Chin. Ann. of Math.
14B: 3(1993),279-286.

GLOBAL SOLUTIONS TO THE EVOLUTION EQUATION OF SCHRÖDINGER TYPE WITH NONLOCAL TERM**

TAN SHAOBIN*

Abstract

The existence of weak and smooth solutions for the nonlocal nonlinear Schrödinger equation is solved by parabolic regularization. In addition, the continuous dependence on the initial data of smooth solution is also discussed.

Keywords Nonlinear Schrödinger equation, Parabolic regularization, Existence and Uniqueness

1991 MR Subject Classification 35Q20

§1. Introduction and Results

In the present paper we shall study the initial-value problem for a nonlinear Schrödinger equation of the following type[1]

$$q_t + iq_{xx} + \sigma \left(q ⨍_{-\infty}^{\infty} \frac{|q|^2(x')}{x-x'} dx' \right)_x = 0 \quad \text{for } x \in R,\ t \geq 0, \tag{1.1}$$

$$q(x,0) = q_0(x) \quad \text{for } x \in R, \tag{1.2}$$

where $i = \sqrt{-1}$, $\sigma > 0$ is a real constant, $⨍_{-\infty}^{\infty}$ denotes the principal-value integral.

Without loss of generality, we assume $\sigma\pi = 1$. Under some conditions on the given initial data, the global existence of solutions to the problem (1.1), (1.2) is derived by the following parabolic regularization

$$q_t + iq_{xx} - \left(q\mathbf{H}(|q|^2)\right)_x - \varepsilon q_{xx} = 0 \quad \text{for } x \in R,\ t \geq 0, \tag{1.3}$$

$$q(x,0) = q_{0\varepsilon}(x) \quad \text{for } x \in R, \tag{1.4}$$

where $\varepsilon > 0$ is a constant,

$$\mathbf{H}(f) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(x')}{x-x'} dx'$$

is Hilbert transform, $q_{0\varepsilon} \in H^\infty(R) = \bigcap_{k\geq 0} H^k(R)$ is such that

$$\|q_{0\varepsilon}\|_{H^m(R)} \leq \|q_0\|_{H^m(R)},$$

and $q_{0\varepsilon} \to q_0$ in $H^m(R)$ as $\varepsilon \downarrow 0$ for $m \geq 0$.

For $\varepsilon > 0$ fixed, by a similar strategy as in [2-4], we can easily show that (1.3),(1.4) possesses a unique solution $q = q^\varepsilon(x,t)$ in the class $C^\infty(0, T^*; H^\infty(R))$, where $T^* > 0$ is a constant depending on q_0.

Manuscript received January 27, 1991. Revised January 3, 1992.
*Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.
**Project supported by the National Natural Science Foundation of China .

Theorem 1.1. *Suppose that* $q_0 \in H^1(R)$, *and* $\|q_0\|_2 \leq \frac{1}{\sqrt{3\sigma\pi}}$. *Then there exists a weak solution* $q(x,t)$ *of the problem* (1.1),(1.2) *such that*

(i) $q(x,t) \in L^\infty(R_+; H^1(R)) \cap C^{(\frac{1}{2},\frac{1}{4})}(R \times R_+)$,

(ii) $\int_{R_+\times R} \left(q\varphi_t + iq_x\varphi_x - \varphi_x q\mathbf{H}(|q|^2)\right) dxdt + \int_R q_0\varphi(x,0)dx = 0$,

for any function $\varphi(x,t) \in C_0^1(R \times R_+)$.

Theorem 1.2. *Suppose that* $q_0 \in H^k(R)$ $(k \geq 2)$, *and* $\|q_0\|_2 \leq \frac{1}{\sqrt{3\sigma\pi}}$. *Then there exists a unique solution* $q(x,t)$ *of the problem* (1.1),(1.2) *such that* $q \in C^r(R_+;\ H^{k-2r}(R))$ *for all* $r, k \in N$ *with* $k - 2r \geq 0$, *and* q *is continuously depending on the initial data* q_0.

§2. Proofs of Theorem 1.1 and Theorem 1.2

In order to prove the conclusion of the Theorems, the most important procedure is to establish certain a priori estimates for the solution of problem (1.3), (1.4). For this purpose, we first give the following lemmas.

