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# 一个新的涉及一个多重可变上限函数和一个部分和的半离散 Hilbert 型不等式*

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**提要**　应用权函数的方法, Euler-Maclaurin 求和公式, Abel 部分求和公式及实分析技巧, 求出了一个新的涉及一个多重可变上限函数和一个部分和的半离散 Hilbert 型不等式. 作为应用, 考虑了特殊参数下不等式中最佳常数因子联系多参数的等价条件及一些特殊不等式.

**关键词**　权函数, Euler-Maclaurin 求和公式, Abel 部分求和公式, 半离散 Hilbert 型不等式, 多重可变上限函数, 部分和

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## §1　引　言

若 $p>1, \frac{1}{p}+\frac{1}{q}=1, a_m, b_n \geqslant 0, 0<\sum\limits_{m=1}^{\infty} a_m^p<\infty$ 及 $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, 则有如下著名的具有最佳常数因子 $\frac{\pi}{\sin(\frac{\pi}{p})}$ 的 Hardy-Hilbert 不等式 (见 [1, 定理 315]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}\Big(\sum_{m=1}^{\infty}a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}b_n^q\Big)^{\frac{1}{q}}. \tag{1.1}$$

2006 年, 通过引入参数 $\lambda_i \in (0,2]$ $(i=1,2), \lambda_1+\lambda_2=\lambda\in(0,4]$, 应用 Euler-Maclaurin 求和公式及实分析技巧, 文 [2] 建立了 (1.1) 的如下推广式:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}}, \tag{1.2}$$

这里常数因子 $B(\lambda_1,\lambda_2)$ 是最佳的,

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\mathrm{d}t, \quad u,v>0 \tag{1.3}$$

为 Beta 函数. 当 $\lambda=1, \lambda_1=\frac{1}{q}, \lambda_2=\frac{1}{p}$, 由 (1.2) 可导出 (1.1); 当 $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, 由 (1.2) 可导出杨 [3] 的一个早期结果. 2019 年, 应用 (1.2) 的方法及 Abel 部分求和公式, Adiyasuren 等, [4] 给出了核为 $\frac{1}{(m+n)^{\lambda}}$ 的涉及两个部分和的 Hilbert 型不等式.

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不等式 (1.1) 及其积分类在分析学及其应用中作用颇为重要 (见 [5–15]).

1934 年, 文 [1] 中定理 3.5.1 给出了如下半离散 Hilbert 型不等式: 设 $K(t)\,(t>0)$ 为递减函数, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(t)t^{s-1}\mathrm{d}t<\infty, a_n\geqslant 0$ , 使得 $0<\sum\limits_{n=1}^{\infty}a_n^p<\infty$, 则有

$$\int_0^\infty x^{p-2}\Big(\sum_{n=1}^{\infty}K(nx)a_n\Big)^p\mathrm{d}x<\phi^p\Big(\frac{1}{p}\Big)\sum_{n=1}^{\infty}a_n^p. \tag{1.4}$$

(1.4) 的一些近期推广应用见之于文 [16–20].

2016 年, 应用实分析技巧, 洪等 [21] 给出了 (1.1) 的推广式中最佳常数因子联系多参数的一个等价陈述. 其他类似的工作可见文 [22–28]. 最近, 文 [29–30] 给出了逆向半离散 Hilbert 型不等式的一些新结果.

本文基于文 [2,4,21] 的研究思想, 综合应用权函数的方法, Euler-Maclaurin 求和公式,Abel 部分求和公式及实分析技巧, 求出一个新的核为 $\frac{1}{(x+n^\alpha)^\lambda}$ 的涉及一个多重可变上限函数和一个部分和的半离散 Hilbert 型不等式. 作为应用, 考虑了特殊参数 $(\alpha=1)$ 下不等式中最佳常数因子联系多参数的等价条件及一些特殊不等式. 本文的方法对同类问题的研究具有多方面的借鉴作用.

## §2 一些引理

为避免重复陈述, 本文下设: $p>1, \frac{1}{p}+\frac{1}{q}=1$, $m\in\{0,1,2,3,4\}, \alpha\in(0,1]$, $\lambda\in(0,5-m], \lambda_1\in(0,\lambda+1), \lambda_2\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda+m), \widetilde{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widetilde{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}, f(x):=F_0(x)$ 在 $\mathbf{R}_+=(0,\infty)$ 中 (除有限点外) 连续. 定义多重可变上限函数为: $F_i(x):=\int_0^\infty F_{i-1}(x)\mathrm{d}x$ $(x\geqslant 0)$, 且满足条件 $F_i(x)=o(\mathrm{e}^{tx})(t>0, i=1,\cdots,m); x\to\infty)$. 对于 $a_k\geqslant 0$, 定义部分和为: $A_n:=\sum\limits_{k=1}^{n}a_k$ $(n\in\mathbf{N}=\{1,2,\cdots\}$, 且满足条件 $A_n=o(\mathrm{e}^{tn^\alpha})$ $(t>0; n\to\infty)$, 还设

$$0<\int_0^\infty x^{p(1-m-\widetilde{\lambda}_1)-1}F_m^p(x)\mathrm{d}x<\infty \quad 及 \quad 0<\sum_{n=1}^{\infty}n^{q[1-\alpha(1+\widetilde{\lambda}_2)]-1}A_n^q<\infty. \tag{2.1}$$

**引理 2.1** (见 [5, (2.2.3)]) (i) 若 $(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}g(t)>0, t\in[m,\infty)(m\in\mathbf{N}), g^{(i)}(\infty)=0\,(i=0,1,2,3), P_i(t)$ 和 $B_i(i\in\mathbf{N})$ 分别为 $i$- 阶 Bernoulli 函数和 Bernoulli 数, 则有

$$\int_m^\infty P_{2q-1}(t)g(t)\mathrm{d}t=-\varepsilon_q\frac{B_{2q}}{2q}g(m),\quad 0<\varepsilon_q<1; q=1,2,\cdots. \tag{2.2}$$

