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ON THE APPROXIMATELY SIMILARITY OF OPERATORS**

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Abstract

The author shows that a rank-preserving $*$-isomorphism between separable C^*-algebras with unity is approximately equivalent to the identity representation. Some applications are made for approximately similarity of n-tuples of operators.

Keywords C^*-algebra, Rank-preserving $*$-isomorphism, Approximately equevalent, Approximately similar, n-tuples of operators
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§1. Introduction

In what follows, all of Hilbert spaces appeared in this paper are infinite-dimensional separable complex Hilbert space. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{L}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$.

Two operators S and T are called approximately equivalent if there is a sequenc $\{U_n\}$ of unitary operators such that $\|U_n^*SU_n - T\| \to 0$ $(n \to \infty)$.

Similarly two operatos S and T are approximately similar if there is a sequence $\{V_n\}$ of invertible operators with sup $(\|V_n\|, \|V_n^{-1}\|) < \infty$ (such a sequence will be called invertibly bounded) and $\|V_n^{-1}SV_n - T\| \to 0$ $(n \to \infty)$.

D. W. Hadwin[5] initiated a study of an asymptotic version of unitary equivalence of operators. All of the questions raised in [5] were answered in a paper of D. Voiculescu[9] which contains a complete characterization of approximately equivalent represention s of a separable C^*-algebra. D. W. Hadwin has shown that two operators are approximately equivalent if and only if there is a rank-preserving $*$-isomorphism between the C^*-algebras they generate that sends one of the operators onto the other. Although, in general, there seems to be no analogous result for approximately similarity, but there is one in the case when one of the operators is normal (see [7, Proposition 3.6]). It is the purpose of this paper to prove a rank-preserving $*$-isomorphism between separable C^*-algebras with unity is approximately equivalent to the identity representation. Some applications are made for approximately similarity of n-tuples of operators.

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Following [9], let $\mathcal{A}$ be a separable C^*- algebra in $\mathcal{L}(\mathcal{H})$; $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are two separable representations of $\mathcal{A}$. $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are called approximately equivalent, if there is a sequence of unitary operators $\{U_n\}$, satisfying : for any $A \in \mathcal{A}$,

(1) $U_n\pi_1(A)U_n^* - \pi_2(A)$ is a compact operator, $n = 1, 2, \cdots$;

(2) $\lim\limits_{n\to\infty} \|U_n\pi_1(A)U_n^* - \pi_2(A)\| = 0$.

In fact, it was shown by D. Voiculescu[9] that (2) contains (1).

Definition 1.1. *Let $\mathcal{A}$ be a subspace in $\mathcal{L}(\mathcal{H})$. A linear mapping $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}^1)$ is called* rank*-preserving if* $\operatorname{rank} \pi(A) = \operatorname{rank} A$ *for any $A \in \mathcal{A}$.*

Evidently, a rank-preserving mapping must be injective and its inverse $\pi^{-1} : \pi(\mathcal{A}) \to \mathcal{A}$ is rank-preserving too.

Let $\mathcal{A}$ be a C^*-algebra in $\mathcal{L}(\mathcal{H})$. A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}^1)$ is called a rank-preserving $*$-isomorphism if π is rank-preserving.

§2. Representation of Rank-preserving $*$-isomorphism

Proposition 2.1. *If π is a continuous rank-preserving mapping from C^*-algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}^1)$, then $\pi(\mathcal{A} \cap \mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H}^1)$.*

Proof. Since $\mathcal{A} \cap \mathcal{K}(\mathcal{H})$ is a closed idea in $\mathcal{A}$, it suffices to prove $\pi(K) \in \mathcal{K}(\mathcal{H}^1)$ for any self -adjoint operator K in $\mathcal{A} \cap \mathcal{K}(\mathcal{H})$.

Let K be a self-adjoint in $\mathcal{A} \cap \mathcal{K}(\mathcal{H})$ with spectrum $\sigma(K) = \{\lambda_n\}$. P_n is the projection from $\mathcal{H}$ onto $\ker(K - \lambda_n I_{\mathcal{H}})$. According to the spectral decomposition theorem, the series $\sum\limits_n \lambda_n P_n$ is convergent uniformly to K, where $\lambda_n \neq 0$ and $\lambda_n \to 0$ $(n \to \infty)$, $P_n \in \mathcal{A}$, $n = 1, 2, \cdots$.

