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加权双正则函数的 Cauchy 积分公式及其边界性质*

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提要 正则函数是单复分析中解析函数在高维空间的推广, 加权正则函数是正则函数的进一步发展. 加权正则函数在解决各项异性介质的热传导问题上发挥着重要作用. 加权双正则函数又是加权正则函数的进一步发展, 加权双正则函数是 Clifford 分析中的又一类新的函数类, 具有一定的研究意义. 作者首先证明了加权双正则函数的 Cauchy-Pompeiu 公式, 进而得到了加权双正则函数的 Cauchy 积分公式, 最后证明了加权双正则函数的 Cauchy 型积分算子的边界性质.

关键词 实 Clifford 分析, 加权双正则函数, Cauchy-Pompeiu 公式, Cauchy 积分公式, Cauchy 型积分算子的边界性质

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§1 引 言

1878 年, Clifford[1] 建立了一个新的数学框架, 创造出了一种可结合不可交换的几何代数结构, 并以他的名字将其命名为 Clifford 代数. 1982 年, Brackx, Delanghe 和 Sommen[2] 研究了正则函数的 Cauchy-Pompeiu 公式、Cauchy 积分公式等, 为 Clifford 分析的理论奠定了基础. 随后, 实 Clifford 分析作为新兴的分析分支得到了发展. 国内外有很多学者如 Gürlebeck[3]、Begher[4] 和 Kähler[5] 等以及黄沙 [6–7]、乔玉英 [8–9]、杜金元 [10–11] 等都致力于 Clifford 分析的研究.

2008 年, 王海燕 [12] 对双正则函数的 Cauchy 积分公式进行了研究. 2009 年, 边小丽 [13] 研究了双超正则函数的 Cauchy 积分公式. 2010 年, 许娜 [14] 对泛 Clifford 分析中无界域上的 Cauchy-Pompeiu 公式和 Cauchy 积分公式进行探究. 2016 年, García 等 [15] 探究了第一类 multi-meta-weighted-monogenic 函数的 Cauchy-Pompeiu 公式, 并且给出了非齐次 meta-n-weighted-monogenic 方程的分布解. 2018 年, Vanegas 等 [16] 研究了带有 Clifford 常数权的 Dirac 算子的基本解, 并给出了该类加权正则函数的 Cauchy-Pompeiu 公式与 Cauchy 积分公式. 2022 年, 罗利萍等 [17] 对 Vanegas 等研究的加权 Dirac 算子进行了进一步探究, 并给出了该类加权正则函数的平均值定理、最大模原理、Weierstrass 定理、非欧氏距离下的 Hile 引理等性质. 2024 年, 王丽萍等 [18] 又给出了加权正则函数的 Taylor 展式、唯一性定理、Laurent 展式和留数定理等.

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在上述研究的基础上, 本文主要研究了 Clifford 分析中加权双正则函数的 Cauchy-Pompeiu 公式、Cauchy 积分公式以及 Cauchy 型积分算子的边界性质.

§2 预备知识

设 $\{e_1, e_2, \cdots, e_n\}$是 n 维欧氏空间 $\mathbb{R}^n$ 的一组标准正交基, $\mathcal{A}_n(R)$ 是 2^n 维实 Clifford 代数空间, 它的基为 $\beta = \{e_N \mid N \in \Gamma_n\}$, 其中 $\Gamma_n = \{0, 1, \cdots, n, 12, 13, \cdots, 123\cdots n\}$, $\mathcal{A}_n(R)$ 中的基元素一般可以写为 $e_N = e_{N_1}e_{N_2}\cdots e_{N_r}$, 其中 $N = \{N_1, \cdots, N_r\} \subseteq \{1, 2, \cdots, n\}$, 且 $1 \leqslant N_1 < \cdots < N_r \leqslant n$. 当 $N = \varnothing$ 时, $e_N = e_0 = 1$. 而且对于任意元素 $a \in \mathcal{A}_n(R)$, 都可表示为 $a = \sum\limits_N a_N e_N$, 其中 $a_N \in R$. Clifford 代数中乘法运算有如下法则:

$$\begin{cases} e_i^2 = -1, & i = 1, 2, \cdots, n, \\ e_ie_j = -e_je_i, & i, j = 1, 2, \cdots, n,\ i < j, \\ \overline{a \cdot b} = \bar{b} \cdot \bar{a}, & a, b \in \mathcal{A}_n(R). \end{cases}$$

$\mathcal{A}_n(R)$ 中任一元素 a 的模及共轭分别定义为

$$|a| = \sqrt{[a, a]_0} = \Big(\sum_N |a_N|^2\Big)^{\frac{1}{2}}, \quad \overline{a} = \sum_N a_N \overline{e}_N,$$

其中 $\overline{e}_N = (-1)^{\#N(\#N+1)/2} e_N$, $\#N$ 记为 N 的指标.

设 $\Omega_1 \subset \mathbb{R}^n\,(\Omega_2 \subset \mathbb{R}^m)$ 是一非空连通开集, 则定义在 Ω_1 中取值于 $\mathcal{A}_n(R)$ 的函数 f 可表示为 $f(x) = \sum\limits_N f_N(x)e_N$, 其中 $f_N(x)$ 为实值函数. $F_{\Omega_1}^{(r)}$ 表示 Ω_1 中 C^r 函数的全体, 即

$$F_{\Omega_1}^{(r)} = \{f \mid f : \Omega_1 \to \mathcal{A}_n(R),\ f(x) = \sum_N f_N(x)e_N,\ f_N(x) \in C^r,\ x \in \Omega_1\}.$$

经典 Dirac 算子定义如下

$$D : F_{\Omega_1}^{(r)} \to F_{\Omega_1}^{(r-1)},$$

$$Df = \sum_{i=1}^n e_i \frac{\partial f}{\partial x_i} = \sum_{i,N} e_i e_N \frac{\partial f_N}{\partial x_i},$$

$$fD = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i = \sum_{i,N} \frac{\partial f_N}{\partial x_i} e_N e_i.$$

加权 Dirac 算子 $\mathcal{D}_\omega$ 及共轭 $\overline{\mathcal{D}}_\omega$ 分别定义为

$$\mathcal{D}_\omega = \sum_{i=1}^n \psi_i \partial_i, \quad \overline{\mathcal{D}}_\omega = \sum_{i=1}^n \overline{\psi_i} \partial_i, \tag{2.1}$$

其中 $\psi_i\,(i = 1, 2, \cdots, n)$ 是一个 Clifford 常数. ψ_i 的构造方法如文 [16] 所述.

二阶椭圆微分算子 $\widetilde{\Delta}_n$ 的定义为

$$\widetilde{\Delta}_n = \sum_{i=1}^n B_{1_{ii}} \partial_i^2 + 2 \sum_{1 \leqslant i < j \leqslant n} B_{1_{ij}} \partial_i \partial_j, \tag{2.2}$$

其中 $B_{1_{ij}}$ 是矩阵 B_1 的元素. 由于矩阵 B_1 是对称的正定矩阵, 从而它的逆矩阵 $A_1 = B_1^{-1}$ 和它的平方根 $B_1^{\frac{1}{2}}$ 也是对称的正定矩阵, 且满足 $A_1^{\frac{1}{2}} = B_1^{-\frac{1}{2}}$. 此外, 矩阵 B_1 还满足 Cholesky 分解 $B_1 = L_1 L_1^{\mathrm{T}}$ (L_1 是对角线上元素都大于 0 的下三角矩阵).

二阶椭圆微分算子 $\widetilde{\Delta}_m$ 的定义为

$$\widetilde{\Delta}_m = \sum_{i=1}^{m} B_{2_{ii}} \partial_i^2 + 2 \sum_{1 \leqslant i < j \leqslant m} B_{2_{ij}} \partial_i \partial_j, \tag{2.3}$$

其中 $B_{2_{ij}}$ 是矩阵 B_2 的元素. 矩阵 B_2 与矩阵 B_1 有类似的性质.