特别地, 在 (2.2) 中取 $q=1, B_2=\frac{1}{6}$, 有

$$-\frac{1}{12}g(m)<\int_m^\infty P_1(t)g(t)\mathrm{d}t<0; \tag{2.3}$$

当 $q=2, B_4=-\frac{1}{30}$, 有

$$0<\int_m^\infty P_3(t)g(t)\mathrm{d}t<\frac{1}{120}g(m); \tag{2.4}$$

(ii) 由文 [5, (2.2.16)], 若 $h(t)(>0)\in C^3[m,\infty), h^{(i)}(\infty)=0$ $(i=0,1,2,3)$, 则有如下

Euler-Maclaurin 求和公式:

$$\sum_{k=m}^{\infty} h(k) = \int_m^{\infty} h(t)\mathrm{d}t + \frac{1}{2}h(m) + \int_m^{\infty} P_1(t)h'(t)\mathrm{d}t, \tag{2.5}$$

$$\int_m^{\infty} P_1(t)h'(t)\mathrm{d}t = -\frac{1}{12}h'(m) + \frac{1}{6}\int_m^{\infty} P_3(t)h''(t)\mathrm{d}t. \tag{2.6}$$

**引理 2.2**　设 $s \in (0,6], s_2 \in (0, \frac{2}{\alpha}]\cap(0,s), k_s(s_2) := B(s_2, s-s_2)$. 定义下面权函数:

$$\varpi(s_2, x) := x^{s-s_2}\sum_{n=1}^{\infty}\frac{n^{\alpha s_2-1}}{(x+n^{\alpha})^s}, \quad x \in \mathbf{R}_+. \tag{2.7}$$

我们有如下不等式:

$$k_s(s_2)\Big(1 - O\Big(\frac{1}{x^{s_2}}\Big)\Big) < \varpi_s(s_2, x) < k_s(s_2), \quad x \in \mathbf{R}_+, \tag{2.8}$$

这里 $O(\frac{1}{x^{s_2}}) := \frac{1}{k_s(s_2)}\int_0^{\frac{1}{x}}\frac{u^{s_2-1}}{(1+u)^s}\mathrm{d}u > 0$.

**证**　固定 $x \in \mathbf{R}_+$, 定义函数 $g_x(t) = \frac{\alpha t^{\alpha s_2-1}}{(x+t^{\alpha})^s}$ $(t>0)$, 由 (2.5), 则有

$$\begin{aligned}\sum_{n=1}^{\infty} g_x(n) &= \int_1^{\infty} g_x(t)\mathrm{d}t + \frac{1}{2}g_x(1) + \int_1^{\infty} P_1(t)g_x'(t)\mathrm{d}t\\ &= \int_0^{\infty} g_x(t)\mathrm{d}t - h(x),\end{aligned}$$

这里 $h(x) := \int_0^1 g_x(t)\mathrm{d}t - \frac{1}{2}g_x(1) - \int_1^{\infty} P_1(t)g_x'(t)\mathrm{d}t$.

可求得 $-\frac{1}{2}g_x(1) = \frac{-\alpha}{2(x+1)^s}$. 由分部积分法, 得

$$\begin{aligned}\int_0^1 g_x(t)\mathrm{d}t &= \alpha\int_0^1 \frac{t^{\alpha s_2-1}}{(x+t^{\alpha})^s}\mathrm{d}t = \int_0^1 \frac{u^{s_2-1}}{(x+u)^s}\mathrm{d}u = \frac{1}{s_2}\int_0^1 \frac{du^{s_2}}{(x+u)^s}\\ &= \frac{1}{s_2}\frac{u^{s_2}}{(x+u)^s}\Big|_0^1 + \frac{s}{s_2}\int_0^1 \frac{u^{s_2}\mathrm{d}u}{(x+u)^{s+1}}\\ &= \frac{1}{s_2}\frac{1}{(x+1)^s} + \frac{s}{s_2(s_2+1)}\int_0^1 \frac{du^{s_2+1}}{(x+u)^{s+1}}\\ &> \frac{1}{s_2(x+1)^s} + \frac{s}{s_2(s_2+1)}\Big[\frac{u^{s_2+1}}{(x+u)^{s+1}}\Big]_0^1\\ &\quad + \frac{s(s+1)}{s_2(s_2+1)(x+1)^{s+2}}\int_0^1 u^{s_2+1}\mathrm{d}u\\ &= \frac{1}{s_2(x+1)^s} + \frac{s}{s_2(s_2+1)}\frac{1}{(x+1)^{s+1}}\\ &\quad + \frac{s(s+1)}{s_2(s_2+1)(s_2+2)(x+1)^{s+2}},\end{aligned}$$

$$\begin{aligned}-g_x'(t) &= -\frac{\alpha(\alpha s_2-1)t^{\alpha s_2-2}}{(x+t^{\alpha})^s} + \frac{\alpha^2 s t^{\alpha(1+s_2)-2}}{(x+t^{\alpha})^{s+1}}\\ &= -\frac{\alpha(\alpha s_2-1)t^{\alpha s_2-2}}{(x+t^{\alpha})^s} + \frac{\alpha^2 s(x+t^{\alpha}-x)t^{\alpha s_2-2}}{(x+t^{\alpha})^{s+1}}\\ &= \frac{\alpha(\alpha s-\alpha s_2+1)t^{\alpha s_2-2}}{(x+t^{\alpha})^s} - \frac{\alpha^2 s x t^{\alpha s_2-2}}{(x+t^{\alpha})^{s+1}},\end{aligned}$$

