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拟凸域上沿某些方向的边界 Schwarz 引理*

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提要 本文直接利用全纯映照的性质研究边界 Schwarz 引理, 建立了拟凸域上沿某些满足正定条件的方向的边界 Schwarz 引理. 文章推广了强拟凸域情形的主要结果, 但是证明的方法是不一样的.

关键词 全纯映照, 边界 Schwarz 引理, 拟凸域, 强拟凸域

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§1 引 言

Schwarz 引理在复分析和复几何中扮演着非常重要的角色, 已经被广泛研究. 对于这方面的结果, 可以见文 [1–3]. 边界 Schwarz 引理也吸引着很多数学家的注意, 建立了许多全纯映照的刚性边界定理, 相关结果见文 [4–13]. 本文研究 $\mathbb{C}^n$ 中具有 C^2 边界的一般的 Levi 拟凸域上的边界 Schwarz 引理.

设 Ω 是 $\mathbb{C}^n$ 中具有 C^1 边界的区域, 那么存在 Ω 的一个 C^1 定义函数 $\rho:\mathbb{C}^n\to\mathbb{R}$, 满足: (1) $\Omega=\{z\in\mathbb{C}^n:\rho(z)<0\}$; (2) $\overline{\Omega}^c=\{\rho(z)>0\}$; (3) 对任意的 $P\in\partial\Omega$, $\nabla\rho(P)\neq 0$. 对于从 Ω 到 $\mathbb{C}^n$ 的全纯映照 $f=(f_1,f_2,\cdots,f_n)$, f 在 $z\in\Omega$ 处的复 Jacobian 矩阵为 $J_f(z)=\big(\frac{\partial f^i}{\partial z^j}(z)\big)_{n\times n}$, 是一个从 $\mathbb{C}^n$ 到 $\mathbb{C}^n$ 的线性映照.

回顾如果 $\Omega\subset\mathbb{C}^n$ 是一个 Levi 拟凸域, 那么存在一个 C^2 定义函数 ρ, 满足对于任意的 $P\in\partial\Omega$ 和 $\xi=(\xi^1,\xi^2,\cdots,\xi^n)\in T_P^{1,0}(\partial\Omega)$, 有

$$\sum_{j,k=1}^n\frac{\partial^2\rho}{\partial z^j\partial\overline{z}^k}(P)\xi^j\overline{\xi}^k\geqslant 0.$$

此处,

$$T_P^{1,0}(\partial\Omega)=\Big\{\xi\in\mathbb{C}^n:\sum_{j=1}^n\frac{\partial\rho}{\partial z^j}(P)\xi^j=0\Big\},$$

见文 [14]. 如果对于 $\xi\in T_P^{1,0}(\partial\Omega)\setminus\{0\}$, 上述不等式是严格的, 则称 Ω 为强 Levi 拟凸域.

在文 [8, 13] 中, 作者得到了强拟凸域上的边界 Schwarz 引理的下列重要结果.

定理 1.1 [8,13] 设 $\Omega\subset\subset\mathbb{C}^n$ 是一个强 Levi 拟凸域, $P\in\partial\Omega$, $z_0\in\Omega$. 如果 $f:\Omega\to\Omega$ 是一个全纯映照, $f(z_0)=z_0$, f 在 P 处全纯且 $f(P)=P$, 那么 $J_f(P)$ 的特征值 $\lambda,\mu_2,\cdots,\mu_n$ 有下列性质:

(1) $\partial\Omega$ 在 P 处的法向量是 $\overline{J_f(P)}^t$ 的一个特征向量, 对应的特征值是 λ, 即

$$\overline{J_f(P)}^t\nabla\rho(P)=\lambda\nabla\rho(P),$$

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或者 $\frac{\partial\rho}{\partial z}(P)J_f(P)=\lambda\frac{\partial\rho}{\partial z}(P)$, 这里 $\frac{\partial\rho}{\partial z}(P)=\big(\frac{\partial\rho}{\partial z^1},\frac{\partial\rho}{\partial z^2},\cdots,\frac{\partial\rho}{\partial z^n}\big)\big|_P$.

(2) $\lambda\geqslant 1$.