定义 2.1　用 $F_{\Omega}^{(r)}$ 表示 $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ 中 $C^r\ (r \geqslant 1)$ 函数的全体:

$$F_{\Omega}^{(r)} = \Big\{ f \mid f : \Omega \to \mathcal{A}_n(R),\ f(x,y) = \sum_N f_N(x,y) e_N,$$
$$f_N(x,y) \in C^r(\Omega),\ x \in \Omega_1,\ y \in \Omega_2 \Big\}.$$

定义 2.2　设 $f \in F_{\Omega}^{(r)}$, $r \geqslant 1$. 若

$$\begin{cases} \mathcal{D}_{\omega_x} f(x,y) = \displaystyle\sum_{i=1}^{n} \psi_i \frac{\partial f}{\partial x_i} = \sum_{i,N} \psi_i e_N \frac{\partial f_N}{\partial x_i} = 0, \\ f(x,y) \mathcal{D}_{\omega_y} = \displaystyle\sum_{j=1}^{m} \frac{\partial f}{\partial y_j} \psi_j = \sum_{j,N} \frac{\partial f_N}{\partial y_j} e_N \psi_j = 0, \end{cases}$$

则称 $f(x,y)$ 为加权双正则函数.

定义 2.3　对于 $\mathbb{R}^n$ 内任意两点 $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ 和 $x = (x_1, x_2, \cdots, x_n)$, 定义 x 和 ξ 之间的非欧氏距离 ρ_1 为

$$\rho_1^2(x,\xi) = \sum_{i,j=1}^{n} A_{1_{ij}} (x_i - \xi_i)(x_j - \xi_j) = \langle (x-\xi), A_1(x-\xi) \rangle, \tag{2.4}$$

其中 $A_{1_{ij}}$ 是矩阵 A_1 的元素. 当 $x \neq \xi$ 时, 设它们之间的欧氏距离为 r_1, 即 $r_1 = |x - \xi|$, 则有 $x - \xi = r_1 x^* (|x^*| = 1)$, 把此点 x^* 和 $(0, \cdots, 0)$ 之间的非欧氏距离记作 ρ_0, 则有 $\rho_0 \geqslant c_1 > 0$. 还可证明

$$\rho_1 = r_1 \rho_0, \quad \rho_1 \geqslant c_1 r_1.$$

定义 2.4　对于 $\mathbb{R}^m$ 内任意两点 $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$ 和 $y = (y_1, y_2, \cdots, y_m)$, 定义 y 和 τ 之间的非欧氏距离 ρ_2 为

$$\rho_2^2(y,\tau) = \sum_{i,j=1}^{m} A_{2_{ij}} (y_i - \tau_i)(y_j - \tau_j) = \langle (y-\tau), A_2(y-\tau) \rangle, \tag{2.5}$$

其中 $A_{2_{ij}}$ 是矩阵 A_2 的元素. 类似地有 $\rho_2 = r_2 \rho_0^*$ 和 $\rho_2 \geqslant c_2 r_2$.

用类似于文 [19–20] 中的方法可证得二阶椭圆微分算子 $\widetilde{\Delta}_n$, $\widetilde{\Delta}_m$ 的基本解分别为

$$\widetilde{K}(x,\xi) = \frac{1}{\det(B_1)^{\frac{1}{2}} \omega_n} \begin{cases} \ln \rho_1, & n = 2, \\ \dfrac{-1}{n-2} \dfrac{1}{\rho_1^{n-2}}, & n \geqslant 3, \end{cases} \quad x \neq \xi, \tag{2.6}$$

其中 ω_n 表示 $\mathbb{R}^n$ 中单位球的表面积, $\det(B_1)$ 表示矩阵 B_1 的行列式,

$$\widetilde{K}(y,\tau)=\frac{1}{\det(B_2)^{\frac{1}{2}}\omega_m}\begin{cases}\ln\rho_2, & m=2,\\ \dfrac{-1}{m-2}\dfrac{1}{\rho_2^{m-2}}, & m\geqslant 3,\end{cases}\quad y\neq\tau, \tag{2.7}$$

其中 ω_m 表示 $\mathbb{R}^m$ 中单位球的表面积, $\det(B_2)$ 表示矩阵 B_2 的行列式.

将 $\overline{D}_\omega$ 作用到 $\widetilde{K}(x,\xi)$ 上, 得到

$$E_\omega(x,\xi)=\overline{D}_\omega\widetilde{K}(x,\xi)=\frac{1}{\det(B_1)^{\frac{1}{2}}\omega_n}\frac{1}{\rho_1^n}\sum_{i,j=1}^{n}\overline{\psi_i}A_{1_{ij}}(x_j-\xi_j). \tag{2.8}$$

由于 $\widetilde{\Delta}_n=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega$, 从而 $\mathcal{D}_\omega E_\omega(x,\xi)=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega\widetilde{K}(x,\xi)=\widetilde{\Delta}_n\widetilde{K}(x,\xi)=0$. 故 $E_\omega(x,\xi)$ 是加权左正则函数. 同样可证 $E_\omega(x,\xi)$ 是加权右正则函数.

同理将 $\overline{D}_\omega$ 作用到 $\widetilde{K}(y,\tau)$ 上, 得到

$$E_\omega(y,\tau)=\overline{D}_\omega\widetilde{K}(y,\tau)=\frac{1}{\det(B_2)^{\frac{1}{2}}\omega_m}\frac{1}{\rho_2^m}\sum_{i,j=1}^{m}\overline{\psi_i}A_{2_{ij}}(y_j-\tau_j). \tag{2.9}$$

由于 $\widetilde{\Delta}_m=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega$, 从而 $\mathcal{D}_\omega E_\omega(y,\tau)=\mathcal{D}_\omega\overline{\mathcal{D}}_\omega\widetilde{K}(y,\tau)=\widetilde{\Delta}_m\widetilde{K}(y,\tau)=0$. 故 $E_\omega(y,\tau)$ 是加权左正则函数. 同样可证 $E_\omega(y,\tau)$ 是加权右正则函数.

引理 2.1 (见 [16], Stokes 公式) 设 Ω_1 如上所述, $\partial\Omega_1$ 足够光滑, $u,v:\Omega_1\to\mathcal{A}_n(R)$ 是 Ω_1 上的连续可微函数, 则对于加权 Dirac 算子 $\mathcal{D}_\omega$, 有如下公式成立

$$\int_{\Omega_1}(v\mathcal{D}_\omega\cdot u+v\cdot\mathcal{D}_\omega u)\mathrm{d}x=\int_{\partial\Omega_1}v\mathrm{d}\sigma_x u, \tag{2.10}$$

其中 $\mathrm{d}\sigma_x=\sum\limits_{i=1}^{n}\psi_i\mathcal{N}_{1_i}\mathrm{d}\mu_1$ 是在坐标系 $\{\psi_1,\cdots,\psi_n\}$ 下 $\mathcal{A}_n(R)$ 值的面积微元, $\mathcal{N}_1=(\mathcal{N}_{1_1},\cdots,\mathcal{N}_{1_n})$ 为单位外法向量, $\mathrm{d}\mu_1$ 为标量面积微元, 则有 $\mathrm{d}\sigma_x=\mathcal{N}_1\mathrm{d}\mu_1$. $\mathrm{d}x^n$ 为体积微元, 且 $\mathrm{d}x^n=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$.