Lemma 2.1. *For any function* $f(x), g(x) \in L^2(R)$, *we have*

$$\int_R f\mathbf{H}(g)dx = -\int_R g\mathbf{H}(f)dx, \quad \int_R \mathbf{H}(f)\mathbf{H}(g)dx = \int_R fgdx.$$

Lemma 2.2. *Let* $f(x) \in H^2(R)$ *be a real function. Then we have the following inequalities*

$$\int_R f_x\mathbf{H}(f)dx \geq 0,$$

$$\|f_x\|_2^2 \leq \left(\int_R f_x\mathbf{H}(f)dx\right)^{1/2}\left(\int_R f_{xx}\mathbf{H}(f_x)dx\right)^{1/2},$$

$$\|f_x\|_2^2 \leq \|f\|_2^{2/3}\left(\int_R f_{xx}\mathbf{H}(f_x)dx\right)^{2/3},$$

where $\|\cdot\|_p$ *denotes the usual* $L^p(R)$*–norm for* $p \geq 1$.

Proof. Setting $\tilde{f} = \int_R f(x)e^{-ix\zeta}dx$, we obtain $\widetilde{\mathbf{H}}(f_x) = -|\zeta|\tilde{f}$. Then by using Parseval identity, we deduce that

$$\int_R f_x\mathbf{H}(f)dx = -\frac{1}{2\pi}\int_R \bar{\tilde{f}}\widetilde{\mathbf{H}}(f_x)d\zeta = \frac{1}{2\pi}\int_R |\zeta||\tilde{f}|^2 d\zeta \geq 0.$$

Furthermore, by Höder's inequality we obtain

$$\begin{aligned}
\|f_x\|_2^2 =& \frac{1}{2\pi}\int_R \bar{\tilde{f}}_x\tilde{f}_x d\zeta = \frac{1}{2\pi}\int_R |\zeta|^2|\tilde{f}|^2 d\zeta \\
\leq& \frac{1}{2\pi}\left(\int_R |\zeta||\tilde{f}|^2 d\zeta\right)^{1/2}\left(\int_R |\zeta|^3|\tilde{f}|^2 d\zeta\right)^{1/2} \\
=& \left(\int_R f_x\mathbf{H}(f)dx\right)^{1/2}\left(\int_R f_{xx}\mathbf{H}(f_x)dx\right)^{1/2}.
\end{aligned}$$

To verify the last inequality of the lemma, we need merely to notice that

$$\int_R f_x\mathbf{H}(f)dx \leq \|f_x\|_2\|\mathbf{H}(f)\|_2 = \|f_x\|\|f\|_2.$$

This completes the proof of the lemma.

Lemma 2.3. *Let* $\varepsilon > 0$,

$$q = q^\varepsilon(x,t) \in C^\infty(R_+; H^\infty(R))$$

be the solution of (1.3),(1.4)*. Then we have the following identities*

(i) $\frac{d}{dt}\int_R |q|^2dx + \int_R (|q|^2)_x \mathbf{H}(|q|^2)dx + 2\varepsilon\int_R |q_x|^2dx = 0,$

(ii)

$$\begin{aligned}
&\frac{d}{dt}\int_R \left(|q_x|^2 + \frac{3}{2}\mathrm{Im}(\bar{q}q_x)\mathbf{H}(|q|^2)\right)dx + 2\varepsilon\int_R |q_{xx}|^2dx\\
&+\frac{1}{4}\int_R (|q|^2)_{xx}\mathbf{H}(|q|^2)_x + 3\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx\\
=&\frac{3}{2}\mathrm{Im}\left(\int_R \bar{q}q_x\mathbf{H}(|q|^2)\mathbf{H}(|q|^2)_x dx + 2\int_R \bar{q}q_x\mathbf{H}(|q|^2\mathbf{H}(|q|^2)_x)dx\right.\\
&\left.+\int_R \bar{q}q_x\mathbf{H}(\mathbf{H}(|q|^2)_x\mathbf{H}(|q|^2))dx\right) + \frac{3\varepsilon}{2}\mathrm{Im}\left(\int_R q_x\bar{q}_{xx}\mathbf{H}(|q|^2)dx\right.\\
&\left.-\int_R q_{xx}(\bar{q}\mathbf{H}(|q|^2))_x dx + 2\int_R \bar{q}q_x\mathbf{H}(\mathrm{Re}(\bar{q}q_{xx}))dx\right).
\end{aligned}$$

Proof. The above two identities can be checked out simply by performing several integrations by parts. We shall give the detailed demonstration in Appendix.

