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On the Decomposition Theorems for C^*-Algebras

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Abstract Elliott dimension drop interval algebra is an important class among all C^*-algebras in the classification theory. Especially, they are building stones of $\mathcal{AHD}$ algebra and the latter contains all AH algebras with the ideal property of no dimension growth. In this paper, the authors will show two decomposition theorems related to the Elliott dimension drop interval algebra. Their results are key steps in classifying all AH algebras with the ideal property of no dimension growth.

Keywords C^*-algebra, Elliott dimension drop interval algebra, Decomposition theorem, Spectral distribution property
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1 Introduction

Classification theorems have been obtained for AH algebras—the inductive limits of cut downs of matrix algebras over compact metric spaces by projections—and AD algebras—the inductive limits of Elliott dimension drop interval algebras in two special cases:

(1) Real rank zero case: All such AH algebras with no dimension growth and such AD algebras (see [1–4, 7–8, 12–17]).

(2) Simple case: All such AH algebras with no dimension growth (which includes all simple AD algebras by [9]) (see [5–6, 11, 18, 27–32, 39–40]).

In [11], the authors pointed out two important possible next steps after the completion of classification of simple AH algebras (with no dimension growth). One of these is the classification of simple ASH algebras—the simple inductive limits of subhomogeneous algebras (with no dimension growth). The other is to generalize and unify the above-mentioned classification theorems for simple AH algebras and real rank zero AH algebras by classifying AH algebras with the ideal property. The ideal properties in the classification theory are intensively studied previously (see [35–36, 41–42])In particular, ASI and AI algebras with the ideal property are classified by the Stevens-Jiang invariant (see [22, 26, 41])In this article, we have achieved several key results for the second goal by providing two decomposition theorems.

As in [8], let $T_{II,k}$ be the 2-dimensional connected simplicial complex with $H^1(T_{II,k}) = 0$ and $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$, and let I_k be the subalgebra of $M_k(C[0,1])$ defined by

$$I_k = \{f \in M_k(C[0,1]) : \ f(0) \in \mathbb{C} \cdot 1_k \text{ and } f(1) \in \mathbb{C} \cdot 1_k\}.$$

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This algebra is called an Elliott dimension drop interval algebra. Denote by $\mathcal{HD}$ the class of algebras consisting of direct sums of building blocks of the forms $M_l(I_k)$ and $PM_n(C(X))P$, with X being one of the spaces $\{pt\}$, $[0,1]$, S^1 and $T_{II,k}$, and with $P \in M_n(C(X))$ being a projection. (In [2], this class is denoted by $SH(2)$, and in [24], this class is denoted by $\mathcal{B}$). We will call a C^*-algebra an $A\mathcal{HD}$ algebra, if it is an inductive limit of algebras in $\mathcal{HD}$. In [19–20, 23, 28], it is proved that all AH algebras with the ideal property of no dimension growth are inductive limits of algebras in the class $\mathcal{HD}$—that is, they are $A\mathcal{HD}$ algebras. By this reduction theorem, to classify AH algebras with the ideal property, we must study the properties of homomorphisms between those basic building blocks.

In the local uniqueness theorem for classification, it requires the homomorphisms involved to satisfy a certain spectral distribution property, called the sdp property (more specifically, sdp(η,δ) property introduced in [18] and [11] for some positive real numbers η and δ). This property automatically holds for the homomorphisms $\phi_{n,m}$ (provided that m is large enough) giving rise to a simple inductive limit procedure. But for the case of general inductive limit C^*-algebras with the ideal property, to obtain this sdp property, we must pass to certain good quotient algebras which correspond to simplicial sub-complexes of the original spaces; a uniform uniqueness theorem, that does not depend on the choice of simplicial sub-complexes involved, is required. For the case of an interval, whose simplicial sub-complexes are finite unions of subintervals and points, such a uniform uniqueness theorem is proved in [23] (see [5, 26] also). But for the general case, there is no uniqueness theorem for the general case involving arbitrary finite subsets of $M_n(C(T_{II,k}))$ (or $M_l(I_k)$). In this paper, we prove decomposition theorems between such building blocks or between a building block of this kind and a homogeneous building block. And we will compare the decompositions of two different homomorphisms in the last part of Section 4. Such decomposition and comparison results will be used in the proof of the uniqueness theorem for AH algebras with the ideal property in [19].

2 Notation and Terminology

In this section, we will introduce some notation and terminology. We can assume all connecting maps in the inductive system are injective (see [10]).

Definition 2.1 *Let X be a compact metric space and $\psi : C(X) \to PM_{k_1}(C(Y))P$* (with rank$(P) = k$) *be a unital homomorphism. For any point $y \in Y$, there are k mutually orthogonal rank 1 projections $p_1, p_2, \cdots, p_k$ with $\sum_{i=1}^{k} p_i = P(y)$ and $\{x_1(y), x_2(y), \cdots, x_k(y)\} \subset X$* (*may be repeat*) *such that*

$$\psi(f)(y) = \sum_{i=1}^{k} f(x_i(y))p_i, \quad \forall f \in C(X).$$

We denote the set $\{x_1(y), x_2(y), \cdots, x_k(y)\}$ (*counting multiplicities*) *by* $\mathrm{Sp}\psi_y$. *We call* $\mathrm{Sp}\psi_y$ *the spectrum of ψ at the point y.*

2.1 For any $f \in I_k \subset M_k(C[0,1]) = C([0,1], M_k(\mathbb{C}))$ as in [13, 3.2], let function $\underline{f} : [0,1] \to \mathbb{C} \sqcup M_k(\mathbb{C})$ (disjoint union) be defined by

$$\underline{f}(t) = \begin{cases} \lambda, & \text{if } t = 0 \text{ and } f(0) = \lambda \mathbf{1}_k, \\ \mu, & \text{if } t = 1 \text{ and } f(1) = \mu \mathbf{1}_k, \\ f(t), & \text{if } 0 < t < 1. \end{cases}$$

That is, $\underline{f}(t)$ is the value of irreducible representation of f corresponding to the point t. Similarly, for $f \in M_l(I_k)$, we can define $\underline{f} : [0,1] \to M_l(\mathbb{C}) \sqcup M_{lk}(\mathbb{C})$ by

$$\underline{f}(t) = \begin{cases} a, & \text{if } t = 0 \text{ and } f(0) = a \otimes \mathbf{1}_k, \\ b, & \text{if } t = 1 \text{ and } f(1) = b \otimes \mathbf{1}_k, \\ f(t), & \text{if } 0 < t < 1. \end{cases}$$

2.2 Suppose that $\phi : I_k \to PM_n(C(Y))P$ is a unital homomorphism. Let $r = \mathrm{rank}(P)$. For each $y \in Y$, there are $t_1, t_2, \cdots, t_m \in [0,1]$ and a unitary $u \in M_n(\mathbb{C})$ such that

$$P(y) = u \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P)} & 0 \\ 0 & 0 \end{pmatrix} u^*$$

and

$$\phi(f)(y) = u \begin{pmatrix} \underline{f}(t_1) & & & & \\ & \underline{f}(t_2) & & & \\ & & \ddots & & \\ & & & \underline{f}(t_m) & \\ & & & & \mathbf{0}_{n-r} \end{pmatrix} u^* \in P(y)M_n(\mathbb{C})P(y) \qquad (2.1)$$

for all $f \in I_k$.

2.3 Let ϕ be the homomorphism defined by (2.1) above with $t_1, t_2, \cdots, t_m$ as appeared in the diagonal of the matrix. We define the set $\mathrm{Sp}\phi_y$ to be the points $t_1, t_2, \cdots, t_m$ with possible fraction multiplicity. If $t_i = 0$ or 1, we assume that the multiplicity of t_i is $\frac{1}{k}$; if $0 < t_i < 1$, we assume that the multiplicity of t_i is 1. For example if we assume

$$t_1 = t_2 = t_3 = 0 < t_4 \leq t_5 \leq \cdots \leq t_{m-2} < 1 = t_{m-1} = t_m,$$

then $\mathrm{Sp}\phi_y = \{0^{\sim\frac{1}{k}}, 0^{\sim\frac{1}{k}}, 0^{\sim\frac{1}{k}}, t_4, t_5, \cdots, t_{m-2}, 1^{\sim\frac{1}{k}}, 1^{\sim\frac{1}{k}}\}$, which can also be written as

$$\mathrm{Sp}\phi_y = \{0^{\sim\frac{3}{k}}, t_4, t_5, \cdots, t_{m-2}, 1^{\sim\frac{2}{k}}\}.$$

Here we emphasize that, for $t \in (0,1)$, we do not allow the multiplicity of t to be non-integral. Also for 0 or 1, the multiplicity must be multiple of $\frac{1}{k}$ (other fraction numbers are not allowed).

Let $\psi : C[0,1] \to PM_n(C(Y))P$ be defined by the following composition:

$$\psi : \ C[0,1] \hookrightarrow I_k \xrightarrow{\phi} PM_n(C(X))P,$$

where the first map is the canonical inclusion. Then we have $\mathrm{Sp}\psi_y = \{\mathrm{Sp}\phi_y\}^{\sim k}$—that is, for each element $t \in (0,1)$, its multiplicity in $\mathrm{Sp}\psi_y$ is exactly k times of the multiplicity in ϕ_y.

2.4 (a) We use $\sharp(.)$ to denote the cardinal number of a set. Very often, the sets under consideration will be sets with multiplicity, in which case we will also count multiplicity when we use the notation $\sharp$. The set may also contain fractional point. For example,

$$\sharp\{0_1, 1_2, 0, 0, 1\} = 5.$$

(b) We use $a^{\sim k}$ to denote $\underbrace{a, a, \cdots, a}_{k}$. For example $\{a^{\sim 3}, b^{\sim 2}\} = \{a, a, a, b, b\}$.

(c) For any metric space X, any $x_0 \in X$ and $c > 0$, let $B_c(x_0) \triangleq \{x \in X \mid d(x, x_0) < c\}$, the open ball with radius c and center x_0.

(d) Suppose that A is a C^*-algebra, $B \subset A$ is a subset (often a subalgebra), $F \subset A$ is a finite subset and $\varepsilon > 0$. If for each element $f \in F$, there is an element $g \in B$ such that $\|f - g\| < \varepsilon$, then we will say that F is approximately contained in B within ε, and denote this by $F \subset_\varepsilon B$.

(e) Let X be a compact metric space. For any $\delta > 0$, a finite set $\{x_1, x_2, \cdots, x_n\}$ is said to be δ-dense in X if for any $x \in X$, there is $x_i \in \{x_1, x_2, \cdots, x_n\}$ such that $\mathrm{dist}(x, x_i) < \delta$.

(f) We will use $\bullet$ or $\bullet\bullet$ to denote any possible positive integers.

(g) For any two projections $p, q \in A$, by $[p] \leq [q]$ we mean that p is unitarily equivalent to a sub-projection of q. And we use $p \sim q$ to denote that p is unitarily equivalent to q.

2.5 Let $A = M_l(I_k)$. Then every point $t \in (0, 1)$ corresponds to an irreducible representation π_t, defined by $\pi_t(f) = f(t)$. The representations π_0 and π_1 defined by

$$\pi_0 = f(0), \quad \pi_1 = f(1)$$

are no longer irreducible. We use $\underline{0}$ and $\underline{1}$ to denote the corresponding points for the irreducible representations. That is,

$$\pi_{\underline{0}}(f) = \underline{f}(0), \quad \pi_{\underline{1}}(f) = \underline{f}(1).$$

Or we can also write $\underline{f}(0) \triangleq f(\underline{0})$ and $\underline{f}(1) \triangleq f(\underline{1})$. Then (2.1) could be written as

$$\phi(f)(y) = u \begin{pmatrix} f(t_1) & & & & \\ & f(t_2) & & & \\ & & \ddots & & \\ & & & f(t_m) & \\ & & & & \mathbf{0}_{n-r} \end{pmatrix} u^*,$$

where some of t_i may be $\underline{0}$ or $\underline{1}$. In this notation, up to unitary equivalence, $f(0)$ is equal to $\mathrm{diag}(\underbrace{f(\underline{0}), f(\underline{0}), \cdots, f(\underline{0})}_{k})$.

Under this notation, we can also write $0^{\sim\frac{1}{k}}$ as $\underline{0}$. Then the example of $\mathrm{Sp}\phi_y$ in 2.3 can be written as

$$\mathrm{Sp}\phi_y = \{0^{\sim\frac{1}{k}}, 0^{\sim\frac{1}{k}}, 0^{\sim\frac{1}{k}}, t_4, t_5, \cdots, t_{m-2}, 1^{\sim\frac{1}{k}}, 1^{\sim\frac{1}{k}}\} = \{\underline{0}, \underline{0}, \underline{0}, t_4, t_5, \cdots, t_{m-2}, \underline{1}, \underline{1}\}.$$

2.6 For a homomorphism $\phi : A \to M_n(I_k)$, where $A = I_k$ or $C(X)$, and for any $t \in [0, 1]$, define $\mathrm{Sp}\phi_t = \mathrm{Sp}\psi_t$, where ψ is defined by the composition

$$\psi : A \xrightarrow{\phi} M_n(I_l) \to M_{nl}(C[0, 1]).$$

Also $\mathrm{Sp}\phi_{\underline{0}} = \mathrm{Sp}(\pi_{\underline{0}} \circ \phi)$. Hence, $\mathrm{Sp}\phi_0 = \{\mathrm{Sp}\phi_{\underline{0}}\}^{\sim k}$.

2.7 Let $\phi : M_n(A) \to B$ be a unital homomorphism. It is well known (see [8, 1.34, 2.6]) that there is an identification of B with $(\phi(e_{11})B\phi(e_{11})) \otimes M_n(\mathbb{C})$ such that

$$\phi = \phi_1 \otimes \mathrm{id}_n : M_n(A) = A \otimes M_n(\mathbb{C}) \to (\phi(e_{11})B\phi(e_{11})) \otimes M_n(\mathbb{C}) = B,$$

where e_{11} is the matrix unit of upper left corner of $M_n(A)$ and $\phi_1 = \phi|_{e_{11}M_n(A)e_{11}} : A \to \phi(e_{11})B\phi(e_{11})$.

If we further assume that $A = I_k$ or $C(X)$ (with X being a connected CW complex) and B is either $QM_n(C(Y))Q$ or $M_l(I_{k_1})$, then for any $y \in \mathrm{Sp}B$, define $\mathrm{Sp}\phi_y \triangleq \mathrm{Sp}(\phi_1)_y$. Here, we use the standard notation that if $B = PM_m(C(Y))P$ then $\mathrm{Sp}B = Y$; and if $B = M_l(I_k)$, then $\mathrm{Sp}(B) = [0,1]$.