(3) 对于任意的 μ_i, 存在 $\alpha_i\in T_P^{1,0}(\partial\Omega)\setminus\{0\}$, 使得

$$J_f(P)\alpha_i=\mu_i\alpha_i,\quad i=2,3,\cdots,n.$$

并且 $|\mu_i|\leqslant\sqrt{\lambda},\ i=2,3,\cdots,n$.

(4) $|\mathrm{det}J_f(P)|\leqslant\lambda^{\frac{n+1}{2}}$, $|\mathrm{tr}J_f(P)|\leqslant\lambda+\sqrt{\lambda}(n-1)$.

而且, 不等式 (2)–(4) 是精确的.

为了方便, 这里引入一些记号. $\frac{\partial\rho}{\partial z}(P)J_f(P)=\lambda\frac{\partial\rho}{\partial z}(P)$ 意味着

$$\frac{\partial\rho}{\partial z^j}(P)\frac{\partial f^j}{\partial z^k}(P)=\lambda\frac{\partial\rho}{\partial z^k}(P),\quad k=1,2,\cdots,n.$$

此处用到了 Einstein 求和记号. 对于定理 1.1 的 (3) 中的任意

$$\alpha=(\xi^1,\xi^2,\cdots,\xi^n)\in\{\alpha_2,\alpha_3,\cdots,\alpha_n\},$$

以及对应的特征值 μ, $J_f(P)\alpha=\mu\alpha$ 意味着 $\frac{\partial f^k}{\partial z^j}(P)\xi^j=\mu\xi^k,\ k=1,2,\cdots,n$.

定理 1.1 中最关键的结果是 (3), 即

$$|\mu_i|\leqslant\sqrt{\lambda},\quad i=2,3,\cdots,n. \tag{1.1}$$

定理 1.1 中的不等式 (2) 和 (4) 由 (1.1) 得到. 在文 [8, 13] 中, (1.1) 的证明主要是非常巧妙地利用了强拟凸域上 Carathéodory 度量或者 Kobayashi 度量的边界行为, 即如下定理.

定理 1.2[15] 设 $\Omega\subset\subset\mathbb{C}^n$ 是一个强 Levi 拟凸域, $P\in\partial\Omega$, 那么对于任意的 $\xi\in T_P^{1,0}(\partial\Omega)$,

$$\lim_{\substack{z\to P\\ z\in\Gamma(P)}}(F(z,\xi))^2d(z,\partial\Omega)=\frac{1}{2\|\nabla\rho(P)\|}\sum_{j,k=1}^n\frac{\partial^2\rho}{\partial z^j\partial\overline{z}^k}(P)\xi^j\overline{\xi}^k.$$

这里 $F(\cdot,\cdot)$ 是 Carathéodory 度量或者 Kobayashi 度量, $\Gamma(P)\subset\mathbb{C}^n$ 是 P 处的一个非切向区域.

在本文中, 考虑一般的拟凸域上沿某些方向的边界 Schwarz 引理, 其中 ρ 在 $P\in\partial\Omega$ 处的复 Hessian 矩阵关于该方向是正定的. 强拟凸域上的边界 Schwarz 引理可看成这里所得结果的特别情形. 相比于文 [8, 13], 本文采取了一种不同的方法证明 (1.1). 本文的证明是直接利用全纯映照 f 的微分性质和映照特点, 以及 ρ 的复 Hessian 矩阵在某些方向的正定性. 这也给研究边界 Schwarz 引理提供了新的思路和方法.

§2 拟凸域上的边界 Schwarz 引理

定理 2.1 设 $\Omega\subset\subset\mathbb{C}^n$ 是一个 Levi 拟凸域, $P\in\partial\Omega$. $f:\Omega\to\Omega$ 是全纯映照, f 在 P 处全纯, 且 $f(P)=P$. 对于 $J_f(P)$ 的特征值 $\lambda,\mu_2,\cdots,\mu_n$ (见定理 1.1), 如果存在 $\mu\in\{\mu_2,\cdots,\mu_n\}$, 使得对应的特征向量 $\alpha=(\xi^1,\xi^2,\cdots,\xi^n)\in T_P^{1,0}(\partial\Omega)\setminus\{0\}$ 满足

$$\sum_{j,k=1}^n\frac{\partial^2\rho}{\partial z^j\partial\overline{z}^k}(P)\xi^j\overline{\xi}^k>0, \tag{2.1}$$

那么有 $|\mu|\leqslant\sqrt{\lambda}$.

