omega_\varepsilon)).\end{array}\right\}.\end{aligned}$$

给定先验假设

$$\begin{cases}E^2:=\sup\limits_{0\leqslant\tau\leqslant\tau_1(\varepsilon)}\|\nabla\phi\|_1^2\leqslant\dfrac{1}{160}\nu^{\alpha+3},\\ \sup\limits_{0\leqslant\tau\leqslant\tau_1(\varepsilon)}\|\phi\|_\infty\leqslant\dfrac{\nu}{4},\end{cases}\tag{3.4}$$

其中第一个式子是为了估计 (4.9), 第二个式子表明主要考虑小扰动情形, 这是为了 Φ 的下界控制且根据 (2.13) 可知 $\rho\sim\widetilde{\rho}\sim\overline{\rho}$.

注 3.1 $\overline{\rho}$ 在真空附近截断能保证动量方程 $(1.1)_2$ 是严格抛物的, 这对于证明 NS 方程经典解的局部存在性和全局存在性是至关重要的.

命题 3.1 (全局存在性和一致估计) 存在正常数 $\varepsilon_1<1$, 如果 $0<\varepsilon\leqslant\varepsilon_1$ 和取 ν,δ 满足 (3.9), 则扰动方程 (3.2)–(3.3) 有唯一整体解 $(\phi,\Psi)\in\Pi\big(0,\frac{T}{\varepsilon}\big)$, 且满足

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant\frac{T}{\varepsilon}}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac12}\Psi)(\tau)\|^2+\|(\nabla\phi,\nabla\Psi)(\tau)\|_1^2]\\ &+\int_0^{\frac{T}{\varepsilon}}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1x_1}^{\frac12}\phi,\overline{\rho}^{\frac12}\overline{u}_{1x_1}^{\frac12}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}(\nabla\phi,\nabla^2\phi)\|^2+\|\nabla\Psi\|_2^2]\mathrm{d}\tau\\ \leqslant&\,E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4},\end{aligned}$$

其中 C_T 是与时间 T 有关但与 ε,δ,ν 无关的正常数.

注 3.2 如果命题 3.1 成立, 结合扰动方程 (3.2) 的局部存在性 (证明是标准的, 因此一般情形下省略), 可得到扰动方程的整体光滑解的存在性, 即定理 1.1 的光滑解的整体存在性. 需要指出的是, 命题 3.1 的衰减率 $\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}$ 取决于引理 4.1 对粘性误差项 I_1 的估计 (4.5).

由命题 3.1 的一致估计有 (ϕ,Ψ) 的 H^2 估计为

$$\begin{cases}\|(\phi,\Psi)\|\leqslant C_T\dfrac{\varepsilon^{\frac12}}{\nu^{\alpha+s_0}\delta^2},\\ \|(\nabla\phi,\nabla\Psi)\|_1\leqslant C_T\dfrac{\varepsilon^{\frac12}}{\nu^{\alpha+1}\delta^2},\end{cases}\tag{3.5}$$

其中

$$s_0=\begin{cases}\dfrac32, & 1<\gamma\leqslant 3,\\ \dfrac{\gamma}{2}, & \gamma>3.\end{cases}$$

为了满足先验假设 $(3.4)_1$, 联合 (3.5) 需满足下式:

$$C_T\frac{\varepsilon}{\nu^{2(\alpha+1)}\delta^4}\leqslant\frac{1}{160}\nu^{\alpha+3}.\tag{3.6}$$

同理为了满足先验假设 $(3.4)_2$, 需要有下式成立:

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\|(\phi,\Psi)(t,x_1,x_2)\|_{L^\infty(\Omega)}=\sup_{0\leqslant\tau\leqslant\frac{T}{\varepsilon}}\|(\phi,\Psi)(\tau,y_1,y_2)\|_{L^\infty(\Omega_\varepsilon)}\\ &\leqslant C(\|(\phi,\Psi)\|^{\frac12}_{L^2(\Omega_\varepsilon)}\|(\nabla\phi,\nabla\Psi)\|^{\frac12}_{L^2(\Omega_\varepsilon)}+\|(\nabla\phi,\nabla\Psi)\|^{\frac12}_{L^2(\Omega_\varepsilon)}\|(\nabla^2\phi,\nabla^2\Psi)\|^{\frac12}_{L^2(\Omega_\varepsilon)})\\ &\leqslant C_T\frac{\varepsilon^{\frac12}}{\nu^{(\alpha+\frac{s_0+1}{2})}\delta^2}\leqslant\frac{\nu}{4}.\end{aligned}\tag{3.7}$$

第一个不等号成立是由于 Sobolev 不等式, 其中的 C 与 ε 无关, 虽然积分区域与 ε 有关, 具体说明见文 [34, 43], 第二个不等号成立是根据估计 (3.5) 所得. 上式表明, 扰动量 (ϕ,Ψ) 与截断参数 ν 同阶, 甚至是更低阶.

利用 (3.6)–(3.7), (2.9), (2.7), (2.3) 可以得到, 对于任意给定的正常数 $0<h<T$, 有

$$\begin{aligned}&\sup_{h\leqslant t\leqslant T}\|\rho-\rho^{r_2}\|_{L^\infty(\Omega)}\\ &\leqslant\sup_{0\leqslant t\leqslant T}\|\rho-\widetilde{\rho}\|_{L^\infty(\Omega)}+\sup_{0\leqslant t\leqslant T}\|\widetilde{\rho}-\overline{\rho}\|_{L^\infty(\Omega)}\\ &\quad+\sup_{h\leqslant t\leqslant T}\|\overline{\rho}-\rho^{r_2}_\nu\|_{L^\infty(\Omega)}+\sup_{h\leqslant t\leqslant T}\|\rho^{r_2}_\nu-\rho^{r_2}\|_{L^\infty(\Omega)}\\ &\leqslant\frac{\nu}{4}+C_T\frac{\varepsilon}{\nu^{\frac34\alpha}\delta^{\frac32}}+C_{h,T}\delta\ln|\delta|+C_{h,T}\nu\\ &\leqslant C_{h,T}\delta\ln|\delta|+C_{h,T}\nu,\\ &\sup_{h\leqslant t\leqslant T}\|m_1-m_1^{r_2}\|_{L^\infty(\Omega)}\\ &\leqslant\sup_{0\leqslant t\leqslant T}\|m_1-\widetilde{m}_1\|_{L^\infty(\Omega)}+\sup_{0\leqslant t\leqslant T}\|\widetilde{m}_1-\overline{m}_1\|_{L^\infty(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{h\leqslant t\leqslant T}\|\overline{m_1} - m_{1\nu}^{r_2}\|_{L^\infty(\Omega)} + \sup_{h\leqslant t\leqslant T}\|m_{1\nu}^{r_2} - m_1^{r_2}\|_{L^\infty(\Omega)}\\
&\leqslant C\sup_{0\leqslant t\leqslant T}\|\phi,\Psi\|_{L^\infty(\Omega)} + C\sup_{0\leqslant t\leqslant T}\Big\|\Big(d_1, \frac{-d_1\overline{u}_1 + d_2}{\widetilde{\rho}}\Big)\Big\|_{L^\infty(\Omega)}\\
&\quad + C\sup_{h\leqslant t\leqslant T}\|(\overline{\rho} - \rho_\nu^{r_2}, \overline{u}_1 - u_{1\nu}^{r_2})\|_{L^\infty(\Omega)} + C_{h,T}\nu\\
&\leqslant \frac{\nu}{4} + C_T\frac{\varepsilon}{\nu^{\frac34\alpha+1}\delta^{\frac32}} + C_{h,T}\delta\ln|\delta| + C_{h,T}\nu\\
&\leqslant C_{h,T}\delta\ln|\delta| + C_{h,T}\nu,
\end{aligned}
$$

$$
\sup_{0\leqslant t\leqslant T}\|m_2\| \leqslant C\sup_{0\leqslant t\leqslant T}\|\psi_2\|_{L^\infty(\Omega)} \leqslant C_T\frac{\varepsilon^{\frac12}}{\nu^{(\alpha+\frac{s_0+1}{2})}\delta^2} \leqslant C_T\nu.
$$

注 3.3 以上估计在 $t=0$ 附近截断, 是因为引理 2.1 中 (2.3) 和引理 2.3 中 (2.7) 在远离初始时刻 $t=0$ 的情形下成立. 但 (d_1,d_2) 和 (ϕ,Ψ) 不需要在 $t=0$ 附近截断, 因为引理 2.4 和命题 3.1 在 $t=0$ 处仍成立.