2.8 Let A and B be either of form $PM_n(C(X))P$ (with X path connected) or of form $M_l(I_k)$. Let $\phi : A \to B$ be a unital homomorphism. We say that ϕ has property $\mathrm{sdp}(\eta, \delta)$ (spectral distribution property with respect to η and δ) if for any η-ball

$$B_\eta(x) = \{x' \in X \mid \mathrm{dist}(x', x) < \eta\} \subset X (= \mathrm{Sp}(A))$$

and any point $y \in \mathrm{Sp}(B)$,

$$\sharp(\mathrm{Sp}\phi_y \cap B_\eta(x)) \geq \delta \cdot \sharp\mathrm{Sp}\phi_y,$$

counting multiplicity. If ϕ is not unital, we say that ϕ has $\mathrm{sdp}(\eta, \delta)$ if the corresponding unital homomorphism $\phi : A \to \phi(\mathbf{1}_A)B\phi(\mathbf{1}_A)$ has property $\mathrm{sdp}(\eta, \delta)$.

2.9 Set $P^nX = \underbrace{X \times X \times \cdots \times X}_{n} / \sim$, where the equivalence relation $\sim$ is defined by

$$(x_1, x_2, \cdots, x_n) \sim (x'_1, x'_2, \cdots, x'_n)$$

if there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $x_i = x'_{\sigma(i)}$ for each $1 \leq i \leq n$. A metric d on X can be extended to a metric on P^nX by

$$d([x_1, x_2, \cdots, x_n], [x'_1, x'_2, \cdots, x'_n]) = \min_{\sigma} \max_{1 \leq i \leq n} d(x_i, x'_{\sigma(i)}),$$

where σ is taken from the set of all permutations, and $[x_1, x_2, \cdots, x_n]$ denotes the equivalence class of $(x_1, x_2, \cdots, x_k)$ in P^kX.

2.10 Let X be a metric space with metric d. Two k-tuple of (possible repeating) points $\{x_1, x_2, \cdots, x_n\} \subset X$ and $\{x'_1, x'_2, \cdots, x'_n\} \subset X$ are said to be paired within η if there is a permutation σ such that

$$d(x_i, x'_{\sigma(i)}) < \eta, \quad i = 1, 2, \cdots, k.$$

This is equivalent to the following statement. If one regards $[x_1, x_2, \cdots, x_n]$ and $[x'_1, x'_2, \cdots, x'_n]$ as points in P^nX, then

$$d([x_1, x_2, \cdots, x_n], [x'_1, x'_2, \cdots, x'_n]) < \eta.$$

2.11 For $X = [0,1]$, let $P^{(n,k)}X$, where $n, k \in \mathbb{Z}_+ \backslash \{0\}$, denote the set of $\frac{n}{k}$ elements from X, in which only 0 or 1 may appear fractional times. That is, each element in X is of the form

$$\{0^{\sim \frac{n_0}{k}}, t_1, t_2, \cdots, t_m, 1^{\sim \frac{n_1}{k}}\} \tag{2.2}$$

with $0 < t_1 \leq t_2 \leq \cdots \leq t_m < 1$ and $\frac{n_0}{k} + m + \frac{n_1}{k} = \frac{n}{k}$.

An element in $P^{(n,k)}X$ can always be written as

$$\{0^{\sim \frac{k_0}{k}}, t_1, t_2, \cdots, t_i, 1^{\sim \frac{k_1}{k}}\}, \tag{2.3}$$

where $0 \le k_0 < k$, $0 \le k_1 < k$, $0 \le t_1 \le t_2 \le \cdots \le t_i \le 1$ and $\frac{k_0}{k} + i + \frac{k_1}{k} = \frac{n}{k}$. (Here t_i could be 0 or 1.) In the above Representations (2.2)–(2.3), we know that

$$k_0 \equiv n_0 \pmod{k}, \quad k_1 \equiv n_1 \pmod{k}.$$

Let

$$y = [0^{\sim\frac{k_0}{k}}, t_1, t_2, \cdots, t_i, 1^{\sim\frac{k_1}{k}}] \in P^{(n,k)}X,$$
$$y' = [0^{\sim\frac{k_0'}{k}}, t_1', t_2', \cdots, t_i', 1^{\sim\frac{k_1'}{k}}] \in P^{(n,k)}X$$

with $k_0, k_1, k_0', k_1' \in \{0, 1, \cdots, k-1\}$.

We define $\mathrm{dist}(y, y')$ as the following: If $k_0 \ne k_0'$ or $k_1 \ne k_1'$, then $\mathrm{dist}(y, y') = 1$; if $k_0 = k_0'$ and $k_1 = k_1'$ (consequently $i = i'$), then

$$\mathrm{dist}(y, y') = \max_{1\le j\le i} |t_j - t_j'|$$

as we order the $\{t_j\}$ and $\{t_j'\}$ as $t_1 \le t_2 \le \cdots \le t_i$ and $t_1' \le t_2' \le \cdots \le t_i'$, respectively.

Note that $P^{(n,1)}X = P^nX$ with the same metric. Let $\phi, \varphi : I_k \to M_n(\mathbb{C})$ be two unital homomorphisms. Then $\mathrm{Sp}\phi$ and $\mathrm{Sp}\psi$ define two elements in $P^{(n,k)}[0,1]$. We say that $\mathrm{Sp}\phi$ and $\mathrm{Sp}\psi$ can be paired within η, if $\mathrm{dist}(\mathrm{Sp}\phi, \mathrm{Sp}\psi) < \eta$.

Note that if $\mathrm{dist}(\mathrm{Sp}\phi, \mathrm{Sp}\psi) < 1$, then $KK(\phi) = KK(\psi)$.

2.12 Let $A = PM_k(C(X))P$, or $M_l(I_k)$ and $X_1 \subset \mathrm{Sp}(A)$ be a closed subset—that is, X_1 is a closed subset of X or of $[0, 1]$. We define $A|_{X_1}$ to be the quotient algebra A/I, where $I = \{f \in A, f|_{X_1} = 0\}$. Evidently $\mathrm{Sp}(A|_{X_1}) = X_1$.

If $B = QM_k(C(Y))Q$, $\phi : A \to B$ is a homomorphism, and $Y_1 \subset \mathrm{Sp}(B) (= Y$ or $[0, 1])$ is a closed subset, then we use $\phi|_{Y_1}$ to denote the composition:

$$\phi|_{Y_1} : A \xrightarrow{\phi} B \to B|_{Y_1}.$$

If $\mathrm{Sp}(\phi|_{Y_1}) \subset X_1 \cup X_2 \cup \cdots \cup X_k$, where $X_1, X_2, \cdots, X_k$ are mutually disjoint closed subsets of X, then the homomorphism $\phi|_{Y_1}$ factors as

$$A \to A|_{X_1\cup X_2\cup\cdots\cup X_n} = \bigoplus_{i=1}^{n} A|_{X_i} \to B|_{Y_1}.$$

We use $\phi|_{Y_1}^{X_i}$ to denote the part of $\phi|_{Y_1}$ corresponding to the map $A|_{X_i} \to B|_{Y_1}$. Hence $\phi|_{Y_1} = \bigoplus_i \phi|_{Y_1}^{X_i}$.

3 Decomposition Theorem I

In this section, we will prove the following theorem.

Theorem 3.1 *Let $F \subset I_k$ be a finite set, $\varepsilon > 0$. There is an $\eta > 0$, satisfying that if*

$$\phi : I_k \to PM_\bullet(C(X))P \quad (\dim(X) \le 2)$$

is a unital homomorphism such that for any $x \in X$,

$$\sharp\Big(\mathrm{Sp}\phi_x' \cap \Big[0, \frac{\eta}{4}\Big]\Big) \ge k, \quad \sharp\Big(\mathrm{Sp}\phi_x' \cap \Big[1 - \frac{\eta}{4}, 1\Big]\Big) \ge k,$$

where

$$\phi' : C[0,1] \xrightarrow{\imath} I_k \xrightarrow{\phi} PM(C(X))P,$$

then there are three mutually orthogonal projections

$$Q_0,\ Q_1,\ P_1\ \in PM_\bullet(C(X))P$$

with

$$Q_0 + Q_1 + P_1 = P$$

and a unital homomorphism

$$\psi_1 :\ M_k(C[0,1]) \to P_1 M_\bullet(C(X))P_1,$$

such that

(1) *write* $\psi(f) = f(\underline{0})Q_0 + f(\underline{1})Q_1 + (\psi_1 \circ \imath)(f)$, *then*

$$\| \phi(f) - \psi(f) \| < \varepsilon$$

for all $f \in F \subset I_k \subset M_k(C[0,1])$, *and*

(2) $\mathrm{rank}(Q_0) \leq k$ *and* $\mathrm{rank}(Q_1) \leq k$.

We divide the proof into several steps.

3.1 Let $\eta > 0$ (and $\eta < 1$) be such that if $| t - t' | < \eta$, then $\| f(t) - f(t') \| < \frac{\varepsilon}{6}$ for all $f \in F$. We will prove that this η is as desired. Let a unital homomorphism $\phi : I_k \to PM_\bullet C(X)P$ satisfy that $\sharp\big(\mathrm{Sp}\phi_x \cap \big[0, \frac{\eta}{4}\big]\big) \geq k$ and $\sharp\big(\mathrm{Sp}\phi_x \cap \big[1 - \frac{\eta}{4}, 1\big]\big) \geq k$ for each $x \in X$. We will prove such ϕ has the decomposition as desired.

3.2 Let $\mathrm{rank}(P) = n$, and let $e_{i,j} \in M_n(\mathbb{C})$ be the matrix units. For any closed set $Y \subset [0,1]$, define $h_Y \in C[0,1] \subset I_k$ (considering C[0,1] as in the center of I_k) as

$$h_Y(t) = \begin{cases} 1, & \text{if } t \in Y, \\ 1 - \dfrac{12n}{\eta}\mathrm{dist}(t,x), & \text{if } \mathrm{dist}(t,x) \leq \dfrac{\eta}{12n}, \\ 0, & \text{if } \mathrm{dist}(t,x) \geq \dfrac{\eta}{12n}. \end{cases}$$

Define $H' = \{h_Y \mid Y \text{ is closed}\} \cup \{h_Y e_{ij} \mid Y \subset [\frac{\eta}{12n}, 1 - \frac{\eta}{12n}] \text{ is closed}\}$. Note that for a closed set $Y \subset [\frac{\eta}{12n}, 1 - \frac{\eta}{12n}]$, $h_Y(0) = h_Y(1) = 0$, and therefore $h_Y e_{ij} \in I_k$. Note also that the family H' is equally continuous. There is a finite set $H \subset H'$ satisfying that for any $h' \in H'$, $\exists h \in H$ such that

$$\| h - h' \| \leq \frac{\varepsilon}{12(n+1)^2}.$$

For finite set $H \cup F$, $\varepsilon > 0$ and $\phi : I_k \to PM_\bullet(C(X))P$, there is a $\tau > 0$ such that the following are true:

(a) For $x, x' \in X$ with $\mathrm{dist}(x', x) < \tau$, $\mathrm{Sp}\phi|_x$ and $\mathrm{Sp}\phi|_{x'}$ can be paired within $\frac{\eta}{24n^2}$. This is equivalent to the condition that $\mathrm{Sp}\phi'|_x$ can be paired with $\mathrm{Sp}\phi'|_{x'}$ within $\frac{\eta}{24n^2}$ (since $KK(\phi|_x) = KK(\phi|_{x'})$), where $\phi' = \phi \circ \imath$ is as the above.

(b) For $x, x' \in X$ with $\mathrm{dist}(x', x) < \tau$,

$$\|\phi(h)(x) - \phi(h)(x')\| \leq \frac{\varepsilon}{12(n+1)^2}\ ,$$

regarding $\phi(h)(x) \in P(x)M_\bullet(\mathbb{C})P(x) \subset M_\bullet(\mathbb{C})$ and $\phi(h)(x') \in P(x')M_\bullet(\mathbb{C})P(x') \subset M_\bullet(\mathbb{C})$. In particular, $\|P(x) - P(x')\| < \frac{\varepsilon}{12(n+1)^2}$ since $1 \in H$.

3.3 Choose any simplicial decomposition on X such that for any simplex $\Delta \subset X$, the set

$$\mathrm{Star}(\triangle) = \cup\{\overset{\circ}{\Delta'} \mid \Delta' \text{ is a simplex of } X \text{ with } \Delta' \cap \Delta \neq \emptyset\}$$

has diameter at most $\frac{\tau}{2}$, where $\overset{\circ}{\Delta'}$ is the interior of the simplex Δ'.

3.4 We will construct the homomorphism $\psi : I_k \to PM_\bullet(C(X))P$ which is of the form

$$\psi(f) = \underline{f}(0)Q_0 + \underline{f}(1)Q_1 + \psi_1(f)$$

as described in the theorem. Our construction will be carried out simplex by simplex.

First, define the restriction of map ψ to $PM_\bullet(C(X))P|_v = P(v)M_\bullet(\mathbb{C})P(v)$ for each vertex $v \in X$. The homomorphism is denoted by

$$\psi|_{\{v\}} : I_k \to P(v)M_\bullet(\mathbb{C})P(v).$$

(Here and below, we refer the reader to 2.12 for the notation $\psi|_{X_1}$ for a subset $X_1 \subset X$.)

Next, we will define, for each 1-simplex $[a,b] \subset X$, the homomorphisms

$$\psi|_{[a,b]} : I_k \to P|_{[a,b]}M_\bullet(C([a,b]))P|_{[a,b]}$$

which will give the same maps as the previously defined maps $\psi|_{\{a\}}$ and $\psi|_{\{b\}}$ on the boundary $\{a,b\}$. Finally, we will define, for each 2-simplex $\Delta \subset X$, the homomorphism

$$\psi|_\Delta : I_k \to P|_\Delta M_\bullet(C(\Delta))P|_\Delta$$

such that $\psi|_{\partial\triangle}$ should be the same as what previously defined.

3.5 For each simplex Δ of any dimension, let C_Δ denote the center of the simplex. That is, if Δ is a vertex v, then $C_\Delta = v$; if Δ is a 1-simplex identified with [a,b], then $C_\Delta = \frac{a+b}{2}$; and if Δ is a 2-simplex identified with a triangle in $\mathbb{R}^2$ with vertices $\{a,b,c\} \subseteq \mathbb{R}^2$, then $C_\Delta = \frac{a+b+c}{3} \in \mathbb{R}^2$ which is barycenter of Δ.