证 在证明中, 利用 Einstein 求和记号. 设 $\alpha\in T_P^{1,0}(\partial\Omega)\setminus\{0\}$ 满足 (2.1), $J_f(P)\alpha=\mu\alpha$. 取 $\gamma=(z^1,z^2,\cdots,z^n)=(z^1(t),z^2(t),\cdots,z^n(t)):[-1,1]\to\partial\Omega$, 使得

$$\gamma(0)=P,\quad \gamma'(0)=\Big(\frac{\mathrm{d}z^1}{\mathrm{d}t},\frac{\mathrm{d}z^2}{\mathrm{d}t},\cdots,\frac{\mathrm{d}z^n}{\mathrm{d}t}\Big)\Big|_{t=0}=(\xi^1,\xi^2,\cdots,\xi^n)=\alpha,$$

并且 $\gamma([-1,1])$ 包含于 P 的一个充分小邻域内, 那么有

$$\rho(\gamma(t))\equiv 0,\quad t\in[-1,1].\tag{2.2}$$

在 (2.2) 两边关于 t 求导数, 得

$$\frac{\partial\rho}{\partial z^i}\frac{\mathrm{d}z^i}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{z}^i}\frac{\mathrm{d}\overline{z}^i}{\mathrm{d}t}\equiv 0,\quad t\in[-1,1].$$

再求导数, 得

$$\begin{aligned}&\frac{\partial^2\rho}{\partial z^i\partial z^j}(P)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)\frac{\mathrm{d}z^j}{\mathrm{d}t}(0)+2\frac{\partial^2\rho}{\partial z^i\partial\overline{z}^j}(P)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}(0)\\&+\frac{\partial^2\rho}{\partial\overline{z}^i\partial\overline{z}^j}(P)\frac{\mathrm{d}\overline{z}^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}(0)+\frac{\partial\rho}{\partial z^i}(P)\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial\overline{z}^i}(P)\frac{\mathrm{d}^2\overline{z}^i}{\mathrm{d}t^2}(0)=0.\end{aligned}\tag{2.3}$$

又取 $\widetilde{\gamma}=(z^1,z^2,\cdots,z^n)=(\widetilde{z}^1(t),\widetilde{z}^2(t),\cdots,\widetilde{z}^n(t)):[-1,1]\to\partial\Omega$, 使得

$$\widetilde{\gamma}(0)=P,\quad \widetilde{\gamma}'(0)=\Big(\frac{\mathrm{d}\widetilde{z}^1}{\mathrm{d}t},\frac{\mathrm{d}\widetilde{z}^2}{\mathrm{d}t},\cdots,\frac{\mathrm{d}\widetilde{z}^n}{\mathrm{d}t}\Big)\Big|_{t=0}=\sqrt{-1}\gamma'(0)=\sqrt{-1}\alpha.$$

类似于上述求导, 得到

$$\frac{\partial\rho}{\partial z^i}\frac{\mathrm{d}\widetilde{z}^i}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{z}^i}\frac{\mathrm{d}\overline{\widetilde{z}}^i}{\mathrm{d}t}\equiv 0,\quad t\in[-1,1],$$

以及

$$\begin{aligned}&\frac{\partial^2\rho}{\partial z^i\partial z^j}(P)\frac{\mathrm{d}\widetilde{z}^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\widetilde{z}^j}{\mathrm{d}t}(0)+2\frac{\partial^2\rho}{\partial z^i\partial\overline{z}^j}(P)\frac{\mathrm{d}\widetilde{z}^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t}(0)\\&+\frac{\partial^2\rho}{\partial\overline{z}^i\partial\overline{z}^j}(P)\frac{\mathrm{d}\overline{\widetilde{z}}^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t}(0)+\frac{\partial\rho}{\partial z^i}(P)\frac{\mathrm{d}^2\widetilde{z}^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial\overline{z}^i}(P)\frac{\mathrm{d}^2\overline{\widetilde{z}}^i}{\mathrm{d}t^2}(0)=0.\end{aligned}\tag{2.4}$$