Since π is a rank-preserving mapping, thus $\pi(P_n) \in \mathcal{K}(\mathcal{H}^1)$, it follows from the continuity of π that $\pi(K) \in \mathcal{K}(\mathcal{H}^1)$.

Corollary 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be separately C^*-subalgebras in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}^1)$. If $\pi : \mathcal{A} \to \mathcal{B}$ is a rank-preserving $*$-isomorphism, then*

$$\pi(\mathcal{A} \cap \mathcal{K}(\mathcal{H})) = \mathcal{B} \cap \mathcal{K}(\mathcal{H}^1), \quad \pi^{-1}(\mathcal{B} \cap \mathcal{K}(\mathcal{H}^1)) = \mathcal{A} \cap \mathcal{K}(\mathcal{H}).$$

In what follows, we assume $\mathcal{A}$ and $\mathcal{B}$ are two separable C^*- subalgebras in $\mathcal{L}(\mathcal{H})$.

Let $\hat{\mathcal{A}}$ (resp. $\hat{\mathcal{B}}$) denote the C^*-subalgebra $\mathcal{A} \cap \mathcal{K}(\mathcal{H})$ (resp. $\mathcal{B} \cap \mathcal{K}(\mathcal{H})$).

A projection P in $\hat{\mathcal{A}}$ is called minimal if $P \neq 0$ and the only subprojections of P in $\hat{\mathcal{A}}$ are 0 and P.

Lemma 2.1. *Let P be a nonzero projection in $\hat{\mathcal{A}}$. Then P is a minimal projection if and only if $P\hat{\mathcal{A}}P = \{\lambda P, \lambda \in \mathbb{C}\}$. Every nonzero projection in $\hat{\mathcal{A}}$ is finite-dimensional, and is a finite sum of orthogonal minimal projections.*

Proof. If $P\hat{\mathcal{A}}P$ consists of scalar multiples of P, then P is minimal. Conversely, assume P is minimal. It suffices to show that PTP is a scalar multiple of P, for every self-adjoint operator $T \in \hat{\mathcal{A}}$. Considering the spectral formula for PTP, we have $PTP = \sum\limits_n \lambda_n P_n$, where the P_n are mutually orthogonal spectral projections of PTP. Since PTP annihilates $P^\perp\mathcal{H}$, so does each P_n, and hence $P_n \leq P$. Thus each nonzero P_n must be P, and hence $PTP = \lambda P$ has the required form. It is plain that every projection in $\hat{\mathcal{A}}$ is finite-dimensional, by compactness, and the last phrase follows from the usual sort of finite induction.

Corollary 2.2. *Let $\mathcal{A}$ be a C^*-subalgebra in $\mathcal{L}(\mathcal{H})$ and $\mathcal{A}\cap\mathcal{K}(\mathcal{H})\neq 0$. Then there is a minimal projection $P\in\hat{\mathcal{A}}$. If $\mathcal{F}(\hat{\mathcal{A}})$ denotes the set of all minimal projections in $\hat{\mathcal{A}}$, then $\overline{\mathrm{span}}\,\mathcal{F}(\hat{\mathcal{A}})=\hat{\mathcal{A}}$, where $\overline{\mathrm{span}}\,\mathcal{F}(\hat{\mathcal{A}})$ denotes the uniformly closure of all linear combinations of $\mathcal{F}(\hat{\mathcal{A}})$.*

Proof. Since $\mathcal{A}\cap\mathcal{K}(\mathcal{H})\neq 0$, choosing a nonzero self-adjoint operator $T\in\hat{\mathcal{A}}$, by the spectral theorem about the compact operators, we see that there is a nonzero spectral projection P of T. From the proof of Lemma 2.1, we easily know that P is a minimal projection in $\hat{\mathcal{A}}$. The last assertion now follows from Proposition 2.1 and the spectral theorem of compact operators again.