3.6 According to each simplex Δ (of possible dimensions 0, 1 or 2), we will divide the set $\mathrm{Sp}\phi'|_\Delta \subset [0,1]$ into pieces, where $\phi' : C[0,1] \hookrightarrow I_k \xrightarrow{\phi} PM_\bullet C(X)P$. (Recall $\mathrm{Sp}\phi'|_x = \{\mathrm{Sp}\phi|_x\}^{\sim k}$, and $\mathrm{Sp}\phi'|_x$ has no fractional multiplicity.) So for each $x \in X$,

$$\mathrm{Sp}\phi'|_x = n = \mathrm{rank}(P) \quad \text{(counting multiplicity)}.$$

If we order $\mathrm{Sp}\phi'|_x$ as

$$0 \le \lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x) \le 1,$$

then all functions λ_i are continuous functions. By path connectedness of simplex Δ, the set $\mathrm{Sp}\phi|_\Delta$ can be written as

$$\mathrm{Sp}\phi|_\Delta = [a_0,b_0] \cup [a_1,b_1] \cup \cdots \cup [a_{k'-1},b_{k'-1}] \cup [a_{k'},b_{k'}]$$

with

$$0 \le a_0 \le b_0 < a_1 \le b_1 < a_2 \le b_2 < \cdots < a_{k'-1} \le b_{k'-1} < a_{k'} \le b_{k'} \le 1.$$

(Note that, if $a_i = b_i$, then $[a_i, b_i] = \{a_i\}$ is a degenerated interval.)

We will group the above intervals into groups $T_0 \cup T_1 \cup \cdots \cup T_{\rm last}$ such that $\mathrm{Sp}\phi|_\Delta = \cup T_j$, with the condition that for any $\lambda \in T_j, \mu \in T_{j+1}$, we have $\lambda < \mu$, according to the following procedure:

(i) $\mathrm{Sp}\phi|_\Delta \cap \big[0, \frac{\eta}{4} + \frac{\eta}{12n}\big] \subset T_0$, that is, all the above intervals $[a_i, b_i]$ with $a_i \le \frac{\eta}{4} + \frac{\eta}{12n}$ should be in the group T_0; and $\mathrm{Sp}\phi|_\Delta \cap \big[1 - \big(\frac{\eta}{4} + \frac{\eta}{12n}\big), 1\big] \subset T_{\rm last}$, that is all $[a_i, b_i]$ with $b_i \ge 1 - \big(\frac{\eta}{4} + \frac{\eta}{12n}\big)$ will be grouped into the last group $T_{\rm last}$.

(ii) If $a_i - b_{i-1} \le \frac{\eta}{12n}$, then $[a_{i-1}, b_{i-1}]$ and $[a_i, b_i]$ are in the same group, say T_j.

(iii) If $a_i - b_{i-1} > \frac{\eta}{12n}$, $a_i > \frac{\eta}{4} + \frac{\eta}{12n}$ and $b_{i-1} < 1 - \big(\frac{\eta}{4} + \frac{\eta}{12n}\big)$, then $[a_{i-1}, b_{i-1}]$ and $[a_i, b_i]$ are in different groups, say T_j and T_{j+1}.

Denote $T_{\rm last}$ by T_{l_Δ} (i.e., $l_\Delta = $ last) — if there is no confusion, we call T_{l_Δ} by T_l. Let $t_0 = 0$, $s_0 = \max\{\frac{\eta}{4}, \max T_0\}$, $t_l = \min\{1 - \frac{\eta}{4}, \min T_l\}$, $s_l = 1$; and for $1 < i < l$, let $t_i = \min T_i$, and $s_i = \max T_i$. Then $T_i \subset [t_i, s_i]$. With the above notation, we have the following lemma.

Lemma 3.1 *With the above notation, we have the following:*

(a) $\mathrm{Length}[t_0, s_0] \le \frac{\eta}{4} + \frac{\eta}{6}$;

(b) $\mathrm{Length}[t_l, s_l] \le \frac{\eta}{4} + \frac{\eta}{6}$;

(c) $\mathrm{Length}[t_i, s_i] \le \frac{\eta}{6}$ *for* $i \in \{1, 2, \cdots, l-1\}$;

(d) $t_{i+1} - s_i > \frac{\eta}{12n}$ *for* $i \in \{0, 1, 2, \cdots, l-1\}$.

Proof From (ii) of 3.6, we know that $\min T_{i+1} - \max T_i > \frac{\eta}{12n}$; and from (i), we know that $\min T_1 > \frac{\eta}{4} + \frac{\eta}{12n}$ and $\max T_{l-1} < 1 - \big(\frac{\eta}{4} + \frac{\eta}{12n}\big)$. Hence (d) holds.

The following fact is well known:

For any two sequences $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 1$ and $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n \le 1$, $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$ can be paired within σ if and only if $|\lambda_i - \mu_i| < \sigma$ for all $i \in \{1, 2, \cdots, n\}$.

Note that Δ is path connected and $\mathrm{Sp}\phi|_\Delta = \bigcup_{i=1}^{k'} [a_i, b_i]$ with $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ if $i \ne j$. We conclude that for any $z, z' \in \Delta$ and i,

$$\sharp(\mathrm{Sp}\phi|_z \cap [a_i, b_i]) = \sharp(\mathrm{Sp}\phi|_{z'} \cap [a_i, b_i])$$

counting multiplicity. In our construction, we know that $\mathrm{Sp}\phi|_z$ and $\mathrm{Sp}\phi|_{z'}$ can be paired within $\frac{\eta}{24n^2}$, using the above mentioned fact. We know also that

$$\mathrm{Sp}\phi|_z \cap [a_i, b_i], \quad \mathrm{Sp}\phi|_{z'} \cap [a_i, b_i]$$

can be paired within $\frac{\eta}{24n^2}$. Consequently

$$[a_i, b_i] \subset_{\frac{\eta}{24n^2}} [a_i, b_i] \cap \mathrm{Sp}\phi|_{C_\Delta},$$

where C_Δ is the center of simplex Δ. Note that $\mathrm{Sp}\phi|_{C_\Delta} \cap [a_i, b_i]$ is a finite set with at most n points in $[0, 1]$ and $\frac{\eta}{24n^2}$-neighborhood of each point is a closed interval of length at most $\big(\frac{\eta}{24n^2}\big) \cdot 2 = \frac{\eta}{12n^2}$. Hence we have

$$\mathrm{length}[a_i, b_i] \le \Big(\frac{\eta}{12n^2}\Big) \cdot n = \frac{\eta}{12n}.$$

Furthermore, each T_j contains at most n intervals $[a_i, b_i]$. And for each consecutive pair of intervals in T_j $(0 < j < l)$, we have

$$[a_i, b_i] \cup [a_{i+1}, b_{i+1}] \subset \Big(\frac{\eta}{4} + \frac{\eta}{12n}, 1 - \Big(\frac{\eta}{4} + \frac{\eta}{12n}\Big)\Big)$$

and the distance between them $a_{i+1} - b_i \leq \frac{\eta}{12n}$. That is, the gap between them is at most $\frac{\eta}{12n}$. Hence for each $i \in \{1, 2, \cdots, l-1\}$, the length of $[t_i, s_i]$ is at most

$$n \cdot \frac{\eta}{12n} + (n-1) \cdot \frac{\eta}{12n} < \frac{\eta}{6}$$

(at most n possible intervals and $n-1$ gaps).

Also,

$$\text{length}[t_0, s_0] < \frac{\eta}{4} + \frac{\eta}{6}$$

and

$$\text{length}[t_l, s_l] < \frac{\eta}{4} + \frac{\eta}{6}.$$

3.7 For each simplex Δ with face $\Delta' \subset \Delta$, we use $T_i(\Delta)$ and $T_j(\Delta')$ to denote the sets $[t_i(\Delta), s_i(\Delta)]$ or $[t_j(\Delta'), s_j(\Delta')]$ as in 3.6, corresponding to Δ and Δ'. Then evidently, the decomposition

$$\text{Sp}\phi|_{\Delta'} = \bigcup_j (T_j(\Delta') \cap \text{Sp}\phi|_{\Delta'})$$

is a refinement of the decomposition $\text{Sp}\phi|_\Delta = \cup(T_i(\Delta) \cap \text{Sp}\phi|_\Delta)$— that is, if two elements λ, $\mu \in \text{Sp}\phi|_{\Delta'}$ are in the set $T_j(\Delta')$ for a same index j, then they are in the set $T_i(\Delta)$ for a same index i.

3.8 For each simplex Δ, consider the homomorphism

$$\phi : I_k \to PM_\bullet(C(\Delta))P = A|_\Delta.$$

Since $\text{Sp}\phi|_\Delta \subset \bigcup_{j=0}^{l} T_j(\Delta) = \bigcup_{j=1}^{l} [t_j, s_j]$, ϕ factors through as

$$I_k \to \bigoplus_{j=0}^{l} I_k|_{[t_j, s_j]} \xrightarrow{\oplus \phi_j} PM_\bullet(C(\Delta))P.$$

Let $P_j(x) = \phi_j(\mathbf{1}_k|_{[t_j, s_j]})(x)$ for each $x \in \Delta$. Then $P_j(x)$ are mutually orthogonal projections satisfying

$$\sum_{j=0}^{l} P_j(x) = P(x).$$

By the assumption of Theorem 3.1, we have $\text{rank}(P_0) \geq k$ and $\text{rank}(P_l) \geq k$.

3.9 Now we define $\psi : I_k \to A|_\Delta$ simplex by simplex, starting with vertices — the zero dimensional simplices.

Let $v \in X$ be a vertex. As in 3.6, we write

$$\text{Sp}\phi|_{\{v\}} = \bigcup_{i=0}^{l} [t_i, s_i] \cap \text{Sp}\phi|_{\{v\}},$$

where $0 = t_0 < s_0 < t_1 \le s_1 < \cdots < t_{l-1} \le s_{l-1} < t_l < s_l = 1$, with

$$\begin{aligned} &\left[0, \frac{\eta}{4}\right] \subset [t_0, s_0] \subset \left[0, \frac{\eta}{2}\right], \\ &\left[1 - \frac{\eta}{4}, 1\right] \subset [t_l, s_l] \subset \left[1 - \frac{\eta}{2}, 1\right], \\ &0 \le s_i - t_i < \frac{\eta}{6} \quad \text{for each } i \in \{1, 2, \cdots, l-1\}, \\ &t_{i+1} - s_i > \frac{\eta}{12n} \quad \text{for each } i \in \{0, 1, 2, \cdots, l-1\}. \end{aligned}$$

Recall that $\phi|_{\{v\}} : I_k \to P(v)M_\bullet(\mathbb{C})P(v)$ (as in 3.8) can be written as

$$\phi|_{\{v\}} = \operatorname{diag}(\phi_0, \phi_1, \cdots, \phi_l) : I_k \to \bigoplus_{i=0}^{l} P_i M_\bullet(\mathbb{C}) P_i \subset P(v)M_\bullet(\mathbb{C})P(v),$$

where $\phi_i = \phi|_{\{v\}}^{[t_i,s_i]} : I_k \to I_k|_{[t_i,s_i]} \to P_i M_\bullet(\mathbb{C}) P_i$ and $P(v) = \sum_{i=0}^{l} P_i$. (Here and below, we refer the reader to 2.12 for the notation $\phi|_{X_1}^{Z_j}$ $(X_1 \subset X)$, which makes sense, provided that $\operatorname{Sp}(\phi|_{X_1}) \subset \bigcup_j Z_j$, where $\{Z_j\}$ are mutually disjoint closed subsets of the spectrum of the domain algebra of ϕ.)

From now on, we will use $\operatorname{diag}_{0 \le i \le l}(\phi_i)$ to denote $\operatorname{diag}(\phi_0, \phi_1, \cdots, \phi_l)$.

Define $\psi_i : I_k|_{[t_i,s_i]} \to P_i M_\bullet(\mathbb{C}) P_i$ by

$$\psi_i = \phi_i \quad \text{if } 1 \le i \le l-1.$$

(That is, we do not modify ϕ_i for $1 \le i \le l-1$.) For $i = 0$ (the case $i = l$ is similar) we do the following modification. There is a unitary $u \in M_\bullet(\mathbb{C})$ such that

$$\phi_0(f)(v) = u \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & & \\ & f(\underline{0}) & & \\ & & \ddots & \\ & & & f(\underline{0}) \end{pmatrix}_{j \times j} & & & & & & & \\ & f(\xi_1) & & & & & & \\ & & f(\xi_2) & & & & & \\ & & & \ddots & & & & \\ & & & & f(\xi_{\bullet\bullet}) & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} u^*,$$

where $\xi_i \in (0, s_1]$, $0 < \xi_1 \le \xi_2 \le \cdots \le \xi_{\bullet\bullet} \le s_1$. Or write it as

$$\phi_0(f)(v) = u \operatorname{diag}(f(\underline{0})^{\sim j}, f(\xi_1), f(\xi_2), \cdots, f(\xi_{\bullet\bullet}), 0, \cdots, 0)u^*.$$

If $0 < j \le k$ then we do not do any modification and just let $\psi_0 = \phi_0$. If $j > k$, then write $j = kk' + j'$ with $0 < j' \le k$, choose $\xi' \in (0, \xi_1)$, and define

$$\psi_0(f)(v) = u \operatorname{diag}(f(\underline{0})^{\sim j'}, f(\xi')^{\sim k'}, f(\xi_1), f(\xi_2), \cdots, f(\xi_{\bullet\bullet}), 0, \cdots, 0)u^*.$$

That is, change kk' terms of $f(\underline{0})$ in the diagonal of the definition of ϕ_0 to k' terms of the form $f(\xi')$. If $j=0$, then we change ξ_1 to 0, that is,

$$\psi_0(f)(v) = u\,\mathrm{diag}(f(\underline{0})^{\sim k}, f(\xi_2), \cdots, f(\xi_{\bullet\bullet}), 0, \cdots, 0)u^*.$$

Since $|\xi' - 0| < \frac{\eta}{2}$ and $|\xi_1 - 0| < \frac{\eta}{2}$, we have $\|\phi_0(f) - \psi_0(f)\| < \frac{\varepsilon}{6}$ for all $f \in F$ (see 3.1). We modify ϕ_l in a similar way to define ψ_l. Let

$$\psi|_{\{v\}} = \mathrm{diag}(\psi_0, \psi_1, \cdots, \psi_l) : I_k \to P(v)M_\bullet(\mathbb{C})P(v),$$

where $\psi_i = \psi|_{\{v\}}^{[t_i,s_i]}$. Then $\|\phi(f) - \psi(f)\| < \frac{\varepsilon}{6}$ for all $f \in F$.