注 3.4 上述讨论表明, 在 (3.7) 成立的前提下, 最终衰减率是由引理 2.4 的双曲波估计, 磨光的截断 2-稀疏波的待定参数 δ 的 $\delta|\ln\delta|$, 截断参数 ν 共同决定. 结合 (3.6) 易知

$$
\max\Big\{\frac{\varepsilon}{\nu^{\frac34\alpha}\delta^{\frac32}}, \frac{\varepsilon}{\nu^{\frac34\alpha+1}\delta^{\frac32}}\Big\} \leqslant \nu,
$$

因此粘性消失极限问题的最终的衰减率由 $\delta|\ln\delta|$ 和 ν 共同决定. 需要指出的是, (3.6)–(3.7) 的成立是为了使先验假设 (3.4) 成立. 考虑到若设 $\delta=\varepsilon^a$, 则相应的衰减率决定项中有 $\delta|\ln\delta| = a\varepsilon^a|\ln\varepsilon|$, 取 $\nu=\varepsilon^a|\ln\varepsilon|$, 这样的话 $\delta|\ln\delta|$ 与 ν 同阶, 可使衰减率取到最优. 下面证明 $a=\frac{1}{3(\gamma+2)}$ 时, 衰减率达到最优.

设 $\nu=\varepsilon^a|\ln\varepsilon|$, $\delta=\varepsilon^a$, 结合 (3.6), 可得

$$
1-(2\alpha+2+4)a \geqslant (\alpha+3)a,
$$

根据引理 2.4 中 $\alpha=\gamma-1$ $(1<\gamma<19)$, 得到 $a\leqslant\frac{1}{3(\gamma+2)}$, 取最大值

$$
a=\frac{1}{3(\gamma+2)}, \tag{3.8}
$$

因此有

$$
\nu=\varepsilon^{\frac{1}{3(\gamma+2)}}|\ln\varepsilon|, \quad \delta=\varepsilon^{\frac{1}{3(\gamma+2)}}. \tag{3.9}
$$

故而得到

$$
\sup_{h\leqslant t\leqslant T}\|(\rho-\rho^{r_2}, m_1-m_1^{r_2}, m_2)\|_{L^\infty(\Omega)} \leqslant C_{h,T}\varepsilon^{\frac{1}{3(\gamma+2)}}|\ln\varepsilon|,
$$

即证带真空平坦稀疏波的粘性消失并得到关于粘性一致的收敛率.

命题 3.2 (先验估计) 设 $(\phi,\Psi)\in\Pi(0,\tau(\varepsilon))$ 是 (3.2)–(3.3)的解, $\forall\tau(\varepsilon)\in(0,\frac{T}{\varepsilon})$, 则存在与 $\varepsilon,\delta,\nu,\tau_1(\varepsilon)$ 无关的常数 $\varepsilon_2>0$, 使得对任意 $0<\varepsilon\leqslant\varepsilon_2$, 满足以下估计

$$
\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi, \overline{\rho}^{\frac12}\Psi)(\tau)\|^2 + \|(\nabla\phi,\nabla\Psi)(\tau)\|_1^2]\\
&\quad + \int_0^{\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1x_1}^{\frac12}\phi, \overline{\rho}^{\frac12}\overline{u}_{1x_1}^{\frac12}\psi_1)\|^2 + \|\overline{\rho}^{\frac{\gamma-2}{2}}(\nabla\phi,\nabla^2\phi)\|^2 + \|\nabla\Psi\|_2^2]\mathrm{d}\tau
\end{aligned}
$$

$$\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4},$$

其中 C_T 是与时间 T 有关但与 ε,δ,ν 无关的正常数.

下面主要给出命题 3.2 的证明.

4 先验估计

引理 4.1　存在正常数 C_T, 对任意 $\tau(\varepsilon)\in\big(0,\frac{T}{\varepsilon}\big)$, 有以下估计

$$\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac{1}{2}}\Psi)(\tau)\|^2+\int_0^{\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}\phi,\overline{\rho}^{\frac{1}{2}}\overline{u}_{1y_1}\psi_1)\|^2+\|\nabla\Psi\|^2]\mathrm{d}\tau\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.$$

证　将 $(3.2)_2$ 乘以 Ψ, 有

$$\begin{aligned}&\Big(\rho\frac{|\Psi|^2}{2}\Big)_\tau+\mathrm{div}\Big(\rho u\frac{|\Psi|^2}{2}-\mu\psi_i\nabla\psi_i+(\mu+\lambda)\Psi\mathrm{div}\,\Psi\Big)+\rho\widetilde{u}_{1y_1}\psi_1^2\\&+\mu|\nabla\Psi|^2+(\mu+\lambda)|\mathrm{div}\,\Psi|^2+p'(\rho)\nabla\phi\cdot\Psi+\Big(p'(\rho)-\frac{\rho}{\widetilde{\rho}}p'(\widetilde{\rho})\Big)\widetilde{\rho}_{y_1}\psi_1\\=&(2\mu+\lambda)\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\psi_1-(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\psi_1\\&-(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\psi_1-(2\mu+\lambda)\frac{\overline{u}_{1y_1y_1}\phi\psi_1}{\widetilde{\rho}}\\&-\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\phi\psi_1}{\widetilde{\rho}}-\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\phi\psi_1}{\widetilde{\rho}}.\end{aligned}\tag{4.1}$$

定义位势函数

$$\Phi(\rho,\widetilde{\rho}):=\int_{\widetilde{\rho}}^{\rho}\frac{p(s)-p(\widetilde{\rho})}{s^2}\mathrm{d}s=\frac{1}{(\gamma-1)\rho}(p(\rho)-p(\widetilde{\rho})-p'(\overline{\rho})\phi).$$

结合 $(3.2)_1$, 直接推导出

$$\begin{aligned}&(\rho\Phi)_\tau+\mathrm{div}(\rho u\Phi+(p(\rho)-p(\widetilde{\rho}))\Psi)+\widetilde{u}_{1y_1}(p(\rho)-p(\widetilde{\rho})-p'(\widetilde{\rho})\phi)\\&-p'(\rho)\nabla\phi\cdot\Psi-\Big(p'(\rho)-\frac{\rho}{\widetilde{\rho}}p'(\widetilde{\rho})\Big)\widetilde{\rho}_{y_1}\varphi_1=0.\end{aligned}\tag{4.2}$$

其中注意到

$$\rho\psi_1^2\widetilde{u}_{1y_1}=\rho\psi_1^2\overline{u}_{1y_1}+\rho\psi_1^2\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1},$$

根据 (2.13) 中 $\widetilde{\rho}\sim\overline{\rho}$ 以及先验假设 $\|\phi\|_\infty\leqslant\frac{\nu}{4}\leqslant\overline{\rho}$, 有下式成立

$$\Phi(\rho,\widetilde{\rho})=\frac{1}{(\gamma-1)}(p(\rho)-p(\widetilde{\rho})-p'(\widetilde{\rho})\phi)\geqslant C\widetilde{\rho}^{\gamma-2}\phi^2\sim C\overline{\rho}^{\gamma-2}\phi^2,\tag{4.3}$$

结合上述关系式, 对 (4.1)–(4.2) 在 $[0,\tau]\times\Omega$ 上积分, 有

$$\begin{aligned}&\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac{1}{2}}\Psi)(\tau)\|^2+\int_0^\tau(\|\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}^{\frac{1}{2}}\phi\|^2+\|\overline{\rho}^{\frac{1}{2}}\overline{u}_{1y_1}^{\frac{1}{2}}\Psi\|^2+\|\nabla\Psi\|^2)\mathrm{d}\tau\\\leqslant&E_0^2+\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big[(2\mu+\lambda)\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\psi_1\end{aligned}$$

$$
\begin{aligned}
&-(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\psi_1-(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\psi_1\\
&-(2\mu+\lambda)\frac{\overline{u}_{1y_1y_1}\phi\psi_1}{\widetilde{\rho}}-\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\phi\psi_1}{\widetilde{\rho}}\\
&-\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\phi\psi_1}{\widetilde{\rho}}-\rho\psi_1^2\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}\\
&-\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}(p(\rho)-p(\widetilde{\rho})-p'(\widetilde{\rho})\phi)\Big]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
:=\;&E_0^2+\Big|\sum_{i=1}^{8}I_i\Big|.
\end{aligned}\tag{4.4}
$$

利用 Hölder 不等式, Young 不等式和 Sobolev 不等式, 下面分别对 I_i 进行估计. 先估计 I_1, 有

$$
\begin{aligned}
|\mathrm{I}_1|&=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}(2\mu+\lambda)\Big(\frac{-d_1\overline{u}_1+d_2}{\overline{\rho}}\Big)_{y_1y_1}\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}(2\mu+\lambda)\Big(\frac{-d_1\overline{u}_1+d_2}{\overline{\rho}}\Big)_{y_1}\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&\leqslant C\int_0^\tau\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}\Big\|\cdot\|\psi_{1y_1}\|\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\psi_{1y_1}\|^2\mathrm{d}\tau+\int_0^\tau\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}\Big\|^2\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\psi_{1y_1}\|^2\mathrm{d}\tau+C_T\varepsilon^{-1}\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1}\Big\|^2\\
&\leqslant\frac{1}{160}\int_0^\tau\|\psi_{1y_1}\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.
\end{aligned}\tag{4.5}
$$