及对于 $0 < s_2 \leqslant \frac{2}{\alpha}(0 < \alpha \leqslant 1), s_2 < s \leqslant 6$, 有

$$(-1)^i \frac{\partial^i}{\partial t^i}\Big[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^s}\Big] > 0, (-1)^i \frac{\partial^i}{\partial t^i}\Big[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}\Big] > 0, \quad i = 0, 1, 2, 3.$$

由 (2.3)–(2.6) 可求得

$$\begin{aligned}
&\alpha(\alpha s - \alpha s_2 + 1)\int_1^\infty P_1(t)\frac{t^{\alpha s_2-2}\mathrm{d}t}{(x+t^\alpha)^s}\\
&> -\frac{\alpha(\alpha s - \alpha s_2 + 1)}{12(x+1)^s}, -\alpha^2 sx\int_1^\infty P_1(t)\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}\mathrm{d}t\\
&= \frac{\alpha^2 sx}{12(x+1)^{s+1}} - \frac{\alpha^2 sx}{6}\int_1^\infty P_3(t)\Big[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}\Big]_t''\mathrm{d}t\\
&> \frac{\alpha^2 sx}{12(x+1)^{s+1}} - \frac{\alpha^2 sx}{720}\Big[\frac{t^{\alpha s_2-2}}{(x+t^\alpha)^{s+1}}\Big]_{t=1}''\\
&> \frac{\alpha^2 s(x+1-1)}{12(x+1)^{s+1}} - \frac{\alpha^2 s(x+1)}{720}\Big[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+3}}\\
&\quad + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+2}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^{s+1}}\Big]\\
&= \frac{\alpha^2 s}{12(x+1)^s} - \frac{\alpha^2 s}{12(x+1)^{s+1}} - \frac{\alpha^2 s}{720}\Big[\frac{(s+1)(s+2)\alpha^2}{(x+1)^{s+2}}\\
&\quad + \frac{\alpha(s+1)(5-\alpha-2\alpha s_2)}{(x+1)^{s+1}} + \frac{(2-\alpha s_2)(3-\alpha s_2)}{(x+1)^s}\Big].
\end{aligned}$$

故有 $h(x) > \frac{1}{(x+1)^s}h_1 + \frac{s}{(x+1)^{s+1}}h_2 + \frac{s(s+1)}{(x+1)^{s+2}}h_3$, 这里

$$\begin{aligned}
h_1 &:= \frac{1}{s_2} - \frac{\alpha}{2} - \frac{\alpha - \alpha^2 s_2}{12} - \frac{\alpha^2 s(2-\alpha s_2)(3-\alpha s_2)}{720},\\
h_2 &:= \frac{1}{s_2(s_2+1)} - \frac{\alpha^2}{12} - \frac{\alpha^3(s+1)(5-\alpha-2\alpha s_2)}{720},\\
h_3 &:= \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\alpha^4(s+2)}{720}.
\end{aligned}$$

可求得 $h_1 = \frac{g(s_2)}{720 s_2}$, 这里定义函数 $g(\sigma)\ (\sigma \in (0, \frac{2}{\alpha}])$ 如下：

$$g(\sigma) := 720 - (420\alpha + 6s\alpha^2)\sigma + (60\alpha^2 + 5s\alpha^3)\sigma^2 - s\alpha^4\sigma^3.$$

对于 $\alpha \in (0, 1], s \in (0, 6]$, 可求得

$$\begin{aligned}
g'(\sigma) &= -(420\alpha + 6s\alpha^2) + 2(60\alpha^2 + 5s\alpha^3)\sigma - 3s\alpha^4\sigma^2\\
&\leqslant -420\alpha - 6s\alpha^2 + 2(60\alpha^2 + 5s\alpha^3)\frac{2}{\alpha} = (14s\alpha - 180)\alpha < 0,
\end{aligned}$$

因而有 $h_1 \geqslant \frac{g\left(\frac{2}{\alpha}\right)}{720 s_2} = \frac{1}{6 s_2} > 0$.

对于 $s_2 \in (0, \frac{2}{\alpha}]$, 有

$$h_2 > \frac{\alpha^2}{6} - \frac{\alpha^2}{12} - \frac{5\alpha^2(s+1)}{720} = \Big(\frac{1}{12} - \frac{s+1}{140}\Big)\alpha^2 > 0,$$

及 $h_3 \geqslant \big(\frac{1}{24} - \frac{s+2}{720}\big)\alpha^3 > 0\ (0 < s \leqslant 6)$.

因此有 $h(x)>0$. 作变换 $t=x^{\frac{1}{\alpha}}u^{\frac{1}{\alpha}}$, 有

$$\begin{aligned}\varpi(s_2,x)&=x^{s-s_2}\sum_{n=1}^{\infty}g_x(n)<x^{s-s_2}\int_0^{\infty}g_x(t)\mathrm{d}t\\&=\alpha x^{s-s_2}\int_0^{\infty}\frac{t^{\alpha s_2-1}\mathrm{d}t}{(x+t^{\alpha})^s}=\int_0^{\infty}\frac{u^{s_2-1}\mathrm{d}u}{(1+u)^s}\\&=B(s_2,s-s_2)=k_s(s_2).\end{aligned}$$

另外, 由 (2.5), 我们有

$$\begin{aligned}\sum_{n=1}^{\infty}g_x(n)&=\int_1^{\infty}g_x(t)\mathrm{d}t+\frac{1}{2}g_x(1)+\int_1^{\infty}P_1(t)g_x'(t)\mathrm{d}t\\&=\int_1^{\infty}g_x(t)\mathrm{d}t+H(x),\end{aligned}$$