将

$$\frac{\mathrm{d}\widetilde{z}^i}{\mathrm{d}t}(0)=\sqrt{-1}\frac{\mathrm{d}z^i}{\mathrm{d}t}(0),\quad i=1,2,\cdots,n$$

代入 (2.4), 得到

$$\begin{aligned}&-\frac{\partial^2\rho}{\partial z^i\partial z^j}(P)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)\frac{\mathrm{d}z^j}{\mathrm{d}t}(0)+2\frac{\partial^2\rho}{\partial z^i\partial\overline{z}^j}(P)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}(0)\\&-\frac{\partial^2\rho}{\partial\overline{z}^i\partial\overline{z}^j}(P)\frac{\mathrm{d}\overline{z}^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}(0)+\frac{\partial\rho}{\partial z^i}(P)\frac{\mathrm{d}^2\widetilde{z}^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial\overline{z}^i}(P)\frac{\mathrm{d}^2\overline{\widetilde{z}}^i}{\mathrm{d}t^2}(0)=0.\end{aligned}\tag{2.5}$$

(2.3) 和 (2.5) 给出

$$\begin{aligned}&\frac{\partial\rho}{\partial z^i}(P)\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial\overline{z}^i}(P)\frac{\mathrm{d}^2\overline{z}^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial z^i}(P)\frac{\mathrm{d}^2\widetilde{z}^i}{\mathrm{d}t^2}(0)+\frac{\partial\rho}{\partial\overline{z}^i}(P)\frac{\mathrm{d}^2\overline{\widetilde{z}}^i}{\mathrm{d}t^2}(0)\\&=-4\frac{\partial^2\rho}{\partial z^i\partial\overline{z}^j}(P)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}(0).\end{aligned}\tag{2.6}$$

根据 f 的映照特点, 有 $\max\limits_{t\in[-1,1]}\rho(f(\gamma(t)))=\rho(f(\gamma(0)))=0$, 以及

$$\max_{t\in[-1,1]}\rho(f(\widetilde{\gamma}(t)))=\rho(f(\widetilde{\gamma}(0)))=0.$$

这得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(f(\gamma(t)))\Big|_{t=0}=0,\quad \frac{\mathrm{d}}{\mathrm{d}t}\rho(f(\widetilde{\gamma}(t)))\Big|_{t=0}=0,$$

以及

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(f(\gamma(t)))\Big|_{t=0}\leqslant 0,\quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(f(\widetilde{\gamma}(t)))\Big|_{t=0}\leqslant 0.$$

再由

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(f(\gamma(t)))=\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}$$

和

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(f(\widetilde{\gamma}(t)))=\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}\widetilde{z}^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t},$$

得到

$$\begin{aligned}0&\geqslant\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(f(\gamma(t)))\Big|_{t=0}\\&=\frac{\partial^2\rho}{\partial w^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}z^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial^2\rho}{\partial w^i\partial\overline{w}^k}\frac{\partial\overline{f}^k}{\partial\overline{z}^l}\frac{\mathrm{d}\overline{z}^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}\\&\quad+\frac{\partial\rho}{\partial w^i}\frac{\partial^2 f^i}{\partial z^j\partial z^m}\frac{\mathrm{d}z^m}{\mathrm{d}t}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}^2z^j}{\mathrm{d}t^2}\\&\quad+\frac{\partial^2\rho}{\partial\overline{w}^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}z^l}{\mathrm{d}t}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}+\frac{\partial^2\rho}{\partial\overline{w}^i\partial\overline{w}^k}\frac{\partial\overline{f}^k}{\partial\overline{z}^l}\frac{\mathrm{d}\overline{z}^l}{\mathrm{d}t}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}\\&\quad+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial^2\overline{f}^i}{\partial\overline{z}^j\partial\overline{z}^m}\frac{\mathrm{d}\overline{z}^m}{\mathrm{d}t}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}^2\overline{z}^j}{\mathrm{d}t^2},\end{aligned}\tag{2.7}$$