推论 2.1 在引理 2.1 的条件下, 如果满足 v 是加权右正则函数且 u 是加权左正则函数, 则有

$$\int_{\partial\Omega_1}v\mathrm{d}\sigma_x u=0.$$

引理 2.2 (见 [16], 加权正则函数的 Cauchy-Pompeiu 公式) 设 $\Omega_1,\partial\Omega_1$ 如上所述, $u:\overline{\Omega}_1\to\mathcal{A}_n(R)$, 且 u 在 $\overline{\Omega}_1$ 上连续可微, 则有

$$\int_{\partial\Omega_1}E_\omega(x,\xi)\mathrm{d}\sigma_x u-\int_{\Omega_1}E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x}u\mathrm{d}x=\begin{cases}u(\xi), & \xi\in\Omega_1,\\ 0, & \xi\in\overline{\Omega}_1^c.\end{cases} \tag{2.11}$$

推论 2.2 设 $\Omega_2,\partial\Omega_2$ 如上所述, $v:\overline{\Omega}_2\to\mathcal{A}_m(R)$, 且 v 在 $\overline{\Omega}_2$ 上连续可微, 则有

$$\int_{\partial\Omega_2}v\mathrm{d}\sigma_y E_\omega(y,\tau)-\int_{\Omega_2}v\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y=\begin{cases}v(\tau), & \tau\in\Omega_2,\\ 0, & \tau\in\overline{\Omega}_2^c,\end{cases}$$

其中 $\mathrm{d}\sigma_y=\sum\limits_{j=1}^{m}\psi_j\mathcal{N}_{2_j}\mathrm{d}\mu_2$ 是在坐标系 $\{\psi_1,\cdots,\psi_m\}$ 下 $\mathcal{A}_m(R)$ 值的面积微元, $\mathcal{N}_2=(\mathcal{N}_{2_1},\cdots,\mathcal{N}_{2_m})$ 为单位外法向量, $\mathrm{d}\mu_2$ 为标量面积微元, 则有 $\mathrm{d}\sigma_y=\mathcal{N}_2\mathrm{d}\mu_2$. $\mathrm{d}y^m$ 为体积微元, 且 $\mathrm{d}y^m=\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_m$.

设 $\Omega_1 \subset \mathbb{R}^n$ 是一个有界域且边界 $\partial\Omega_1$ 充分光滑, 对于任意 $\xi \in \Omega_1$, 以 ξ 为中心, $\varepsilon > 0$ 为半径, 作 n 维非欧氏距离超球 $U_{1\varepsilon}(\xi) = \{x \in \Omega_1 : \rho(\xi, x) < \varepsilon\}$, $\partial U_{1\varepsilon}(\xi)$ 的法向量取外法向量, 则曲面 $\partial U_{1\varepsilon}(\xi)$ 的参数方程可表示为

$$x(t) = \varepsilon B_1^{\frac{1}{2}} r_1(t) + \xi, \quad t \in \mathbb{R}^{n-1}, \tag{2.12}$$

其中 $r_1(t)$ 是 $\mathbb{R}^n$ 中欧氏距离下单位球的参数方程. 由文 [16] 可知下式成立

$$E_\omega(x, \xi) \cdot \mathrm{d}\sigma_x = \frac{1}{\omega_n} \mathrm{d}\mu_{r_1}. \tag{2.13}$$

设 $\Omega_2 \subset \mathbb{R}^m$ 是一个有界域且边界 $\partial\Omega_2$ 充分光滑, 对于任意 $\tau \in \Omega_2$, 以 τ 为中心, $\varepsilon > 0$ 为半径, 作 m 维非欧氏距离超球 $U_{2\varepsilon}(\tau) = \{y \in \Omega_2 : \rho(\tau, y) < \varepsilon\}$, $\partial U_{2\varepsilon}(\tau)$ 的法向量取外法向量, 则曲面 $\partial U_{2\varepsilon}(\tau)$ 的参数方程可表示为

$$y(t) = \varepsilon B_2^{\frac{1}{2}} r_2(t) + \tau, \quad t \in \mathbb{R}^{m-1}, \tag{2.14}$$

其中 $r_2(t)$ 是 $\mathbb{R}^m$ 中欧氏距离下单位球的参数方程. 由文 [16] 可知下式成立

$$\mathrm{d}\sigma_y \cdot E_\omega(y, \tau) = \frac{1}{\omega_m} \mathrm{d}\mu_{r_2}. \tag{2.15}$$

§3　加权双正则函数的 Cauchy 积分公式

引理 3.1 (见 [16])　设 $\Omega_1, \partial\Omega_1$ 如上所述, $u : \overline{\Omega}_1 \to \mathcal{A}_n(R)$, 且 u 在 $\overline{\Omega}_1$ 上连续可微, $U_{1\varepsilon}(\xi) = \{x \in \Omega_1 : \rho(x, \xi) < \varepsilon\}$, 则有

$$\lim_{\varepsilon \to 0} \int_{\partial U_{1\varepsilon}(\xi)} E_\omega(x, \xi) \mathrm{d}\sigma_x u = u(\xi). \tag{3.1}$$

推论 3.1　设 $\Omega_2, \partial\Omega_2$ 如上所述, $v : \overline{\Omega}_2 \to \mathcal{A}_m(R)$, 且 v 在 $\overline{\Omega}_2$ 上连续可微, $U_{2\varepsilon}(\tau) = \{y \in \Omega_2 : \rho(y, \tau) < \varepsilon\}$, 则有

$$\lim_{\varepsilon \to 0} \int_{\partial U_{2\varepsilon}(\tau)} v \mathrm{d}\sigma_y E_\omega(y, \tau) = v(\tau).$$

引理 3.2　设 $\Omega = \Omega_1 \times \Omega_2$, $K_1 \subset \Omega_1 \subset R^n$, Ω_1 为非空连通开集, K_1 为 n 维可微紧的定向流形, $n \geqslant 2$, $K_2 \subset \Omega_2 \subset R^m$, Ω_2 为非空连通开集, K_2 为 m 维可微紧的定向流形, $m \geqslant 2$, $f(x, y) \in F_\Omega^r$, $g(x) \in F_{\Omega_1}^r$, $h(y) \in F_{\Omega_2}^r$, $r \geqslant 2$, ∂K_1, ∂K_2 具有给定的诱导定向, 则

$$\int_{\partial K_1} g(x) \mathrm{d}\sigma_x \Big[\int_{\partial K_2} f(x, y) \mathrm{d}\sigma_y h(y) \Big] = \int_{\partial K_2} \Big[\int_{\partial K_1} g(x) \mathrm{d}\sigma_x f(x, y) \Big] \mathrm{d}\sigma_y h(y). \tag{3.2}$$

证　令 $g(x) = \sum\limits_C g_C(x) e_C$, $f(x, y) = \sum\limits_H f_H(x, y) e_H$, $h(y) = \sum\limits_G h_G(y) e_G$, 则

$$\begin{aligned}
&\int_{\partial K_1} g(x) \mathrm{d}\sigma_x \Big[\int_{\partial K_2} f(x, y) \mathrm{d}\sigma_y h(y) \Big] \\
&= \int_{\partial K_1} \sum_C g_C(x) e_C \sum_{i=1}^n \psi_i N_{1_i} \mathrm{d}\mu_1 \Big[\int_{\partial K_2} \sum_H f_H(x, y) e_H \sum_{j=1}^m \psi_j N_{2_j} \mathrm{d}\mu_2 \sum_G h_G(y) e_G \Big] \\
&= \sum_C \sum_{i=1}^n \sum_H \sum_{j=1}^m \sum_G e_C \psi_i e_H \psi_j e_G \int_{\partial K_1} g_C(x) N_{1_i} \mathrm{d}\mu_1 \Big[\int_{\partial K_2} f_H(x, y) N_{2_j} \mathrm{d}\mu_2 h_G(y) \Big].
\end{aligned}$$

记

$$S_{C,i,H,j,G}=\sum_{C}\sum_{i=1}^{n}\sum_{H}\sum_{j=1}^{m}\sum_{G}e_C\psi_ie_H\psi_je_G,$$