Lemma 2.4. *Under the conditions of Theroem* 1.1, *we denote by* $q = q^\varepsilon(x,t)$ *the solution of* (1.3), (1.4) *with* $\varepsilon > 0$. *Then we have the following a priori estimates*

$$\|q^\varepsilon(\cdot,t)\|_{H^1} + \varepsilon\int_0^1 \|q^\varepsilon(\cdot,\tau)\|_{H^2}^2 d\tau \le C,$$

$$\int_0^t\int_R (|q|^2)_{xx}\mathbf{H}(|q|^2)_x dxdt + \int_0^t\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx \le C$$

for all $t \ge 0$, *where* C *is a positive constant depending only on the norm* $\|q_0\|_{H^1}$.

Proof. On account of the identity (i) of Lemma 2.3, one can easily adduce

$$\|q^\varepsilon(\cdot,t)\|_2^2 + \int_0^t\int_R (|q|^2)_x\mathbf{H}(|q|^2)dxdt + 2\varepsilon\int_0^1 \|q^\varepsilon(\cdot,\tau)\|_2^2 d\tau \le \|q_0\|_2^2 \tag{2.1}$$

for all $t \ge 0$.

Denoting the terms in the right hand side of (ii) of Lemma 2.3 by $R_0(t)$, and by using the following Sobolev interpolation inequalities

$$\|f\|_\infty \le C\|f_x\|_2^{3/2}\|f\|_1^{1/3}, \quad \|f\|_\infty \le \sqrt{2}\|f_x\|_2^{1/2}\|f\|_2^{1/2},$$

$$\|f\|_2 \le C\|f_x\|_2^{1/3}\|f\|_1^{2/3}, \quad \|f\|_4 \le \sqrt[4]{2}\|f_x\|_2^{1/4}\|f\|_2^{3/4}, \quad \|f\|_\infty \le C\|f_{xx}\|_2^{1/4}\|f\|_2^{3/4},$$

and Höder's inequality, we compute as follows:

$$\begin{aligned}|R_0(t)| \le & C(\|q\|_\infty\|q_x\|_2\|\mathbf{H}(|q|^2)\|_\infty\|(|q|^2)_x\|_2+\|q\|_\infty^3\|q_x\|_2\|(|q|^2)_x\|_2)\\ & +\varepsilon(2\|q_x\|_\infty\|q_{xx}\|_2\|q\|_4^2+4\|q\|_\infty^2\|q_x\|_2\|q_{xx}\|_2)\\ \le & C\|q\|_2\|q_x\|_2\|(|q|^2)_x\|_2^2+\varepsilon C\|q\|_2^{3/2}\|q_{xx}\|_2^{3/2}\|q_x\|_2\\ \le & C\|q_0\|_2\|q_x\|_2\left(\int_R(|q|^2)_x\mathbf{H}(|q|^2)dx\right)^{1/2}\left(\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdx\right)^{1/2}\\ & +\varepsilon C\|q_0\|_2^{3/2}\|q_x\|_2\|q_{xx}\|_2^{3/2}\\ \le & \frac{1}{8}\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdx+C\left(\int_R(|q|^2)_x\mathbf{H}(|q|^2)dx\right)\|q_x\|_2^2\\ & +\varepsilon\|q_{xx}\|_2^2+C\varepsilon\|q_x\|_2^4, \end{aligned}\tag{2.2}$$

where we have used Lemma 2.2 and Young's inequality.