Lemma 2.2. *If $\pi:\mathcal{A}\to\mathcal{B}$ is a rank-preserving $*$-isomorphism, and $\mathcal{A}\cap\mathcal{K}(\mathcal{H})\neq 0$, then there is a unitary map $U:[\hat{\mathcal{A}}\mathcal{H}]\mapsto[\hat{\mathcal{B}}\mathcal{H}]$ such that $UT=\pi(T)U$, for any $T\in\hat{\mathcal{A}}$, where $[\hat{\mathcal{A}}\mathcal{H}]$ denotes the closed linear span of all vectors of the form $T\xi, T\in\hat{\mathcal{A}},\xi\in\mathcal{H}$ (similarly $[\hat{\mathcal{B}}\mathcal{H}]$).*

Proof. By Corollary 2.2, $\mathcal{F}(\hat{\mathcal{A}})\neq\phi$. For any $P\in\mathcal{F}(\hat{\mathcal{A}})$, by Lemma 2.1, there is a linear functional f such that $PTP=f(T)P$, for any $T\in\hat{\mathcal{A}}$.

Since π is a rank-preserving $*$-isomorphism, $P\mathcal{H}$ and $\pi(P)\mathcal{H}$ have the same finite dimension.

Choose arbitrarily a unit vector η (resp. ξ) in $\pi(P)\mathcal{H}$ (resp. in $P\mathcal{H}$). Then $\mathcal{H}_0=[\pi(\hat{\mathcal{A}})\eta]$ defines a rank-preserving $*$-subisomorphism π_0 of π. For any $T\in\hat{\mathcal{A}}$, we have

$$\begin{aligned}\|\pi(T)\eta\|^2&=\|\pi(T)\pi(P)\eta\|^2=\|\pi(TP)\eta\|^2\\&=(\pi(PT^*TP)\eta,\eta)=f(T^*T)(\pi(P)\eta,\eta)\\&=f(T^*T)=(PT^*TP\xi,\xi)=\|T\xi\|^2.\end{aligned}$$

This shows that map $U:T\xi\mapsto\pi(T)\eta$ extends uniquely to a untary map of $[\hat{\mathcal{A}}\xi]$ onto $[\pi(\hat{\mathcal{A}})\eta]$, and the formula $UT=\pi_0(T)U$ is immediate from the definition of π_0 and U.

Since $\mathcal{H}$ is separable, by the Zorn lemma, we can choose a maximal sequence $\{\pi_i\}$ of orthogonal rank-preserving $*$-subisomorphisms of π and a sequence $\{U_i\}$ of unitary maps such that $\pi=\oplus\pi_i$ and $U_iT=\pi_i(T)U_i$, for any $T\in\hat{\mathcal{A}}$.

It follows that there is a unitary map $U:[\hat{\mathcal{A}}\mathcal{H}]\mapsto[\pi(\hat{\mathcal{A}})\mathcal{H}]$, such that $UT=\pi(T)U$ for all $T\in\hat{\mathcal{A}}$.

By Corollary 2.1, $\pi(\hat{\mathcal{A}})=\hat{\mathcal{B}}$, the proof of the Lemma now is completed .

Theorem 2.1. *If $\pi:\mathcal{A}\to\mathcal{B}$ is a rank-preserving $*$-isomorphism, where $\mathcal{A}$ and $\mathcal{B}$ are separable C^*-algebras with unity in $\mathcal{L}(\mathcal{H})$, then π is approximately equivalent to the identity representation id.*

Proof. First, if $\mathcal{A}\cap\mathcal{K}(\mathcal{H})=0$, then by Collorary of Theorem 5 in [2], π is approximately equivalent to *id*.

If $\mathcal{A}\cap\mathcal{K}(\mathcal{H})\neq 0$, following the notation in [2], let

$$\mathcal{H}_e=\chi_{\mathcal{H}}^{\pi(\ker\dot{\pi})}=\overline{\mathrm{span}}\Big\{\sum_{i=1}^{n}\pi(A_i)x_i|A_i\in\ker\dot{\pi},x_i\in\mathcal{H}\Big\},$$

where

$$\ker\dot{\pi}=\{A\in\mathcal{A}|\ \pi(A)\in\mathcal{K}(\mathcal{H})\}=\{A\in\mathcal{A}|\ \pi(A)\in\mathcal{B}\cap\mathcal{K}(\mathcal{H})\}.$$