最后一个不等号成立是因为

$$
\begin{aligned}
&\Big\|\frac{d_{2x_1}}{\widetilde{\rho}}\Big\|\leqslant\frac{1}{\nu}\|d_{2x_1}\|\leqslant\frac{\varepsilon}{\nu^{\alpha+1}\delta^2},\\
&\Big\|\frac{d_1\overline{u}_{1x_1}}{\widetilde{\rho}}\Big\|\leqslant\frac{1}{\nu}\|\overline{u}_{1x_1}\|_\infty\|d_1\|\leqslant\frac{\varepsilon}{\nu^{\frac{\alpha}{2}+1}\delta^2},\\
&\Big\|\frac{d_2\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\|\leqslant\frac{1}{\nu^{\frac{\alpha}{2}+1}}\|\overline{u}_{1x_1}\|_\infty\|d_2\|+\frac{1}{\nu^2}\|d_{1x_1}\|_\infty\|d_2\|\leqslant\frac{\varepsilon}{\nu^{\alpha+1}\delta^2},
\end{aligned}
$$

故有 $\big\|\big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\big)_{x_1}\big\|\leqslant\frac{\varepsilon}{\nu^{\alpha+1}\delta^2}$. 再估计 I_2, 有

$$
\begin{aligned}
|\mathrm{I}_2|&=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\overline{\rho}}\Big)_{y_1}\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&\leqslant\frac{C}{\nu^{\frac12}}\int_0^\tau\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\overline{\rho}}\Big)_{y_1}\Big\|\cdot\|\overline{\rho}^{\frac12}\psi_1\|\mathrm{d}\tau\\
&\leqslant\frac{C}{\nu^{\frac12}}\int_0^\tau\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\overline{\rho}}\Big)_{y_1}\Big\|\mathrm{d}\tau\cdot\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac12}\psi_1\|
\end{aligned}
$$

$$\leqslant \frac{1}{160}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2+\frac{C_T}{\nu\varepsilon^2}\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\overline{\rho}}\Big)_{x_1}\Big\|^2$$
$$\leqslant \frac{1}{160}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2+\frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.$$

I_2 估计的最后一个不等号用到下面结果

$$\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{x_1}=\frac{(-d_1\overline{u}_1+d_2)^2_{x_1}}{\widetilde{\rho}}-\frac{(-d_1\overline{u}_1+d_2)^2\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2},$$
$$\Big\|\frac{d_2d_{2x_1}}{\widetilde{\rho}}\Big\|\leqslant\frac{1}{\nu}\|d_2\|_\infty\|d_{2x_1}\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{7}{2}}},$$
$$\Big\|\frac{d_1^2\overline{u}_1\overline{u}_{1x_1}}{\widetilde{\rho}}\Big\|\leqslant\frac{1}{\nu}\|\overline{u}_{1x_1}\|_\infty\|d_1\|_\infty\|d_1\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{7}{2}}},$$
$$\Big\|\frac{d_2^2\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\|\leqslant\frac{1}{\nu^{\frac{\alpha}{2}+1}}\|\overline{u}_{1x_1}\|_\infty\|d_2\|_\infty\|d_2\|+\frac{1}{\nu^2}\|d_2\|^2_\infty\|d_{2x_1}\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{7}{2}}},$$

因此有 $\big\|\big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\big)_{x_1}\big\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{7}{2}}}$. 接着估计 I_3, 有

$$|\mathrm{I}_3|=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$\leqslant\frac{C}{\nu^{\frac{1}{2}}}\int_0^\tau\|(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\|\mathrm{d}\tau\cdot\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|$$
$$\leqslant\frac{1}{160}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2+\frac{C_T}{\nu\varepsilon^2}\sup_{0\leqslant t\leqslant T}\|(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{x_1}\|^2$$
$$\leqslant\frac{1}{160}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2+\frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.$$

最后一个不等号是因为

$$(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{x_1}=(p'(\widetilde{\rho})-p'(\overline{\rho})-p''(\overline{\rho})d_1)\widetilde{\rho}_{x_1}+p''(\overline{\rho})d_1d_{1x_1},$$
$$\|(p'(\widetilde{\rho})-p'(\overline{\rho})-p''(\overline{\rho})d_1)\widetilde{\rho}_{x_1}\|\leqslant\frac{1}{\nu}\|\overline{u}_{1x_1}\|_\infty\|d_1\|_\infty\|d_1\|+\|d_{1x_1}\|_\infty\|d_1\|_\infty\|d_1\|$$
$$\leqslant\frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{7}{2}}},$$
$$\|p''(\overline{\rho})d_1d_{1x_1}\|\leqslant\frac{1}{\nu}\|d_{1x_1}\|_\infty\|d_1\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{7}{2}}},$$

因而 $\|(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{x_1}\|\leqslant\frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{7}{2}}}$. 然后估计 I_4, 得到

$$|\mathrm{I}_4|=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}(2\mu+\lambda)\frac{\overline{u}_{1y_1y_1}\phi\psi_1}{\widetilde{\rho}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$\leqslant\frac{C}{\nu^{\frac{\alpha}{2}+1}}\int_0^\tau\|\overline{u}_{y_1y_1}\|_\infty\cdot\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|\cdot\|\overline{\rho}^{\frac{1}{2}}\psi_1\|\mathrm{d}\tau$$
$$\leqslant\frac{C_T\varepsilon}{\nu^{\frac{\alpha}{2}+1}\delta^2}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|\cdot\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|$$
$$\leqslant\frac{1}{160}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2+\frac{C_T\varepsilon^2}{\nu^{\alpha+2}\delta^4}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2.$$

由先验假设 $(3.4)_2$ 知 $\|\phi\|_\infty \leqslant \frac{\nu}{4} \leqslant \overline{\rho} \sim \widetilde{\rho} \sim \rho$, 及 $\mathrm{I}_2, \mathrm{I}_3$ 的估计, 则有 $|\mathrm{I}_5| + |\mathrm{I}_6| \leqslant \frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}$. 然后对于 I_7, 有以下估计

$$\begin{aligned}|\mathrm{I}_7| &= \Big|\int_0^\tau \iint_{\Omega_\varepsilon} \rho\psi_1^2\Big(\frac{-d_1\overline{u}_1+d_2}{\rho}\Big)_{y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\ &\leqslant C_T\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\rho}\Big)_{x_1}\Big\|_\infty\cdot\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac12}\psi_1\|^2\\ &\leqslant \frac{C_T\varepsilon}{\nu^{\frac54\alpha+1}\delta^{\frac52}}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac12}\psi_1\|^2.\end{aligned}$$

最后一个不等号成立是因为

$$\begin{aligned}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1} &= \frac{(-d_1\overline{u}_1+d_2)_{x_1}}{\widetilde{\rho}} - \frac{(-d_1\overline{u}_1+d_2)\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2},\\ \Big\|\frac{d_{2x_1}}{\widetilde{\rho}}\Big\|_\infty &\leqslant \frac{1}{\nu}\|d_{2x_1}\|_\infty \leqslant \frac{\varepsilon}{\nu^{\frac54\alpha+1}\delta^{\frac52}},\\ \Big\|\frac{d_1\overline{u}_{1x_1}}{\widetilde{\rho}}\Big\|_\infty &\leqslant \frac{1}{\nu}\|\overline{u}_{1x_1}\|_\infty\|d_1\|_\infty \leqslant \frac{\varepsilon}{\nu^{\frac34\alpha+1}\delta^{\frac52}},\\ \Big\|\frac{d_2\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\|_\infty &\leqslant \frac{1}{\nu^{\frac{\alpha}{2}+1}}\|\overline{u}_{1x_1}\|_\infty\|d_2\|_\infty + \frac{1}{\nu^2}\|d_{1x_1}\|_\infty\|d_2\|_\infty \leqslant \frac{\varepsilon}{\nu^{\frac54\alpha+1}\delta^{\frac52}},\end{aligned}$$

因此有 $\big\|\big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\big)_{x_1}\big\|_\infty \leqslant \frac{\varepsilon}{\nu^{\frac54\alpha+1}\delta^{\frac52}}$. 最后估计 I_8, 可得

$$\begin{aligned}|\mathrm{I}_8| &= \Big|\int_0^\tau \iint_{\Omega_\varepsilon}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}(p(\rho)-p(\widetilde{\rho})-p'(\widetilde{\rho})\phi)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\ &\leqslant C\int_0^\tau \Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1}\Big\|_\infty\cdot\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2\mathrm{d}\tau\\ &\leqslant C_T\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1}\Big\|_\infty\sup_{0\leqslant t\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2\\ &\leqslant \frac{C_T\varepsilon}{\nu^{\frac54\alpha+1}\delta^{\frac52}}\sup_{0\leqslant t\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2.\end{aligned}$$

综上 $\mathrm{I}_i, 1\leqslant i\leqslant 8$ 的估计, 如果 $\frac{\varepsilon}{\nu^{2\alpha+2}\delta^4} \leqslant \varepsilon^{\alpha'}$, α' 是任意小的正常数, 则有粘性误差项 I_1 确定最终的收敛率, 结合(4.4), 引理得证.

注 4.1 以上关于 $\mathrm{I}_i, 1\leqslant i\leqslant 8$ 的估计都需要 $\widetilde{\rho}$ 的正下界, 因此在真空附近对 2-稀疏波截断, 这样就有正下界 ν. 积分不等式 (4.4) 需要对 $(\overline{\rho}, \overline{u}_1)$ 的导数做估计, 但截断 2-稀疏波只是 Lipschitz 连续的, 故将 2-稀疏波磨光是有必要的. 这里引入双曲波 d_1, d_2 也是至关重要的, 因为稀疏波粘性项的存在会导致衰减率不足.