由实 Stokes 公式可得

$$\begin{aligned}&\int_{\partial K_1}g(x)\mathrm{d}\sigma_x\Big[\int_{\partial K_2}f(x,y)\mathrm{d}\sigma_yh(y)\Big]\\
&=S_{C,i,H,j,G}\Big\{\int_{\partial K_1}g_C(x)N_{1_i}\mathrm{d}\mu_1\int_{K_2}\Big[\frac{\partial f_H}{\partial y_j}(x,y)h_G(y)+f_H(x,y)\frac{\partial h_G}{\partial y_j}(y)\Big]\mathrm{d}y\Big\}\\
&=S_{C,i,H,j,G}\Big\{\int_{K_1}\frac{\partial g_C}{\partial x_i}(x)\mathrm{d}x\int_{K_2}\Big[\frac{\partial f_H}{\partial y_j}(x,y)h_G(y)+f_H(x,y)\frac{\partial h_G(y)}{\partial y_j}(y)\Big]\mathrm{d}y\\
&\quad+\int_{K_1}g_C(x)\mathrm{d}x\int_{K_2}\Big[\frac{\partial^2 f_H}{\partial x_i\partial y_j}(x,y)h_G(y)+\frac{\partial f_H}{\partial x_i}(x,y)\frac{\partial h_G}{\partial y_j}(y)\Big]\mathrm{d}y\Big\}\\
&=S_{C,i,H,j,G}\Big\{\int_{K_1\times K_2}\frac{\partial g_C}{\partial x_i}(x)\Big[\frac{\partial f_H}{\partial y_j}(x,y)h_G(y)+f_H(x,y)\frac{\partial h_G}{\partial y_j}(y)\Big]\\
&\quad+g_C(x)\Big[\frac{\partial^2 f_H}{\partial x_i\partial y_j}(x,y)h_G(y)+\frac{\partial f_H}{\partial x_i}(x,y)\frac{\partial h_G}{\partial y_j}(y)\Big]\mathrm{d}x\mathrm{d}y\Big\}\\
&=S_{C,i,H,j,G}\int_{K_1\times K_2}\frac{\partial^2(g_Cf_Hh_G)}{\partial x_i\partial y_j}(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

同理可得

$$\begin{aligned}&\int_{\partial K_2}\Big[\int_{\partial K_1}g(x)\mathrm{d}\sigma_xf(x,y)\Big]\mathrm{d}\sigma_yh(y)\\
&=S_{C,i,H,j,G}\int_{K_2\times K_1}\frac{\partial^2(g_Cf_Hh_G)}{\partial y_j\partial x_i}(x,y)\mathrm{d}y\mathrm{d}x\\
&=S_{C,i,H,j,G}\int_{K_1\times K_2}\frac{\partial^2(g_Cf_Hh_G)}{\partial x_i\partial y_j}(x,y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

综上, (3.2) 成立, 即

$$\int_{\partial K_1}g(x)\mathrm{d}\sigma_x\Big[\int_{\partial K_2}f(x,y)\mathrm{d}\sigma_yh(y)\Big]=\int_{\partial K_2}\Big[\int_{\partial K_1}g(x)\mathrm{d}\sigma_xf(x,y)\Big]\mathrm{d}\sigma_yh(y).$$

定理 3.1 (加权双正则函数的 Cauchy-Pompeiu 公式) 设 $\Omega,\partial\Omega$ 如上所述, $\Omega=\Omega_1\times\Omega_2$, $f:\overline{\Omega}\to\mathcal{A}_n(R)$, 且 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 则

$$\begin{aligned}&\int_{\partial\Omega_1\times\partial\Omega_2}E_\omega(x,\xi)\mathrm{d}\sigma_xf(x,y)\mathrm{d}\sigma_yE_\omega(y,\tau)-\int_{\Omega_1\times\Omega_2}E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}x\mathrm{d}y\\
&=\begin{cases}\displaystyle\int_{\Omega_1}E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x}f(x,\tau)\mathrm{d}x+\int_{\Omega_2}f(\xi,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y+f(\xi,\tau), & (\xi,\tau)\in\Omega,\\ 0, & (\xi,\tau)\in\overline{\Omega}^c.\end{cases}\end{aligned}\tag{3.3}$$

证 当 $(\xi,\tau)\in\overline{\Omega}^c$ 时, 由引理 3.2 和引理 2.1, 得

$$\begin{aligned}&\int_{\partial\Omega_1\times\partial\Omega_2}E_\omega(x,\xi)\mathrm{d}\sigma_xf(x,y)\mathrm{d}\sigma_yE_\omega(y,\tau)\\
&=\int_{\partial\Omega_1}E_\omega(x,\xi)\mathrm{d}\sigma_x\Big[\int_{\partial\Omega_2}f(x,y)\mathrm{d}\sigma_yE_\omega(y,\tau)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x\Big[\int_{\Omega_2} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y\Big]\\
&= \int_{\Omega_1} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x}\Big[\int_{\Omega_2} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}y\Big]\mathrm{d}x\\
&= \int_{\Omega_1\times\Omega_2} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}x\mathrm{d}y. \qquad (3.4)
\end{aligned}
$$

当 $(\xi,\tau)\in\Omega$ 时, 以 ξ 为中心, ε 为半径, 作球 $U_{1\varepsilon}=\{x\in\Omega_1:\rho(x,\xi)<\varepsilon\}$, 以 τ 为中心, ε 为半径, 作球 $U_{2\varepsilon}=\{y\in\Omega_2:\rho(y,\tau)<\varepsilon\}$, 令 $\Omega_{1\varepsilon}=\Omega_1\backslash\overline{U}_{1\varepsilon}$, $\Omega_{2\varepsilon}=\Omega_2\backslash\overline{U}_{2\varepsilon}$, 此时, $(\xi,\tau)\in\overline{\Omega_{1\varepsilon}\times\Omega_{2\varepsilon}}^{c}$, 因此, 由 (3.4), 可得

$$
\begin{aligned}
&\int_{\Omega_{1\varepsilon}\times\Omega_{2\varepsilon}} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}x\mathrm{d}y\\
&= \int_{\partial\Omega_{1\varepsilon}\times\partial\Omega_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \int_{\partial(\Omega_1\backslash\overline{U}_{1\varepsilon})\times\partial(\Omega_2\backslash\overline{U}_{2\varepsilon})} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&\quad -\int_{\partial\Omega_1\times\partial U_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&\quad -\int_{\partial U_{1\varepsilon}\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&\quad +\int_{\partial U_{1\varepsilon}\times\partial U_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \mathrm{I}_1-\mathrm{I}_2-\mathrm{I}_3+\mathrm{I}_4. \qquad (3.5)
\end{aligned}
$$

首先计算 I_2, 由引理 3.2、推论 3.1 和引理 2.2, 得

$$
\begin{aligned}
\lim_{\varepsilon\to0}\mathrm{I}_2 &= \lim_{\varepsilon\to0}\int_{\partial\Omega_1\times\partial U_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \lim_{\varepsilon\to0}\int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x\Big[\int_{\partial U_{2\varepsilon}} f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\Big]\\
&= \int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x\Big[\lim_{\varepsilon\to0}\int_{\partial U_{2\varepsilon}} f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\Big]\\
&= \int_{\partial\Omega_1} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,\tau)\\
&= \int_{\Omega_1} E_\omega(x,\xi)\cdot\mathcal{D}_{\omega_x} f(x,\tau)\mathrm{d}x+f(\xi,\tau).
\end{aligned}
$$