Integrating (ii) of Lemma 2.3 over the temporal interval $[0,t]$, with the inequality (2.2), and noticing that

$$\frac{3}{2}\mathrm{Im}\int_R(\bar{q}q_x)\mathbf{H}(|q|^2)dx\le\frac{3}{2}\|q\|_\infty\|q_x\|_2\||q|^2\|_2\le 3\|q_0\|_2^2\|q_x\|_2^2,$$

from (ii) of Lemma 2.3 we obtain

$$\begin{aligned}&(1-3\|q_0\|_2^2)\|q_x(\cdot,t)\|_2^2+\varepsilon\int_0^t\|q_{xx}^\varepsilon(\cdot,\tau)\|_2^2d\tau+\frac{1}{8}\int_0^t\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdxdt\\ &+3\int_0^t\int_R(\mathrm{Im}(\bar{q}q_x))_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dxdt\\ \le&\left(\|q_{0x}\|_2^2+\frac{3}{2}\int_R|\bar{q}_0q_{0x}\mathbf{H}(|q_0|^2)|dx\right)\\ &+\int_0^t\left(C\int_R(|q|^2)_x\mathbf{H}(|q|^2)dx+C\varepsilon\|q_x^\varepsilon(\cdot,t)\|_2^2\right)\|q_x^\varepsilon(\cdot,t)\|_2^2dt.\end{aligned}\tag{2.3}$$

In virtue of the facts: $1-3\|q_0\|_2^2>0$, $q_0(x)\in H^1(R)$ and

$$\frac{1}{8}\int_0^t\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdxdt+3\int_0^t\int_R\mathrm{Im}(\bar{q}q_x))_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dxdt\ge 0,$$

(2.3) leads to the inequality

$$\|q_x^\varepsilon(\cdot,t)\|_2^2\le C+\int_0^t\left(C\int_R(|q|^2)_x\mathbf{H}(|q|^2)dx+C\varepsilon\|q_x^\varepsilon(\cdot,t)\|_2^2\right)\|q_x^\varepsilon(\cdot,t)\|_2^2dt.$$

By Gronwall's lemma, the above inequality

$$\begin{aligned}\|q_x^\varepsilon(\cdot,t)\|_2^2&\le C\exp\left(\int_0^t\left(C\int_R(|q|^2)_x\mathbf{H}(|q|^2)dx+\varepsilon\|q_x^\varepsilon(\cdot,t)\|_2^2\right)dt\right)\\ &\le C\exp(\|q_0\|_2^2),\end{aligned}$$

where we have used the inequality (2.1).

Corollary 2.1. *Under the conditions of Lemma* 2.3,*we have*

$$\|q_x^\varepsilon(\cdot,t)\|_{H^{-1}}\le C\quad for\ all\ t\ge 0,$$

where the constant C *is independent of* ε.

Proof of Theorem 1.1. With the results of Lemma 2.4 and Corollary 2.1, we can now easily pass to the limit by extracting a subsequence, still denoted by q^ε, such that

$$q^\varepsilon \to q \quad \text{in} \quad L^\infty(R_+; H^1(R)) \quad \text{weak*, as } \varepsilon \to 0, \tag{2.4}$$

$$q_t^\varepsilon \to q_t \quad \text{in} \quad L^\infty(R_+; H^{-1}(R)) \quad \text{weak*, as } \varepsilon \to 0. \tag{2.5}$$

By using classical compactness arguments and a standard calculation[4], one can deduce from (2.4),(2.5) that the limiting function $q = q(x,t)$ is a weak solution of (1.1),(1.2), and $q(x,t) \in C^{(\frac{1}{2},\frac{1}{4})}(R \times R_+)$.

The proof of Theorem 1.1 is now complete.

Proof of Theorem 1.2. Again we consider the parabolic regularization (1.3),(1.4). The proof is born out by establishing a priori estimates for high Sobolev norms. To this end, we give the following lemmas to bound $\|q_{xx}^\varepsilon(\cdot,t)\|_2$ and $\|q^\varepsilon(\cdot,t)\|_{H^k}$ for $k > 2$.