这里 $H(x):=\frac{1}{2}g_x(1)+\int_1^{\infty}P_1(t)g_x'(t)\mathrm{d}t$. 前面已求得 $\frac{1}{2}g_x(1)=\frac{\alpha}{2(x+1)^s}$ 及

$$g_x'(t)=-\frac{\alpha(\alpha s-\alpha s_2+1)t^{\alpha s_2-2}}{(x+t^{\alpha})^s}+\frac{\alpha^2sxt^{\alpha s_2-2}}{(x+t^{\alpha})^{s+1}}.$$

对于 $s_2\in(0,\frac{2}{\alpha}]\cap(0,s),0<s\leqslant 6$, 由 (2.3), 可得

$$-\alpha(\alpha s-\alpha s_2+1)\int_1^{\infty}P_1(t)\frac{t^{\alpha s_2-2}}{(x+t^{\alpha})^s}\mathrm{d}t>0$$

及

$$\alpha^2sx\int_1^{\infty}P_1(t)\frac{t^{\alpha s_2-2}}{(x+t^{\alpha})^{s+1}}\mathrm{d}t>-\frac{\alpha^2sx}{12(x+1)^{s+1}}>-\frac{\alpha^2s}{12(x+1)^s}.$$

因而有

$$H(x)>\frac{\alpha}{2(x+1)^s}-\frac{\alpha^2s}{12(x+1)^s}\geqslant\frac{\alpha}{2(x+1)^s}-\frac{6\alpha^2}{12(x+1)^s}=0,$$

同时可求得

$$\begin{aligned}\varpi(s_2,x)&=x^{s-s_2}\sum_{n=1}^{\infty}g_x(n)>x^{s-s_2}\int_1^{\infty}g_x(t)\mathrm{d}t\\&=x^{s-s_2}\Big[\int_0^{\infty}g_x(t)\mathrm{d}t-\int_0^1g_x(t)\mathrm{d}t\Big]\\&=k_s(s_2)\Big[1-\frac{1}{k_s(s_2)}\int_0^{\frac{1}{x}}\frac{u^{s_2-1}\mathrm{d}u}{(1+u)^s}\Big]>0,\end{aligned}$$

这里置 $O(\frac{1}{x^{s_2}})=\frac{1}{k_s(s_2)}\int_0^{\frac{1}{x}}\frac{u^{s_2-1}\mathrm{d}u}{(1+u)^s}$, 满足

$$0<\int_0^{\frac{1}{x}}\frac{u^{s_2-1}\mathrm{d}u}{(1+u)^s}<\int_0^{\frac{1}{x}}u^{s_2-1}\mathrm{d}u=\frac{1}{s_2x^{s_2}}.$$

因此,(2.8) 成立. 引理得证.

**引理 2.3**　对于 $s\in(0,6],s_1\in(0,s),s_2\in(0,\frac{2}{\alpha}]\cap(0,s)$, 我们有如下推广的半离散 Hardy -Hilbert 不等式:

$$I:=\int_0^{\infty}\sum_{n=1}^{\infty}\frac{a_nf(x)\mathrm{d}x}{(x+n^{\alpha})^s}<\Big(\frac{1}{\alpha}k_s(s_2)\Big)^{\frac{1}{p}}(k_s(s_1))^{\frac{1}{q}}$$

$$\times\Big[\int_0^\infty x^{p[1-(\frac{s-s_2}{p}=\frac{s_1}{q})]-1}f^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^\infty n^{q[1-\alpha(\frac{s_2}{p}=\frac{s-s_1}{q})]-1}a_n^q\Big]^{\frac{1}{q}}. \tag{2.9}$$

**证** 作变换 $u=\frac{x}{n^\alpha}$, 可求得如下另一权函数表示式:

$$\begin{aligned}\omega_\alpha(s_1,n):&=n^{\alpha(s-s_1)}\int_0^\infty\frac{x^{s_1-1}\mathrm{d}x}{(x+n^\alpha)^s}\\&=\int_0^\infty\frac{u^{s_1-1}\mathrm{d}u}{(u+1)^s}=k_s,(s_1),\quad n\in\mathbf{N}.\end{aligned} \tag{2.10}$$

由 Höllder 不等式 (见 [31]), 得

$$\begin{aligned}I&=\int_0^\infty\sum_{n=1}^\infty\frac{1}{(x+n^\alpha)^s}\Big[\frac{x^{\frac{1-s_1}{q}}f(x)}{n^{\frac{1-\alpha s_2}{p}}}\Big]\Big[\frac{n^{\frac{1-\alpha s_2}{p}}a_n}{x^{\frac{1-s_1}{q}}}\Big]\mathrm{d}x\\&\leqslant\Big[\int_0^\infty\sum_{n=1}^\infty\frac{1}{(x+n^\alpha)^s}\frac{x^{(1-s_1)(p-1)}f^p(x)}{n^{1-\alpha s_2}}\mathrm{d}x\Big]^{\frac{1}{p}}\\&\quad\times\Big[\sum_{n=1}^\infty\int_0^\infty\frac{1}{(x+n^\alpha)^s}\frac{n^{(1-\alpha s_2)(q-1)}}{x^{1-s_1}}a_n^q\mathrm{d}x\Big]^{\frac{1}{q}}\\&=\Big\{\frac{1}{\alpha}\int_0^\infty\varpi(s_2,x)x^{p\big[1-\big(\frac{s-s_2}{p}=\frac{s_1}{q}\big)\big]-1}f^p(x)\mathrm{d}x\Big\}^{\frac{1}{p}}\\&\quad\times\Big\{\sum_{n=1}^\infty\omega_\alpha(s_1,n)n^{q\big[1-\alpha\big(\frac{s_2}{p}=\frac{s-s_1}{q}\big)\big]-1}a_n^q\Big\}^{\frac{1}{q}}.\end{aligned}$$

再由 (2.8) 及 (2.10), 得 (2.9). 引理得证.