Remark 3.1 Let us emphasize that the homomorphisms ψ_i are the same as ϕ_i for $i \in \{1, 2, \cdots, l_{\{v\}} - 1\}$. We modify ϕ_0 and ϕ_l $(l = l_{\{v\}})$ to get ψ_0 and ψ_l.

Also, we have

$$\mathrm{Sp}(\psi_0) \subset [0, s_0], \quad \mathrm{Sp}(\psi_{l_{\{v\}}}) \subset [t_{l_{\{v\}}}, 1].$$

Furthermore, $\psi_i(1) = \phi_i(1)$ for any i, and consequently $\psi(1) = \phi(1)$.

3.10 Now consider 1-simplex $\Delta = [a, b] \subset X$. We need to define $\psi|_\Delta = \psi|_{[a,b]}$ from previously defined $\psi|_{\{a\}}$ and $\psi|_{\{b\}}$. According to 3.6, write $\mathrm{Sp}\phi|_\Delta = \bigcup_{j=1}^{l_\Delta} \mathrm{Sp}\phi|_\Delta \cap T_j(\Delta)$ with $T_0(\Delta) = [0, s_0(\Delta)]$ and $T_{l_\Delta}(\Delta) = [t_{l_\Delta}(\Delta), 1]$. Recall that in the definition of $\psi|_{\{a\}}$, $\psi|_{\{b\}}$, we use the decomposition

$$\phi|_{\{a\}} = \mathrm{diag}_{1\le j\le l_{\{a\}}}(\phi|_{\{a\}}^{T_j(\{a\})})$$

and

$$\phi|_{\{b\}} = \mathrm{diag}_{1\le j\le l_{\{b\}}}(\phi|_{\{b\}}^{T_j(\{b\})}),$$

and only modified $\phi_0 = \phi|_{\{a\}}^{[0,s_0\{a\}]}$ (or $\phi|_{\{b\}}^{[0,s_0\{b\}]}$) and $\phi_{l\{a\}} = \phi|_{\{a\}}^{[t_{l_{\{a\}}}(\{a\}),1]}$ (or $\phi|_{\{b\}}^{[t_{l_{\{b\}}}(\{b\}),1]}$).

For $\Delta = [a, b]$, let us consider the decomposition

$$\phi|_\Delta = \bigoplus_{j=1}^{l_\Delta} \phi|_\Delta^{[t_j(\Delta), s_j(\Delta)]}.$$

From the above, we know that for any $0 < j < l_\Delta$, the definition of $\psi|_{\{a\}}^{[t_j(\Delta),s_j(\Delta)]}$ is the same as $(\phi|_\Delta^{[t_j(\Delta),s_j(\Delta)]})|_{\{a\}}$, since the decomposition

$$\mathrm{Sp}\phi|_{\{a\}} = \bigcup_{j=1}^{l_{\{a\}}} T_j(\{a\}) \cap \mathrm{Sp}\phi|_{\{a\}}$$

is finer than the decomposition

$$\mathrm{Sp}\phi|_{\{a\}} = \bigcup_{j=1}^{l_\Delta} T_j(\Delta) \cap \mathrm{Sp}\phi|_{\{a\}}$$

(see 3.7) and only partial maps involving $[0, s_1\{a\}]$ $(\subset [0, s_1(\Delta)])$ and $[t_{l_{\{a\}}}(\{a\}), 1]$ $(\subset [t_{l_\Delta}(\Delta), 1])$ are modified. The same is true for $\phi|_{\{b\}}$ and $\psi|_{\{b\}}$. Therefore, we can define the partial maps

$$\psi|_\Delta^{[t_j(\Delta),s_j(\Delta)]} = \phi|_\Delta^{[t_j(\Delta),s_j(\Delta)]}$$

for $0 < j < l_\Delta$. The only parts need to be modified are $\phi|_\Delta^{[0,s_0(\Delta)]}$ and $\phi|_\Delta^{[t_l(\Delta),1]}$.

3.11 Now denote $\phi|_\Delta^{[0,s_0(\Delta)]}$ $(\Delta = [a,b])$ by ϕ_0 and $\phi|_\Delta^{[t_l(\Delta),1]}$ by ϕ_l, and $s_0(\Delta)$ by s_0, $t_{l(\Delta)}(\Delta)$ by t_l. Now we have two unital homomorphisms

$$\phi_0 : I_k|_{[0,s_0]} \to P_0 M_\bullet C(\Delta) P_0$$

and

$$\phi_l : I_k|_{[t_l,1]} \to P_l M_\bullet C(\Delta) P_l,$$

where P_0, P_l are defined as in 3.8. We will do the modification of ϕ_0 to get ψ_0 (the one for ϕ_l is completely the same).

We already have the definitions of $\psi_0|_{\{a\}}$ and $\psi_0|_{\{b\}}$. Note that $P_0 \in M_\bullet(C(\Delta))$ can be written as $\phi(h_{[0,s_0]})$, where $h_{[0,s_0]}$ is the test function appeared in 3.2, which is equal to 1 on $[0, s_0]$ and 0 on $\left[s_0 + \frac{\eta}{12n}, 1\right]$. $\big($Note that $\phi(h_{[0,s_0]})$ is a projection since $\mathrm{Sp}\phi \subset [0, s_0] \cup [t_1, 1]$ and $t_1 > s_0 + \frac{\eta}{12n}$.$\big)$ Consequently,

$$\|P_0(x) - P_0(y)\| < \frac{\varepsilon}{12(n+1)^2}$$

for all $x, y \in [a, b] = \Delta$ (see (b) of 3.2).

There exists a unitary $W \in M_\bullet(C(\Delta))$ such that

$$P_0(x) = W(x) \begin{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{\mathrm{rank}(P_0)\times\mathrm{rank}(P_0)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} W^*(x)$$

for all $x \in \Delta$ and $\|W(x) - W(y)\| < \dfrac{\varepsilon}{6(n+1)^2}$.

To define

$$\psi_0 : I_k|_{[0,s_0]} \to P_0 M_\bullet(C(\Delta)) P_0,$$

it suffices to define

$$AdW \circ \psi_0 : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(C(\Delta)),$$

since

$$W^* P_0 W = \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$\sharp(\mathrm{Sp}\widetilde{\psi}_0|_{\{a\}} \cap \{0\}) = \mathrm{rank}(P_0) \pmod{k},$$

where

$$\widetilde{\psi}_0 : C[0, s_0] \hookrightarrow I_k|_{[0,s_0]} \xrightarrow{\psi_0} P_0(\{a\}) M_\bullet.(\mathbb{C}) P_0(\{a\}).$$

(This is true since the multiplicities of all the spectra other than 0 are multiples of k.) Similarly,

$$\sharp(\mathrm{Sp}\widetilde{\psi}_0|_{\{b\}} \cap \{0\}) = \mathrm{rank}(P_0) \pmod{k}.$$

Also, from the definition of ψ on the vertices (namely on $\{a\}$ and $\{b\}$) from 3.9, we know that

$$\sharp(\mathrm{Sp}\widetilde{\psi}_0|_{\{b\}} \cap \{0\}) = \sharp(\mathrm{Sp}\widetilde{\psi}_0|_{\{a\}} \cap \{0\}) \triangleq k' \le k.$$

Lemma 3.2 *Suppose that two unital homomorphisms*

$$\alpha', \alpha'' : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(\mathbb{C})$$

satisfy that

$$0 < \sharp(\mathrm{Sp}\widetilde{\alpha}' \cap \{0\}) = \sharp(\mathrm{Sp}\widetilde{\alpha}'' \cap \{0\}) \le k$$

counting multiplicity, where $\widetilde{\alpha}'$ *(or* $\widetilde{\alpha}''$*) is the composition*

$$C[0,s_0] \hookrightarrow I_k|_{[0,s_0]} \xrightarrow{\alpha'} M_{\mathrm{rank}(P_0)}(\mathbb{C}) \ (\mathit{or}\ C[0,s_0] \hookrightarrow I_k|_{[0,s_0]} \xrightarrow{\alpha''} M_{\mathrm{rank}(P_0)}(\mathbb{C})),$$

then there is a homomorphism

$$\alpha : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(C[a,b]),$$

such that $0 < \sharp(\mathrm{Sp}\widetilde{\alpha}|_t \cap \{0\}) \le k$ *for all* $t \in [a,b]$ *and* $\alpha|_{\{a\}} = \alpha'$, $\alpha|_{\{b\}} = \alpha''$, *where again* $\widetilde{\alpha}$ *is the composition*

$$C[0,s_0] \hookrightarrow I_k|_{[0,s_0]} \xrightarrow{\alpha} M_{\mathrm{rank}(P_0)}(C[a,b]).$$

Proof We can regard $[a,b] = [0,1]$. There are two unitaries $u, v \in M_{\mathrm{rank}(P_0)}(\mathbb{C})$, a number $k' \in \{1,2,\cdots,k\}$, and two finite sequences of numbers:

$$0 < \xi_1 \le \xi_2 \le \cdots \le \xi_\bullet \le s_0,$$
$$0 < \xi' \le \xi_2' \le \cdots \le \xi_\bullet' \le s_0$$

such that

$$\alpha'(f) = u \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & \\ & f(\xi_1) & & \\ & & \ddots & \\ & & & f(\xi_\bullet) \end{pmatrix} u^*$$

and

$$\alpha''(f) = v \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & \\ & f(\xi_1') & & \\ & & \ddots & \\ & & & f(\xi_\bullet') \end{pmatrix} v^*.$$

Let $u(t)$, $0 \le t \le \frac{1}{2}$ be any unitary path with $u(0) = u$, $u\big(\frac{1}{2}\big) = v$. Define α as follows. For $0 \le t \le \frac{1}{2}$,

$$\alpha(f)(t) = u(t) \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & & \\ & f(\xi_1) & & & \\ & & f(\xi_2) & & \\ & & & \ddots & \\ & & & & f(\xi_\bullet) \end{pmatrix} u^*(t);$$

and for $\frac{1}{2} \le t \le 1$,

$$\alpha(f)(t) = v \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k' \times k'} & & & \\ & f((2-2t)\xi_1 + (2t-1)\xi_1') & & \\ & & \ddots & \\ & & & f((2-2t)\xi_\bullet + (2t-1)\xi_\bullet') \end{pmatrix} v^*.$$

Then α is a desired homomorphism.

3.12 Applying the above lemma, we can define

$$\alpha : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(C[a,b])$$

such that

$$\imath \circ \alpha|_{\{a\}} = AdW(a) \circ \psi_0|_{\{a\}}$$

and

$$\imath \circ \alpha|_{\{b\}} = AdW(b) \circ \psi_0|_{\{b\}},$$

where $\imath : M_{\mathrm{rank}(P_0)}(\mathbb{C}) \to M_\bullet(\mathbb{C})$ is defined by

$$\imath(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Define

$$\psi_0 : I_k|_{[0,s_0]} \to P_0 M_\bullet(C(\Delta)) P_0$$

by $\psi_0 = AdW^* \circ (\imath \circ \alpha)$ — that is, for any $t \in [a,b] = \Delta$,

$$\psi_0(f)(t) = W(t) \begin{pmatrix} \alpha(f)(t) & 0 \\ 0 & 0 \end{pmatrix} W^*(t).$$

As mentioned in 3.10, when we modify $\phi|_{[a,b]}$ to obtain $\psi|_{[a,b]}$, we only need to modify $\phi_0 = \phi|_{[a,b]}^{[0,s_0]}$ and $\phi_l = \phi|_{[a,b]}^{[t_l,1]}$. The modifications of ϕ_l to ψ_l are the same as the one from ϕ_0 to ψ_0. Thus we have the definition of $\psi|_{[a,b]} = \mathrm{diag}_{0 \le i \le l}(\psi_i)$.

3.13 Let us estimate the difference of $\phi|_{[a,b]}$ and $\psi|_{[a,b]}$ on the finite set $F \subset I_k$. Note that

$$\phi|_{[a,b]} = \mathrm{diag}_{0 \le i \le l}(\phi_i), \quad \psi|_{[a,b]} = \mathrm{diag}_{0 \le i \le l}(\psi_i)$$

and $\phi_i = \psi_i$ for $0 < i < l$. So we only need to estimate $\|\phi_0(f) - \psi_0(f)\|$ and $\|\phi_l(f) - \psi_l(f)\|$.

Note that ϕ_0 and ψ_0 are from $I_k|_{[0,s_0]}$ to $P_0 M_\bullet(C[a,b]) P_0$, where P_0 is as in 3.11. And both $AdW \circ \phi_0$ and $AdW \circ \psi_0$ can be regarded as $\imath \circ \phi'$ and $\imath \circ \psi'$ for

$$\phi', \psi' : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(C[a,b]),$$

where

$$\imath : M_{\mathrm{rank}(P_0)}(C[a,b]) \to M_\bullet(C[a,b])$$

is given by

$$\imath(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Claim Let $\alpha : I_k|_{[0,s_0]} \to M_{\mathrm{rank}(P_0)}(C[a,b])$ be any unital homomorphism. Then we have

$$\left\| \alpha(f) - \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{\mathrm{rank}(P_0)} \right\| \leq \sup_{0<\xi\leq s_0} \|f(\xi) - f(0)\|.$$

In fact, for each $x \in [a,b]$, there exist $u_x \in U(M_{\mathrm{rank}(P_0)}(\mathbb{C}))$, $k' \in \{1, 2, \cdots, k\}$ and $0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{\bullet\bullet} \leq s_0$ such that

$$\alpha(f)(x) = u_x \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & \\ & f(\xi_1) & & \\ & & \ddots & \\ & & & f(\xi_{\bullet\bullet}) \end{pmatrix} u_x^*.$$

It follows that

$$\begin{aligned} &\|\alpha(f)(x) - f(\underline{0}) \cdot \mathbf{1}_{\mathrm{rank}(P_0)}\| \\ &= \left\| u_x \left[\begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & \\ & f(\xi_1) & & \\ & & \ddots & \\ & & & f(\xi_{\bullet\bullet}) \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k'\times k'} & & & \\ & f(0) & & \\ & & \ddots & \\ & & & f(0) \end{pmatrix} \right] u_x^* \right\| \\ &= \left\| \begin{pmatrix} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}_{k'\times k'} & & & \\ & f(\xi_1) - f(0) & & \\ & & \ddots & \\ & & & f(\xi_{\bullet\bullet}) - f(0) \end{pmatrix} \right\| \\ &\leq \sup_{0\leq\xi\leq s_0} \|f(\xi) - f(0)\| . \end{aligned}$$

Thus, the claim is true.