Evidently, $\ker\pi = \ker(id)$. On the other hand, since

$$\ker\dot{\pi} = \{A \in \mathcal{A} |\ \pi(A) \in \mathcal{B} \cap \mathcal{K}(\mathcal{H})\} = \pi^{-1}(\mathcal{B} \cap \mathcal{K}(\mathcal{H})),$$

it follows from Colloray 2.1 that

$$\ker\dot{\pi} = \mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \ker(\dot{id}),$$

hence $\mathcal{H}_e = \chi_{\mathcal{H}}^{\mathcal{B}\cap\mathcal{K}(\mathcal{H})}$. Similarly,

$$\mathcal{H}_e^1 = \chi_{\mathcal{H}}^{id(\ker(\dot{id}))} = \chi_{\mathcal{H}}^{\mathcal{A}\cap\mathcal{K}(\mathcal{H})}.$$

Let P_e (resp. P_e^1) be the projection on $\mathcal{H}_e$ (resp. $\mathcal{H}_e^1$). From the definition of U in Lemma 2.2, we have $UP_e^1 = P_eU$. Since $\mathcal{H}_e$ and $\mathcal{H}_e^1$ are invariable separably with respect to $\pi(\mathcal{A})$ and $id(\mathcal{A})$, we see that corresponding to the space decomposition

$$\mathcal{H} = \mathcal{H}_e \oplus H_e^{\perp} = \mathcal{H}_e^1 \oplus \mathcal{H}_e^{1\perp},$$

π and id have separablly decomposition : $\pi = \pi_e \oplus \pi', id = (id)_e \oplus (id)'$, where $\pi_e = \pi|_{\mathcal{H}_e}, (id)_e = id|_{\mathcal{H}_e}$.

By [2, Theorem 5], we need only to show that π_e is unitarily equvalent to $(id)_e$.

Let $W = P_eUP_e^1|_{\mathcal{H}_e^1}$. Then W is a unitary operator from $\mathcal{H}_e^1$ onto $\mathcal{H}_e$. Thus for any $K \in \mathcal{A} \cap \mathcal{K}(\mathcal{H})$, we have

$$\begin{aligned}\pi_e(K) = \pi(K)|_{\mathcal{H}_e} &= UKU^{-1}|_{\mathcal{H}_e} = UKP_e^1U^{-1}|_{\mathcal{H}_e}\\ &= UP_eKP_e^1U^{-1}|_{\mathcal{H}_e} = P_eUP_e^1KP_e^1U^{-1}|_{\mathcal{H}_e}\\ &= W(K|_{\mathcal{H}_e^1}) = W(id)_e(K)W^{-1}.\end{aligned}$$

Because $\mathcal{A}\cap\mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{A}$ and the restrictions of π_e and $(id)_e$ are non-degenerate, we have for any $A \in \mathcal{A}$ and $K \in \mathcal{A} \cap \mathcal{K}(\mathcal{H})$, $\pi(AK) = W(id)_e(AK)W^{-1}$. It follows that

$$\pi_e(A)\pi_e(K) = W(id)_e(A)W^{-1}\pi_e(K).$$

Since $\chi_{\mathcal{H}_e}^{\pi_e(\mathcal{A})\cap\mathcal{K}(\mathcal{H})} = \mathcal{H}_e$, we have

$$\pi_e(A) = W(id)_e(A)W^{-1},$$

that is, π_e is unitarily equivalent to $(id)_e$.

Corollary 2.3. *Let $(\pi, \mathcal{H}_\pi)$ be a non-degenerate representation of a separable C^*-algebra $\mathcal{A}$, where $\mathcal{H}_\pi$ is a separable Hilbert space. If $\pi|_{\mathcal{A}\cap\mathcal{K}(\mathcal{H})} = 0$, then $id\oplus\pi$ approximatly equvalent to id.*

Proof. Since $\mathcal{H}$ and $\mathcal{H}_\pi$ both are separable, there exists a unitary operator U from $\mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}_\pi$. Let $\rho : T \longmapsto U^*(T \oplus \pi(T))U$. We can easily justify that $\rho : \mathcal{A} \longmapsto \rho(\mathcal{A})$ is a rank-preserving $*$-isomorphism. Theorem 2.1 guarantees that $id \oplus \pi$ is approximatly equivalent to id.