**注 2.1** 对于 $m\in\{0,1,2,3,\lambda4\}$, 在 (2.9) 中, 设

$$\begin{aligned}&s=\lambda+m+1\in(m+1,6],\quad\lambda\in(0,5-m],\\&s_1=\lambda_1+m\in(m,s),\lambda_1\in(0,\lambda+1),\\&s_2=\lambda_2+1\in\Big(1,\frac{2}{\alpha}\Big]\cap(1,s),\lambda_2\in\Big(0,\frac{2}{\alpha}-1\Big]\cap(0,\lambda+m).\end{aligned}$$

置换 $f(x)$ 和 $(a_n)$ 分别为 $F_m(x)$ 和 $(A_n)$. 由条件式 (2.1), 可得如下含新参数的不等式:

$$\begin{aligned}&\int_0^\infty\sum_{n=1}^\infty\frac{A_n}{(x+n^\alpha)^{\lambda+m+1}}F_m(x)\mathrm{d}x\\&<\Big(\frac{1}{\alpha}k_{\lambda+m+1}(\lambda_2+1)\Big)^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}\\&\quad\times\Big[\int_0^\infty x^{p(1-m-\widetilde{\lambda}_1)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^\infty n^{q[1-\alpha(1+\widetilde{\lambda}_2)]-1}A_n^q\Big]^{\frac{1}{q}}.\end{aligned} \tag{2.11}$$

**引理 2.4** 对于 $t>0$, 我们有

$$\int_0^\infty\mathrm{e}^{-tx}f(x)\mathrm{d}x=t^m\int_0^\infty\mathrm{e}^{-tx}F_m(x)\mathrm{d}x, \tag{2.12}$$

$$\sum_{n=1}^\infty\mathrm{e}^{-tn^\alpha}a_n\leqslant t\sum_{n=1}^\infty\mathrm{e}^{-tn^\alpha}A_n. \tag{2.13}$$

**证**　对于 $m=0$, (2.13) 自然成立; 对于 $m\in\{1,2,3,4\}$, 由分部积分法及条件 $F_i(x)=o(\mathrm{e}^{tx})(t>0;x\to\infty)$, 故得

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-tx}F_{i-1}(x)\mathrm{d}x&=\int_0^\infty \mathrm{e}^{-tx}dF_i(x)=\mathrm{e}^{-tx}F_i(x)|_0^\infty-\int_0^\infty F_i(x)\mathrm{d}\mathrm{e}^{-tx}\\&=\lim_{x\to\infty}\mathrm{e}^{-tx}F_i(x)+t\int_0^\infty \mathrm{e}^{-tx}F_i(x)\mathrm{d}x=t\int_0^\infty \mathrm{e}^{-tx}F_i(x)\mathrm{d}x.\end{aligned}$$

代以 $i=1,\cdots,m$, 易得 (2.12) .

由条件 $\mathrm{e}^{-tn^\alpha}A_n=o(1)$ $(n\to\infty)$, 再利用 Abel 部分求和公式, 得

$$\begin{aligned}\sum_{n=1}^\infty \mathrm{e}^{-tn^\alpha}a_n&=\lim_{n\to\infty}\mathrm{e}^{-tn^\alpha}A_n+\sum_{n=1}^\infty A_n[\mathrm{e}^{-tn^\alpha}-\mathrm{e}^{-t(n+1)^\alpha}]\\&=\sum_{n=1}^\infty A_n[\mathrm{e}^{-tn^\alpha}-\mathrm{e}^{-t(n+1)^\alpha}].\end{aligned}$$

由不等式 $1-\mathrm{e}^{-t}<t$ $(t>0)$, 对于 $\alpha\in(0,1]$,

$$\begin{aligned}&\mathrm{e}^{-t(n+1)^\alpha}\geqslant \mathrm{e}^{-t(n^\alpha+1)}<=>\mathrm{e}^{t[(n+1)^\alpha-n^\alpha-1]}\leqslant 1\\&\qquad<=>(n+1)^\alpha-n^\alpha-1=\alpha(n+\theta_n)^{\alpha-1}-1\leqslant 0,\quad \theta_n\in(0,1).\end{aligned}$$

我们有

$$\begin{aligned}\sum_{n=1}^\infty \mathrm{e}^{-tn^\alpha}a_n&=\sum_{n=1}^\infty A_n[\mathrm{e}^{-tn^\alpha}-\mathrm{e}^{-t(n+1)^\alpha}]\\&=(1-\mathrm{e}^{-t})\sum_{n=1}^\infty A_n\mathrm{e}^{-tn^\alpha}\leqslant t\sum_{n=1}^\infty A_n\mathrm{e}^{-tn^\alpha}.\end{aligned}$$

即 (2.13) 成立. 引理得证.

## §3　主要结果及应用

**定理 3.1**　我们有如下涉及多重可变上限函数和部分和的半离散 Hilbert 型不等式:

$$\begin{aligned}I:&=\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)\mathrm{d}x}{(x+n^\alpha)^\lambda}\\&<\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\Big(\frac{1}{\alpha}k_{\lambda+m+1}(\lambda_2+1)\Big)^{\frac1p}(k_{\lambda+m+1}(\lambda_1+m))^{\frac1q}\\&\quad\times\Big[\int_0^\infty x^{p(1-m-\widetilde{\lambda}_1)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac1p}\Big[\sum_{n=1}^\infty n^{q[1-\alpha(1+\widetilde{\lambda}_2)]-1}A_n^q\Big]^{\frac1q}.\end{aligned}\tag{3.1}$$

特别地, 若 $\lambda_1+\lambda_2=\lambda\in(0,5-m],\lambda_1\in(0,\lambda),\lambda_2\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$, 我们有

$$0<\int_0^\infty x^{p(1-m-\lambda_1)-1}F_m^p(x)\mathrm{d}x<\infty,\quad 0<\sum_{n=1}^\infty n^{q[1-\alpha(1+\lambda_2)]-1}A_n^q<\infty$$