It follows from the claim that

$$\|\phi(f)(t) - \psi(f)(t)\| \leq 2\max\Big(\sup_{0\leq\xi\leq s_0} \|f(\xi) - f(0)\|, \sup_{t_l\leq\xi\leq 1} \|f(\xi) - f(1)\| \Big) \leq 2 \cdot \frac{\varepsilon}{6}$$

for all $t \in [a,b]$, and $f \in F$, as $|s_0 - 0| < \frac{\eta}{2}$ and $|t_l - 1| < \frac{\eta}{2}$. Hence we have the definition of ψ on the 1-skeleton $X^{(1)} \subset X$ satisfying

$$\|\phi(f)(t) - \psi(f)(t)\| < \frac{\varepsilon}{3}$$

for all $t \in X^{(1)}$ and $f \in F$.

3.14 Now fix a 2-simplex $\Delta \subset X$. We define

$$\psi|_{\Delta} : I_k \to PM_{\bullet}(C(\Delta))P$$

based on the previous definition of

$$\psi|_{\partial\Delta} : I_k \to PM_{\bullet}C(\partial\Delta)P.$$

Again, write

$$\phi|_{\Delta} = \mathrm{diag}_{0 \le i \le l(\Delta)}(\phi_i),$$

where

$$\phi_i = \phi|_{\Delta}^{[t_i(\Delta), s_i(\Delta)]} = I_k|_{[t_i, s_i]} \to P_i M_{\bullet}(C(\Delta))P_i$$

and P_i are projections defined on Δ with

$$\sum_{i=0}^{l(\Delta)} P_i(x) = P(x), \quad \forall x \in \Delta.$$

For each face $\Delta' \subset \partial\Delta$, we know that the decomposition

$$\mathrm{Sp}\phi|_{\Delta'} = \bigcup_{j=0}^{l_{\Delta'}} T_j(\Delta') \cap \mathrm{Sp}\phi|_{\Delta'} = \bigcup_{j=0}^{l_{\Delta'}} [t_j(\Delta'), s_j(\Delta')] \cap \mathrm{Sp}\phi|_{\Delta'}$$

is finer than the decomposition

$$\mathrm{Sp}\phi|_{\Delta'} = \bigcup_{j=0}^{l_{\Delta}} T_j(\Delta) \cap \mathrm{Sp}\phi|_{\Delta'} = \bigcup_{j=0}^{l_{\Delta}} [t_j(\Delta), s_j(\Delta)] \cap \mathrm{Sp}\phi|_{\Delta'}.$$

Consequently,

$$[0, s_0(\Delta')] \subset [0, s_0(\Delta)], \quad [t_{l(\Delta')}, 1] \subset [t_{l(\Delta)}, 1].$$

Note that when we define $\psi|_{\Delta'}$ by modifying $\phi|_{\Delta'}$, we only modify the parts of $\phi|_{\Delta'}^{[0, s_0(\Delta')]}$ and $\phi|_{\Delta'}^{[t_{l(\Delta')}, 1]}$ — that is

$$\phi|_{\Delta'}^{[s_0(\Delta')+\delta, t_{l(\Delta')}(\Delta')-\delta]} = \psi|_{\Delta'}^{[s_0(\Delta')+\delta, t_{l(\Delta')}(\Delta')-\delta]},$$

where $\delta \in (0, \frac{\eta}{12n})$. Hence

$$\phi|_{\Delta'}^{[t_1(\Delta), s_{l(\Delta)-1}(\Delta)]} = \psi|_{\Delta'}^{[t_1(\Delta), s_{l(\Delta)-1}(\Delta)]}$$

since

$$t_1(\Delta) > s_0(\Delta) + \frac{\eta}{12n} \ge s_0(\Delta') + \delta$$

and

$$s_{l(\Delta)-1} < t_{l(\Delta)}(\Delta) - \frac{\eta}{12n} < t_{l(\Delta')}(\Delta') - \delta.$$

Because $\Delta' \subset \partial\Delta$ is an arbitrary face, we have

$$\phi|_{\partial\Delta}^{[t_1(\Delta), s_{l(\Delta)-1}(\Delta)]} = \psi|_{\partial\Delta}^{[t_1(\Delta), s_{l(\Delta)-1}(\Delta)]}.$$

Therefore similar to what we did on 1-simplexes, define

$$\psi|_{\Delta}^{[t_j(\Delta),s_j(\Delta)]} = \phi|_{\Delta}^{[t_j(\Delta),s_j(\Delta)]}$$

for $j \in \{1, 2, \cdots, l(\Delta) - 1\}$. Then we only need to modify $\phi|_{\Delta}^{[0,s_0(\Delta)]} = \phi_0$ and $\phi|_{\Delta}^{[t_{l(\Delta)},1]} = \phi_l$. We will only do it for ϕ_0.

3.15 We have the definition of unital homomorphism

$$\psi_0|_{\partial\Delta} : I_k|_{[0,s_0]} \to P_0 M_{\bullet}(C(\partial\Delta))P_0$$

such that

$$\sharp(\mathrm{Sp}\widetilde{\psi}_0|_x \cap \{0\}) = k' \in \{1, 2, \cdots, k\}$$

for any $x \in \partial\Delta$, where Δ is a 2-simplex and $\widetilde{\psi}_0$ is defined as the composition

$$C[0, s_0] \hookrightarrow I_k|_{(0,s_0]} \xrightarrow{\psi_0} P_0 M_{\bullet}(C(\partial\Delta))P_0.$$

We need to extend it to a homomorphism

$$\psi_0|_{\Delta} : I_k|_{[0,s_0]} \to P_0 M_{\bullet}(C(\Delta))P_0$$

such that $\sharp(\mathrm{Sp}\widetilde{\psi}_0|_{\Delta} \cap \{0\}) = k'$ for all $x \in \Delta$. Once this extension is obtained, as in 3.13, we can use the claim in 3.13 to prove that $\phi|_{\Delta}^{[0,s_0]}$ and $\psi|_{\Delta}^{[0,s_0]}$ are approximately equal within $\frac{\varepsilon}{3}$ for all $f \in F$. (Note that in the argument of 3.13, the estimation is true which do not depend on the choice of the extension. It only uses $|s_0 - 0| < \frac{\eta}{2} < \eta$, and $\|f(t) - f(t')\| < \frac{\varepsilon}{6}$ whenever $|t - t'| < \eta$.)

There is a $W \in U(M_{\bullet}(C(\Delta)))$ such that

$$P_0(x) = W(x) \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix} W^*(x)$$

for all $x \in \Delta$. Again, if we can extend

$$(AdW \circ \psi_0)|_{\partial\Delta} : I_k|_{[0,s_0]} \to \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix} M_{\bullet}(C(\Delta)) \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix}$$

to

$$\alpha|_{\Delta} : I_k|_{[0,s_0]} \to \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix} M_{\bullet}(C(\Delta)) \begin{pmatrix} \mathbf{1}_{\mathrm{rank}(P_0)} & 0 \\ 0 & 0 \end{pmatrix},$$

then we can set $\psi_0|_{\Delta} = AdW^* \circ \alpha|_{\Delta}$ to obtain our extension. But $(AdW \circ \psi_0)|_{\partial\Delta}$ (or $\alpha|_{\Delta}$) should be regarded as a homomorphism from $I_k|_{[0,s_0]}$ to $M_{\mathrm{rank}(P_0)}(C(\partial\Delta))$ (or to $M_{\mathrm{rank}(P_0)}(C(\Delta))$). Hence the construction of $\psi_0|_{\Delta}$ follows from the following lemma.

Lemma 3.3 *Let $\beta : I_k|_{[0,s_0]} \to M_{n'}(C(S^1))$ be a unital homomorphism such that for any $x \in S^1$,*

$$\sharp(\mathrm{Sp}(\beta \circ \imath)_x \cap \{0\}) = k' \in \{1, 2, \cdots, k\}$$

for some fixed k' (not depending on x), where $\imath : C[0, s_0] \to I_k|_{[0,s_0]}$. Then there is a homomorphism

$$\overline{\beta} : I_k|_{[0,s_0]} \to M_{n'}(C(D)),$$

where D is the disk with boundary S^1, such that

$$\sharp \mathrm{Sp}(\overline{\beta} \circ \imath)_x \cap \{0\} = k'$$

for all $x \in D$ and $\pi \circ \overline{\beta} = \beta$, where

$$\pi : M_{n'}(C(D)) \to M_{n'}(C(S^1))$$

is the restriction.

Proof Let $h(t) = t \cdot \mathbf{1}_k$ be the function in the center of $I_k|_{[0,s_0]}$. Then $\beta(h)$ is a self adjoint element in $M_{n'}(C(S^1))$. For each $z \in S^1$, write the eigenvalue of $\beta(h)(z)$ in increasing order

$$0 = \lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_{n'}(z) \leq s_0.$$

Then $\lambda_1, \lambda_2, \cdots, \lambda_{n'}$ are continuous functions from S^1 to $[0, s_0]$. From the assumption, we know that $\lambda_1(z) = \lambda_2(z) = \cdots = \lambda_{k'}(z) = 0$ and for all $j > k'$, $\lambda_j(z) > 0$. (Note that each λ_j $(j > k')$ repeats some multiple of k times.) Consequently, there is $\xi \in (0, s_0]$ such that $\lambda_j(z) \geq \xi$ for all $j > k'$. Hence β factors through as

$$I_k|_{[0,s_0]} \to I_k|_{\{0\}} \oplus I_k|_{[\xi,s_0]} \xrightarrow{\mathrm{diag}(\beta_0,\beta_1)} M_{n'}(C(S^1)),$$

where

$$\beta_0 : I_k|_{\{0\}}(= \mathbb{C}) \to Q_0 M_{n'}(C(S^1))Q_0$$

and

$$\beta_1 : I_k|_{[\xi,s_0]}(= M_k(C[\xi, s_0])) \to Q_1 M_{n'}(C(S^1))Q_1$$

with

$$Q_0 + Q_1 = \mathbf{1}_{n'} \in M_{n'}(C(S^1)).$$

Note that $\mathrm{rank}(Q_0) = k'$, and $\mathrm{rank}(Q_1) = n' - k'$, which is a multiple of k. Write $\mathrm{rank}(Q_1) = n' - k' = kk''$. There is a unitary $u \in M_n(C(S^1))$ such that

$$uQ_0u^* = \begin{pmatrix} \mathbf{1}_{k'} & 0 \\ 0 & 0 \end{pmatrix}, \quad uQ_1u^* = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_{n'-k'} \end{pmatrix}.$$

Hence

$$Adu^* \circ \beta = \mathrm{diag}(\beta_0', \beta_1')$$

with

$$\beta_0' : I_k|_{\{0\}}(= \mathbb{C}) \to M_{k'}(C(S^1)),$$
$$\beta_1' : I_k|_{[\xi,s_0]}(= M_k(C[\xi, s_0])) \to M_{kk''}(C(S^1)).$$

Evidently,

$$\beta_0'(c) = \begin{pmatrix} c & & \\ & \ddots & \\ & & c \end{pmatrix} = c \cdot \mathbf{1}_{k'} \in M_{k'}(C(S^1)), \quad \forall c \in \mathbb{C}.$$

For β_1', there exist $\beta'' : C[\xi, s_0] \to M_{k''}(C(S^1))$ and a unitary $V \in M_{kk''}(C(S^1))$ such that

$$V\beta_1'(f)V^* = \beta'' \otimes \mathrm{id}_k(f), \quad \forall f \in M_k(C[\xi, s_0]).$$

Let

$$W = \begin{pmatrix} \mathbf{1}_{k'} & 0 \\ 0 & V \end{pmatrix} \cdot u.$$

Then

$$(AdW^* \circ \beta)(f) = \begin{pmatrix} f(\underline{0}) & & & \\ & \ddots & & \\ & & f(\underline{0}) & \\ & & & \beta'' \otimes \mathrm{id}_k(f) \end{pmatrix}.$$

Let m be the winding number of the map

$$S^1 \ni z \mapsto \det(W(z)) \in \mathbb{T} \subseteq \mathbb{C}.$$

Then $W \in U(M_{n'}(C(S^1)))$ is homotopic to $W' \in M_{n'}(C(S^1))$ defined by

$$W'(z) = \begin{pmatrix} z^m & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \forall z \in S^1 = \mathbb{T}.$$

Let $\{w_r\}_{\frac{1}{2} \le r \le 1}$ be a unitary path in $M_{n'}(C(S^1))$ with

$$w_{\frac{1}{2}}(z) = W'(z), \quad w_1(z) = W(z), \quad \forall z \in S^1.$$

Evidently the homomorphism

$$\beta'' : C[\xi, s_0] \to M_{k''}(C(S^1))$$

is homotopic to the homomorphism

$$\beta''' : C[\xi, s_0] \to M_{k''}(C(S^1))$$

defined by

$$\beta'''(f)(\mathrm{e}^{2\pi \mathrm{i}\theta}) = f(\xi)\mathbf{1}_{k''}$$

— that is $\beta'''(f)(\mathrm{e}^{\mathrm{i}\theta})$ is the constant matrix $f(\xi)\mathbf{1}_{k''}$ (which does not depend on θ). There is a path $\{\beta_r\}_{0 \le r \le \frac{1}{2}}$ of homomorphisms

$$\beta_r : C([\xi, s_0]) \to M_{k''}(C(S^1))$$

such that $\beta_{\frac{1}{2}} = \beta''$ and $\beta_0 = \beta'''$.

Finally, regard $D = \{r\mathrm{e}^{\mathrm{i}\theta}, 0 \le r \le 1\}$, and define $\overline{\beta} : I_k|_{[0,s_0]} \to M_{n'}(C(D))$ by

$$\overline{\beta}(f)(r\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} w_r^*(\mathrm{e}^{\mathrm{i}\theta}) \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k' \times k'} & \\ & (\beta'' \otimes \mathrm{id}_k)(f)(\mathrm{e}^{\mathrm{i}\theta}) \end{pmatrix} w_r(\mathrm{e}^{\mathrm{i}\theta}), & \text{if } \frac{1}{2} \le r \le 1, \\ \begin{pmatrix} \begin{pmatrix} f(\underline{0}) & & \\ & \ddots & \\ & & f(\underline{0}) \end{pmatrix}_{k' \times k'} & \\ & (\beta_r \otimes \mathrm{id}_k)(f)(\mathrm{e}^{\mathrm{i}\theta}) \end{pmatrix}, & \text{if } 0 \le r \le \frac{1}{2}. \end{cases}$$

This homomorphism is as desired.