同理计算 I_3, 由引理 3.2、引理 3.1 和推论 2.2, 得

$$
\begin{aligned}
\lim_{\varepsilon\to0}\mathrm{I}_3 &= \lim_{\varepsilon\to0}\int_{\partial U_{1\varepsilon}\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \lim_{\varepsilon\to0}\int_{\partial\Omega_2}\Big[\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \int_{\partial\Omega_2}\Big[\lim_{\varepsilon\to0}\int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau)
\end{aligned}
$$

$$= \int_{\partial\Omega_2} f(\xi, y)\mathrm{d}\sigma_y E_\omega(y,\tau)$$
$$= \int_{\Omega_2} f(\xi, y)\mathcal{D}_{\omega_y} \cdot E_\omega(y,\tau)\mathrm{d}y + f(\xi,\tau).$$

最后计算 I_4, 由引理 3.2 以及 (2.12)–(2.15), 得

$$\begin{aligned}
\mathrm{I}_4 &= \int_{\partial U_{1\varepsilon}\times\partial U_{2\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\\
&= \int_{\partial U_{1\varepsilon}} E_\omega(x,\xi)\mathrm{d}\sigma_x\Big[\int_{\partial U_{2\varepsilon}} f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau)\Big]\\
&= \int_{|r_1|=1}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}[f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\Big]\\
&\quad+\int_{|r_1|=1}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}\frac{1}{\omega_m}f(\xi,\tau)\mathrm{d}\mu_{r_2}\Big]\\
&= \mathrm{J}_1+\mathrm{J}_2.
\end{aligned}$$

首先讨论 J_1,

$$\begin{aligned}
|\mathrm{J}_1| &= \Big|\int_{|r_1|=1}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}[f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\Big]\Big|\\
&\leqslant \frac{1}{\omega_n\omega_m}\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|\mathrm{d}\mu_{r_2}\Big]\\
&\leqslant \frac{1}{\omega_n\omega_m}\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|\mathrm{d}\mu_{r_2}\Big]\\
&= \frac{c}{\omega_n\omega_m}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|\\
&\quad\cdot\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\cdot\int_{|r_2|=1}\mathrm{d}\mu_{r_2}.
\end{aligned}$$

因为 $f:\overline{\Omega}\to\mathcal{A}_n(R)$, 且 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 所以

$$\lim_{\varepsilon\to 0}\sup_{|r_1|=1,|r_2|=1}|f(\varepsilon B_1^{\frac12}r_1(t)+\xi, \varepsilon B_2^{\frac12}r_2(t)+\tau)-f(\xi,\tau)|=0.$$

故

$$\lim_{\varepsilon\to 0}\mathrm{J}_1=0.$$

接着讨论 J_2,

$$\begin{aligned}
\mathrm{J}_2 &= \int_{|r_1|=1}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}f(\xi,\tau)\Big]\\
&= \frac{1}{\omega_n\omega_m}\int_{|r_1|=1}\mathrm{d}\mu_{r_1}\Big[\int_{|r_2|=1}\mathrm{d}\mu_{r_2}\Big]\cdot f(\xi,\tau)\\
&= f(\xi,\tau).
\end{aligned}$$

故

$$\lim_{\varepsilon\to 0}\mathrm{J}_2=f(\xi,\tau).$$

因此

$$\lim_{\varepsilon\to 0} \mathrm{I}_4 = f(\xi,\tau).$$

因为 $E_\omega(x,\xi)$, $E_\omega(y,\tau)$ 分别在 ξ, τ 处具有弱奇异性, 故可以得到 (3.5) 左侧

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_{1\varepsilon}\times\Omega_{2\varepsilon}} E_\omega(x,\xi)\cdot \mathcal{D}_{\omega_x} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}x\mathrm{d}y\\ =&\int_{\Omega_1\times\Omega_2} E_\omega(x,\xi)\cdot \mathcal{D}_{\omega_x} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

对 (3.5) 两边取 $\varepsilon\to 0$, 结合上述讨论可得到加权双正则函数的 Cauchy-Pompeiu 公式.

定理 3.2 (Cauchy 积分公式)　若 f 是 Ω 中的加权双正则函数, 那么上述结论变为

$$\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau) = \begin{cases} f(\xi,\tau), & (\xi,\tau)\in\Omega,\\ 0, & (\xi,\tau)\in\overline{\Omega}^c.\end{cases} \tag{3.6}$$

§4　加权双正则函数的 Cauchy 型积分算子的边界性质

定义 4.1　若设 $\Omega,\partial\Omega$ 如上所述, 对任意的 $\xi_0\in\partial\Omega_1$, $\tau_0\in\partial\Omega_2$, 构造一个以 ξ_0 为中心, $\delta>0$ 为半径的球 G_1, 以 τ_0 为中心, $\delta>0$ 为半径的球 G_2, 并设球 G_1 与 $\partial\Omega_1$ 的交为 $\lambda_{1\delta}$, 球 G_2 与 $\partial\Omega_2$ 的交为 $\lambda_{2\delta}$, 记

$$\Phi_\delta(\xi_0,\tau_0) = \int_{(\partial\Omega_1-\lambda_{1\delta})\times(\partial\Omega_2-\lambda_{2\delta})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0). \tag{4.1}$$

若 $\lim\limits_{\delta\to 0}\Phi_\delta = I$ 存在, 我们称上述奇异积分在主值意义下是收敛的, 且称 I 为上述奇异积分的 Cauchy 主值, 并记 $I=\Phi(\xi_0,\tau_0)$.

定理 4.1　设 $\Omega,\partial\Omega$ 如上所述, $f:\overline{\Omega}\to\mathcal{A}_n(R)$, 且 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, $\xi_0\in\partial\Omega_1$, $\tau_0\in\partial\Omega_2$, 则有

$$\begin{aligned}&\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\ =&\frac{1}{2}\int_{\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x} f(x,\tau_0)\mathrm{d}x + \frac{1}{2}\int_{\Omega_2} f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\\ &+\int_{\Omega_1\times\Omega_2} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x} f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y + \frac{1}{4}f(\xi_0,\tau_0).\end{aligned} \tag{4.2}$$

证　设 $\lambda_{1\delta}$, $\lambda_{2\delta}$ 如上所述, 则先来考虑

$$\int_{(\partial\Omega_1-\lambda_{1\delta})\times(\partial\Omega_2-\lambda_{2\delta})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0).$$

令 $D_{1\mathrm{out}} = \{\partial[(G_1(\xi_0,\delta))\cup\overline{\Omega}_1]\}\cap(\mathbb{R}^n-\overline{\Omega}_1)$, $D_{2\mathrm{out}} = \{\partial[(G_2(\tau_0,\delta))\cup\overline{\Omega}_2]\}\cap(\mathbb{R}^m-\overline{\Omega}_2)$, 设曲面 $D_{1\mathrm{out}}$, $D_{2\mathrm{out}}$ 的方向为逆时针方向, 则由引理 3.2, 得

$$\begin{aligned}&\int_{(\partial\Omega_1-\lambda_{1\delta})\times(\partial\Omega_2-\lambda_{2\delta})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\ =&\int_{\partial\Omega_2-\lambda_{2\delta}}\Big[\int_{\partial\Omega_1-\lambda_{1\delta}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega_2-\lambda_{2\delta}} \Big[\int_{\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y) - \int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&= \int_{\partial\Omega_2-\lambda_{2\delta}} \Big[\int_{\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{\partial\Omega_2-\lambda_{2\delta}} \Big[\int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&= \int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}}} \Big[\int_{\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{D_{2\text{out}}} \Big[\int_{\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}}} \Big[\int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad + \int_{D_{2\text{out}}} \Big[\int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&= \int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}})\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x [f(x,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad + \int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}})\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}})\times D_{2\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x [f(x,y)-f(x,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\text{out}})\times D_{2\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{D_{1\text{out}}\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x [f(x,y)-f(\xi_0,y)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad - \int_{D_{1\text{out}}\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\text{out}})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad + \int_{D_{1\text{out}}\times D_{2\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x [f(x,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&\quad + \int_{D_{1\text{out}}\times D_{2\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&= \mathrm{I}_5+\mathrm{I}_6-\mathrm{I}_7-\mathrm{I}_8-\mathrm{I}_9-\mathrm{I}_{10}+\mathrm{I}_{11}+\mathrm{I}_{12}.
\end{aligned}
$$