以及

$$\begin{aligned}0&\geqslant\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(f(\widetilde{\gamma}(t)))\Big|_{t=0}\\&=\frac{\partial^2\rho}{\partial w^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}\widetilde{z}^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}\widetilde{z}^j}{\mathrm{d}t}+\frac{\partial^2\rho}{\partial w^i\partial\overline{w}^k}\frac{\partial\overline{f}^k}{\partial\overline{z}^l}\frac{\mathrm{d}\overline{\widetilde{z}}^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}\widetilde{z}^j}{\mathrm{d}t}\\&\quad+\frac{\partial\rho}{\partial w^i}\frac{\partial^2 f^i}{\partial z^j\partial z^m}\frac{\mathrm{d}\widetilde{z}^m}{\mathrm{d}t}\frac{\mathrm{d}\widetilde{z}^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}^2\widetilde{z}^j}{\mathrm{d}t^2}\\&\quad+\frac{\partial^2\rho}{\partial\overline{w}^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}\widetilde{z}^l}{\mathrm{d}t}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t}+\frac{\partial^2\rho}{\partial\overline{w}^i\partial\overline{w}^k}\frac{\partial\overline{f}^k}{\partial\overline{z}^l}\frac{\mathrm{d}\overline{\widetilde{z}}^l}{\mathrm{d}t}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t}\\&\quad+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial^2\overline{f}^i}{\partial\overline{z}^j\partial\overline{z}^m}\frac{\mathrm{d}\overline{\widetilde{z}}^m}{\mathrm{d}t}\frac{\mathrm{d}\overline{\widetilde{z}}^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial\overline{w}^i}\frac{\partial\overline{f}^i}{\partial\overline{z}^j}\frac{\mathrm{d}^2\overline{\widetilde{z}}^j}{\mathrm{d}t^2}.\end{aligned}\tag{2.8}$$

为简单起见, 这里 P 和 0 省略了.

将

$$\frac{\mathrm{d}\widetilde{z}^i}{\mathrm{d}t}(0)=\sqrt{-1}\frac{\mathrm{d}z^i}{\mathrm{d}t}(0),\quad i=1,2,\cdots,n,$$

代入 (2.8), 得到

$$0\geqslant\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho(f(\widetilde{\gamma}(t)))\Big|_{t=0}$$

$$
\begin{aligned}
&= -\frac{\partial^2\rho}{\partial w^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}z^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}\frac{\partial \overline{f}^k}{\partial \overline{z}^l}\frac{\mathrm{d}\overline{z}^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}\\
&\quad -\frac{\partial\rho}{\partial w^i}\frac{\partial^2 f^i}{\partial z^j\partial z^m}\frac{\mathrm{d}z^m}{\mathrm{d}t}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}^2\widetilde{z}^j}{\mathrm{d}t^2}\\
&\quad +\frac{\partial^2\rho}{\partial \overline{w}^i\partial w^k}\frac{\partial f^k}{\partial z^l}\frac{\mathrm{d}z^l}{\mathrm{d}t}\frac{\partial \overline{f}^i}{\partial \overline{z}^j}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}-\frac{\partial^2\rho}{\partial \overline{w}^i\partial \overline{w}^k}\frac{\partial \overline{f}^k}{\partial \overline{z}^l}\frac{\mathrm{d}\overline{z}^l}{\mathrm{d}t}\frac{\partial \overline{f}^i}{\partial \overline{z}^j}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}\\
&\quad -\frac{\partial\rho}{\partial \overline{w}^i}\frac{\partial^2 \overline{f}^i}{\partial \overline{z}^j\partial \overline{z}^m}\frac{\mathrm{d}\overline{z}^m}{\mathrm{d}t}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial \overline{w}^i}\frac{\partial \overline{f}^i}{\partial \overline{z}^j}\frac{\mathrm{d}^2\overline{\widetilde{z}}^j}{\mathrm{d}t^2}. \qquad (2.9)
\end{aligned}
$$