及

$$I=\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)\mathrm{d}x}{(x+n^\alpha)^\lambda}<\frac{\Gamma(\lambda+m+1)}{\alpha^{\frac1p}\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)$$

$$\times\Big[\int_0^\infty x^{p(1-m-\lambda_1)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^\infty n^{q[1-\alpha(1+\lambda_2)]-1}A_n^q\Big]^{\frac{1}{q}}. \tag{3.2}$$

**证** 因

$$\frac{1}{(x+n^\alpha)^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\mathrm{e}^{-(x+n^\alpha)t}\mathrm{d}t,$$

由 L 逐项积分定理 (见 [32]),(2.12) 及 (2.13), 得如下不等式:

$$\begin{aligned}
I&=\frac{1}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty a_nf(x)\int_0^\infty t^{\lambda-1}\mathrm{e}^{-(x+n^\alpha)t}\mathrm{d}t\mathrm{d}x\\
&=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\Big(\int_0^\infty \mathrm{e}^{-xt}f(x)\mathrm{d}x\Big)\Big(\sum_{n=1}^\infty \mathrm{e}^{-tn^\alpha}a_n\Big)\mathrm{d}t\\
&\leqslant\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\Big(t^m\int_0^\infty \mathrm{e}^{-xt}F_m(x)\mathrm{d}x\Big)\Big(t\sum_{n=1}^\infty \mathrm{e}^{-tn^\alpha}A_n\Big)\mathrm{d}t\\
&=\frac{1}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty A_nF_m(x)\Big[\int_0^\infty t^{\lambda+m}\mathrm{e}^{-(x+n^\alpha)t}\mathrm{d}t\Big]\mathrm{d}x\\
&=\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}\int_0^\infty\sum_{n=1}^\infty\frac{A_nF_m(x)\mathrm{d}x}{(x+n^\alpha)^\lambda}.
\end{aligned}$$

再由 (2.11), 得 (3.1). 定理得证.

**注 3.1** 在 (3.1) 中, 设 $\alpha=1,\lambda_1\in(0,\lambda+1),\lambda_2\in(0,1]\cap(0,\lambda+m)$, 我们有如下半离散 Hilbert 型不等式:

$$\begin{aligned}
&\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)\mathrm{d}x}{(x+n)^\lambda}\\
&<\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}\\
&\quad\times\Big[\int_0^\infty x^{p(1-m-\widetilde{\lambda}_1)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^\infty n^{-q\widetilde{\lambda}_2-1}A_n^q\Big]^{\frac{1}{q}}.
\end{aligned}\tag{3.3}$$

**定理 3.2** 设 $\lambda_1\in(0,\lambda),\lambda_2,\lambda-\lambda_1\in(0,1]\cap(0,\lambda)$, 则 (3.3) 的常数因子

$$\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

为最佳值的充要条件是 $\lambda_1+\lambda_2=\lambda\ (\in(0,5-m])$.

**证** "$\Leftarrow$" 若 $\lambda_1+\lambda_2=\lambda\ (\in(0,5-m])$, 则 (3.3) 变为

$$\begin{aligned}
\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{(x+n)^\lambda}\mathrm{d}x&<\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)\\
&\quad\times\Big[\int_0^\infty x^{p(1-m-\lambda_1)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big(\sum_{n=1}^\infty n^{-q\lambda_2-1}A_n^q\Big)^{\frac{1}{q}}.
\end{aligned}\tag{3.4}$$

任给 $0<\varepsilon<\min\{p\lambda_1,q\lambda_2\}$, 令

$$\widetilde{f}(x):=\begin{cases}0, & 0<x<1,\\ x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x\geqslant 1,\end{cases}$$

$$\widetilde{a}_n:=n^{\lambda_2-\frac{\varepsilon}{q}-1},\quad n\in\mathbf{N},$$

我们有 $\widetilde{F}_0(x):=\widetilde{f}(x)$,

$$\begin{aligned}\widetilde{F}_i(x)&:=\int_0^x\widetilde{F}_{i-1}(t)\mathrm{d}t\\&\leqslant\begin{cases}0, & 0<x<1,\\ \Big[\prod_{i=0}^{m-1}\Big(\lambda_1+i-\dfrac{\varepsilon}{p}\Big)\Big]^{-1}x^{\lambda_1+i-\frac{\varepsilon}{p}-1}, & x\geqslant 1, i\in\{1,\cdots,m\},\end{cases}\\ \widetilde{A}_n&:=\sum_{k=1}^n\widetilde{a}_k=\sum_{k=1}^n k^{\lambda_2-\frac{\varepsilon}{q}-1}<\int_0^n t^{\lambda_2-\frac{\varepsilon}{q}-1}\mathrm{d}t=\frac{n^{\lambda_2-\frac{\varepsilon}{q}}}{\lambda_2-\frac{\varepsilon}{q}},\quad n\in\mathbf{N},\end{aligned}$$

这里, 规定当 $m=0,\varepsilon\geqslant 0$ 时, $\prod\limits_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})=1$. 上设满足条件 $\widetilde{F}_i(x)=o(\mathrm{e}^{tx})$ $(t>0,i=1,\cdots,m;x\to\infty)$, $\widetilde{A}_n=o(\mathrm{e}^{tn^{\alpha}})$ $(t>0;n\to\infty)$.