Proof of Theorem 3.1 From the above arguments, we have constructed

$$\psi : I_k \to PM_\bullet(C(X))P$$

with the property

$$\|\phi(f) - \psi(f)\| < \frac{\varepsilon}{3}$$

for all $f \in F$. And importantly, for each $x \in X$, $\sharp(\mathrm{Sp}\widetilde{\psi}|_x \cap \{0\})$ is a constant $k' \in \{1, 2, \cdots, k\}$ and $\sharp(\mathrm{Sp}\widetilde{\psi}|_x \cap \{1\})$ is also a constant $k_1' \in \{1, 2, \cdots, k\}$, where $\widetilde{\psi}$ is the composition

$$C[0,1] \hookrightarrow I_k \xrightarrow{\psi} PM_\bullet(C(X))P.$$

Let $h(t) = t \cdot \mathbf{1}_k \in I_k$ be the canonical function in the center of I_k. Then $\psi(h) \in PM_\bullet(C(X))P$ is a self-adjoint element. For each $x \in X$, denote the eigenvalues of $\psi(h)(x)$ by

$$0 \le \lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_{\mathrm{rank}(P)}(x) \le 1.$$

Then all $\lambda_i(x)$ are continuous functions from X to $[0,1]$. Furthermore,

$$\begin{aligned} &\lambda_1(x) = \lambda_2(x) = \cdots = \lambda_{k'}(x) = 0, \\ &0 < \lambda_{k'+1}(x) \le \lambda_{k'+2}(x) \le \cdots \le \lambda_{\mathrm{rank}(P)-k_1'}(x) < 1 \end{aligned}$$

and

$$\lambda_{\mathrm{rank}(P)-k_1'+1}(x) = \lambda_{\mathrm{rank}(P)-k_1'+2}(x) = \cdots = \lambda_{\mathrm{rank}(P)}(x) = 1.$$

Let

$$\xi_1 = \min_{x \in X} \lambda_{k'+1}(x) > 0, \quad \xi_2 = \max \lambda_{\mathrm{rank}(P)-k_1'}(x) < 1.$$

Then

$$\mathrm{Sp}\psi \subset \{0\} \cup [\xi_1, \xi_2] \cup \{1\}.$$

That is, ψ factors through as

$$I_k \to \mathbb{C} \oplus M_k(C[\xi_1, \xi_2]) \oplus \mathbb{C} \xrightarrow{\mathrm{diag}(\alpha_0, \psi_1, \alpha_1)} PM_\bullet(C(X))P,$$

where we identify $I_k|_{\{0\}} = \mathbb{C}$ and $I_k|_{\{1\}} = \mathbb{C}$.

Let $Q_0 = \alpha_0(1)$, $Q_1 = \alpha_1(1)$ and $P_1 = \psi_1(\mathbf{1}_{M_k(C([\xi_1, \xi_2]))})$. Finally, regarding ψ_1 as

$$M_k(C[0,1]) \xrightarrow{\text{restriction}} M_k(C([\xi_1, \xi_2])) \xrightarrow{\psi_1} P_1 M_\bullet(C(X)) P_1,$$

we finish the proof of Theorem 3.1.

3.16 From the definition of ψ in the above procedure, for every $x \in X$, the map

$$\psi|_x : I_k \xrightarrow{\psi} PM_\bullet(C(X))P \xrightarrow{\text{evaluate at } x} P(x)M_\bullet(\mathbb{C})P(x)$$

is defined when the construction of

$$\psi|_\Delta : I_k \to PM_\bullet(C(\Delta))P$$

is carried out for the unique simplex Δ such that $x \in \overset{\circ}{\Delta}$ (the interior of Δ). And when we define $\psi|_\Delta$ by modifying $\phi|_\Delta$, the only modifications are made on the two parts $\phi|_\Delta^{[0,s_0(\Delta)]}$ and $\phi|_\Delta^{[t_l(\Delta),1]}$. Consequently,

$$\mathrm{Sp}\phi|_x \cap (s_0(\Delta), t_l(\Delta)) = \mathrm{Sp}\psi|_x \cap (s_0(\Delta), t_l(\Delta))$$

as sets with multiplicity. On the other hand for any simplex Δ, $s_0(\Delta) < \frac{\eta}{2}$ and $t_{l(\Delta)}(\Delta) > 1-\frac{\eta}{2}$. Hence

$$\mathrm{Sp}\phi|_x \cap \Big[\frac{\eta}{2}, 1-\frac{\eta}{2}\Big] = \mathrm{Sp}\psi|_x \cap \Big[\frac{\eta}{2}, 1-\frac{\eta}{2}\Big].$$

If we further assume that ϕ has property $\mathrm{sdp}\big(\frac{\eta}{4}, \delta\big)$, then ψ has property $\mathrm{sdp}(\eta, \delta)$. As a consequence, we can use the decomposition theorem for

$$\psi_1 : M_k(C[0,1]) \to P_1 M_\bullet(C(X)) P_1$$

to study the homomorphisms $\phi, \psi : I_k \to PM_\bullet(C(X))P$. Note that the homomorphisms $f \mapsto f(\underline{0})Q_0$ and $f \mapsto f(\underline{1})Q_1$ factor through the C^*-algebra $\mathbb{C}$.

3.17 Lemma 3.3 is not true for the case $k' = 0$. In fact, there exists a unital homomorphism $\alpha : M_k(\mathbb{C}) \to M_k(C(S^1))$, which can not be extended to a homomorphism $\overline{\alpha} : M_k(\mathbb{C}) \to M_k(C(D))$. Let $\pi_{s_0} :\ I_k|_{[0,s_0]} \to M_k(\mathbb{C})$ be the map defined by evaluating at the point s_0. Then $\beta = \alpha \circ \pi_{s_0} :\ I_k|_{[0,s_0]} \to M_k(C(S^1))$ can not be extended to $\overline{\beta} :\ I_k|_{[0,s_0]} \to M_k(C(D))$ such that $\sharp\mathrm{Sp}(\overline{\beta} \circ \imath)_x \cap \{0\} = k' = 0$ for all $x \in D$, where $\imath$ is the canonical map from $M_k(\mathbb{C})$ to $I_k|_{[\zeta,s_0]}$ for some $0 < \zeta < s_0$.

4 Decomposition Theorem II

Our next task is to study the possible decomposition of $\phi : C(X) \to M_l(I_{k_2})$ for X being $[0,1]$, S^1 or $\mathrm{T}_{II,k}$. The cases of $[0,1]$, S^1 are more or less known (see [4, 9]). Let us assume that X is a 2-dimensional connected simplicial complex.

The following lemma is essentially due to Su [38]. The case of X =graph was stated in [27].

Lemma 4.1 *For any connected simplicial complex X, a finite set $F \subset C(X)$ which generates $C(X)$, $\eta > 0$ and a positive interger $n > 0$, there is a $\delta > 0$, such that for any two unital homomorphisms $\phi, \psi : C(X) \to M_n(\mathbb{C})$, if $\|\phi(f) - \psi(f)\| < \delta$ for all $f \in F$, then* $\mathrm{Sp}(\phi)$ *and* $\mathrm{Sp}(\psi)$ *can be paired within η.*

This is a consequence of [39, Lemmas 2.2–2.3]; also see [25, argument 2.1.3]. For the case of graphs, it was stated in [25, 2.1.9].

Lemma 4.2 *For any connected simplicial complex X, a finite generating set $F \subset C(X)$, $\varepsilon > 0$ and positive integer $n > 0$, there is $\delta > 0$ with the following property: If $x_1, x_2, \cdots, x_n \in X$ are n points (possibly repeating), $u, v \in M_n(\mathbb{C})$ are two unitaries such that*

$$\left\| u \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_n) \end{pmatrix} u^* - v \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_n) \end{pmatrix} v^* \right\| < \delta$$

for all $f \in F$, then there is a path of unitaries $u_t \in M_n(\mathbb{C})$ connecting u and v (i.e, $u_0 = u, u_1 = v$) with the property that

$$\left\| u_t \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_n) \end{pmatrix} u_t^* - u_{t'} \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_n) \end{pmatrix} u_{t'}^* \right\| < \varepsilon$$

for all $f \in F$ and $t, t' \in [0,1]$ (of course δ depends on both ε and n).

This was proved in the steps 2–3 of the proof of [39, Theorem 3.1].

The following lemma reduces the study of $\phi : C(X) \to M_l(I_k)$ to the study of homomorphism $\phi_1 : C(\Gamma) \to M_l(I_k)$, where $\Gamma \subset X$ is 1-skeleton of X under a certain simplicial decomposition. Since Γ is a graph, we will use the technique in [25–26] to obtain the decomposition of ϕ_1.

Lemma 4.3 *Let X be a 2-dimensional simplicial complex. For any $F \subset C(X)$, $\varepsilon > 0$, $\eta > 0$, and any unital homomorphism $\phi : C(X) \to M_l(I_k)$, there is a simpicial decomposition of X with 1-skeleton $X^{(1)} = \Gamma$ and a homomorphism $\phi_1 : C(\Gamma) \to M_l(I_k)$ such that*

(1) $\|\phi(f) - \phi_1 \circ \pi(f)\| < \varepsilon$, *where $\pi : C(X) \to C(\Gamma)$ is given by $\pi(f) = f|_\Gamma$.*

(2) *For any $t \in [0,1]$, $\mathrm{Sp}\phi|_t$ and $\mathrm{Sp}(\phi_1 \circ \pi)_t$ can be paired within η.*

Proof By Lemma 4.1, we only need to prove that there exists a homomorphism ϕ_1 to satisfy condition (1). Without loss of generality, we assume that F generates $C(X)$. By Lemma 4.2, there is an $\varepsilon' > 0$ such that for any $x_1, x_2, \cdots, x_{kl} \in X$ and unitaries $u, v \in M_{kl}(\mathbb{C})$, if

$$\left\| u \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_{kl}) \end{pmatrix} u^* - v \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_{kl}) \end{pmatrix} v^* \right\| < \varepsilon',$$

then there is a continuous path u_t with $u_0 = u, u_1 = v$ satisfying

$$\left\| u_t \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_{kl}) \end{pmatrix} u_t^* - u_{t'} \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_{kl}) \end{pmatrix} u_{t'}^* \right\| < \frac{\varepsilon}{3}.$$

Recall for the simplicial complex, a continuous path $\{x(t)\}_{0 \le t \le 1}$ is called piecewise linear if there is a sequence of points

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that $\{x(t)\}_{t_i \le t \le t_{i+1}}$ fall in the same simplex of X and are linear there. Note that the property of piecewise linear is preserved under any subdivision of the simplicial complex. For the simplicial complex X, we endow the standard metric on X, briefly described as below (see [18, 1.4.1] for detail). Identify each n-simplex with an n-simplex in $\mathbb{R}^n$ whose edges are of length 1, preserving affine structure of the simplexes. Such identifications give rise to a unique metric on the simplex Δ. For any two points $x, y \in X$, $d(x, y)$ is defined to be the length of the shortest path connecting x and y. (The length is measured in individual simplex, by breaking the path into small pieces.) With this metric, if $x_0, x_1 \in X$ with $d(x_0, x_1) = d$, then there is a piecewise linear path $x(t)$ with length d such that $x(0) = x_0,\ x(1) = x_1$. Furthermore, $d(x(t), x(t)') \leq d$ for all $t, t' \in [0, 1]$. In fact, we can choose $x(t)$, such that

$$d(x(t), x(t')) = |t' - t| \cdot d.$$

There is an $\eta' < \frac{\eta}{4}$ such that the following is true: For any $x, x' \in X$ with $d(x, x') < 2\eta'$,

$$|f(x) - f(x)'| < \frac{\varepsilon'}{3}.$$

Let $\delta > 0$, such that if $|t - t'| \leq \delta$, then

$$\|\phi(f)(t) - \phi(f)(t')\| < \frac{\varepsilon'}{3}, \quad \forall f \in F,$$

and $\mathrm{Sp}\phi_t$ and $\mathrm{Sp}\phi_{t'}$ can be paired within η'.

Divide the interval $[0, 1]$ into pieces $0 = t_0 < t_1 < t_2 < \cdots < t_\bullet = 1$, with $|t_{i+1} - t_i| < \delta$. We first define $\psi : C(X) \to M_l(I_k)$ such that ψ is close to ϕ on F within $\frac{\varepsilon}{3}$, $\mathrm{Sp}\phi_t$ and $\mathrm{Sp}\psi_t$ can be paired within η', and with extra property that on each interval $[t_i, t_{i+1}]$, $\mathrm{Sp}\psi_t = \{\alpha_1(t), \alpha_2(t), \cdots, \alpha_{lk}(t)\}$ with all $\alpha_j : [t_i, t_{i+1}] \to X$ being piecewise linear.

Set $\psi|_{\{t_i\}} = \phi|_{\{t_i\}}$ for each t_i $(i = 0, 1, 2, \cdots, \bullet)$— that is

$$\psi(f)(t_i) = \phi(f)(t_i) \quad \text{for } i = 0, 1, 2, \cdots, \bullet.$$

We will define $\psi|_{\{t\}}$ for $t \in (t_i, t_{i+1})$ by interpolating the definitions between $\psi|_{\{t_i\}}$ and $\psi|_{\{t_i+1\}}$. (Note that we do not change the definitions of $\phi|_{\{0\}}$ and $\phi|_{\{1\}}$, hence ψ is a homomorphism into $M_l(I_k)$ instead of $M_{lk}(C[0, 1])$.)