结合 $\frac{\partial f^k}{\partial z^l}(P)\frac{\mathrm{d}z^l}{\mathrm{d}t}(0)=\mu\frac{\mathrm{d}z^k}{\mathrm{d}t}(0)(k=1,2,\cdots,n)$, $\frac{\partial\rho}{\partial w^i}(P)\frac{\partial f^i}{\partial z^j}(P)=\lambda\frac{\partial\rho}{\partial z^j}(P)(j=1,2,\cdots,n)$, 以及 (2.6), (2.7) 和 (2.9) 给出

$$
\begin{aligned}
0 &\geqslant 4\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}\frac{\partial \overline{f}^k}{\partial \overline{z}^l}\frac{\mathrm{d}\overline{z}^l}{\mathrm{d}t}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}z^j}{\mathrm{d}t}+\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}^2 z^j}{\mathrm{d}t^2}+\frac{\partial\rho}{\partial \overline{w}^i}\frac{\partial \overline{f}^i}{\partial \overline{z}^j}\frac{\mathrm{d}^2\overline{z}^j}{\mathrm{d}t^2}\\
&\quad +\frac{\partial\rho}{\partial w^i}\frac{\partial f^i}{\partial z^j}\frac{\mathrm{d}^2\widetilde{z}^j}{\mathrm{d}t^2}+\frac{\partial\rho}{\partial \overline{w}^i}\frac{\partial \overline{f}^i}{\partial \overline{z}^j}\frac{\mathrm{d}^2\overline{\widetilde{z}}^j}{\mathrm{d}t^2}\\
&= 4|\mu|^2\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}\frac{\mathrm{d}\overline{z}^k}{\mathrm{d}t}\frac{\mathrm{d}z^i}{\mathrm{d}t}+\lambda\frac{\partial\rho}{\partial z^j}\frac{\mathrm{d}^2 z^j}{\mathrm{d}t^2}+\lambda\frac{\partial\rho}{\partial \overline{z}^j}\frac{\mathrm{d}^2\overline{z}^j}{\mathrm{d}t^2}\\
&\quad +\lambda\frac{\partial\rho}{\partial z^j}\frac{\mathrm{d}^2\widetilde{z}^j}{\mathrm{d}t^2}+\lambda\frac{\partial\rho}{\partial \overline{z}^j}\frac{\mathrm{d}^2\overline{\widetilde{z}}^j}{\mathrm{d}t^2}\\
&= 4|\mu|^2\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}\frac{\mathrm{d}\overline{z}^k}{\mathrm{d}t}\frac{\mathrm{d}z^i}{\mathrm{d}t}-4\lambda\frac{\partial^2\rho}{\partial z^i\partial \overline{z}^j}\frac{\mathrm{d}z^i}{\mathrm{d}t}\frac{\mathrm{d}\overline{z}^j}{\mathrm{d}t}.
\end{aligned}
$$

再由条件 (2.1) $\sum\limits_{i,k=1}^{n}\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}(P)\frac{\mathrm{d}\overline{z}^k}{\mathrm{d}t}(0)\frac{\mathrm{d}z^i}{\mathrm{d}t}(0)=\sum\limits_{i,k=1}^{n}\frac{\partial^2\rho}{\partial w^i\partial \overline{w}^k}(P)\overline{\xi}^k\xi^i>0$, 推出 $|\mu|^2\leqslant\lambda$, 或者 $|\mu|\leqslant\sqrt{\lambda}$. 证毕.

注 2.1 显然, 定理 2.1 的结论能应用于强 Levi 拟凸域情形. 定理 2.1 是定理 1.1 的推广, 但是证明方法是不同的.

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Boundary Schwarz Lemma of Pseudoconvex Domains along Some Directions

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Abstract In this paper, the author studies the boundary Schwarz lemma directly by using the properties of holomorphic mappings. A boundary Schwarz lemma along some directions satisfying positive definite conditions on pseudoconvex domains is established. This generalizes the main results in the case of strongly pseudoconvex domains, but the methods of proof are different.

Keywords Holomorphic mapping, Boundary Schwarz lemma, Pseudoconvex domains, Strongly pseudoconvex domains

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