We obtain again the result of [3, Corollary 2].

Corollary 2.4. *Let $A = (A_1, A_2, \cdots, A_n)$ and $B = (B_1, B_2, \cdots, B_n)$ be two n-tuples of operators in $\mathcal{L}(\mathcal{H})$. Then A and B are approximatly equivalent if and only if there is a rank-preserving $*$- ismorphism $\rho : C^*(A_1, A_2, \cdots, A_n) \longmapsto C^*(B_1, B_2, \cdots, B_n)$ such that $\rho(I) = I$ and $\rho(A_i) = B_i, i = 1, 2, \cdots, n$, where $C^*(A_1, A_2, \cdots, A_n)$ (similarly $C^*(B_1, B_2, \cdots, B_n)$) is the C^*-algebra generated by I and $A_1, A_2, \cdots, A_n$.*

Evidently, Corollary 2.4 is the generalization of [6, Corollary 3.7].

Theorem 2.2. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Von-Neumann algbras in $\mathcal{L}(\mathcal{H})$. $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a rank-preserving $*$-isomorphism. If $\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H}) \neq 0$, then there is a unitary operator U in $\mathcal{L}(\mathcal{H})$, such that $\pi(A) = UAU^*$ for any $A \in \mathcal{M}_1$.*

Proof. Since $\mathcal{H}$ is separable, the unit ball is second numerable by strong topology, so is the unit ball of $\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})$ too. It follows that $\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})$ is strongly separable. Let $\{A_n\}$ be a strongly dense sequece of operators in $\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})$, and $\mathcal{A}$ is the C^*-algebra which is generated by I and $\{A_n\}$. Since

$$[\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})]' = [\mathcal{M}_1{}' \cap \mathcal{K}(\mathcal{H})']'' = \mathcal{M}_1{}',$$

we have $[\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})]'' = \mathcal{M}_1$. Write $\mathcal{N} = \overline{\mathcal{M}_1 \cap \mathcal{K}(\mathcal{H})}^s$. Then by [3, Theorem 2],

$$\mathcal{M}_1 = \{\lambda I_{\mathcal{H}} + T |\ \lambda \in \mathbb{C}, T \in \mathcal{N}\},$$

and it follows that $\mathcal{A}$ is strongly dense in $\mathcal{M}_1$. The fact that an isomorphism between two Von-Neumann algebras must be strongly continuous guarantees that $\pi(\mathcal{A})$ is strongly dense in $\mathcal{M}_2$.

Since $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) \neq 0$, by Lemma 2.2 there exists a unitary map U from $[\hat{\mathcal{A}}\mathcal{H}]$ onto $[\hat{\mathcal{B}}\mathcal{H}]$, such that $\pi(A) = UAU^{-1}$, for any $A \in \hat{\mathcal{A}}$.

Let P and Q be separably the projections on $[\hat{\mathcal{A}}\mathcal{H}]$ and $[\hat{\mathcal{B}}\mathcal{H}]$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are Von-Neumann algebras, we have $P \in \mathcal{M}_1,\ Q \in \mathcal{M}_2$. Because $\pi(P) = Q$, we have

$$\operatorname{rank} \pi(I - P) = \operatorname{rank}(I - P) = \operatorname{rank}(I - Q),$$

and therefore U can be extented to the unitary operator (still be denoted by U) on $\mathcal{H}$.

Since

$$\{[\mathcal{A} \cap \mathcal{K}(\mathcal{H})]\}'' = \mathcal{A}'' = \mathcal{M}_1$$

and π is strongly continuous, we have $\pi(A) = UAU^*$ for any $A \in \mathcal{M}_1$.

§3. Some Applications

In this section, we discuss some applications of Theorem 2.1 for n-tuples of operators.