Lemma 2.5. *Under the conditions of Lemma 2.3, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_R \left(|q_{xx}|^2 + \frac{15}{8}\mathrm{Im}(\bar{q}_{xxx}q)\mathbf{H}(|q|^2) - \frac{5}{8}\mathrm{Im}(\bar{q}_{xx}q_x)\mathbf{H}(|q|^2)\right)dx + 2\varepsilon\int_R |q_{xxx}|^2 dx \\
&+ \frac{1}{16}\int_R (|q|^2)_{xxx}\mathbf{H}(|q|^2)_{xx} + \frac{15}{4}\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))_x dx \\
=&5\mathrm{Im}\int_R \bar{q}_x q_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx - \left(\frac{15}{8}\mathrm{Im}\int_R \bar{q}_{xx}(\mathbf{H}(|q|^2)(q\mathbf{H}(|q|^2))_x)_x dx\right. \\
&+ \frac{15}{4}\mathrm{Im}\int_R \bar{q}_{xx}(q\mathbf{H}(|q|^2\mathbf{H}(|q|^2)_x))_x dx + \frac{5}{8}\mathrm{Im}\int_R \bar{q}_{xx}(\mathbf{H}(|q|^2)(q\mathbf{H}(|q|^2))_{xx}dx \\
&- \frac{5}{8}\mathrm{Im}\int_R (q_x\mathbf{H}(|q|^2))_x(\bar{q}\mathbf{H}(|q|^2))_{xx}dx \\
&\left.+ \frac{5}{8}\mathrm{Im}\int_R \bar{q}_{xx}q_x\mathbf{H}(2|q|^2\mathbf{H}(|q|^2)_x + (|q|^2)_x\mathbf{H}(|q|^2))dx\right) \\
&+ \varepsilon\left(\frac{5}{2}\mathrm{Im}\int_R \bar{q}_{xxx}q_{xx}\mathbf{H}(|q|^2)dx + \frac{15}{8}\mathrm{Im}\int_R \bar{q}_{xxx}(q\mathbf{H}(|q|^2))_{xx}dx\right. \\
&+ \frac{15}{4}\mathrm{Im}\int_R \bar{q}_{xxx}q\mathbf{H}(\mathrm{Re}(\bar{q}q_{xx}))dx + \frac{5}{8}\mathrm{Im}\int_R \bar{q}_{xxx}(q_x\mathbf{H}(|q|^2))_x dx \\
&\left.- \frac{5}{4}\mathrm{Im}\int_R \bar{q}_{xx}q_x\mathbf{H}(\bar{q}q_{xx})dx\right).
\end{aligned}\tag{2.6}$$

Proof. The above identity can be verified by several suitable integrations by parts, which are similar to the proof of Lemma 2.3. For the sake of shortness, we omit the proof.

Lemma 2.6. *Let $T > 0$. Under the conditions of Lemma 2.4, we have the a priori estimates*

$$\|q_{xx}^\varepsilon(\cdot,t)\|_2 + \varepsilon\int_0^t \|q_{xxx}^\varepsilon(\cdot,\tau)\|_2^2 d\tau \le C,$$

$$\int_0^t\int_R (|q|^2)_{xxx}\mathbf{H}(|q|^2)_{xx}dxdt + \int_0^t\int_R \mathrm{Im}(\bar{q}q_x)_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx \le C$$

for all $t \in [0,T]$, where the constant C depends only on the norm $\|q_0\|_{H^2}$ and T.

Proof. By several standard Sobolev interpolation inequalities and the $H^1(R)$–estimate of Lemma 2.4, the terms (denoted here by $R_1(t)$) in the right hand side of (2.6) can be bounded as follows:

$$|R_1(t)| \leq 5\|q_x\|_\infty\|q_{xx}\|_2\|\mathrm{Im}(\bar{q}q_x)_x\|_2 + C(1+\|q_{xx}\|_2^2) + \varepsilon\|q_{xxx}\|_2^2, \tag{2.7}$$

where we have used Höder's and Young's inequality.