引理 4.2 存在正常数 C_T, 对任意 $\tau(\varepsilon)\in\big(0,\frac{T}{\varepsilon}\big)$, 有

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi, \overline{\rho}^{\frac12}\Psi)(\tau)\|^2 + \|(\nabla\phi,\nabla\Psi)(\tau)\|^2]\\ &\quad+\int_0^{\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}\phi, \overline{\rho}^{\frac12}\overline{u}_{1y_1}\psi_1)\|^2 + \|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2 + \|\nabla\Psi\|_1^2]\mathrm{d}\tau\\ &\leqslant E_0^2 + \frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.\end{aligned}$$

证 对 $(3.2)_1$ 求梯度 ∇, 再乘以 $\frac{\nabla\phi}{\rho^2}$, 有

$$\begin{aligned}&\Big(\frac{|\nabla\phi|^2}{2\rho^2}\Big)_\tau+\mathrm{div}\Big(\frac{u|\nabla\phi|^2}{2\rho^2}\Big)+\frac{\nabla\mathrm{div}\,\Psi\cdot\nabla\phi}{\rho}\\&=\frac{|\nabla\phi|^2\mathrm{div}\,\Psi}{2\rho^2}+\frac{\widetilde{u}_{1y_1}|\nabla\phi|^2}{2\rho^2}-\frac{\widetilde{\rho}_{y_1}\phi_{y_1}\mathrm{div}\,\Psi}{\rho^2}-\frac{\phi_{y_i}\nabla\phi\nabla\psi_i}{\rho^2}-\frac{\widetilde{u}_{1y_1}\phi_{y_1}^2}{\rho^2}\\&\quad-\frac{\widetilde{\rho}_{y_1}\nabla\phi\cdot\nabla\psi_1}{\rho^2}-\frac{\widetilde{\rho}_{y_1y_1}\phi_{y_1}\psi_1}{\rho^2}-\frac{\widetilde{u}_{1y_1y_1}\phi_{y_1}\phi}{\rho^2}\\&:=G(\tau,y_1,y_2).\end{aligned}\tag{4.6}$$

对 $(3.2)_2$ 乘以 $\frac{\nabla\phi}{\rho}$, 可得

$$\begin{aligned}&(\Psi\cdot\nabla\phi)_\tau+\mathrm{div}\Big(-\Psi\phi_\tau+u_i\Psi_{y_i}\phi-u\phi\mathrm{div}\,\Psi-\frac{\mu}{\rho}\nabla\Psi_i\phi_{y_i}\Big)\\&\quad+\Big(\frac{\mu}{\rho}\nabla\psi_i\cdot\nabla\phi\Big)_{y_i}+\frac{p'(\rho)}{\rho}|\nabla\phi|^2-\frac{(2\mu+\lambda)\nabla\mathrm{div}\,\Psi\nabla\phi}{\rho}\\&=\widetilde{\rho}|\mathrm{div}\,\Psi|^2+\phi\Psi_{y_i}\nabla\psi_i+\widetilde{\rho}_{y_1}\psi_1\mathrm{div}\,\Psi+\widetilde{u}_{1y_1}\phi\psi_{1y_1}-\widetilde{u}_{1y_1}\psi_1\phi_{y_1}\\&\quad-\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\nabla\psi_1\nabla\phi+\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\Psi_{y_1}\nabla\phi-\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1}\phi_{y_1}\\&\quad-(2\mu+\lambda)\frac{\overline{u}_{1y_1y_1}\phi_{y_1}\phi}{\widetilde{\rho}\rho}-\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\phi_{y_1}\phi}{\widetilde{\rho}\rho}\\&\quad-\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}\phi_{y_1}\phi}{\widetilde{\rho}\rho}+\frac{2\mu+\lambda}{\rho}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\phi_{y_1}\\&\quad-\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}}{\rho}\phi_{y_1}-\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}}{\rho}\phi_{y_1}\\&:=\widetilde{\rho}|\mathrm{div}\,\Psi|^2+H(\tau,y_1,y_2).\end{aligned}\tag{4.7}$$

将 (4.6) 乘以 $2\mu+\lambda$ 加上 (4.7), 最后在 $[0,\tau]\times\Omega_\varepsilon$ 上积分, 可得

$$\begin{aligned}&\iint_{\Omega_\varepsilon}\Big[\frac{(2\mu+\lambda)|\nabla\phi|^2}{2\rho^2}+\Psi\cdot\nabla\phi\Big]\mathrm{d}y_1\mathrm{d}y_2\Big|_0^\tau+\int_0^\tau\iint_{\Omega_\varepsilon}\frac{p'(\rho)}{\rho}|\nabla\phi|^2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\\&=\int_0^\tau\iint_{\Omega_\varepsilon}[\widetilde{\rho}|\mathrm{div}\,\Psi|^2+(2\mu+\lambda)G(\tau,y_1,y_2)+H(\tau,y_1,y_2)]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau.\end{aligned}$$

结合引理 4.1 和上述积分式, 可得

$$\begin{aligned}&\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac12}\Psi,\overline{\rho}^{-1}\nabla\phi)(\tau)\|^2\\&\quad+\int_0^\tau[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}^{\frac12}\phi,\overline{\rho}^{\frac12}\overline{u}_{1y_1}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2+\|\nabla\Psi\|^2]\mathrm{d}\tau\\&\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}+\int_0^\tau\iint_{\Omega_\varepsilon}[(2\mu+\lambda)G(\tau,y_1,y_2)+H(\tau,y_1,y_2)]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau.\end{aligned}\tag{4.8}$$

利用 Hölder 不等式, Young 不等式, Sobolev 不等式估计 $G(\tau,y_1,y_2),H(\tau,y_1,y_2)$, 由于 $G(\tau,y_1,\ y_2),H(\tau,y_1,y_2)$ 很多项估计方法类似, 因此只估计其中的特殊项, 下面先估计 $G(\tau,y_1,y_2)$ 中的项

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{|\nabla\phi|^2}{2\rho^2}\mathrm{div}\,\Psi\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$

$$
\begin{aligned}
&\leqslant \frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau \|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|\cdot\|\nabla\phi\|_{L^4}\|\nabla\Psi\|_{L^4}\mathrm{d}\tau\\
&\leqslant \frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau \|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|\cdot\|\nabla\phi\|_1\|\nabla\Psi\|_1\mathrm{d}\tau\\
&\leqslant \frac{1}{160}\int_0^\tau \|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau+\frac{C}{\nu^{\alpha+3}}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\nabla\phi\|_1^2\int_0^\tau\|\nabla\Psi\|_1^2\mathrm{d}\tau\\
&\leqslant \frac{1}{160}\int_0^\tau \|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau\|\nabla\Psi\|_1^2\mathrm{d}\tau, \qquad (4.9)
\end{aligned}
$$

上述估计需要注意的是第二个不等号利用 Sobolve 不等式 $\|f\|_{L^4(\mathbb{R}\times\mathbb{T}_\varepsilon)}\leqslant C\|f\|_1(\mathbb{R}\times\mathbb{T}_\varepsilon)$, 其中 C 是与 ε 无关的常数, 同时有先验假设 $(3.4)_1$ 知道右端项可控. 同理可以估计 $H(\tau,y_1,y_2)$ 中的项

$$
\begin{aligned}
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\phi\Psi_{y_i}\nabla\psi_i\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\int_0^\tau\|\nabla\Psi\|\cdot\|\phi\|_{L^4}\|\nabla\psi_i\|_{L^4}\mathrm{d}\tau\\
&\leqslant\int_0^\tau\|\nabla\Psi\|\cdot\|\phi\|_1\|\nabla\psi_i\|_1\mathrm{d}\tau\leqslant\frac{1}{160}\int_0^\tau\|\nabla\Psi\|^2\mathrm{d}\tau+C\cdot E^2\int_0^\tau\|\nabla\Psi\|_1^2\mathrm{d}\tau,\\
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{(2\mu+\lambda)}{\rho}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\phi_{y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&\leqslant\frac{C}{\nu^{\frac{\alpha+1}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|\cdot\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\Big\|^2\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau+\frac{C}{\nu^{\alpha+1}}\int_0^\tau\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\|^2\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^{\alpha+1}}\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1x_1}\Big\|^2\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^2}{\nu^{2\alpha+1}\delta^2}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.
\end{aligned}
$$