Let $S = (S_1, \cdots, S_n)$ and $T = (T_1, \cdots, T_n)$ be two n-tuples of operators. S and T are called approxinately equivalent or approximately similar if there is a sequence of unitary operators $\{U_n\}$ or invertibly bounded operators $\{V_n\}$ such that $\|U_n^* S_k U_n - T_k\| \to 0$ or $\|V_n^{-1} S_k V_n - T_k\| \to 0$ $(n \to \infty)$, $k = 1, 2, \cdots, n$.

First we discuss the approximately similar problem of n-tuples of operators.

Let $T = (T_1, T_2, \cdots, T_n)$ be an n-tuple of operators in $\mathcal{L}(\mathcal{H})$. For any $x \in \mathcal{H}$, write $\operatorname{col}(T)x = (T_1 x, T_2 x, \cdots, T_n x)$. Then $\operatorname{col}(T) : \mathcal{H} \to \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n}$ is a linear continuous operator. Let $\ker T = \bigcap_{i=1}^{n} \ker T_i$. Then $\ker \operatorname{col}(T) = \ker T$.

It is easily known that $\operatorname{col}(T)$ has closed range if and only if there is an $r > 0$ such that $\sum_{i=1}^{n} T_i^* T_i \geq r(I - P)$, where P is the projection from $\mathcal{H}$ onto $\ker T$.

Lemma 3.1. *Suppose $T = (T_1, \cdots, T_n)$ is an n-tuple of operators and V is an invertible operator in $\mathcal{L}(\mathcal{H})$. Write*

$$V^{-1} T V = (V^{-1} T_1 V, \cdots, V^{-1} T_n V).$$

If there is an $\epsilon > 0$ *such that* $\sum_{i=1}^{n} \|T_i x\|^2 \geq \epsilon \|x\|^2$ *for any* $x \in (\ker T)^{\perp}$*, then for any* $x \in (\ker V^{-1}TV)^{\perp}$

$$\sum_{i=1}^{n} \|V^{-1}T_i V x\|^2 \geq \frac{\epsilon}{\|V\|^2 \|V^{-1}\|^2} \|x\|^2.$$

Proof. Let P be the projection from $\mathcal{H}$ onto $(\ker T)^{\perp}$ and $Q = V^{-1}PV$. Then Q is an idempotent operator. If $\mathcal{H} = (\ker Q)^{\perp} \oplus \ker Q$, then $Q = \begin{pmatrix} I & O \\ A & O \end{pmatrix}$. Let $Q_1 = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$. Then

$$\ker Q_1 = \ker Q = \ker V^{-1}PV = \ker V^{-1}TV.$$

It follows that Q^{-1} is the projection from $\mathcal{H}$ onto $(\ker V^{-1}TV)^{\perp}$. Since $Q_1 x = x$ for any $x \in (\ker V^{-1}TV)^{\perp}$ and $T_i P = T_i, i = 1, 2, \cdots, n$, $QQ_1 = Q, VQ = PV$, hence we have

$$\begin{aligned}\sum_{i=1}^{n} \|V^{-1}T_i V x\|^2 &= \sum_{i=1}^{n} \|V^{-1}T_i V Q_1 x\|^2 = \sum_{i=1}^{n} \|V^{-1}T_i PVQ_1 x\|^2 \\ &= \sum_{i=1}^{n} \|V^{-1}T_i VQx\|^2 \geq \frac{1}{\|V\|^2} \sum_{i=1}^{n} \|T_i VQx\|^2.\end{aligned}$$

Since $VQ = PV$, thus $VQx \in (\ker T)^{\perp}$, we have

$$\sum_{i=1}^{n} \|V^{-1}T_i V x\|^2 \geq \frac{\epsilon}{\|V\|^2} \|VQx\|^2 \geq \frac{\epsilon}{\|V\|^2 \|V^{-1}\|^2} \|Qx\|^2 \geq \frac{\epsilon}{\|V\|^2 \|V^{-1}\|^2} \|x\|^2,$$

where the last inequality is due to $x = Q_1 x$.

Proposition 3.1. *Let* $S = (S_1, \cdots, S_n)$ *and* $T = (T_1, \cdots, T_n)$ *be two* n*-tuples of operators in* $\mathcal{L}(\mathcal{H})$*. If* S *is approximately similar to* T *and* $\mathrm{col}(S)$ *has closed range, then* $\mathrm{col}(T)$ *has closed range too and* $\dim\ker S = \dim\ker T$.