首先讨论 I_5, 由定理 3.1, 得

$$
\begin{aligned}
I_5 &= \int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1out})\times(\partial\Omega_2-\lambda_{2\delta}+D_{2out})} E_\omega(x,\xi_0)\mathrm{d}\sigma_x [f(x,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&= [f(\xi_0,\tau_0)-f(\xi_0,\tau_0)] + \int_{G_1(\xi_0,\delta)\cup\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,\tau_0)-f(\xi_0,\tau_0)]\mathrm{d}x\\
&\quad + \int_{G_2(\tau_0,\delta)\cup\Omega_2} [f(\xi_0,y)-f(\xi_0,\tau_0)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\\
&\quad + \int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,y)-f(\xi_0,\tau_0)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y\\
&= \int_{G_1(\xi_0,\delta)\cup\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x} f(x,\tau_0)\mathrm{d}x - \int_{G_1(\xi_0,\delta)\cup\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x} f(\xi_0,\tau_0)\mathrm{d}x
\end{aligned}
$$

$$+\int_{G_2(\tau_0,\delta)\cup\Omega_2} f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y-\int_{G_2(\tau_0,\delta)\cup\Omega_2} f(\xi_0,\tau_0)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y$$
$$+\int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y$$
$$-\int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(\xi_0,\tau_0)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y.$$

由于 $\mathcal{D}_{\omega_x}f(\xi_0,\tau_0)=0$, $f(\xi_0,\tau_0)\mathcal{D}_{\omega_y}=0$, $\mathcal{D}_{\omega_x}f(\xi_0,\tau_0)\mathcal{D}_{\omega_y}=0$, 可得

$$\mathrm{I}_5=\int_{G_1(\xi_0,\delta)\cup\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x+\int_{G_2(\tau_0,\delta)\cup\Omega_2} f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y$$
$$+\int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y.$$

又由于 $|x_i-\xi_i|\leqslant|x-\xi|\,(i=1,2,\cdots,n)$, $|y_j-\tau_j|\leqslant|y-\tau|\,(j=1,2,\cdots,m)$, 可得

$$|E_\omega(x,\xi_0)|\leqslant\frac{\mathrm{M}_1}{\det(B_1)^{\frac{1}{2}}\omega_n}\cdot\frac{|x-\xi_0|}{|x-\xi_0|^n}=\frac{\mathrm{M}_1}{\det(B_1)^{\frac{1}{2}}\omega_n}\cdot\frac{1}{|x-\xi_0|^{n-1}},\tag{4.3}$$

$$|E_\omega(y,\tau_0)|\leqslant\frac{\mathrm{M}_2}{\det(B_2)^{\frac{1}{2}}\omega_m}\cdot\frac{|y-\tau_0|}{|y-\tau_0|^m}=\frac{\mathrm{M}_2}{\det(B_2)^{\frac{1}{2}}\omega_m}\cdot\frac{1}{|y-\tau_0|^{m-1}},\tag{4.4}$$

这表明 $E_\omega(x,\xi_0)$ 在点 ξ_0 处具有弱奇异性, $E_\omega(y,\tau_0)$ 在点 τ_0 处具有弱奇异性. 由于 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 可得 $|\mathcal{D}_{\omega_x}f(x,\tau_0)|\leqslant\mathrm{M}_3$, $|f(\xi_0,y)\mathcal{D}_{\omega_y}|\leqslant\mathrm{M}_4$, $|\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}|\leqslant\mathrm{M}_5$. 所以

$$|\mathrm{I}_5|\leqslant\Big|\int_{G_1(\xi_0,\delta)\cup\Omega_1} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x\Big|+\Big|\int_{G_2(\tau_0,\delta)\cup\Omega_2} f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\Big|$$
$$+\Big|\int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)} E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y\Big|$$
$$\leqslant\int_{G_1(\xi_0,\delta)\cup\Omega_1}|E_\omega(x,\xi_0)|\cdot|\mathcal{D}_{\omega_x}f(x,\tau_0)|\mathrm{d}x+\int_{G_2(\tau_0,\delta)\cup\Omega_2}|f(\xi_0,y)\mathcal{D}_{\omega_y}|\cdot|E_\omega(y,\tau_0)|\mathrm{d}y$$
$$+\int_{(G_1(\xi_0,\delta)\cup\Omega_1)\times(G_2(\tau_0,\delta)\cup\Omega_2)}|E_\omega(x,\xi_0)|\cdot|\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}|\cdot|E_\omega(y,\tau_0)|\mathrm{d}x\mathrm{d}y$$
$$\leqslant\mathrm{M}_6\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{1}{|x-\xi_0|^{n-1}}\mathrm{d}x+\mathrm{M}_7\int_{G_2(\tau_0,\delta)\cup\Omega_2}\frac{1}{|y-\tau_0|^{m-1}}\mathrm{d}y$$
$$+\mathrm{M}_8\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{1}{|x-\xi_0|^{n-1}}\mathrm{d}x\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}\frac{1}{|y-\tau_0|^{m-1}}\mathrm{d}y\Big].$$

根据弱奇异性可知上式是收敛的, 因而

$$\lim_{\delta\to 0}\mathrm{I}_5=\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x+\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y$$
$$+\int_{\Omega_1\times\Omega_2}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y.\tag{4.5}$$

再来讨论 I_6. 由于 (ξ_0,τ_0) 是由 $(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}})$ 围成的区域的内点, 根据加权双正则函数的 Cauchy 积分公式, 有

$$\begin{aligned}\mathrm{I}_6&=\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&=f(\xi_0,\tau_0).\end{aligned}\tag{4.6}$$

由 (2.12)–(2.15) 可知, 曲面 $D_{1\mathrm{out}}$, $D_{2\mathrm{out}}$ 的参数化方程为

$$x(t)=\delta B_1^{\frac12}r_1(t)+\xi_0,\quad y(t)=\delta B_2^{\frac12}r_2(t)+\tau_0,$$

且有

$$E_\omega(x,\xi_0)\cdot\mathrm{d}\sigma_x=\frac{1}{\omega_n}\mathrm{d}\mu_{r_1},\quad \mathrm{d}\sigma_y\cdot E_\omega(y,\tau_0)=\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}.$$

下面讨论 I_7, 由引理 3.2 和引理 2.2, 得

$$\begin{aligned}\mathrm{I}_7&=\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})\times D_{2\mathrm{out}}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x[f(x,y)-f(x,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&=\int_{D_{2\mathrm{out}}}\Big[\int_{\partial\Omega_1-\lambda_{1\delta}+D_{1out}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x[f(x,y)-f(x,\tau_0)]\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&=\int_{D_{2\mathrm{out}}}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,y)-f(x,\tau_0)]\mathrm{d}x\Big]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&\quad+\int_{D_{2\mathrm{out}}}[f(\xi_0,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&=\int_{\frac12 G_2(0,1)}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(x,\tau_0)]\mathrm{d}x\Big]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\\&\quad+\int_{\frac12 G_2(0,1)}[f(\xi_0,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(\xi_0,\tau_0)]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\\&=\mathrm{J}_3+\mathrm{J}_4,\end{aligned}$$