Let

$$\mathrm{Sp}\psi|_{\{t_i\}} = \{\alpha_1, \alpha_2, \cdots, \alpha_{lk}\} \subset X,$$
$$\mathrm{Sp}\psi|_{\{t_{i+1}\}} = \{\beta_1, \beta_2, \cdots, \beta_{lk}\} \subset X.$$

Since $\mathrm{Sp}\psi|_{\{t_i\}}$ and $\mathrm{Sp}\psi|_{\{t_{i+1}\}}$ can be paired within η', we can assume $\mathrm{dist}(\alpha_i, \beta_i) < \eta'$. There exist two unitaries $u, v \in M_{lk}(\mathbb{C})$ such that

$$\psi(f)(t_i) = u \begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl}) \end{pmatrix} u^*, \quad \psi(f)(t_{i+1}) = v \begin{pmatrix} f(\beta_1) & & \\ & \ddots & \\ & & f(\beta_{kl}) \end{pmatrix} v^*.$$

Noting that $\|f(\alpha_j) - f(\beta_j)\| < \frac{\varepsilon'}{3}$ for each j, we have

$$\left\| v \begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl}) \end{pmatrix} v^* - v \begin{pmatrix} f(\beta_1) & & \\ & \ddots & \\ & & f(\beta_{kl}) \end{pmatrix} v^* \right\| < \frac{\varepsilon'}{3}.$$

Combining with $\|\psi(f)(t_i) - \psi(f)(t_{i+1})\| < \frac{\varepsilon'}{3}$, we get

$$\left\| u \begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl}) \end{pmatrix} u^* - v \begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl}) \end{pmatrix} v^* \right\| < \frac{2\varepsilon'}{3}.$$

Since ε' is the number δ in Lemma 4.2 for $\frac{\varepsilon}{3}$, applying Lemma 4.2, there is a unitary path $u(t)$, $t_i \le t \le \frac{t_i+t_{i+1}}{2}$ with $u(t_i)=u$, $u(\frac{t_i+t_{i+1}}{2})=v$ such that

$$\left\| u(t)\begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl})\end{pmatrix} u^*(t) - u(t')\begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl})\end{pmatrix} u^*(t')\right\| < \frac{\varepsilon}{3}$$

for all $t, t' \in [t_i, \frac{t_i+t_{i+1}}{2}]$.

There are piecewise linear paths $r_i(t)$ with $r_i\big(\frac{t_i+t_{i+1}}{2}\big)=\alpha_i$ and $r_i(t_{i+1})=\beta_i$ such that

$$d(r_i(t), r_i(t')) \le \mathrm{dist}(\alpha_i,\beta_i) < \eta'.$$

Define $\psi(f)$ as follows: For $t\in[t_i, \frac{t_i+t_{i+1}}{2}]$,

$$\psi(f)(t) = u(t)\begin{pmatrix} f(\alpha_1) & & \\ & \ddots & \\ & & f(\alpha_{kl})\end{pmatrix} u^*(t);$$

for $t\in[\frac{t_i+t_{i+1}}{2}, t_{i+1}]$,

$$\psi(f)(t) = v\begin{pmatrix} f(r_1(t)) & & & \\ & f(r_2(t)) & & \\ & & \ddots & \\ & & & f(r_{kl}(t))\end{pmatrix} v^*.$$

Then $\{\mathrm{Sp}\psi_t, t\in[t_i,t_{i+1}]\}$ is a collection of kl piecewise linear maps from $[t_i,t_{i+1}]$ to X. (Note that for $t\in[t_i, \frac{t_i+t_{i+1}}{2}]$, we use constant maps which are linear.)

Now let's subdivide the simplicial complex X so that each simplex of the subdivision has diameter at most η', and so that all the points in $\mathrm{Sp}\phi|_{\{0\}} = \mathrm{Sp}\psi|_{\{0\}}$ and $\mathrm{Sp}\phi|_{\{1\}} = \mathrm{Sp}\psi|_{\{1\}}$ are vertices. With this simplicial decomposition we have $\mathrm{Sp}\psi\cap\Delta \subsetneqq \Delta$ for every 2-simplex Δ. This is true because $\mathrm{Sp}\psi|_{[t_i,t_{i+1}]}$ is the union of the collection of images of kl piecewise linear maps from $[t_i,t_{i+1}]$ to X, and a finite union of line segments must be 1-dimensional. Hence for each simplex Δ of dimension 2, we can choose a point $x_\Delta \in \mathring{\Delta}$, such that $x_\Delta \notin \mathrm{Sp}\psi$.

There is a $\sigma>0$ such that $\mathrm{Sp}\psi$ has no intersection with $\overline{B_\sigma(x_\Delta)} = \{x\in X, \mathrm{dist}(x,x_\Delta)\le\sigma\}$ for all Δ. Let $Y = X\backslash(\cup\{B_\sigma(x_\Delta) \mid \Delta \text{ is 2-simplex}\})$. Then $\mathrm{Sp}\psi\subset Y$. That is, ψ factors through $C(Y)$ as

$$\psi:\quad C(X)\xrightarrow{\text{restriction}} C(Y)\xrightarrow{\psi_1} M_l(I_k).$$

Let $\alpha: Y\to X^{(1)}$ be the standard retraction defined as a map sending $\Delta\backslash\{x_\Delta\}$ to $\partial\Delta$ for each simplex Δ. Then $d(x,\alpha(x))<\eta'$. Let $\phi_1: C(X^{(1)})\to M_l(I_k)$ be defined by

$$\psi_1\circ\alpha^*: C(X^{(1)})\xrightarrow{\alpha^*} C(Y)\xrightarrow{\psi_1} M_l(I_k).$$

Evidently ϕ_1 is as desired.

Corollary 4.1 *Suppose that $\phi: C(X)\to M_l(I_k)$ is a unital homomorphism. For any finite set $F\subset C(X)$, $\varepsilon>0$ and $\eta>0$, there is a unital homomorphism*

$$\psi: C(X)\to M_l(I_k)$$

such that

(1) $\phi(f)(0) = \psi(f)(0)$, $\phi(f)(1) = \psi(f)(1)$ *for all* $f \in C(X)$;

(2) $\|\phi(f) - \psi(f)\| < \varepsilon$ *for all* $f \in F$;

(3) $\mathrm{Sp}\phi_t$ *and* $\mathrm{Sp}\psi_t$ *can be paired within* η;

(4) *for each* $t \in (0,1)$, *the maximal multiplicity of* $\mathrm{Sp}\psi_t$ *is one — that is,* $\psi|_{\{t\}}$ *has distinct spectra.*

Proof Applying Lemma 4.3, we reduce the case of $C(X)$ to the case of $C(X^{(1)})$, where $X^{(1)}$ is a 1-dimensional simplicial complex. The corollary of this case is almost the same as the special case of [25, Theorem 2.1.6] (where we let $Y = [0,1]$). Note that from the proof of [25, Theorem 2.1.6], if we do not require the homomorphism ψ to have distinct spectrum at the end points 0 and 1, then we do not need to modify the original homomorphism ϕ at these two end points. The proof goes the same way as the proof there with some small modifications. We briefly describe them as below. One divides the interval $Y = [0,1]$ into small pieces $[0,1] = \bigcup_{i=0}^{m-1} [y^i, y^{i+1}]$ with $y^0 = 0 < y^1 < y^2 \cdots < y^m = 1$, as in the proof of [27, Theorem 2.1.6]. Define $\psi|_{y^i}$ with $1 \le i \le m-1$, by slightly modifying $\phi|_{y^i}$ so that $\psi|_{y^i}$ has distinct spectra; but define $\psi|_0 = \phi|_0$ and $\psi|_1 = \phi|_1$ (no modifications are made at the ending points). Therefore, in our case, $\psi|_0$ and $\psi|_1$ do not have distinct spectra—this is the only difference from [27, Theorem 2.1.6]. For all intervals $[y^i, y^{i+1}]$ with $1 \le i \le m-2$, the constructions of $\psi|_{[y^i,y^{i+1}]}$ are the same as in the proof of [27, Theorem 2.1.6]. For the constructions of $\psi|_{[0,y^1]}$ and $\psi|_{[y^{m-1},1]}$, we need to modify [27, Lemma 2.1.1] and [28, Lemma 2.1.2] accordingly, in an obvious way, and then apply these modifications. For example, [27, Lemma 2.1.1] should be modified to the following case: Among two l-element sets $X^0 = \{x_1^0, x_2^0, \cdots, x_l^0\}$ and $X^1 = \{x_1^1, x_2^1, \cdots, x_l^1\}$ — only one of them is distinct. That is, the following statement is true with the same proof.

Let $X = X_1 \vee X_2 \vee \cdots \vee X_k$ be a bunch of k intervals $X_i = [0,1]$ $(1 \le i \le k)$ and $Y = [0,1]$. Suppose that

$$X^0 = \{x_1^0, x_2^0, \cdots, x_l^0\} \subset X, \quad X^1 = \{x_1^1, x_2^1, \cdots, x_l^1\} \subset X$$

with $x_i^1 \ne x_j^1$ if $i \ne j$. Then there are l continuous functions $f_1, f_2, \cdots, f_l : Y \to X$ such that

(1) as sets with multiplicity, we have

$$\{f_1(0), f_2(0), \cdots, f_l(0)\} = X^0, \quad \{f_1(1), f_2(1), \cdots, f_l(1)\} = X^1,$$

(2) for each $t \in (0,1] \subset Y$ and $i \ne j$, we have

$$f_i(t) \ne f_j(t).$$

Remark 4.1 In Corollary 4.1, we can further assume that $\mathrm{Sp}\psi|_{\{0\}}$ and $\mathrm{Sp}\psi|_{\{1\}}$ have eigenvalue multiplicity just k as homomorphisms from $C(X)$ to $M_{lk}(C[0,1])$, or equivalently, both maps

$$C(X) \xrightarrow{\psi} M_l(I_k) \xrightarrow{\text{evaluate at } 0} M_l(\mathbb{C}), \quad C(X) \xrightarrow{\psi} M_l(I_k) \xrightarrow{\text{evaluate at } 1} M_l(\mathbb{C})$$

have distinct spectrum. To do this, we first extend the definition of the original ϕ to a slightly larger interval $[-\delta, 1+\delta]$ as below.

Find $u \in M_l(\mathbb{C})$ and $x_1, x_2, \cdots, x_l \in X$ such that

$$\phi(f)(0) = u \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_l) \end{pmatrix} u^* \otimes \mathbf{1}_k.$$

Since X is path connected and $X \neq \{pt\}$, there are functions $\alpha_i : [-\delta, 0] \to X$ such that $\{\alpha_i(-\delta)\}_{i=1}^l$ is a set of distinct l points, $\alpha_i(0) = x_i$, and $\mathrm{dist}(\alpha_i(t), \alpha_i(0))$ are as small as we want. Define

$$\phi(f)(t) = u \begin{pmatrix} f(\alpha_1(t)) & & \\ & \ddots & \\ & & f(\alpha_l(t)) \end{pmatrix} u^* \otimes \mathbf{1}_k \quad \text{for } t \in [-\delta, 0].$$

Similarly, we can define $\phi(f)(t)$ for $t \in [1, 1+\delta]$, so that $\phi|_{1+\delta}$ as a homomorphism from $C(X)$ to $M_{kl}(\mathbb{C})$ has multiplicity exactly k and $\phi(f)(1+\delta) \in M_l(\mathbb{C}) \otimes \mathbf{1}_k$. One can reparemetrize $[-\delta, 1+\delta]$ to $[0,1]$ so that $\phi|_0$ and $\phi|_1$ as homomorphisms from $C(X)$ to $M_{kl}(\mathbb{C})$ have multiplicity exactly k. Then we apply the corollary to perturb ϕ to ψ without changing the definition at the end points.

Remark 4.2 The same argument can be used to prove the following result. Let $X \neq \{pt\}$ be a connected finite simplicial complex of any dimension. Let Y be a 1-dimensional simplicial complex. Then any homomorphism $\phi : C(X) \to M_n(C(Y))$ can be approximated arbitrarily well by a homomorphism ψ with distinct spectrum. This is a strengthened form of [18, Theorem 2.1] for the case $\dim(Y) = 1$.

The following theorem, for $X =$ gragh, is a slight modification of [29, Theorem 2.7].

Theorem 4.1 *Let X be a connected simplicial complex of dimension at most 2, and $G \subset C(X)$ be a finite set which generates $C(X)$. For any $\varepsilon > 0$, there is an $\eta > 0$ such that the following statement is true.*

Suppose that $\phi : C(X) \to M_{l_1 l_2 + r}(I_k)$ is a unital homomorphism satisfying the following condition: There are l_1 continuous maps

$$a_1, a_2, \cdots, a_{l_1} : [0,1] (= \mathrm{Sp}(I_k)) \to X$$

such that for every $y \in [0,1]$, $\mathrm{Sp}\phi_y$ (considered as a homomorphism from $C(X)$ to $M_{(l_1 l_2 + r)k}(C[0,1])$) and $\Theta(y)$ can be paired within η, where

$$\Theta(y) = \{a_1(y)^{\sim l_2 k}, a_2(y)^{\sim l_2 k}, \cdots, a_{l_1 - 1}(y)^{\sim l_2 k}, a_{l_1}(y)^{\sim (l_2 + r)k}\}.$$

It follows that there are l_1 mutually orthogonal projections $p_1, p_2, \cdots, p_{l_1} \in M_{l_1 l_2 + r}(I_k)$ such that

(i) *for all $g \in G$ and $y \in Y$,*

$$\left\| \phi(g)(y) - p_0 \phi(g)(y) p_0 \oplus \sum_{k=1}^{l_1} g(a_k(y)) p_k \right\| < \varepsilon,$$

where $p_0 = 1 - \sum_{i=1}^{l_1} p_i$;

(ii) $\mathrm{rank}(p_i) = (l_2 - 3)k$ *for* $1 \le i < l_1$, $\mathrm{rank}(p_{l_1}) = (l_2 + r - 3)k$ *(as projections in $M_{(l_1 l_2 + r)k}(C[0,1])$) and* $\mathrm{rank}(p_0) = 3 l_1 k$.

Proof We will apply [29, Theorem 2.7] (using map a_i to replace map $b \circ a_i$ as in [29, Remark 2.8]) and its proof (see [27, 2.9–2.16]) for the case Y in [29, Theorem 2.7] being [0,1]. As a matter of fact, in the proof of [29, Theorem 2.7], Li does use the fact that X is a graph, for only one property that any homomorphism from $C(X)$ to $M_nC(Y)$ (Y graph) can be approximated arbitrarily well by homomorphisms with distinct spectra. By Remark 4.2, [29, Theorem 2.7] holds for the case $X \neq \{pt\}$ being any connected simplicial complex and Y, a graph.

For finite set $G \subset C(X)$, and $\varepsilon > 0$, choose $\eta > 0$ such that $\mathrm{dist}(x_1, x_2) \leq \eta$ implies $|g(x_1) - g(x_2)| < \frac{\varepsilon}{4}$ for all $g \in G$, as in [29, 2.16]. Without lose of generality, we can assume that the $\mathrm{Sp}\phi|_t$ is distinct for any $t \in (0,1)$ and $\mathrm{Sp}\phi|_0$ and $\mathrm{Sp}\phi|_1$ have multiplicities k as in Corollary 4.1 and Remark 4.1 above. When we go through Li's proof in [29], we need to make the projections p_i satisfy the extra condition:

$$p_i(0), p_i(1) \in (M_{l_1 l_2 + r}(\mathbb{C})) \otimes \mathbf{1}_k \subseteq M_{(l_1 l_2 + r)k}(\mathbb{C}).$$

We will repeat part of the proof of [29, Theorem 2.7] and point out how to modify it.