若有正常数 $M\leqslant\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)$, 且用 $M$ 替代 (3.4) 的常数因子

$$\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)$$

后 (3.4) 仍成立, 则有如下不等式:

$$\begin{aligned}\widetilde{I}&:=\int_0^\infty\sum_{n=1}^\infty\frac{\widetilde{a}_n\widetilde{f}(x)\mathrm{d}x}{(x+n)^\lambda}\\&<M\Big[\int_0^\infty x^{p(1-m-\lambda_1)-1}\widetilde{F}_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big(\sum_{n=1}^\infty n^{-q\lambda_2-1}\widetilde{A}_n^q\Big)^{\frac{1}{q}}.\end{aligned}\tag{3.5}$$

由 (3.5) 及级数的递减性质, 可得

$$\begin{aligned}\widetilde{I}&<M\Big[\prod_{i=0}^{m-1}\Big(\lambda_1+i-\frac{\varepsilon}{p}\Big)\Big]^{-1}\Big[\int_1^\infty x^{p(1-m-\lambda_1)-1}x^{p(\lambda_1+m-1)-\varepsilon}\mathrm{d}x\Big]^{\frac{1}{p}}\\&\quad\times\frac{1}{\lambda_2-\frac{\varepsilon}{q}}\Big(\sum_{n=1}^\infty n^{-q\lambda_2-1}n^{q\lambda_2-\varepsilon}\Big)^{\frac{1}{q}}\\&=\frac{M\Big[\prod\limits_{i=0}^{m-1}\Big(\lambda_1+i-\dfrac{\varepsilon}{p}\Big)\Big]^{-1}}{\lambda_2-\frac{\varepsilon}{q}}\Big(\int_1^\infty x^{-\varepsilon-1}\mathrm{d}x\Big)^{\frac{1}{p}}\Big(1+\sum_{n=2}^\infty n^{-\varepsilon-1}\Big)^{\frac{1}{q}}\\&\leqslant\frac{M\Big[\prod\limits_{i=0}^{m-1}\Big(\lambda_1+i-\dfrac{\varepsilon}{p}\Big)\Big]^{-1}}{\lambda_2-\frac{\varepsilon}{q}}\Big(\int_1^\infty x^{-\varepsilon-1}\mathrm{d}x\Big)^{\frac{1}{p}}\Big(1+\int_1^\infty y^{-\varepsilon-1}\mathrm{d}y\Big)^{\frac{1}{q}}\\&=\frac{M\Big[\prod\limits_{i=0}^{m-1}\Big(\lambda_1+i-\dfrac{\varepsilon}{p}\Big)\Big]^{-1}}{\varepsilon(\lambda_2-\frac{\varepsilon}{q})}(\varepsilon+1)^{\frac{1}{q}}.\end{aligned}$$

由 (2.10), 并取 $\alpha=1, s=\lambda$, $s_1=\widetilde{\lambda}_1:=\lambda_1-\frac{\varepsilon}{p}\in(0,\lambda)$, $\widetilde{\lambda}_2:=\lambda_2+\frac{\varepsilon}{p}\in(0,\lambda)$, 我们有

$$\begin{aligned}\widetilde{I}&=\sum_{n=1}^{\infty}\Big[n^{\lambda_2+\frac{\varepsilon}{p}}\int_1^{\infty}\frac{x^{\lambda_1-\frac{\varepsilon}{p}-1}}{(x+n)^{\lambda}}\mathrm{d}x\Big]n^{-\varepsilon-1}\\&=\sum_{n=1}^{\infty}\omega_1(\widetilde{\lambda}_1,n)n^{-\varepsilon-1}-\sum_{n=1}^{\infty}\Big[n^{\lambda_2-\frac{\varepsilon}{q}-1}\int_0^1\frac{x^{\lambda_1-\frac{\varepsilon}{p}-1}}{(x+n)^{\lambda}}\mathrm{d}x\Big]\\&>B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\sum_{n=1}^{\infty}n^{-\varepsilon-1}-\sum_{n=1}^{\infty}\Big(n^{\lambda_2-\frac{\varepsilon}{q}-1}\int_0^1\frac{x^{\lambda_1-\frac{\varepsilon}{p}-1}}{n^{\lambda}}\mathrm{d}x\Big)\\&>B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\int_1^{\infty}y^{-\varepsilon-1}\mathrm{d}y-O(1)\\&=\frac{1}{\varepsilon}\Big(B\Big(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\Big)-\varepsilon O(1)\Big).\end{aligned}$$

基于上面结果, 有

$$\begin{aligned}&B\Big(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\Big)-\varepsilon O(1)\\&<\varepsilon\widetilde{I}<\frac{M\Big[\prod\limits_{i=0}^{m-1}\Big(\lambda_1+i-\frac{\varepsilon}{p}\Big)\Big]^{-1}}{\lambda_2-\frac{\varepsilon}{q}}(\varepsilon+1)^{\frac{1}{q}}.\end{aligned}$$

令 $\varepsilon\to0^+$, 由 Beta 函数的连续性, 得

$$\begin{aligned}&\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)\\&=\frac{\Gamma(\lambda_1+m)\Gamma(\lambda_2+1)}{\Gamma(\lambda)}=\lambda_2B(\lambda_1,\lambda_2)\prod_{i=0}^{m-1}(\lambda_1+i)\leqslant M.\end{aligned}$$

故 $M=\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\lambda_1+m)$ 是 (3.4) (即 (3.3) 当 $\lambda_1+\lambda_2=\lambda$ 时) 的最佳值.