其中 $\frac12 G_1(0,1)$ 是 $D_{1\mathrm{out}}$ 所对应的欧氏距离下的单位半球, $\frac12 G_2(0,1)$ 是 $D_{2\mathrm{out}}$ 所对应的欧氏距离下的单位半球. 首先讨论 J_3, 由 (4.3), 得

$$\begin{aligned}&|\mathrm{J}_3|\\&=\Big|\int_{\frac12 G_2(0,1)}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(x,\tau_0)]\mathrm{d}x\Big]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\Big|\\&\leqslant\int_{\frac12 G_2(0,1)}\Big|\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(x,\tau_0)]\mathrm{d}x\Big|\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\\&\leqslant\int_{\frac12 G_2(0,1)}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}|E_\omega(x,\xi_0)|\cdot|\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(x,\tau_0)]|\mathrm{d}x\Big]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\\&\leqslant\int_{\frac12 G_2(0,1)}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{\mathrm{M}_9}{|x-\xi_0|^{n-1}}\\&\quad\cdot\sup_{x\in G_1(\xi_0,\delta)\cup\Omega_1}|\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac12}r_2(t)+\tau_0)-f(x,\tau_0)]|\mathrm{d}x\Big]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\\&=\int_{\frac12 G_2(0,1)}\Big[\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{\mathrm{M}_9}{|x-\xi_0|^{n-1}}\mathrm{d}x\end{aligned}$$

$$\cdot \sup_{x\in G_1(\xi_0,\delta)\cup\Omega_1} |\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)-f(x,\tau_0)]|\Big]\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}.$$

在 I_5 的证明过程中, 已经证得 $\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{\mathrm{M}_9}{|x-\xi_0|^{n-1}}\mathrm{d}x$ 收敛, 所以存在 $\mathrm{M}_{10}>0$, 使得 $\left|\int_{G_1(\xi_0,\delta)\cup\Omega_1}\frac{\mathrm{M}_9}{|x-\xi_0|^{n-1}}\mathrm{d}x\right|\leqslant \mathrm{M}_{10}$, 又因为 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 可得

$$\lim_{\delta\to 0}\sup_{x\in G_1(\xi_0,\delta)\cup\Omega_1}|\mathcal{D}_{\omega_x}[f(x,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)-f(x,\tau_0)]|=0.$$

因此 $\lim\limits_{\delta\to 0}\mathrm{J}_3=0$. 接着讨论 J_4,

$$0\leqslant|\mathrm{J}_4|\leqslant\int_{\frac{1}{2}G_2(0,1)}|f(\xi_0,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)-f(\xi_0,\tau_0)|\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}.$$

因为 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 所以

$$\lim_{\delta\to 0}|f(\xi_0,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)-f(\xi_0,\tau_0)|=0.$$

因此

$$\lim_{\delta\to 0}\mathrm{J}_4=0.$$

由上可得

$$\lim_{\delta\to 0}\mathrm{I}_7=0. \tag{4.7}$$

讨论 I_8, 由引理 3.2 和引理 2.2, 可得

$$\begin{aligned}
\mathrm{I}_8&=\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})\times D_{2\mathrm{out}}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&=\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{D_{2\mathrm{out}}}f(x,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\Big]\\
&=\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{\frac{1}{2}G_2(0,1)}f(x,\tau_0)\frac{1}{\omega_m}\mathrm{d}\mu_{r_2}\Big]\\
&=\frac{1}{2}\int_{(\partial\Omega_1-\lambda_{1\delta}+D_{1\mathrm{out}})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,\tau_0)\\
&=\frac{1}{2}\Big[f(\xi_0,\tau_0)+\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x\Big]\\
&=\frac{1}{2}f(\xi_0,\tau_0)+\frac{1}{2}\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x.
\end{aligned}$$

在 I_5 的证明过程中, 已经证得 $\int_{G_1(\xi_0,\delta)\cup\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x$ 绝对收敛, 因此

$$\lim_{\delta\to 0}\mathrm{I}_8=\frac{1}{2}f(\xi_0,\tau_0)+\frac{1}{2}\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x. \tag{4.8}$$

讨论 I_9, 由引理 3.2 和推论 2.2, 得

$$\begin{aligned}
\mathrm{I}_9&=\int_{D_{1\mathrm{out}}\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x[f(x,y)-f(\xi_0,y)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\
&=\int_{D_{1\mathrm{out}}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}}}[f(x,y)-f(\xi_0,y)]\mathrm{d}\sigma_y E_\omega(y,\tau_0)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}[f(x,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\Big]\\
&\quad+\int_{D_{1\text{out}}} E_\omega(x,\xi_0)\mathrm{d}\sigma_x[f(x,\tau_0)-f(\xi_0,\tau_0)]\\
&= \int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\Big]\\
&\quad+\int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}[f(\delta B_1^{\frac12}r_1(t)+\xi_0,\tau_0)-f(\xi_0,\tau_0)]\mathrm{d}\mu_{r_1}\\
&= \mathrm{J}_5+\mathrm{J}_6.
\end{aligned}
$$

讨论 J_5, 由 (4.4), 得

$$
\begin{aligned}
|\mathrm{J}_5| &= \Big|\int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\Big]\Big|\\
&\leqslant \int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big|\int_{G_2(\tau_0,\delta)\cup\Omega_2}[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\Big|\\
&\leqslant \int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}|[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}|\cdot|E_\omega(y,\tau_0)|\mathrm{d}y\Big]\\
&\leqslant \int_{\frac12 G_1(0,1)}\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}\sup_{y\in G_2(\tau_0,\delta)\cup\Omega_2}|[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}|\\
&\quad\cdot\frac{\mathrm{M}_{11}}{|y-\tau_0|^{m-1}}\mathrm{d}y\Big]\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}\\
&= \int_{\frac12 G_1(0,1)}\Big[\int_{G_2(\tau_0,\delta)\cup\Omega_2}\frac{\mathrm{M}_{11}}{|y-\tau_0|^{m-1}}\mathrm{d}y\\
&\quad\cdot\sup_{y\in G_2(\tau_0,\delta)\cup\Omega_2}|[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}|\Big]\frac{1}{\omega_n}\mathrm{d}\mu_{r_1}.
\end{aligned}
$$

在 I_5 的证明过程中, 已经证得 $\int_{G_2(\tau_0,\delta)\cup\Omega_2}\frac{\mathrm{M}_{11}}{|y-\tau_0|^{m-1}}\mathrm{d}y$ 收敛, 因此存在 $M_{12}>0$, 使得 $\big|\int_{G_2(\tau_0,\delta)\cup\Omega_2}\frac{\mathrm{M}_{11}}{|y-\tau_0|^{m-1}}\mathrm{d}y\big|\leqslant\mathrm{M}_{12}$. 又因为 $f\in F_\Omega^{(r)}$, $r\geqslant 2$, 则

$$
\lim_{\delta\to0}\sup_{y\in G_2(\tau_0,\delta)\cup\Omega_2}|[f(\delta B_1^{\frac12}r_1(t)+\xi_0,y)-f(\xi_0,y)]\mathcal{D}_{\omega_y}|=0.
$$

因此 $\lim\limits_{\delta\to0}\mathrm{J}_5=0$. 接着讨论 J_6,

$$
0\leqslant|\mathrm{J}_6|\leqslant\int_{\frac12 G_1(0,1)}\frac{1}{\omega_n}|f(\delta B_1^{\frac12}r_1(t)+\xi_0,\tau_0)-f(\xi_0,\tau_0)|\mathrm{d}\mu_{r_1}.
$$

因为 $f\in F_{\Omega}^{(r)}$, $r\geqslant 2$, 所以

$$\lim_{\delta\to 0}|f(\delta B_1^{\frac{1}{2}}r_1(t)+\xi_0,\tau_0)-f(\xi_0,\tau_0)|=0.$$