Using the results of Lemma 2.4 and the interpolation inequalities in Lemma 2.2, we obtain

$$\|\mathrm{Im}(\bar{q}q_x)_x\|_2^2 \leq \left(\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx\right)^{1/2}\left(\int_R \mathrm{Im}(\bar{q}q_x)_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx\right)^{1/2}.$$

Inserting the above inequality into (2.7), we conclude that

$$\begin{aligned}|R_1(t)| \leq&\|q_{xx}\|_2^{3/2}\left(\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx\right)^{1/4}\left(\int_R \mathrm{Im}(\bar{q}q_x)_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx\right)^{1/4}\\ &+C(1+\|q_{xx}\|_2^2)+\varepsilon\|q_{xxx}\|_2^2\\ \leq& C + C\left(1+\int_R \mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx\right)\|q_{xx}\|_2^2+\varepsilon\|q_{xxx}\|_2^2\\ &+\int_R \mathrm{Im}(\bar{q}q_x)_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx.\end{aligned}$$

Therefore, we combine the above inequality with (2.6) and reach the inequality

$$\begin{aligned}&\|q_{xx}^\varepsilon(\cdot,t)\|_2^2+\frac{15}{8}\mathrm{Im}\int_R(\bar{q}_{xxx}q)\mathbf{H}(|q|^2)dx-\frac{5}{8}\mathrm{Im}\int_R(\bar{q}_{xx}q_x\mathbf{H}(|q|^2)dx\\ &+\varepsilon\int_0^t\|q_{xxx}^\varepsilon(\cdot,\tau)\|_2^2d\tau+\frac{1}{16}\int_0^t\int_R(|q|^2)_{xxx}\mathbf{H}(|q|^2)_{xx}dx\\ &+\frac{11}{4}\int_0^t\int_R \mathrm{Im}(\bar{q}q_x)_{xx}\mathbf{H}(\mathrm{Im}(\bar{q}q_x)_x)dx\\ \leq& C+C\int_0^t\left(1+\int_R\mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x)\right)dx\|q_{xx}\|_2^2dt.\end{aligned} \tag{2.8}$$

By using integration by parts and the application of Hölder's inequality, we see that

$$\begin{aligned}&\left|\frac{15}{8}\mathrm{Im}\int_R(\bar{q}_{xxx}q)\mathbf{H}(|q|^2)dx-\frac{5}{8}\mathrm{Im}\int_R(\bar{q}_{xx}q_x)\mathbf{H}(|q|^2)dx\right|\\ \leq& C\|q_{xx}\|_2\leq\frac{1}{2}\|q_{xx}\|_2^2+C.\end{aligned}$$

Whence, with the above inequality, by Lemma 2.4 and Gronwall's lemma, from (2.8) we immediately achieve the results of the lemma.

Lemma 2.7. *Under the conditions of Theorem 1.2,we have the following estimates, $(k>2)$*

$$\|q_{x^k}^\varepsilon(\cdot,t)\|_2^2+\varepsilon\int_0^t\|q_{x^{k+1}}^\varepsilon(\cdot,\tau)\|_2^2d\tau+\int_0^t\int_R(|q|^2)_{x^{k+1}}\mathbf{H}(|q|^2)_{x^k}dxdt\leq C$$

for all $t\in[0,T]$, where the constant C depends only on the norm $\|q_0\|_{H^k}$ and T.

Proof. From the equation (1.3), we deduce that

$$\frac{d}{dt}\|q_{x^k}\|_2^2+2\varepsilon\|q_{x^{k+1}}\|_2^2+\int_R(|q|^2)_{x^{k+1}}\mathbf{H}(|q|^2)_{x^k}dx$$
$$=-\int_R|q_{x^k}|^2\mathbf{H}(|q|^2)_xdx+2\mathrm{Re}\int_R\bar{q}_{x^k}\left((q\mathbf{H}(|q|^2))_{x^{k+1}}-q_{x^{k+1}}\mathbf{H}(|q|^2)\right.$$
$$\left.-q\mathbf{H}(|q|^2)_{x^{k+1}}\right)dx+\int_R\left((|q|^2)_{x^k}-2\mathrm{Re}(\bar{q}_{x^k}q)\right)\mathbf{H}(|q|^2)_{x^k}dx$$
$$\le\|\mathbf{H}(|q|^2)_x\|_\infty\|q_{x^k}\|_2^2+2\|q_{x^k}\|_2\|(q\mathbf{H}(|q|^2))_{x^{k+1}}-q_{x^{k+1}}\mathbf{H}(|q|^2)-q\mathbf{H}(|q|^2)_{x^{k+1}}\|_2$$
$$+\|(|q|^2)_{x^k}\|_2\|((|q|^2)_{x^k}-2\mathrm{Re}(\bar{q}_{x^k}q))_x\|_2$$
$$\le\|\mathbf{H}(|q|^2)_x\|_\infty\|q_{x^k}\|_2^2+2\|q_{x^k}\|_2(\|q\|_\infty\|(|q|^2)_{x^k}\|_2+\|\mathbf{H}(|q|^2)_x\|_\infty\|q_{x^k}\|_2)$$
$$+2\|(|q|^2)_{x^k}\|_2\|q_x\|_\infty\|q_{x^k}\|_2)$$
$$\le C\|q_{x^k}\|_2^2.$$

Thus, by Gronwall's lemma, the above inequality implies the results of the lemma.