最后一个不等号是因为

$$
\begin{aligned}
\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1x_1}&=\frac{(-d_1\overline{u}_1+d_2)_{x_1x_1}}{\widetilde{\rho}}-2\frac{(-d_1\overline{u}_1+d_2)_{x_1}\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\\
&\quad+(-d_1\widetilde{u}_1+d_2)\Big(-\frac{\widetilde{\rho}_{x_1x_1}}{\widetilde{\rho}^2}+2\frac{\widetilde{\rho}_{x_1}^2}{\widetilde{\rho}^3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\Big\|\frac{d_{2x_1x_1}}{\widetilde{\rho}}\Big\|&\leqslant\frac{1}{\nu}\|d_{2x_1x_1}\|\leqslant\frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3},\\
\Big\|\frac{d_{1x_1}\overline{u}_{1x_1}}{\widetilde{\rho}}\Big\|&\leqslant\frac{1}{\nu}\Big\|\overline{u}_{1x_1}\|_\infty\|d_{1x_1}\|\leqslant\frac{\varepsilon}{\nu^{\alpha+1}\delta^3},\\
\Big\|\frac{d_1\overline{u}_{1x_1x_1}}{\widetilde{\rho}}\Big\|&\leqslant\frac{1}{\nu}\|\overline{u}_{1x_1x_1}\|_\infty\|d_1\|\leqslant\frac{\varepsilon}{\nu^{\frac{\alpha}{2}+1}\delta^3},\\
\Big\|\frac{d_{2x_1}\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\|&\leqslant\frac{1}{\nu^{\frac{\alpha}{2}+1}}\|\overline{u}_{1x_1}\|_\infty\|d_{2x_1}\|+\frac{1}{\nu^2}\|d_{1x_1}\|_\infty\|d_{2x_1}\|\leqslant\frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3},\\
\Big\|\frac{d_1\overline{u}_{1x_1}\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\|&\leqslant\frac{1}{\nu^{\frac{\alpha}{2}+1}}\|u_{1x_1}\|_\infty^2\|d_1\|+\frac{1}{\nu^2}\|\overline{u}_{1x_1}\|_\infty\|d_{1x_1}\|_\infty\|d_1\|\leqslant\frac{\varepsilon}{\nu^{\alpha+1}\delta^3},\\
\Big\|\frac{d_2\widetilde{\rho}_{x_1}^2}{\widetilde{\rho}^3}\Big\|&\leqslant\frac{2}{\nu^{\alpha+1}}\|\overline{u}_{1x_1}\|_\infty^2\|d_2\|+\frac{2}{\nu^3}\|d_{1x_1}\|_\infty^2\|d_2\|\leqslant\frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3},
\end{aligned}
$$

$$\Big\|\frac{d_2\widetilde{\rho}_{x_1x_1}}{\widetilde{\rho}^2}\Big\| \leqslant \frac{1}{\nu^{\frac{\alpha}{2}+1}}\|\overline{u}_{1x_1x_1}\|\|d_1\| + \frac{1}{\nu^{\alpha+1}}\|\overline{u}_{1x_1}\|_\infty^2\|d_2\| + \frac{1}{\nu^2}\|d_2\|_\infty\|d_{1x_1x_1}\|$$
$$\leqslant \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3},$$

因此有 $\big\|\big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\big)_{x_1x_1}\big\| \leqslant \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1}\delta^3}$. 结合 (4.8) 及先验假设 (3.4), 可得

$$\begin{aligned}&\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi^2,\overline{\rho}^{\frac{1}{2}}\Psi,\overline{\rho}^{-1}\nabla\phi)(\tau)\|^2\\&+\int_0^\tau[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}^{\frac{1}{2}}\phi,\overline{\rho}^{\frac{1}{2}}\overline{u}_{1y_1}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2+\|\nabla\Psi\|^2]\mathrm{d}\tau\\&\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau\|\nabla^2\Psi\|^2\mathrm{d}\tau.\end{aligned}\tag{4.10}$$

为了封闭上式估计, 需要对 $(3.2)_2$ 乘以 $-\frac{\Delta\Psi}{\rho}$, 可得

$$\begin{aligned}&\Big(\frac{|\nabla\Psi|^2}{2}\Big)_\tau-\mathrm{div}\Big(\psi_{i\tau}\nabla\psi_i+\frac{\mu+\lambda}{\rho}\mathrm{div}\,\Psi\nabla\mathrm{div}\,\Psi-\frac{\mu+\lambda}{\rho}\mathrm{div}\,\Psi\Delta\Psi\Big)\\&+\frac{\mu}{\rho}|\Delta\Psi|^2+\frac{\mu+\lambda}{\rho}|\nabla\mathrm{div}\,\Psi|^2\\=&\,u_i\Psi_{y_i}\cdot\Delta\Psi+\frac{p'(\rho)}{\rho}\nabla\phi\cdot\Delta\Psi+\widetilde{u}_{1y_1}\psi_1\Delta\psi_1+\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1}\Delta\psi_1\\&-\frac{\mu+\lambda}{\rho^2}\mathrm{div}\,\Psi\nabla\phi\cdot\Delta\Psi+\frac{\mu+\lambda}{\rho^2}\mathrm{div}\,\Psi\nabla\phi\cdot\nabla\mathrm{div}\,\Psi\\&-\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\mathrm{div}\,\Psi\Delta\psi_1+\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\mathrm{div}\Psi\mathrm{div}\,\Psi_{y_1}\\&-\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\Delta\psi_1+\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}}{\rho}\Delta\psi_1\\&+\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}}{\rho}\Delta\psi_1+\frac{2\mu+\lambda}{\widetilde{\rho}\rho}\overline{u}_{1y_1y_1}\phi\Delta\psi_1\\&+\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}}{\widetilde{\rho}\rho}\phi\Delta\psi_1+\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}}{\widetilde{\rho}\rho}\phi\Delta\psi_1\\:=&\,K(\tau,y_1,y_2).\end{aligned}$$

再将上式在 $[0,t]\times\Omega_\varepsilon$ 上积分, 可得

$$\|\nabla\Psi(\tau)\|^2+\int_0^\tau\|\rho^{-\frac{1}{2}}\Delta\Psi\|^2\mathrm{d}\tau\leqslant E_0^2+C\Big|\int_0^\tau\iint_{\Omega_\varepsilon}K(\tau,y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|.\tag{4.11}$$

同样只估计 $K(\tau,y_1,y_2)$ 中的某些特殊项

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}u_i\Psi_{y_i}\cdot\Delta\Psi\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{1}{160}\int_0^\tau\Big\|\nabla^2\Psi\Big\|^2\mathrm{d}\tau+C\int_0^\tau\|\nabla\Psi\|^2\mathrm{d}\tau,$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{p'(\rho)}{\rho}\nabla\phi\Delta\Psi\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac{1}{2}}\Delta\Psi\|^2\mathrm{d}\tau+C\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2\mathrm{d}\tau,$$

$$\begin{aligned}&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\widetilde{u}_{1y_1}\psi_1\nabla\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac{1}{2}}\Delta\Psi\|^2\mathrm{d}\tau+C\int_0^\tau\|\widetilde{u}_{1y_1}\|_\infty^2\|\overline{\rho}^{\frac{1}{2}}\psi_1\|\mathrm{d}\tau\\&\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac{1}{2}}\Delta\Psi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\delta^2}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{1}{2}}\psi_1\|^2.\end{aligned}$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1}\Delta\psi_1\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{C}{\nu}\int_0^\tau\|\rho^{-\frac12}\Delta\Psi\|\cdot\|\widetilde{\rho}_{y_1}\|_\infty\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\Delta\Psi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^{\alpha+2}\delta^2}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2.$$
$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{\mu+\lambda}{\rho^2}\mathrm{div}\,\Psi\nabla\phi\cdot\Delta\Psi\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{C}{\nu^{\frac32}}\int_0^\tau\|\nabla\Psi\|_{L^4}\|\nabla\phi\|_{L^4}\|\rho^{-\frac12}\Delta\Psi\|\mathrm{d}\tau$$
$$\leqslant\frac{C}{\nu^{\frac32}}\int_0^\tau\|\nabla\Psi\|_1\|\nabla\phi\|_1\|\rho^{-\frac12}\Delta\Psi\|\mathrm{d}\tau\leqslant\frac{1}{160}\int_0^\tau\|\nabla^2\Psi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^3}\int_0^\tau\|\nabla\Psi\|_1^2\mathrm{d}\tau.$$

由于 ρ 有上界和椭圆估计, 则 $\|\rho^{-\frac12}\Delta\Psi\|^2\geqslant\|\Delta\Psi\|^2\sim\|\nabla^2\Psi\|^2$, $\|\rho^{-\frac12}\nabla\mathrm{div}\,\Psi\|^2\geqslant\|\nabla\mathrm{div}\,\Psi\|^2\sim\|\nabla^2\Psi\|^2$,

结合积分不等式 (4.10)–(4.11), 引理 4.1 和先验假设 (3.4), 引理可证.