“$\Rightarrow$” 对于 $\widetilde{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widetilde{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, 我们有

$$\widetilde{\lambda}_1+\widetilde{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda,$$

$$0<\widetilde{\lambda}_1,\widetilde{\lambda}_2<\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda,\widetilde{\lambda}_2\leqslant\frac{1}{q}+\frac{1}{p}=1,$$

及 $\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\widetilde{\lambda}_1+m)\in\mathbf{R}_+$. 由 Hölder 不等式 (见 [31]), 我们还有

$$\begin{aligned}&k_{\lambda+m+1}(\widetilde{\lambda}_1+m)\\&=k_{\lambda+m+1}\Big(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+m\Big)\\&=\int_0^{\infty}\frac{u^{\frac{\lambda+m-\lambda_2}{p}+\frac{\lambda_1+m}{q}-1}}{(1+u)^{\lambda+m+1}}\mathrm{d}u=\int_0^{\infty}\frac{(u^{\frac{\lambda+m-\lambda_2-1}{p}})}{(1+u)^{\lambda+m+1}}(u^{\frac{\lambda_1+m-1}{q}})\mathrm{d}u\\&\leqslant\Big[\int_0^{\infty}\frac{u^{\lambda+m-\lambda_2-1}}{(1+u)^{\lambda+m+1}}\mathrm{d}u\Big]^{\frac{1}{p}}\Big[\int_0^{\infty}\frac{u^{\lambda_1+m-1}}{(1+u)^{\lambda+m+1}}\mathrm{d}u\Big]^{\frac{1}{q}}\\&=\Big[\int_0^{\infty}\frac{v^{(\lambda_2+1)-1}}{(1+v)^{\lambda+m+1}}\mathrm{d}v\Big]^{\frac{1}{p}}\Big[\int_0^{\infty}\frac{u^{\lambda_1+m-1}}{(1+u)^{\lambda+m+1}}\mathrm{d}u\Big]^{\frac{1}{q}}\\&=(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}.\end{aligned}\tag{3.6}$$

由于常数因子

$$\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}$$

为 (3.3) 的最佳值, 在 (3.4) 中取 $\lambda_1=\widetilde{\lambda}_1$ $(i=1,2)$, 我们有如下不等式:

$$\begin{aligned}&\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}\\ \leqslant{}&\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda)}k_{\lambda+m+1}(\widetilde{\lambda}_1+m),\quad \in\mathbf{R}_+,\end{aligned}$$

即有不等式

$$k_{\lambda+m+1}(\widetilde{\lambda}_1+m)\geqslant(k_{\lambda+m+1}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+m+1}(\lambda_1+m))^{\frac{1}{q}}.$$

因而 (3.6) 取等号.

(3.6) 取等号的充要条件是存在不全为 0 的常数 $A$ 和 $B$, 使得 (见 [31]) $Au^{\lambda+m-\lambda_2-1}=Bu^{\lambda_1+m-1}$ 在 $\mathbf{R}_+$ 上几乎处处成立. 不妨设 $A\neq0$, 则有 $u^{\lambda-\lambda_1-\lambda_2}=B/A$ 在 $\mathbf{R}_+$ 上几乎处处成立, 及 $\lambda-\lambda_1-\lambda_2=0$. 因而有 $\lambda_1+\lambda_2=\lambda$. 定理得证.

**注 3.2**　在 (3.4) 中, 取 $\lambda=1,\lambda_1=\frac{1}{r},\lambda_2=\frac{1}{s}(r>1,\frac{1}{r}+\frac{1}{s}=1)$, 我们有如下具有最佳常数因子 $\frac{\pi}{s\sin(\frac{\pi}{r})}\prod_{i=0}^{m-1}(\frac{1}{r}+i)$ 的 Hilbert 型不等式:

$$\begin{aligned}\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{x+n}\mathrm{d}x<{}&\frac{\pi}{s\sin\left(\frac{\pi}{r}\right)}\prod_{i=0}^{m-1}\left(\frac{1}{r}+i\right)\\&\times\Big[\int_0^\infty x^{p(\frac{1}{s}-m)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big(\sum_{n=1}^\infty n^{-\frac{q}{s}-1}A_n^q\Big)^{\frac{1}{q}}.\end{aligned}\tag{3.7}$$

特别当 $r=p,s=q$ 时, 有

$$\begin{aligned}\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{x+n}\mathrm{d}x<{}&\frac{\pi}{q\sin\left(\frac{\pi}{p}\right)}\prod_{i=0}^{m-1}\left(\frac{1}{p}+i\right)\\&\times\Big[\int_0^\infty x^{p(1-m)-2}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big(\sum_{n=1}^\infty n^{-2}A_n^q\Big)^{\frac{1}{q}};\end{aligned}\tag{3.8}$$

当 $r=q,s=p$ 时, 有

$$\begin{aligned}\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{x+n}\mathrm{d}x<{}&\frac{\pi}{p\sin\left(\frac{\pi}{p}\right)}\prod_{i=0}^{m-1}\left(\frac{1}{q}+i\right)\\&\times\Big[\int_0^\infty x^{-pm}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big(\sum_{n=1}^\infty n^{-q}A_n^q\Big)^{\frac{1}{q}};\end{aligned}\tag{3.9}$$

当 $p=q=2$ 时,

$$\begin{aligned}&\int_0^\infty\sum_{n=1}^\infty\frac{a_nf(x)}{x+n}\mathrm{d}x\\<{}&\frac{\pi}{2}\prod_{i=0}^{m-1}(\frac{1}{2}+i)\Big(\int_0^\infty x^{-2m}F_m^2(x)\mathrm{d}x\sum_{n=1}^\infty n^{-2}A_n^2\Big)^{\frac{1}{2}}.\end{aligned}\tag{3.10}$$

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# A New Half-Discrete Hilbert-Type Inequality Involving One Multiple Upper Limit Function and One Partial Sums

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**Abstract** By means of the weight functions, Euler-Maclaurin summation formula, Abel's summation by parts formula, and the technique of real analysis, a new half-discrete Hilbert-type inequality involving one multiple variable upper limit function and one partial sums is given. As applications, the equivalent conditions of the best possible constant factor in a particular inequality related to a few parameters and several particular inequalities are considered.

**Keywords** Weight function, Euler-Maclaurin summation formula, Abel's summation by parts formula, Half-Discrete Hilbert-type inequality, Multiple variable upper limit function, Partial sums

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