On account of the above a priori estimates obtained in Lemma 2.6 and Lemma 2.7, we state that we have proved the existence of global solution $q(x,t)\in C^r(R_+;H^{k-2r}(R))$ for $r,k\in N,\ k-2r\ge 0$. In order to finish the proof of Theorem 1.2, it remains to prove the continuous dependence of solution with respect to the initial data.

Let $t>0,\ q_{10},\ q_{20}\in H^k(R),\ (k\ge 2)$ be given. Let q_1,q_2 denote the solution of (1.1),(1.2) corresponding to the initial data $q_{10},\ q_{20}$ respectively. We set $W=q_1-q_2$, so that W satisfies the initial value problem

$$W_t+iW_{xx}=(q_1\mathbf{H}(|q_1|^2))_x-(q_2\mathbf{H}(|q_2|^2))_x, \tag{2.9}$$

$$W(x,0)=W_0=q_{10}-q_{20}. \tag{2.10}$$

Of course, $q_1,\ q_2$ both obey the inequalities which are obtained in Lemma 2.4, Lemma 2.6 and Lemma 2.7. Take the product of (2.9) with $\overline{W}$ and integrate over the interval $[0,t]$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_R|W|^2dx=-\mathrm{Re}\int_R\overline{W}_x\left(q_1\mathbf{H}(|q_1|^2)-q_2\mathbf{H}(|q_2|^2)\right)dx$$
$$=-\mathrm{Re}\int_R\overline{W}_x\left(W\mathbf{H}\left(\frac{|q_1|^2+|q_2|^2}{2}\right)+\frac{q_1+q_2}{2}\mathbf{H}(|q_1|^2-|q_2|^2)\right)dx$$
$$=\frac{1}{2}\int_R|W|^2\mathbf{H}\left(\frac{|q_1|^2+|q_2|^2}{2}\right)_xdx-\mathrm{Re}\int_R\overline{W}_x\frac{q_1+q_2}{2}\mathbf{H}(\mathrm{Re}(\overline{W}(q_1+q_2)))dx$$
$$=\frac{1}{2}\int_R|W|^2\mathbf{H}\left(\frac{|q_1|^2+|q_2|^2}{2}\right)_xdx-\frac{1}{2}\int_R\mathrm{Re}(\overline{W}(q_1+q_2))_x\mathbf{H}(\mathrm{Re}(\overline{W}(q_1+q_2)))dx$$
$$+\frac{1}{2}\mathrm{Re}\int_R\overline{W}(q_1+q_2)_x\mathbf{H}(\mathrm{Re}(\overline{W}(q_1+q_2)))dx$$
$$\le C(\|q_1\|_{H^2},\|q_2\|_{H^2})\int_R|W|^2dx-\frac{1}{2}\int_R\mathrm{Re}(\overline{W}(q_1+q_2))_x\mathbf{H}(\mathrm{Re}(\overline{W}(q_1+q_2)))dx$$
$$\le C_0\int_R|W|^2dx.$$

By Gronwall's lemma, this inequality implies that

$$\|W(\cdot,t)\|_2^2\le\|W_0\|_2^2e^{C_0T}\ \text{for}\ t\in[0,T].$$

Namely, we have

$$\|q_1(\cdot,t)-q_2(\cdot,t)\|_2 \le C\|q_{10}(\cdot)-q_{20}(\cdot)\|_2 \quad \text{for } t\in[0,T], \tag{2.11}$$

where the constant C depends only on the norm $\|q_{10},q_{20}\|_{H^2}$ and T.