引理 4.3 存在正常数 C_T, 对任意 $\tau(\varepsilon)\in\big(0,\frac{T}{\varepsilon}\big)$, 有

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac12}\Psi)(\tau)\|^2+\|(\nabla\phi,\nabla\Psi)(\tau)\|_1^2]\\&+\int_0^{\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}\phi,\overline{\rho}^{\frac12}\overline{u}_{1y_1}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}(\nabla\phi,\nabla^2\phi)\|^2+\|\nabla\Psi\|_2^2]\mathrm{d}\tau\\&\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.\end{aligned}$$

证 对 $(3.2)_1$ 中求 2 阶导 ∇^2, 再乘以 $\frac{\nabla^2\phi}{\rho^2}$, 可得

$$\begin{aligned}&\Big(\frac{|\nabla^2\phi|^2}{2\rho^2}\Big)_\tau+\mathrm{div}\Big(\frac{u|\nabla^2\phi|^2}{2\rho^2}\Big)+\frac{\nabla^2\phi\cdot\nabla^2\mathrm{div}\,\Psi}{\rho}\\&=\frac{\mathrm{div}\,\Psi|\nabla^2\phi|^2}{2\rho^2}-\frac{\widetilde{\rho}_{y_1y_1}\mathrm{div}\,\Psi\phi_{y_1y_1}}{\rho^2}-\frac{\phi_{y_i}\nabla\phi_{y_i}\cdot\nabla\mathrm{div}\,\Psi}{\rho^2}-\frac{2\widetilde{\rho}_{y_1}\nabla\phi_{y_1}\cdot\nabla\mathrm{div}\,\Psi}{\rho^2}\\&\quad-\frac{\nabla\phi\cdot\nabla\phi_{y_i}\mathrm{div}\,\Psi_{y_i}}{\rho^2}-\frac{\nabla\psi_j\cdot\nabla\phi_{y_i}\phi_{y_iy_j}}{\rho^2}-\frac{\psi_{jy_i}\nabla\phi_{y_j}\cdot\nabla\phi_{y_i}}{\rho^2}-\frac{\phi_{y_i}\nabla^2\phi\cdot\nabla^2\psi_i}{\rho^2}\\&\quad-\frac{\widetilde{u}_{1y_1y_1}\phi_{y_1}\phi_{y_1y_1}}{\rho^2}-\frac{2\widetilde{u}_{1y_1}|\nabla\phi_{y_1}|^2}{\rho^2}-\frac{\widetilde{\rho}_{y_1y_1y_1}\psi_1\phi_{y_1y_1}}{\rho^2}-\frac{2\widetilde{\rho}_{y_1y_1}\nabla\psi_1\cdot\nabla\phi_{y_1}}{\rho^2}\\&\quad-\frac{\widetilde{\rho}_{y_1}\nabla^2\psi_1\cdot\nabla^2\phi}{\rho^2}-\frac{\widetilde{u}_{1y_1y_1y_1}\phi\phi_{y_1y_1}}{\rho^2}-\frac{2\widetilde{u}_{1y_1y_1}\nabla\phi\cdot\nabla\phi_{y_1}}{\rho^2}+\frac{\widetilde{u}_{1y_1}|\nabla^2\phi|^2}{2\rho^2}\\&:=L(\tau,y_1,y_2).\end{aligned}\tag{4.12}$$

对 $(3.2)_2$ 除以 ρ, 求梯度 ∇, 乘以 $\nabla^2\phi$, 可得

$$\begin{aligned}&(\nabla\Psi\cdot\nabla^2\phi)_\tau-\mathrm{div}\Big(\phi_{y_i\tau}\nabla\psi_i-u_i\phi_{y_j}\nabla\psi_{jy_i}+u\nabla\phi\cdot\Delta\Psi+\frac{\mu}{\rho}\phi_{y_iy_j}\nabla\psi_{iy_j}\Big)\\&+\Big(\frac{\mu}{\rho}\nabla\phi_{y_j}\cdot\nabla\psi_{iy_j}\Big)_{y_i}+\frac{p'(\rho)}{\rho}|\nabla^2\phi|^2-\frac{2\mu+\lambda}{\rho}\nabla^2\phi\cdot\nabla^2\mathrm{div}\,\Psi\\&=\rho\nabla\mathrm{div}\,\Psi\cdot\Delta\Psi+\widetilde{\rho}_{y_1}\mathrm{div}\,\Psi\Delta\psi_1+\phi_{y_j}\nabla\psi_i\cdot\nabla\psi_{jy_i}+\widetilde{u}_{1y_1}\nabla\phi\cdot\Psi_{y_1y_1}\\&\quad+\phi_{y_i}\nabla\psi_i\cdot\Delta\Psi+\widetilde{u}_{1y_1}\phi_{y_1}\Delta\psi_1+\widetilde{\rho}_{y_1y_1}\psi_1\Delta\psi_1+\widetilde{\rho}_{y_1}\nabla\psi_1\cdot\Delta\Psi\\&\quad+\widetilde{u}_{1y_1y_1}\phi\Delta\psi_1-\psi_{jy_i}\nabla\psi_i\cdot\nabla\phi_{y_i}-\widetilde{u}_{1y_1}\psi_{jy_1}\phi_{y_1y_1}-\widetilde{u}_{1y_1y_1}\psi_1\phi_{y_1y_1}\end{aligned}$$

$$
\begin{aligned}
&-\widetilde{u}_{1y_1}\nabla\psi_1\cdot\nabla\phi_{y_1}-\Big(\frac{p'(\rho)}{\rho}\Big)'\phi_{y_i}\nabla\phi\cdot\nabla\phi_{y_i}-2\Big(\frac{p'(\rho)}{\rho}\Big)'\widetilde{\rho}_{y_1}\nabla\phi\cdot\nabla\phi_{y_1}\\
&-\Big[\Big(\frac{p'(\rho)}{\rho}\Big)'-\Big(\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)'\Big]\widetilde{\rho}_{y_1}^2\phi_{y_1y_1}-\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1y_1}\phi_{y_1y_1}\\
&+\frac{\mu}{\rho^2}\nabla\phi\cdot\nabla\psi_{iy_j}\phi_{y_iy_j}-\frac{\mu}{\rho^2}\phi_{y_i}\nabla\phi_{y_j}\cdot\nabla\psi_{iy_j}+\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\psi_{iy_1y_j}\phi_{y_iy_j}\\
&-\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\nabla\phi_{y_j}\cdot\nabla\psi_{1y_j}-\frac{\mu}{\rho^2}\nabla\phi\cdot\nabla\phi_{y_j}\Delta\psi_i-\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\phi_{y_1y_i}\Delta\psi_i\\
&-\frac{\mu+\lambda}{\rho^2}\nabla\phi\cdot\nabla\phi_{y_i}\mathrm{div}\,\Psi_{y_i}-\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\phi_{y_1y_i}\mathrm{div}\Psi_{y_i}\\
&+\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1y_1}\phi_{y_1y_1}-\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\nabla\phi\cdot\nabla\phi_{y_1}\\
&-\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\widetilde{\rho}_{y_1}\phi_{y_1y_1}-\frac{2\mu+\lambda}{\widetilde{\rho}\rho}\overline{u}_{1y_1y_1y_1}\phi\nabla\phi_{y_1y_1}\\
&-\frac{2\mu+\lambda}{\rho^2}\overline{u}_{1y_1y_1}\nabla\phi\cdot\nabla\phi_{y_1}+\frac{(2\mu+\lambda)(\widetilde{\rho}+\rho)}{\widetilde{\rho}^2\rho^2}\widetilde{\rho}_{y_1}\overline{u}_{1y_1y_1}\phi\phi_{y_1y_1}\\
&-\Big(\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}}{\widetilde{\rho}}\Big)_{y_1}\phi_{y_1y_1}-\Big(\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}}{\widetilde{\rho}}\Big)_{y_1}\phi_{y_1y_1}\\
:={}&\rho\nabla\mathrm{div}\,\Psi\cdot\Delta\Psi+M(\tau,y_1,y_2).
\end{aligned}\tag{4.13}
$$

将 (4.12) 乘以 $2\mu+\lambda$ 与 (4.13) 相加, 再在 $[0,\tau]\times\Omega_\varepsilon$ 上积分, 可得

$$
\begin{aligned}
&\iint_{\Omega_\varepsilon}\Big(\frac{2\mu+\lambda}{2\rho^2}|\nabla^2\phi|^2+\nabla\Psi\cdot\nabla^2\phi\Big)\mathrm{d}y_1\mathrm{d}y_2\Big|_0^\tau+\int_0^\tau\iint_{\Omega_\varepsilon}\frac{p'(\rho)}{\rho}|\nabla^2\phi|^2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\\
={}&\int_0^\tau\iint_{\Omega_\varepsilon}[\rho\nabla\mathrm{div}\,\Psi\cdot\Delta\Psi+(2\mu+\lambda)L(\tau,y_1,y_2)+M(\tau,y_1,y_2)]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau.
\end{aligned}
$$

结合上式和引理 4.2, 可得

$$
\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac12}\Psi)\|^2+\|(\overline{\rho}^{-1}\nabla\phi,\nabla\Psi)\|^2+\|\overline{\rho}^{-1}\nabla^2\phi\|^2]\\
&+\int_0^{\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}\phi,\overline{\rho}^{\frac12}\overline{u}_{1y_1}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}(\nabla\phi,\nabla^2\phi)\|^2+\|\nabla\Psi\|_1^2]\mathrm{d}\tau\\
\leqslant{}&E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}+C\Big|\int_0^\tau\iint_{\Omega_\varepsilon}[(2\mu+\lambda)L(\tau,y_1,y_2)+M(\tau,y_1,y_2)]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|.
\end{aligned}\tag{4.14}
$$