As in the proof of [29, Theorem 2.7], we can choose an open cover $U_0, U_1, \cdots, U_\bullet$ of $[0,1]$ with

$$U_0 = [0, b_0), U_1 = (a_1, b_1), U_2 = (a_2, b_2), \cdots, U_{\bullet-1} = (a_{\bullet-1}, b_{\bullet-1}), U_\bullet = (a_\bullet, 1],$$

$$0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \cdots < a_\bullet < b_{\bullet-1} < 1.$$

We will define $P_U^i (i = 1, 2, \cdots, l_1)$ as same as in [29, 2.12] for $U = U_i$ $(0 < i < \bullet)$—note that $\mathrm{Sp}\phi_y$, for $y \in (a_1, b_{\bullet-1}) \subset (0,1)$, are distinct. For U_0 and $U_\bullet$, a special care is needed as follows. We will only do it for U_0 (it is the same for $U_\bullet$). Write $\mathrm{Sp}\phi|_0 = \{\lambda_1^{\sim k}, \lambda_2^{\sim k}, \cdots, \lambda_q^{\sim k}\}$ with $q = l_1 l_2 + r$. Then $\{\lambda_1, \lambda_2, \cdots, \lambda_q\}$ can be paired with $\{a_0(0)^{\sim l_2}, a_2(0)^{\sim l_2}, \cdots, a_{l_1-1}(0)^{\sim l_2}, a_{l_1}(0)^{\sim(l_2+r)}\}$ (note that $\sim l_2 k$ is changed to $\sim l_2$ here) within η. We can divide $\{\lambda_1, \lambda_2, \cdots, \lambda_q\}$ into groups $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} = \bigcup_{j=1}^{l_1} E'^j$ (where $|E'^j| = l_2$ if $1 \leq j \leq l_1 - 1$, and $|E'^j| = l_2 + r$ if $j = l_1$) such that $\mathrm{dist}(\lambda_i, a_j(0)) < \eta$ for all $\lambda_i \in E^j$.

Let σ' satisfy the following conditions:

(1) $\sigma' < \min\{\mathrm{dist}(\lambda_i, \lambda_j),\ i \neq j\}$;

(2) $\sigma' < \eta - \max\{\mathrm{dist}(\lambda_i, a_j(0)),\ \lambda_i \in E^j\}$.

We can choose $b_1 (> b_0 > a_1 > 0)$ being so small that for any $y \in [0, b_1]$, $\mathrm{Sp}\phi_y$ and $\mathrm{Sp}\phi_0$ can be paired within $\frac{\sigma'}{2}$ and $\mathrm{dist}(a_j(y), a_j(0)) < \frac{\sigma'}{2}$. Then for each $y \in [0, b_1]$, $\mathrm{Sp}\phi_y$ can be written as a set of

$$\{\lambda_1^1(y), \lambda_1^2(y), \cdots, \lambda_1^k(y), \lambda_2^1(y), \lambda_2^2(y), \cdots, \lambda_2^k(y), \cdots, \lambda_q^1(y), \cdots, \lambda_q^k(y)\}$$

with $\lambda_i^j(0) = \lambda_i$. Then let $E^j(y)$ be the set $\{\lambda_i^{i'}(y);\ \lambda_i \in E'^j\}$. In this way we have, if $\lambda_i^{i'} \in E^j$, then

$$\mathrm{dist}(\lambda_i^{i'}(y), a_j(y)) < \eta.$$

Let both $P_{U_0}^j(y)$ and $P_{U_1}^j(y)$ (defined on $U_0 = [0, b_0)$ and $U_1 = (a_1, b_1)$) be the spectral projections corresponding to $E_j(y)$. In particular, $P_{U_0}^j(0) \in M_{l_1 l_2 + r}(\mathbb{C}) \otimes \mathbf{1}_k$. We can define $p_j(y)$ as a subprojection of $P_U^j(y)$ (for $U \ni y$) as in [27, 2.9–2.16] for each $y \in [b_0, a_\bullet]$ but with rank $(p_j(y)) = (l_2 - 3)k$ (instead of $l_2 - 3$ in [29]) for $1 \leq j \leq l_1 - 1$ and $\mathrm{rank}(p_{l_1}(y)) = (l_2 + r - 3)k$ (instead of $l_2 + r - 3$ in [29]). Also we can choose an arbitrary subprojection $p_j(0) < P_{U_0}^j(0) \in$

$M_{l_1 l_2+r}(\mathbb{C}) \otimes \mathbf{1}_k$ of form $p_j(0) = p'_j(0) \otimes \mathbf{1}_k \in M_{l_1 l_2+r}(\mathbb{C}) \otimes \mathbf{1}_k$ with $\mathrm{rank}(p'_j(0)) = l_2 - 3$ for $1 \le j \le l_1 - 1$, and $\mathrm{rank}(p'_{l_1}(0)) = l_2 + r - 3$. Consequently,

$$\mathrm{rank}(p_j(0)) = (l_2 - 3)k, \quad \mathrm{rank}(p_{l_1}(0)) = (l_2 + r - 3)k.$$

Finally, connect $p_j(0)$ and $p_j(b_0)$ by $p_j(y)$ for $y \in [0, b_0]$ inside $P^j_{U_0}(y)$. As one can see from [27, 2.16], if the projections $p_j(y)$ are subprojections of $P^j_U(y)$, then all the estimations in that proof hold. After we do similar modifications for $P^j_{U_\bullet}(y)$ and $p_j(y)$ near point 1, we will get $p_j(y) \in M_{l_1 l_2+r}(I_k)$ instead of $M_{(l_1 l_2+r)k}(C[0,1])$. (This method was also used in the proof of [13, Theorem 3.10].)

The following result is a generalization of [18, Proposition 4.42].

Theorem 4.2 *Let X be a connected finite simplicial complex of dimension at most 2, $\varepsilon > 0$ and $F \subset C(X)$, a finite set of generators. Suppose that $\eta \in (0, \varepsilon)$ satisfies that if $\mathrm{dist}(x, x') \le 2\eta$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{4}$ for all $f \in F$.*

For any $\delta > 0$ and positive integer $J > 0$, there exists an integer $L > 0$ and a finite set $H \subseteq AffTC(X)(= C_{\mathbb{R}}(X))$ such that the following holds.

If $\phi, \psi : C(X) \to B = M_K(I_k)$ (or $B = PM_\bullet(C(Y))P$) are unital homomorphisms with the properties:

(a) *ϕ has* $\mathrm{sdp}\big(\frac{\eta}{32}, \delta\big)$;

(b) *$K \ge L$ (or* $\mathrm{rank}(P) \ge L$*);*

(c) *$\|AffT\phi(h) - AffT\psi(h)\| < \frac{\delta}{4}$ for all $h \in H$,*

then there are three orthogonal projections $Q_0, Q_1, Q_2 \in B$, two homomorphisms $\phi_1 \in \mathrm{Hom}(C(X), Q_1 B Q_1)_1$ and $\phi_2 \in \mathrm{Hom}(C(X), Q_2 B Q_2)_1$, and a unitary $u \in B$ such that

(1) $\mathbf{1}_B = Q_0 + Q_1 + Q_2$;

(2) *$\|\phi(f) - (Q_0\phi(f)Q_0 + \phi_1(f) + \phi_2(f))\| < \varepsilon$ and*
$\|(Adu \circ \psi)(f) - (Q_0(Adu \circ \psi)(f)Q_0 + \phi_1(f) + \phi_2(f))\| < \varepsilon$ for all $f \in F$;

(3) *ϕ_2 factors through $C[0,1]$;*

(4) *$Q_1 = p_1 + \cdots + p_n$ with $(rank(Q_0) + 2)J < \mathrm{rank}(p_i)$ $(i = 1, 2, \cdots, n)$, where* rank*: $K_0(B) \to \mathbb{Z}$ is the map induced on K_0 by the evaluation map at 0 or 1 (which is* rank $p_i(\underline{0})$ *for $B = M_K(I_k)$, where* rank $p_i(\underline{0})$ *is regarded as projections in $M_K(\mathbb{C})$ not $M_K(M_k(\mathbb{C}))$), and ϕ_1 is defined by*

$$\phi_1(f) = \sum_{i=1}^{n} f(x_i) p_i, \quad \forall f \in C(X),$$

where $p_1, p_2, \cdots, p_n$ are mutually orthogonal projections and $\{x_1, x_2, \cdots, x_n\} \subset X$ is an ε-dense subset of X.

Proof For the case $B = PM_\bullet(C(Y))P$, this is [18, Proposition 4.42]. The proof for the case $B = M_K(I_k)$ is almost the same as the proof of [18, Proposition 4.42], replacing [18, Theorem 4.1] by Theorem 4.1 above. The only thing one should notice is that, in [18, Lemma 4.33], $\mathrm{rank}\,\phi(1) = K$; the K should be corresponding to K in our theorem (not Kk) and $\Theta(y)$ should be defined as

$$\Theta(y) = \{\alpha \circ \beta_1(y)^{\sim L_2 k}, \alpha \circ \beta_2(y)^{\sim L_2 k}, \cdots, \alpha \circ \beta_{L-1}(y)^{\sim L_2 k}, \alpha \circ \beta_L(y)^{\sim (L_2+L_1)k}\}.$$

(Note that in the above, we use $\sim L_2 k$ and $\sim(L_2 + L_1)k$ to replace $\sim L_2$ and $\sim(L_2 + L_1)$ in [18].) In the proof of this version of [18, Lemma 4.33], one can choose the homomorphism

$\psi' : C(X) \to M_k(C[0,1])$ (not to $M_{Kk}(C[0,1])$) as the map ψ there, with

$$\|AffT\phi(f) - AffT\psi'(f)\| < \frac{\delta}{4}, \quad \forall f \in H(\eta, \delta, x)$$

as in [18, Lemma 4.33]. Then let $\psi = \psi' \otimes \imath_k$, where $\imath_k : \mathbb{C} \to M_k(\mathbb{C})$ is defined by $\imath_k(\lambda) = \lambda \cdot \mathbf{1}_k$. With this modification, we have that $\mathrm{Sp}\psi'_y$ is

$$\Theta'(y) = \{\alpha \circ \beta_1(y)^{\sim L_2}, \alpha \circ \beta_2(y)^{\sim L_2}, \cdots, \alpha \circ \beta_{L-1}(y)^{\sim L_2}, \alpha \circ \beta_L(y)^{\sim (L_2+L_1)}\},$$

and $\mathrm{Sp}\psi_y$ is

$$\Theta(y) = \{\alpha \circ \beta_1(y)^{\sim L_2k}, \alpha \circ \beta_2(y)^{\sim L_2k}, \cdots, \alpha \circ \beta_{L-1}(y)^{\sim L_2k}, \alpha \circ \beta_L(y)^{\sim (L_2+L_1)k}\}$$

as desired. All other parts of the proof are exactly the same.

For the proof of uinqueness theorem in [19], it is important to have a simultaneous decomposition for two homomorphisms as below.

Theorem 4.3 *Let X be a connected finite simplicial complex of dimension at most 2, $\varepsilon > 0$ and $F \subset C(X)$, a finite set of generators. Suppose that $\eta \in (0, \varepsilon)$ satisfies that if $\mathrm{dist}(x, x') \le 2\eta$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{4}$ for all $f \in F$. Let κ be a fixed simplicial structure of X.*

For any $\delta > 0$ and positive integer $J > 0$, there exists an integer $L > 0$ and a finite set $H \subseteq AffTC(X)(= C_{\mathbb{R}}(X))$ such that the following holds.

If X_1 is a connected sub-complex of (X, κ), and if $\phi, \psi : C(X_1) \to B = M_K(I_k)$ (or $B = PM_\bullet(C(Y))P$) are unital homomorphisms with the following properties:

(a) *ϕ has* $\mathrm{sdp}\big(\frac{\eta}{32}, \delta\big)$;

(b) *$K \ge L$ (or* $\mathrm{rank}(P) \ge L$*)*;

(c) *$\|AffT\phi(h|_{X_1}) - AffT\psi(h|_{X_1})\| < \frac{\delta}{4}$ for all $h \in H$,*

then there are three orthogonal projections $Q_0, Q_1, Q_2 \in B$, two homomorphisms $\phi_1 \in \mathrm{Hom}(C(X_1), Q_1BQ_1)_1$ and $\phi_2 \in \mathrm{Hom}(C(X_1), Q_2BQ_2)_1$, and a unitary $u \in B$ such that

(1) $\mathbf{1}_B = Q_0 + Q_1 + Q_2$;

(2) *$\|\phi(f|_{X_1}) - (Q_0\phi(f|_{X_1})Q_0 + \phi_1(f|_{X_1}) + \phi_2(f|_{X_1}))\| < \varepsilon$ and $\|(Adu \circ \psi)(f|_{X_1}) - (Q_0(Adu \circ \psi)(f|_{X_1})Q_0 + \phi_1(f|_{X_1}) + \phi_2(f|_{X_1}))\| < \varepsilon$ for all $f \in F$;*

(3) *ϕ_2 factors through $C[0,1]$;*

(4) *$Q_1 = p_1 + \cdots + p_n$ with $(\mathrm{rank}(Q_0) + 2)J < \mathrm{rank}(p_i)$ $(i = 1, 2, \cdots, n)$, and ϕ_1 is defined by*

$$\phi_1(f) = \sum_{i=1}^{n} f(x_i)p_i, \quad \forall f \in C(X),$$

where $p_1, p_2, \cdots, p_n$ are mutually orthogonal projections and $\{x_1, x_2, \cdots, x_n\} \subset X_1$ is an ε-dense subset of X_1.

Proof Suppose that $\{X_i\}_i$ are all connected sub-complexes of (X, κ) (there are finitely many of them for a fixed simplicial structure of a finite complex). Use Theorem 4.1 to each X_i to obtain L_i and $H_i \subseteq AffT(C(X_i))$ as in the theorem. By Tietze Extension theorem, there are finite sets $\widetilde{H}_i \subseteq AffT(C(X))$ such that $H_i \subseteq \{h|_{X_i} \mid h \in \widetilde{H}_i\}$. Evidently $L = \max_i\{L_i\}$ and $H = \bigcup_i \widetilde{H}_i$ are as desired.

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