Proof. Because S is approximately similar to T, there is an inversely bounded sequence of oprators $\{V_n\}$ such that

$$\|V_n^{-1} S_i V_n - T_i\| \to 0 \ (n \to \infty), \quad i = 1, 2, \cdots, n.$$

Write $S_i^{(k)} = V_k^{-1} S_i V_k$. Since $\mathrm{col}(S)$ has closed range, by Lemma 3.1

$$\sum_{i=1}^{n} \|S_i^{(k)} x\|^2 \geq \epsilon^2 \|x\|^2$$

for any $x \in (\ker S^{(k)})^{\perp}$, where $S^{(k)} = (S_1^{(k)}, \cdots, S_n^{(k)})$, that is

$$\Big\| \Big(\sum_{i=1}^{n} S_i^{(k)*} S_i^{(k)} \Big)^{\frac{1}{2}} x \Big\| \geq \epsilon \|x\|.$$

On the other hand , since $\ker \big(\sum_{i=1}^{n} S_i^{(k)*} S_i^{(k)} \big)^{\frac{1}{2}} = \ker (S^{(k)})$, and

$$\Big(\sum_{i=1}^{n} S_i^{(k)*} S_i^{(k)} \Big)^{\frac{1}{2}} \to \Big(\sum_{i=1}^{n} T_i^* T_i \Big)^{\frac{1}{2}} \quad (k \to \infty),$$

by [1, Lemma 1.9], it follows that $\big(\sum_{i=1}^{n} T_i^* T_i \big)^{\frac{1}{2}}$ has closed range, and therefore $\mathrm{col}(T)$ has closed range too.

Let P_k and P be the projections from $\mathcal{H}$ onto $\ker S^{(k)}$ and $\ker T$ respectively. Since both of $\mathrm{col}(S^{(k)})$ and $\mathrm{col}(T)$ have closed range, there is $r \geq 0$ such that

$$\sum_{i=1}^{n} S_i^{(k)^*} S_i^{(k)} \geq r(I - P_k)$$

and

$$\sum_{i=1}^{n} T_i^* T_i \geq r(I - P).$$

It follows that $\|P_k - P\| \to 0$ $(k \to \infty)$, and therefore

$$\dim\ker T = \dim\mathrm{ran}\, P \leq \liminf_{n\to\infty} \dim\mathrm{ran}\, P_n = \dim\ker S.$$

By symmetry, we also have $\dim\ker T \leq \dim\ker S$, therefore $\dim\ker T = \dim\ker S$.

Corollary 3.1. *Suppose S is approximately similar to T, where S and T are two double commuting n-tuples of operators in $\mathcal{L}(\mathcal{H})$. If S is a Fredholm n-tupl of operators, then T is a Fredholm n-tuple of operators too, and $\sigma_{ja}(S) = \sigma_{ja}(T)$.*

We follow the notation of [7]. If $\mathcal{Q} \subseteq \mathcal{L}(\mathcal{H})$, then the approximately double commutant of $\mathcal{Q}$, denoted by $\mathrm{appr}(\mathcal{Q})$, is the set of those operators T for which $\|A_nT - TA_n\| \to 0$ $(n \to \infty)$ whenever $\{A_n\}$ is a bounded sequence such that $\|A_nS - SA_n\| \to 0$ $(n \to \infty)$ for every $S \in \mathcal{Q}$.

Theorem 3.1. *Let $T = (T_1, \cdots, T_n)$ be an n-tuple of operators in $\mathcal{L}(\mathcal{H})$ and $\mathcal{Q} = \{T_1, \cdots, T_n\}$. Suppose $\mathrm{appr}(\mathcal{Q})'' = C^*(\mathcal{Q})$. If n-tuples of operators $S = (S_1, \cdots, S_n)$ is approximately similar to T, then there is an n-tuple of operators $R = (R_1, \cdots, R_n)$, such that R is approximately equivalent to T and S is similar to R.*