利用 Hölder 不等式, Young 不等式以及 Sobolev 不等式估计 $L(\tau,y_1,y_2)$, $M(\tau,y_1,y_2)$. 先估计 $L(\tau,y_1,y_2)$ 中的特殊项

$$
\begin{aligned}
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{\mathrm{div}\,\Psi\cdot|\nabla^2\phi|^2}{2\rho^2}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
\leqslant{}&\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\nabla^2\phi\|\cdot\|\nabla\Psi\|_\infty\mathrm{d}\tau\\
\leqslant{}&\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\nabla^2\phi\|\cdot\|\nabla\Psi\|_2\mathrm{d}\tau\\
\leqslant{}&\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau\|\nabla\Psi\|_2^2\mathrm{d}\tau,
\end{aligned}
$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{\phi_{y_i}\nabla\phi_{y_i}\nabla\mathrm{div}\,\Psi}{\rho^2}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\|\nabla\phi\|_{L^4}\|\nabla^2\Psi\|_{L^4}\mathrm{d}\tau$$
$$\leqslant\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\nabla\phi\|_1\|\nabla^2\Psi\|_1\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau\|\nabla^2\Psi\|_1^2\mathrm{d}\tau.$$

下面再估计 $M(\tau,y_1,y_2)$ 中某些特殊项

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big(\frac{p'(\rho)}{\rho}\Big)'\phi_{y_i}\nabla\phi\nabla\phi_{y_i}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\leqslant\frac{C}{\nu^{\frac{3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\nabla\phi\|_{L^4}^2\mathrm{d}\tau$$
$$\leqslant\frac{C}{\nu^{\frac{3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\nabla\phi\|_1^2\mathrm{d}\tau\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^3}\int_0^\tau\|\nabla\phi\|_1^2\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau(\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2)\mathrm{d}\tau,$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big[\Big(\frac{p'(\rho)}{\rho}\Big)'-\Big(\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)'\Big]\widetilde{\rho}_{y_1}^2\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$\leqslant\frac{C}{\nu^2}\int_0^\tau\|\widetilde{\rho}_{y_1}\|_\infty^2\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^3}{\nu^4}\sup_{0\leqslant t\leqslant T}\|\widetilde{\rho}_{x_1}\|_\infty^4\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^3}{\nu^{2\alpha+4}\delta^4}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2,$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\widetilde{\rho}_{y_1}\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$\leqslant\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\widetilde{\rho}_{y_1}\|_\infty\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\Big\|\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{C}{\nu^{\alpha+3}}\int_0^\tau\|\widetilde{\rho}_{y_1}\|_\infty^2\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\Big\|^2\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon^3}{\nu^{\alpha+3}}\sup_{0\leqslant t\leqslant T}\Big(\|\widetilde{\rho}_{x_1}\|_\infty^2\Big\|\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{x_1x_1}\Big\|^2\Big)$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^4}{\nu^{3\alpha+3}\delta^4}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4},$$

$$\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1y_1}}{\widetilde{\rho}}\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{1}{\widetilde{\rho}}\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1y_1}\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|$$
$$\leqslant\frac{C}{\nu^{\frac{\alpha+1}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1y_1}\Big\|\mathrm{d}\tau$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^{\alpha+1}}\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{x_1x_1}\Big\|^2$$
$$\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^4}{\nu^{\frac{7}{2}\alpha+1}\delta^5}\frac{\varepsilon}{\nu^{2\alpha+2}\delta^4},$$

其中用到下面结论

$$\Big\|\frac{d_2 d_{2x_1x_1}}{\widetilde{\rho}}\Big\| \leqslant \frac{1}{\nu}\|d_2\|_\infty \|d_{2x_1x_1}\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\Big\|\frac{d_2 d_{1x_1}\overline{u}_{1x_1}}{\widetilde{\rho}}\Big\| \leqslant \frac{1}{\nu\delta}\|d_2\|_\infty \|d_{1x_1}\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\Big\|\frac{d_2 d_1\overline{u}_{1x_1x_1}}{\widetilde{\rho}}\Big\| \leqslant \frac{1}{\nu\delta^2}\|d_2\|_\infty \|d_1\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1}\delta^{9}},$$

$$\Big\|\frac{d_{2x_1}^2}{\widetilde{\rho}}\Big\| \leqslant \frac{1}{\nu}\|d_{2x_1}\|_\infty \|d_{2x_1}\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\Big\|\frac{d_1^2\overline{u}_{1x_1}^2}{\widetilde{\rho}}\Big\| \leqslant \frac{1}{\nu\delta^2}\|d_1\|_\infty \|d_1\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\Big\|\frac{d_2 d_{2x_1}\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\| \leqslant \frac{1}{\nu^{\frac{\alpha}{2}+1}\delta}\|d_2\|_\infty \|d_{2x_1}\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\Big\|\frac{d_2 d_1\overline{u}_{1x_1}\widetilde{\rho}_{x_1}}{\widetilde{\rho}^2}\Big\| \leqslant \frac{1}{\nu^{\frac{\alpha}{2}+1}\delta^2}\|d_2\|_\infty \|d_1\| + \frac{1}{\nu^2\delta}\|d_2\|_\infty\|d_{1x_1}\|_\infty\|d_2\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

$$\begin{aligned}\Big\|\frac{d_2^2\widetilde{\rho}_{x_1x_1}}{\widetilde{\rho}^2}\Big\| &\leqslant \Big(\frac{1}{\nu^{\frac{\alpha}{2}+1}\delta^2}+\frac{1}{\nu^{\alpha+1}\delta^2}\Big)\|d_2\|_\infty\|d_2\| + \frac{1}{\nu^2}\|d_2\|_\infty\|d_{1x_1x_1}\|_\infty\|d_2\|\\ &\leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}},\end{aligned}$$

$$\Big\|\frac{d_2^2\widetilde{\rho}_{x_1}^2}{\widetilde{\rho}^3}\Big\| \leqslant \frac{1}{\nu^{\alpha+1}\delta^2}\|d_2\|_\infty \|d_2\| + \frac{1}{\nu^3}\|d_2\|_\infty\|d_{1x_1}\|_\infty^2\|d_2\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}},$$

因此有 $\|(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}})_{x_1x_1}\| \leqslant \frac{\varepsilon^2}{\nu^{\frac{9}{4}\alpha+1}\delta^{\frac{9}{2}}}$. 接下来, 有

$$\begin{aligned}&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2 d_1-2\overline{u}_1 d_2)_{y_1}\widetilde{\rho}_{y_1}}{\widetilde{\rho}^2}\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&=\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{\widetilde{\rho}_{y_1}}{\widetilde{\rho}^2}\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1}\phi_{y_1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&\leqslant\frac{C}{\nu^{\frac{\alpha+3}{2}}}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|\cdot\|\widetilde{\rho}_{y_1}\|_\infty\cdot\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1}\Big\|\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{C}{\nu^{\alpha+3}}\int_0^\tau\|\widetilde{\rho}_{y_1}\|_\infty^2\cdot\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1}\Big\|^2\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^{\alpha+3}}\sup_{0\leqslant t\leqslant T}\|\widetilde{\rho}_{x_1}\|_\infty^2\cdot\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{x_1}\Big\|^2\\
&\leqslant\frac{1}{160}\int_0^\tau\|\overline{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\|^2\mathrm{d}\tau+\frac{\varepsilon^4}{\nu^{\frac{7}{2}\alpha+3}\delta^5}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.\end{aligned}$$

结合 (4.14), 可得

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\phi,\overline{\rho}^{\frac{1}{2}}\Psi)\|^2+\|(\overline{\rho}^{-1}\nabla\phi,\nabla\Psi)\|^2+\|\overline{\rho}^{-1}\nabla^2\phi\|^2]\\
&\quad+\int_0^{\tau(\varepsilon)}\Big[\|(\overline{\rho}^{\frac{\gamma-2}{2}}\overline{u}_{1y_1}\phi,\overline{\rho}^{\frac{1}{2}}\overline{u}_{1y_1}\psi_1)\|^2+\|\overline{\rho}^{\frac{\gamma-2}{2}}(\nabla\phi,\nabla^2\phi)\|^2+\|\nabla\Psi\|_1^2\Big]\mathrm{d}\tau\\
&\leqslant E_0^2+\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}+\frac{E^2}{\nu^{\alpha+3}}\int_0^\tau\|\nabla^3\Psi\|^2\mathrm{d}\tau.\end{aligned}\tag{4.15}$$