因此 $\lim\limits_{\delta\to 0}\mathrm{J}_6=0$. 由上述可得

$$\lim_{\delta\to 0}\mathrm{I}_9=0. \tag{4.9}$$

讨论 I_{10}, 由引理 3.2 和推论 2.2, 得

$$\begin{aligned}\mathrm{I}_{10}&=\int_{D_{1\mathrm{out}}\times(\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}})}E_{\omega}(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,y)\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\\&=\int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}}}\Big[\int_{D_{1\mathrm{out}}}E_{\omega}(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,y)\Big]\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\\&=\int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}}}\Big[\int_{\frac{1}{2}G_1(0,1)}\frac{1}{\omega_n}f(\xi_0,y)\mathrm{d}\mu_{r_1}\Big]\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\\&=\frac{1}{2}\int_{\partial\Omega_2-\lambda_{2\delta}+D_{2\mathrm{out}}}f(\xi_0,y)\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\\&=\frac{1}{2}f(\xi_0,\tau_0)+\frac{1}{2}\int_{G_2(\tau_0,\delta)\cup\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_{\omega}(y,\tau_0)\mathrm{d}y.\end{aligned}$$

在 I_5 的证明过程中, 已经证得 $\int_{G_2(\tau_0,\delta)\cup\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_{\omega}(y,\tau_0)\mathrm{d}y$ 绝对收敛, 因此

$$\lim_{\delta\to 0}\mathrm{I}_{10}=\frac{1}{2}f(\xi_0,\tau_0)+\frac{1}{2}\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_{\omega}(y,\tau_0)\mathrm{d}y. \tag{4.10}$$

接着讨论 I_{11}, 由引理 3.2, 得

$$\begin{aligned}|\mathrm{I}_{11}|&=\Big|\int_{D_{1\mathrm{out}}\times D_{2\mathrm{out}}}E_{\omega}(x,\xi_0)\mathrm{d}\sigma_x[f(x,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\Big|\\&=\Big|\int_{D_{1\mathrm{out}}}E_{\omega}(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{D_{2\mathrm{out}}}[f(x,y)-f(\xi_0,\tau_0)]\mathrm{d}\sigma_y E_{\omega}(y,\tau_0)\Big]\Big|\\&=\frac{1}{\omega_n\omega_m}\Big|\int_{\frac{1}{2}G_1(0,1)}\mathrm{d}\mu_{r_1}\Big[\int_{\frac{1}{2}G_2(0,1)}[f(\delta B_1^{\frac{1}{2}}r_1(t)+\xi_0,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)\\&\quad-f(\xi_0,\tau_0)]\mathrm{d}\mu_{r_2}\Big]\Big|\\&\leqslant\frac{1}{\omega_n\omega_m}\int_{\frac{1}{2}G_1(0,1)}\mathrm{d}\mu_{r_1}\Big[\int_{\frac{1}{2}G_2(0,1)}|f(\delta B_1^{\frac{1}{2}}r_1(t)+\xi_0,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)\\&\quad-f(\xi_0,\tau_0)|\mathrm{d}\mu_{r_2}\Big].\end{aligned}$$

因为 $f\in F_{\Omega}^{(r)}$, $r\geqslant 2$, 所以

$$\lim_{\delta\to 0}|f(\delta B_1^{\frac{1}{2}}r_1(t)+\xi_0,\delta B_2^{\frac{1}{2}}r_2(t)+\tau_0)-f(\xi_0,\tau_0)|=0.$$

因此

$$\lim_{\delta\to 0}\mathrm{I}_{11}=0. \tag{4.11}$$

最后讨论 I_{12}, 由引理 3.2, 得

$$\begin{aligned}\mathrm{I}_{12}&=\int_{D_{1\mathrm{out}}\times D_{2\mathrm{out}}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(\xi_0,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\&=\int_{D_{1\mathrm{out}}}E_\omega(x,\xi_0)\mathrm{d}\sigma_x\Big[\int_{D_{2\mathrm{out}}}f(\xi_0,\tau_0)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\Big]\\&=\frac{1}{\omega_n\omega_m}\int_{\frac12 G_1(0,1)}\mathrm{d}\mu_{r_1}\Big[\int_{\frac12 G_2(0,1)}f(\xi_0,\tau_0)\mathrm{d}\mu_{r_2}\Big]\\&=\frac14 f(\xi_0,\tau_0).\end{aligned}\tag{4.12}$$

综上所述,

$$\begin{aligned}&\lim_{\delta\to0}\int_{(\partial\Omega_1-\lambda_{1\delta})\times(\partial\Omega_2-\lambda_{2\delta})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\=&\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x+\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\\&+\int_{\Omega_1\times\Omega_2}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y+f(\xi_0,\tau_0)\\&-0-\frac12 f(\xi_0,\tau_0)-\frac12\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x-0-\frac12 f(\xi_0,\tau_0)\\&-\frac12\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y+0+\frac14 f(\xi_0,\tau_0)\\=&\frac12\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x+\frac12\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\\&+\int_{\Omega_1\times\Omega_2}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y+\frac14 f(\xi_0,\tau_0).\end{aligned}\tag{4.13}$$

另一方面,

$$\begin{aligned}&\lim_{\delta\to0}\int_{(\partial\Omega_1-\lambda_{1\delta})\times(\partial\Omega_2-\lambda_{2\delta})}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\=&\int_{\partial\Omega_1\times\partial\Omega_2}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0).\end{aligned}\tag{4.14}$$

由 (4.13) 和 (4.14), 可知

$$\begin{aligned}&\int_{\partial\Omega_1\times\partial\Omega_2}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)\\=&\frac12\int_{\Omega_1}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,\tau_0)\mathrm{d}x+\frac12\int_{\Omega_2}f(\xi_0,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}y\\&+\int_{\Omega_1\times\Omega_2}E_\omega(x,\xi_0)\cdot\mathcal{D}_{\omega_x}f(x,y)\mathcal{D}_{\omega_y}\cdot E_\omega(y,\tau_0)\mathrm{d}x\mathrm{d}y+\frac14 f(\xi_0,\tau_0).\end{aligned}$$

推论 4.1 设 $\Omega,\partial\Omega$ 如上所述, 若 f 是 Ω 内的加权双正则函数, $\xi_0\in\partial\Omega_1$, $\tau_0\in\partial\Omega_2$, 则下式成立

$$\int_{\partial\Omega_1\times\partial\Omega_2}E_\omega(x,\xi_0)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau_0)=\frac14 f(\xi_0,\tau_0).\tag{4.15}$$

推论 4.2 设 $\Omega, \partial\Omega$ 如上所述, 若 f 是 Ω 内的加权双正则函数, 则

$$\int_{\partial\Omega_1\times\partial\Omega_2} E_\omega(x,\xi)\mathrm{d}\sigma_x f(x,y)\mathrm{d}\sigma_y E_\omega(y,\tau) = \begin{cases} f(\xi,\tau), & (\xi,\tau)\in\Omega, \\ 0, & (\xi,\tau)\in\overline{\Omega}^c, \\ \dfrac{1}{4}f(\xi,\tau), & (\xi,\tau)\in\partial\Omega. \end{cases} \tag{4.16}$$

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Cauchy Integral Formula of Weighted Biregular Function and Its Boundary Properties

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Abstract Regular function is a generalization of analytic function in simple complex analysis in high dimensional space, and weighted regular function is a further development of regular function. Weighted regular function plays an important role in solving the heat conduction problem of anisotropic media. Weighted biregular function is a further development of weighted regular function, which is another new class of functions in Clifford analysis. This paper first proves the Cauchy-Pompeiu formula of weighted biregular function, then obtains the Cauchy integral formula of weighted biregular function, and finally proves the boundary properties of Cauchy type integral operator of weighted biregular function.

Keywords Real Clifford analysis, Weighted biregular function, Cauchy-Pompeiu formula, Cauchy integral formula, Boundary properties of Cauchy type integral operators

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