Proof. Since S is approximately similar to T, there is an invertibly bounded sequence of $\{V_n\} \in \mathcal{L}(\mathcal{H})$ such that

$$\|V_n^{-1} T_i V_n - S_i\| \to 0 \ (n \to \infty), \quad i = 1, 2, \cdots.$$

By [7, Theorem 3.4], for any $T \in \mathrm{appr}(\mathcal{Q})'' = C^*(\mathcal{Q})$, $\{V_n^{-1} T V_n\}$ is a convergent sequence in norm. Let $\pi(A) = \lim\limits_{n\to\infty} V_n^{-1} A V_n$ for any $A \in C^*(\mathcal{Q})$. It is known easily that π is a rank-preserving isomorphism and

$$\|\pi\| \leq \sup_n (\|V_n^{-1}\|, \|V_n\|),$$

that is, π is a representation of $C^*(\mathcal{Q})$.

We claim that π is completely bounded, indeed, for any matrix of operators $(A_{ij})_{k\times k}$, where $A_{ij} \in C^*(\mathcal{Q})$, $i, j = 1, 2, \cdots$, define

$$\pi_k((A_{ij}))_{k\times k} = (\pi((A_{ij}))_{k\times k} = \lim_{n\to\infty} (V_n^{-1} A_{ij} V_n)_{k\times k}.$$

It is easily justified that

$$\|\pi_k\| \leq \sup_n (\|V_n^{-1}\|, \|V_n\|),$$

and

$$\sup_{j\geq 1} \|\pi_j\| \leq \sup_n (\|V_n^{-1}\|, \|V_n\|) < +\infty.$$

Therefore π is completely bounded. It follows from [4, Theorem 1.10] that there is an invertible operator $V \in \mathcal{L}(\mathcal{H})$, such that $V^{-1}\pi(\cdot)V$ is a representation of $C^*(\mathcal{Q})$.

Write $\rho(\cdot) = V^{-1}\pi(\cdot)V$. It is known easily that ρ is a rank-preserving $*$-isomorphism. Hence by Theorem 2.1, ρ is approximately equivalent to id, that is, there is a sequence of unitary operators $\{U_n\}$, such that for any positive integer n and $T \in C^*(\mathcal{Q})$, $\rho(T) - U_n^* T U_n \in \mathcal{K}(\mathcal{H})$ and

$$\|\rho(T) - U_n^* T U_n\| \to 0 \quad (n \to \infty).$$

It follows that $S_i = \pi(T_i) = V\rho(T_i)V^{-1}$ and

$$\|\rho(T_i) - U_k^* T_i U_k\| \to 0 \ (k \to \infty), \quad i = 1, 2, \cdots, n,$$

that is, S is approximately equivalent to $(\rho(T_1), \cdots, \rho(T_n))$ which is unitarily equvalent to T. Let $R = (\rho(T_1), \cdots, \rho(T_n))$. The theorem is completed finally.

Corollary 3.2. *Suppose $N = (N_1, \cdots, N_n)$ is an n-tuple of normal operators in $\mathcal{L}(\mathcal{H})$. If an n-tuple of operators S in $\mathcal{L}(\mathcal{H})$, $S = (S_1, \cdots, S_n)$, is approximately similar to N, then there is an n-tuple of operators $R = (R_1, \cdots, R_n)$, which is unitarily equivalent to N, is approximately equivalent to S.*

Proof. Write $\mathcal{Q} = \{N_1, \cdots, N_n\}$. Then $\text{appr}(\mathcal{Q})'' \in C^*(\mathcal{Q})$. On the other hand, for any $T \in C^*(\mathcal{Q})$, if $\{V_k\}$ is a bounded sequence such that

$$\|N_i V_k - V_k N_i\| \to 0 \ (k \to \infty), \quad i = 1, 2, \cdots n,$$

according to the asymptotic Fuglede's theorem, it follows that

$$\|N_i^* V_k - V_k N_i^*\| \to 0 \ (k \to \infty), \quad i = 1, 2, \cdots,$$

and $T \in \text{appr}(\mathcal{Q})''$ by [7, Theorem 2.2], that is, $\text{appr}(\mathcal{Q})'' = C^*(\mathcal{Q})$. The corollary is completed by Theorem 3.1.

The Corollary 3.2 generate partially the result of [7, Theorem 3.5].

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