为了封闭上式, 需要对 (3.2) 中 Ψ 方程除以 ρ, 求梯度 ∇, 然后乘以 $-\nabla\Delta\Psi$, 可得

$$
\begin{aligned}
&\Big(\frac{|\nabla^2\Psi|^2}{2}\Big)_\tau-\mathrm{div}\Big(\psi_{i\tau y_j}\nabla\psi_{iy_j}+\frac{\mu+\lambda}{\rho}\mathrm{div}\,\Psi_{y_i}\nabla\mathrm{div}\,\Psi_{y_i}-\frac{\mu+\lambda}{\rho}\mathrm{div}\,\Psi_{y_i}\Delta\Psi_{y_i}\Big)\\
&\quad+\frac{\mu}{\rho}|\nabla\Delta\Psi|^2+\frac{\mu+\lambda}{\rho}|\nabla^2\mathrm{div}\,\Psi|^2\\
&=u_i\nabla\Psi_{y_i}\cdot\nabla\Delta\Psi+\frac{p'(\rho)}{\rho}\nabla^2\phi\cdot\nabla\Delta\Psi+\psi_{jy_i}\nabla\psi_i\cdot\nabla\Delta\psi_j+\widetilde{u}_{1y_1}\Psi_{y_1}\cdot\Delta\Psi_{y_1}\\
&\quad+\widetilde{u}_{1y_1y_1}\psi_1\Delta\psi_{1y_1}+\widetilde{u}_{1y_1}\nabla\psi_1\cdot\nabla\Delta\psi_1+\Big(\frac{p'(\rho)}{\rho}\Big)'\phi_{y_i}\nabla\phi\cdot\nabla\Delta\psi_i\\
&\quad+\Big(\frac{p'(\rho)}{\rho}\Big)'\widetilde{\rho}_{y_1}\nabla\phi\cdot\Delta\Psi_{y_1}+\Big(\frac{p'(\rho)}{\rho}\Big)'\widetilde{\rho}_{y_1}\nabla\cdot\nabla\Delta\psi_i+\Big[\Big(\frac{p'(\rho)}{\rho}\Big)'-\Big(\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)'\Big]\widetilde{\rho}_{y_1}^2\Delta\psi_{1y_1}\\
&\quad+\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1y_1}\Delta\psi_{1y_1}+\frac{\mu}{\rho^2}\Delta\psi_i\nabla\phi\cdot\nabla\Delta\psi_i+\frac{\mu}{\rho^2}\widetilde{\rho}_{y_1}\Delta\Psi\cdot\Delta\Psi_{y_1}\\
&\quad-\frac{\mu+\lambda}{\rho^2}\phi_{y_i}\nabla\mathrm{div}\,\Psi\cdot\nabla\Delta\psi_i-\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\nabla\mathrm{div}\,\Psi\cdot\nabla\Delta\psi_1+\frac{\mu+\lambda}{\rho^2}\mathrm{div}\,\Psi_{y_i}\nabla\phi\cdot\nabla\mathrm{div}\,\Psi_{y_i}\\
&\quad+\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\nabla\mathrm{div}\,\Psi\cdot\nabla\mathrm{div}\,\Psi_{y_1}+\frac{\mu+\lambda}{\rho^2}\mathrm{div}\,\Psi_{y_i}\nabla\phi\cdot\nabla\Delta\psi_i+\frac{\mu+\lambda}{\rho^2}\widetilde{\rho}_{y_1}\Delta\Psi_{y_1}\cdot\nabla\mathrm{div}\,\Psi\\
&\quad-\frac{2\mu+\lambda}{\rho}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1y_1}\Delta\psi_{1y_1}+\frac{2\mu+\lambda}{\rho}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\nabla\phi\cdot\nabla\Delta\psi_1\\
&\quad+\frac{2\mu+\lambda}{\rho^2}\Big(\frac{-d_1\overline{u}_1+d_2}{\widetilde{\rho}}\Big)_{y_1y_1}\widetilde{\rho}_{y_1}\Delta\psi_{1y_1}+\frac{2\mu+\lambda}{\rho\widetilde{\rho}}\overline{u}_{1y_1y_1y_1}\phi\Delta\psi_{1y_1}\\
&\quad+\frac{2\mu+\lambda}{\rho^2}\overline{u}_{1y_1y_1}\nabla\phi\cdot\nabla\Delta\psi_1-\frac{(2\mu+\lambda)(\widetilde{\rho}+\rho)}{\widetilde{\rho}^2\rho^2}\widetilde{\rho}_{y_1}\overline{u}_{1y_1y_1}\phi\Delta\psi_{1y_1}\\
&\quad+\Big[\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)}{\widetilde{\rho}}\Big]_{y_1}\Delta\psi_{1y_1}+\Big[\frac{(p(\widetilde{\rho})-p(\overline{\rho})-p'(\overline{\rho})d_1)_{y_1}}{\widetilde{\rho}}\Big]_{y_1}\Delta\psi_{1y_1}\\
&:=N(\tau,y_1,y_2).
\end{aligned}
$$

对上式在 $[0,\tau]\times\Omega_\varepsilon$ 上积分, 可得

$$
\begin{aligned}
&\|\nabla^2\Psi\|^2+\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\mathrm{d}\tau\\
&\leqslant\|\nabla^2\Psi_0\|^2+C\Big|\int_0^\tau\iint_{\Omega_\varepsilon}N(\tau,y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|.
\end{aligned}\tag{4.16}
$$

对 $N(\tau,y_1,y_2)$ 中某些特殊项进行估计, 可得

$$
\begin{aligned}
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\widetilde{\rho})}{\widetilde{\rho}}\Big)\widetilde{\rho}_{y_1y_1}\Delta\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
&\leqslant\frac{C}{\nu}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|\cdot\|\widetilde{\rho}_{y_1y_1}\|_\infty^2\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|\mathrm{d}\tau\\
&\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon^3}{\nu^2}\sup_{0\leqslant t\leqslant T}\|\widetilde{\rho}_{x_1x_1}\|_\infty^2\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2\\
&\leqslant\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon^3}{\nu^{2\alpha+2}\delta^4}\sup_{0\leqslant\tau\leqslant\tau(\varepsilon)}\|\overline{\rho}^{\frac{\gamma-2}{2}}\phi\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1y_1}}{\widetilde{\rho}}\Delta\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
=&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{1}{\widetilde{\rho}}\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1y_1}\Delta\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
\leqslant&\frac{C}{\nu^{\frac12}}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\psi_1\|\cdot\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1y_1}\Big\|\mathrm{d}\tau\\
\leqslant&\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\psi_1\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu}\sup_{0\leqslant t\leqslant T}\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{x_1x_1}\Big\|^2\\
\leqslant&\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\psi_1\|^2\mathrm{d}\tau+\frac{\varepsilon^4}{\nu^{\frac52\alpha+1}\delta^5}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4},\\
&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{(\widetilde{\rho}\widetilde{u}_1^2-\overline{\rho}\overline{u}_1^2+\overline{u}_1^2d_1-2\overline{u}_1d_2)_{y_1}\widetilde{\rho}_{y_1}}{\widetilde{\rho}^2}\Delta\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
=&\Big|\int_0^\tau\iint_{\Omega_\varepsilon}\frac{\widetilde{\rho}_{y_1}}{\widetilde{\rho}^2}\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1}\Delta\psi_{1y_1}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}\tau\Big|\\
\leqslant&\frac{1}{\nu^{\frac32}}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|\cdot\|\widetilde{\rho}_{y_1}\|_\infty\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{y_1}\Big\|\mathrm{d}\tau\\
\leqslant&\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\mathrm{d}\tau+\frac{C_T\varepsilon}{\nu^3}\sup_{0\leqslant t\leqslant T}\Big(\|\widetilde{\rho}_{x_1}\|_\infty^2\Big\|\Big(\frac{(-d_1\overline{u}_1+d_2)^2}{\widetilde{\rho}}\Big)_{x_1}\Big\|^2\Big)\\
\leqslant&\frac{1}{160}\int_0^\tau\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\mathrm{d}\tau+\frac{\varepsilon^2}{\nu^{\frac52\alpha+3}\delta^5}\frac{C_T\varepsilon}{\nu^{2\alpha+2}\delta^4}.
\end{aligned}
$$

由于 ρ 有上界和椭圆估计, 则

$$
\|\rho^{-\frac12}\nabla\Delta\Psi\|^2\geqslant\|\nabla\Delta\Psi\|^2\sim\|\nabla^3\Psi\|^2,
$$

$$
\|\rho^{-\frac12}\nabla^2\mathrm{div}\,\Psi\|^2\geqslant\|\nabla^2\mathrm{div}\,\Psi\|^2\sim\|\nabla^3\Psi\|^2.
$$

结合积分不等式 (4.15)–(4.16) 及先验假设 (3.4), 引理得证.

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Vanishing Viscosity Limit to Planar Rarefaction Wave with Vacuum for 2D Compressible Navier-Stokes Equations

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Abstract This paper mainly studies the vanishing viscosity limit of planar rarefaction wave with vacuum for the two-dimensional (2D for short) isentropic compressible Navier-Stokes equations. A uniform convergence rate related to the viscosity coefficients is obtained. The main novelty and difficulty lie in controlling the degeneracy caused by vacuum states of the rarefaction wave in the multidimensional setting. Appropriate cut-off and scaling techniques are applied near the vacuum states. Eventually, the authors obtain the inviscid decay rate around the planar rarefaction wave with vacuum by introducing the cut-off parameter and handling the viscous error terms of the two-dimensional compressible Navier-Stokes equations. The proof is based on the elementary energy method.

Keywords Compressible Navier-Stokes equations, Planar rarefaction wave, Vacuum, Vanishing viscosity limit

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