Obviously, the uniqueness of solution $q(x,t)$ in the space $C^r(R_+;H^{k-2r}(R))$ for $k\ge 2$, $k-2r\ge 2$ is a direct consequence of the inequality (2.11), and this concludes the proof of Theorem 1.2.

Appendix

Proof of Lemma 2.3. The first identity (i) can be easily deduced by taking the product of equation (1.3) with $\bar{q}$ and integrating over the interval $[0,T]$. We now prove the second identity (ii).

Since

$$\begin{aligned}\frac{d}{dt}\int_R |q_x|^2dx =& 2\mathrm{Re}\int_R \bar{q}_x\left(-iq_{xx}+(q\mathbf{H}(|q|^2))_x+\varepsilon q_{xx}\right)_x dx\\ =&-2\varepsilon\int_R|q_{xx}|^2dx+3\int_R|q_x|^2\mathbf{H}(|q|^2)_xdx-\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdx,\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\mathrm{Im}\int_R\bar{q}q_x\mathbf{H}(|q|^2)dx=\mathrm{Im}\int_R q_x\mathbf{H}(|q|^2)\left(i\bar{q}_{xx}+(\bar{q}\mathbf{H}(|q|^2))_x+\varepsilon\bar{q}_{xx}\right)dx\\ &+\mathrm{Im}\int_R\bar{q}\mathbf{H}(|q|^2)\left(-iq_{xx}+(q\mathbf{H}(|q|^2))_x+\varepsilon q_{xx}\right)_x dx\\ &+\mathrm{Im}\int_R\bar{q}q_x\mathbf{H}\left(2\mathrm{Im}(\bar{q}q_{xx})+2|q|^2\mathbf{H}(|q|^2)_x+(|q|^2)_x\mathbf{H}(|q|^2)+2\varepsilon\mathrm{Re}(\bar{q}q_{xx})\right)dx\\ =&-\frac{1}{2}\int_R|q_x|^2\mathbf{H}(|q|^2)_xdx+\mathrm{Im}\int_R\bar{q}q_x\mathbf{H}(|q|^2)\mathbf{H}(|q|^2)_xdx+\varepsilon\mathrm{Im}\int_R q_x\bar{q}_{xx}\mathbf{H}(|q|^2)dx\\ &-\frac{3}{2}\int_R|q_x|^2\mathbf{H}(|q|^2)_xdx+\frac{1}{2}\int_R(|q|^2)_{xx}\mathbf{H}(|q|^2)_xdx-\varepsilon\mathrm{Im}\int_R q_{xx}(\bar{q}\mathbf{H}(|q|^2))_xdx\\ &-2\int_R\mathrm{Im}(\bar{q}q_x)_x\mathbf{H}(\mathrm{Im}(\bar{q}q_x))dx+2\mathrm{Im}\int_R\bar{q}q_x\mathbf{H}(|q|^2\mathbf{H}(|q|^2)_x)dx\\ &+\mathrm{Im}\int_R\bar{q}q_x\mathbf{H}((|q|^2)_x\mathbf{H}(|q|^2))dx+2\varepsilon\int_R\mathrm{Im}(\bar{q}q_x)\mathbf{H}(\mathrm{Re}(\bar{q}q_{xx}))dx,\end{aligned}$$

combining the above two identities we can easily see the identity (ii) of Lemma 2.3.

References

[1] Mjφlhus, E. & Wyller, J., Nonlinear Alfvén wave in a finite-Beta plasma, *J. Plasma Phys.*, **40** (1988), 299-318.

[2] Asano, K., On semilinear parabolic partial differntial equations, *Publ. Res. Inst. Math. Sci.*, **1** (1964), 67-98.

[3] Zhou Yulin & Guo Boling, Existence of weak solution for boundary problem of system of ferromagnetic Chain, *Scientia Sinica*, **27A** (1984), 799-811.

[4] Temam, R., Sur un probléme non linéaire, *J. Math. Pures Appl.*, **48** (1969), 159-170.

[5] Friedman, A., Partial differential equations, 1969.

[6] Guo Boling & Tan Shaobin, On smooth solutions to the initial value problem for the mixed nonlinear Schrödinger equations, *Proc. Roy. Soc. Edinburgh*, **119A** (1991), 31-45.